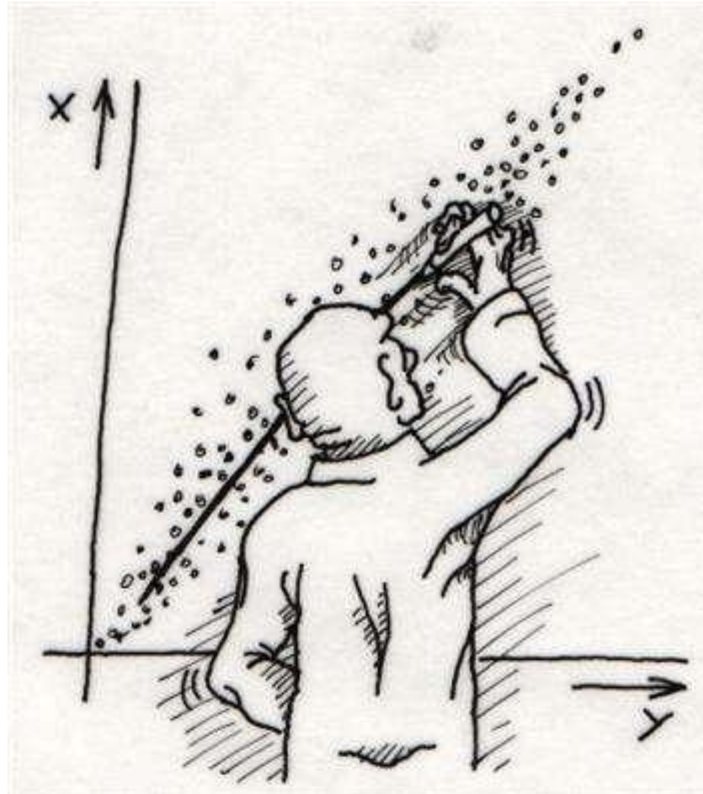

Lecture 2: Multiple Linear Regression

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AGENDA

Multiple Linear Regression

- Regression Example: Predict the selling price of Toyota Corolla


Dependent variable (target)

Independent variables (attributes, features)

Variable	Description
Price	Offer Price in EUROs
Age_08_04	Age in months as in August 2004
KM	Accumulated Kilometers on odometer
Fuel_Type	Fuel Type (Petrol, Diesel, CNG)
HP	Horse Power
Met_Color	Metallic Color? (Yes=1, No=0)
Automatic	Automatic ((Yes=1, No=0)
CC	Cylinder Volume in cubic centimeters
Doors	Number of doors
Quarterly_Tax	Quarterly road tax in EUROs
Weight	Weight in Kilograms

Multiple Linear Regression

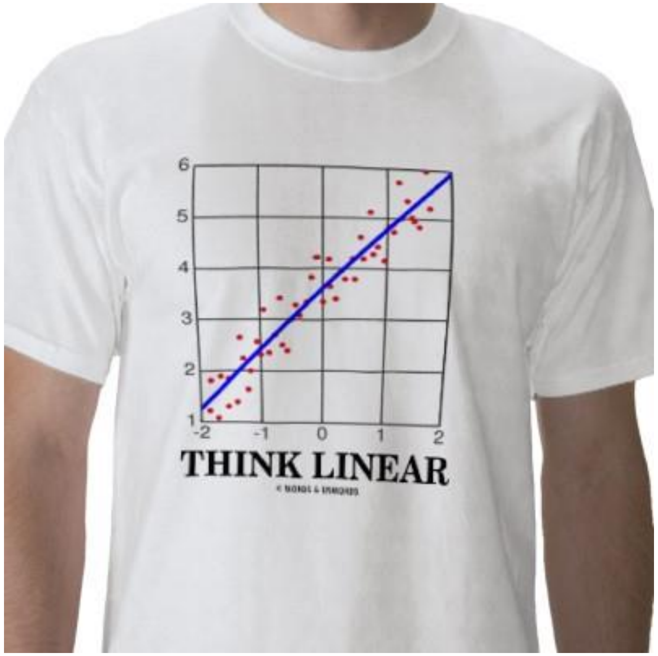
- Goal
  - ✓ Fit a linear relationship between a quantitative dependent variable Y and a set of predictors $X_1, X_2, \ldots, X_p$.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \cdots + \beta_d x_d + \epsilon$$

unexplained

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 \cdots + \hat{\beta}_d x_d$$

coefficients



Multiple Linear Regression

- Explanatory vs. Predictive

Explanatory Regression

- Explain relationship between predictors (explanatory variables) and target.
- Familiar use of regression in data analysis.
- Model Goal: Fit the data well and understand the contribution of explanatory variables to the model.
- "goodness-of-fit": R^2, residual analysis, p-values.

$$Y = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p + \varepsilon$$

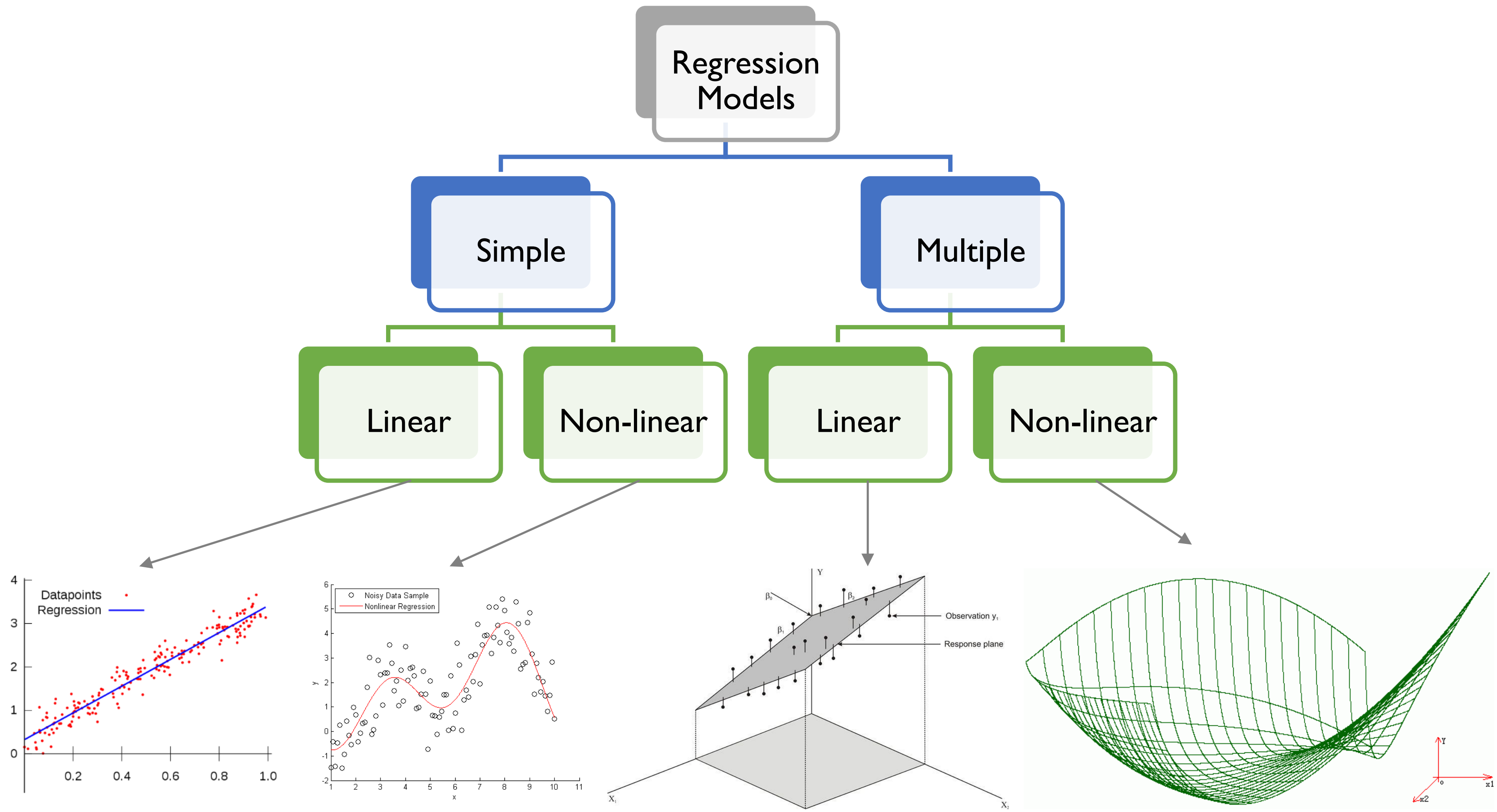
Predictive Regression

- Predict target values in other data where we have predictor values, but not target values.
- Classic data mining context
- Model Goal: Optimize predictive accuracy
- Train model on training data
- Assess performance on validation (hold-out) data
- Explaining role of predictors is not primary purpose (but useful)

$$Y = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p + \varepsilon$$

Multiple Linear Regression

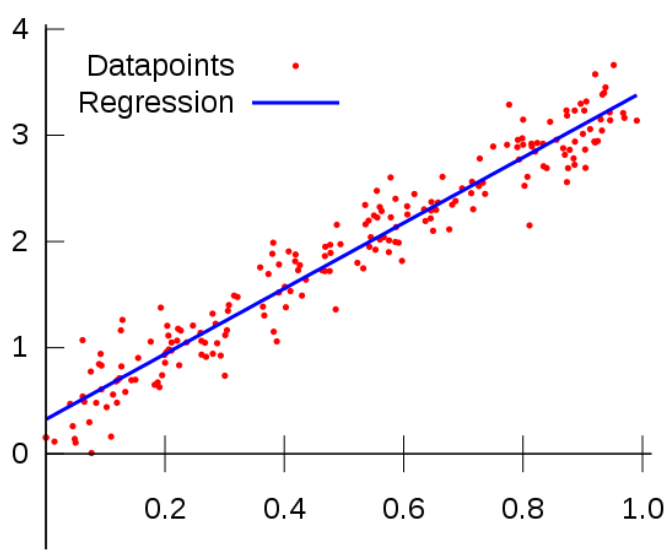
- Type of Regression

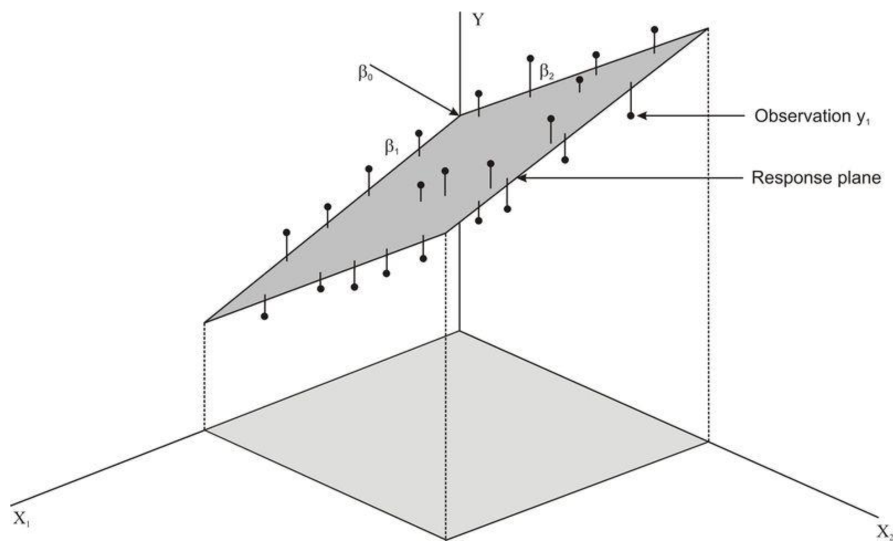


Multiple Linear Regression

- Linear Regression
  - ✓ Assume that the relationship between the input variable and the target variable is always linear.

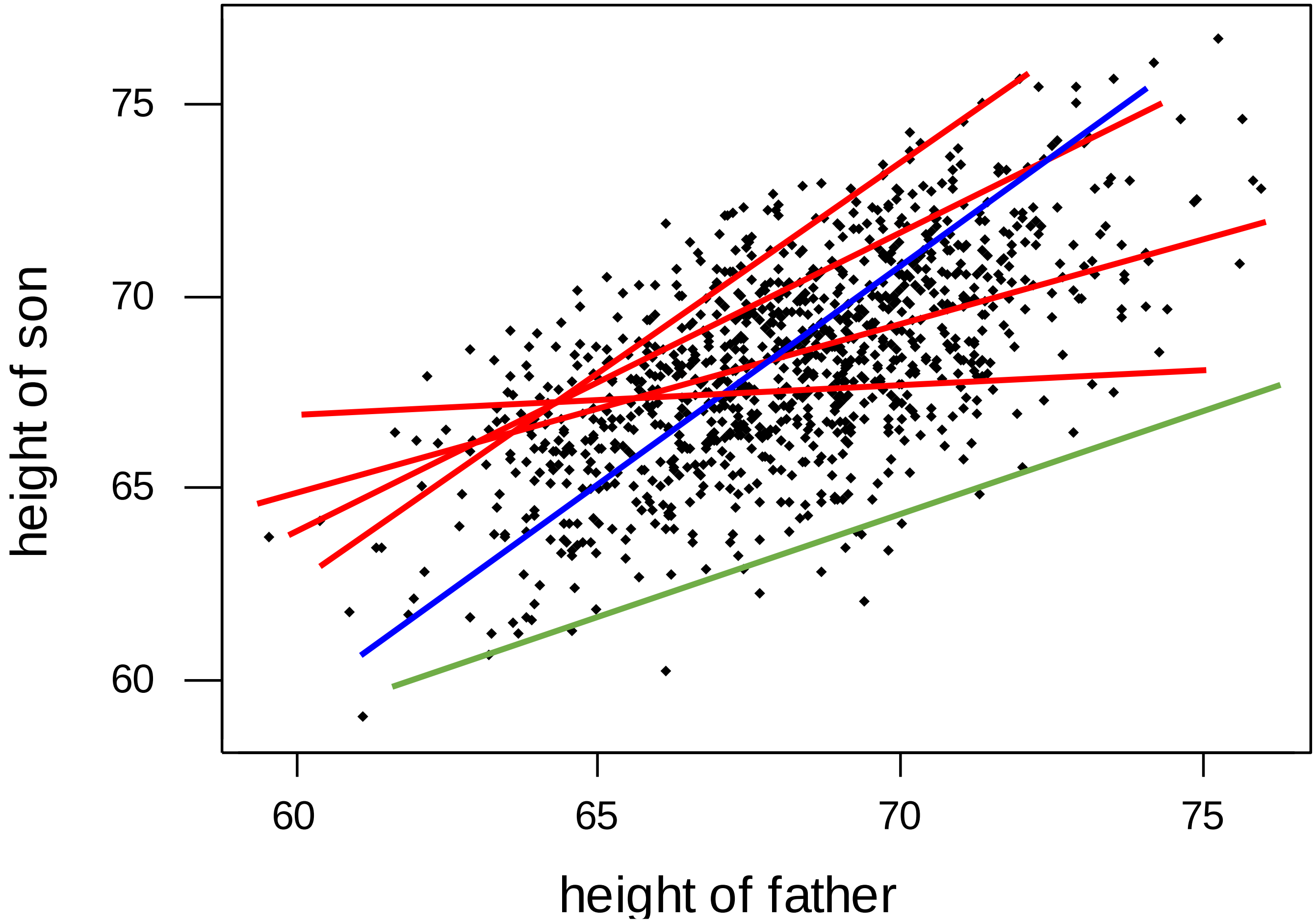
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 \cdots + \hat{\beta}_d x_d$$





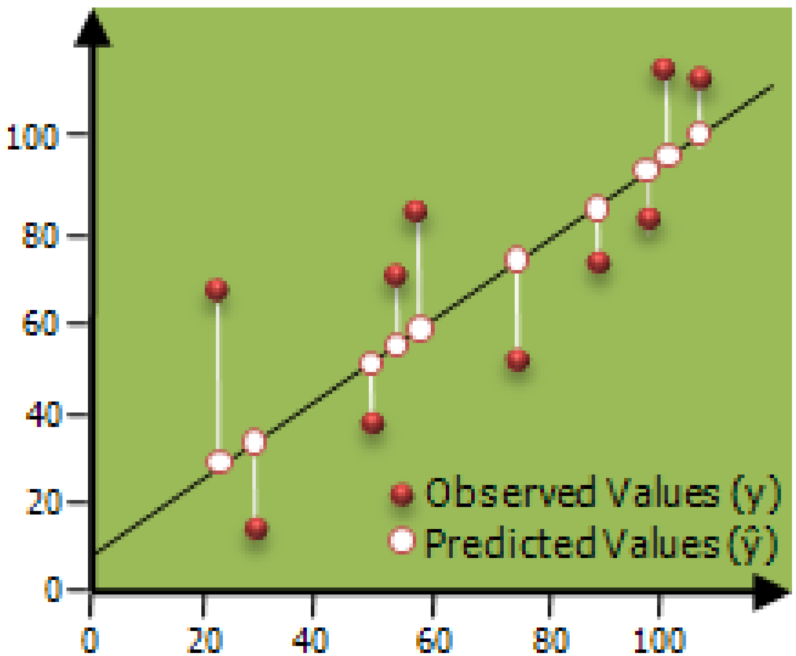
Multiple Linear Regression

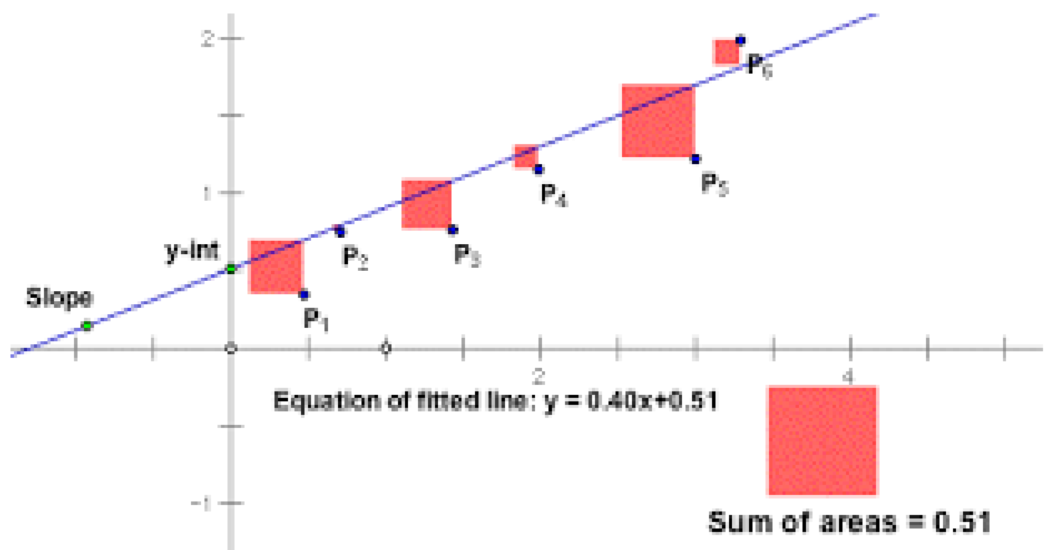
- Which line is optimal?



Multiple Linear Regression

- Estimating the coefficients
  - ✓ Ordinary least square (OLS): Minimize the squared difference between the actual target value and the estimated value by the regression model





Multiple Linear Regression

- Estimating the coefficients
  - ✓ Ordinary least square (OLS)
    - Actual target: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \cdots + \beta_d x_d + \epsilon$
    - Predicted target: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 \cdots + \hat{\beta}_d x_d$
    - Goal: minimize the difference between the actual and predicted target.

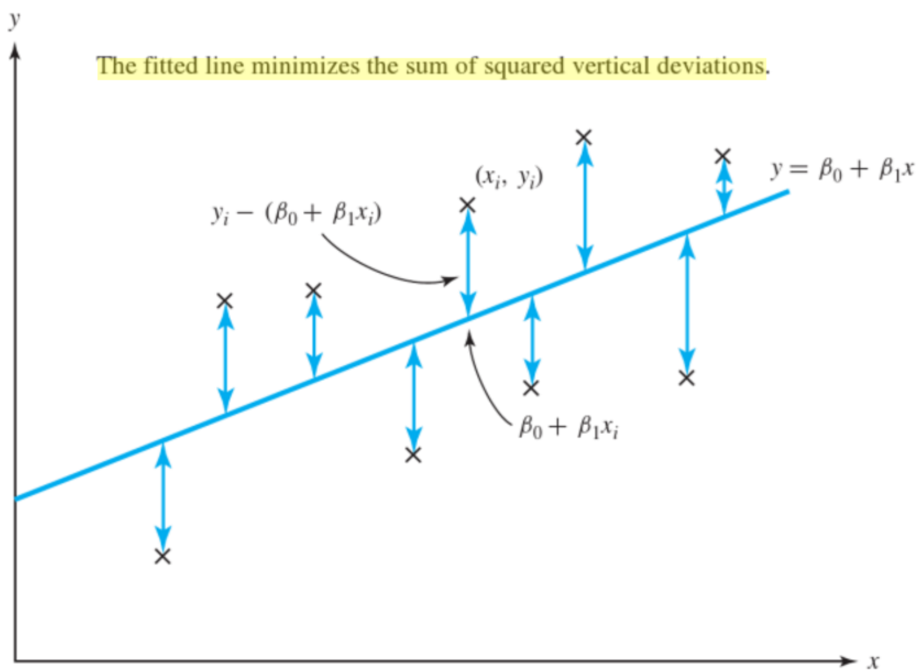
$$\min \frac{1}{2} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

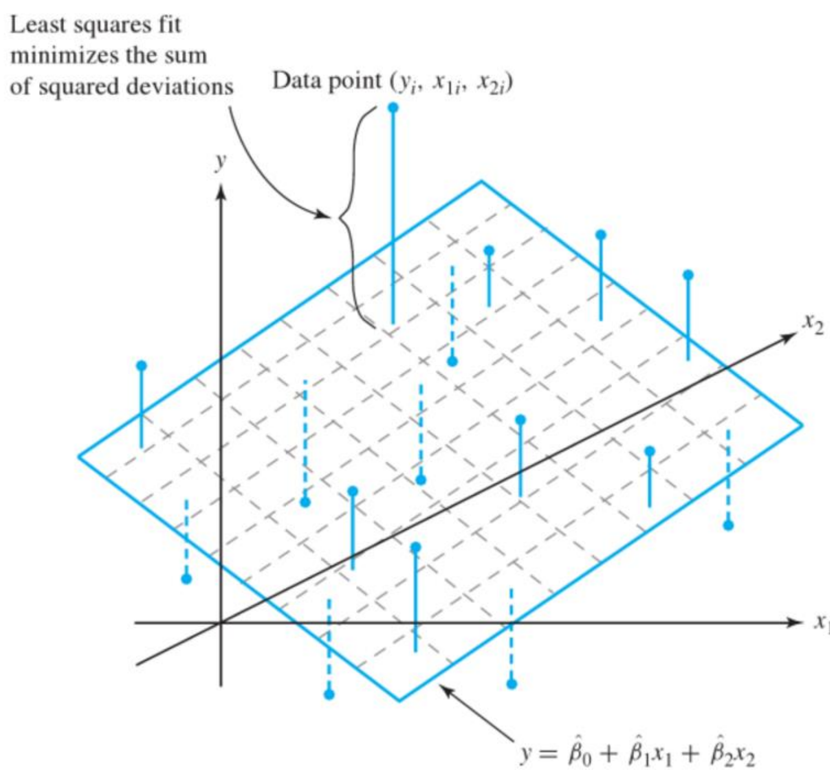
$$= \frac{1}{2} (y_i - \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} \cdots + \hat{\beta}_d x_{id})^2$$

Multiple Linear Regression

- Estimating the coefficients

  ✓ Ordinary least square (OLS)



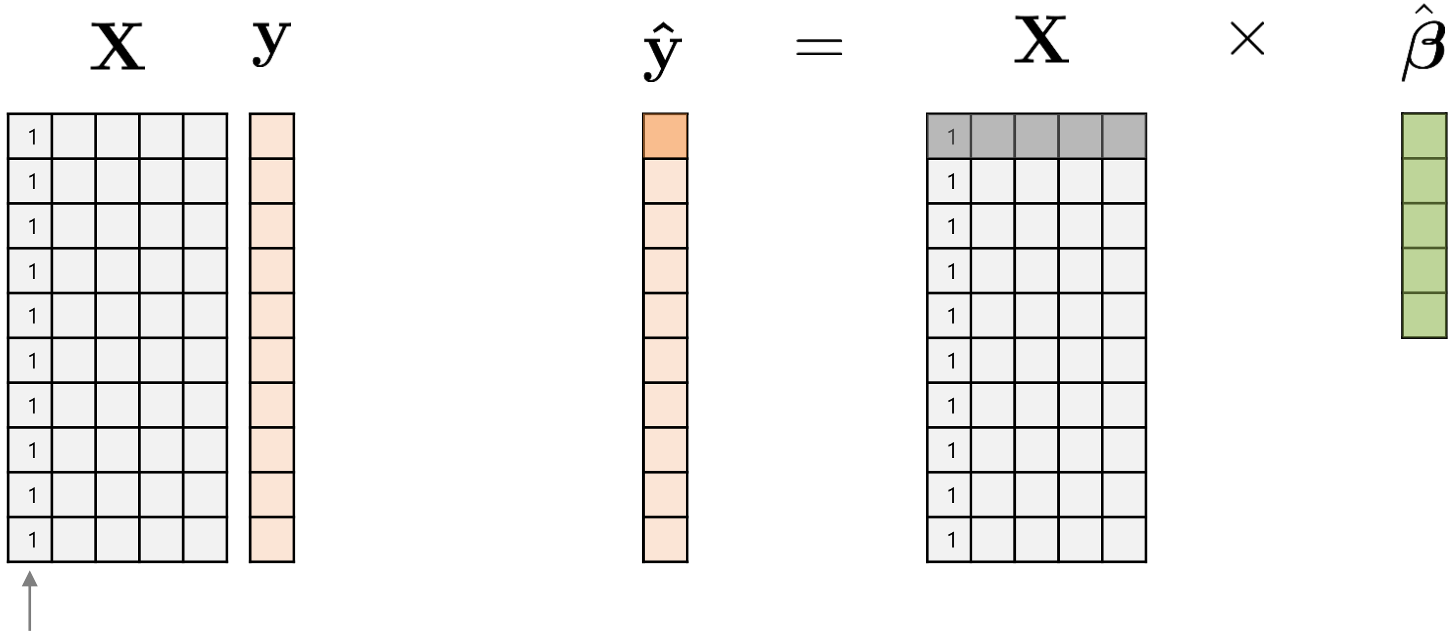


Multiple Linear Regression

- Ordinary least square: Matrix solution

$$\mathbf{X} : \; n \times (d+1) \; matrix, \;\; \mathbf{y} : \; n \times 1 \; vector$$

$$\hat{\boldsymbol{\beta}} : (d+1) \times 1 \; vector$$



For intercept

Multiple Linear Regression

- Ordinary least square: Matrix solution

$$\mathbf{X}: \;\; n \times (d+1) \; matrix, \;\; \mathbf{y}: \;\; n \times 1 \; vector$$

$$\hat{\boldsymbol{\beta}}: (d+1) \times 1 \; vector$$

$$\min E(\mathbf{X}) = \frac{1}{2}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right)^T \left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right)$$

$$\Rightarrow \frac{\partial E(\mathbf{X})}{\partial \hat{\boldsymbol{\beta}}} = -\mathbf{X}^T \left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right) = 0$$

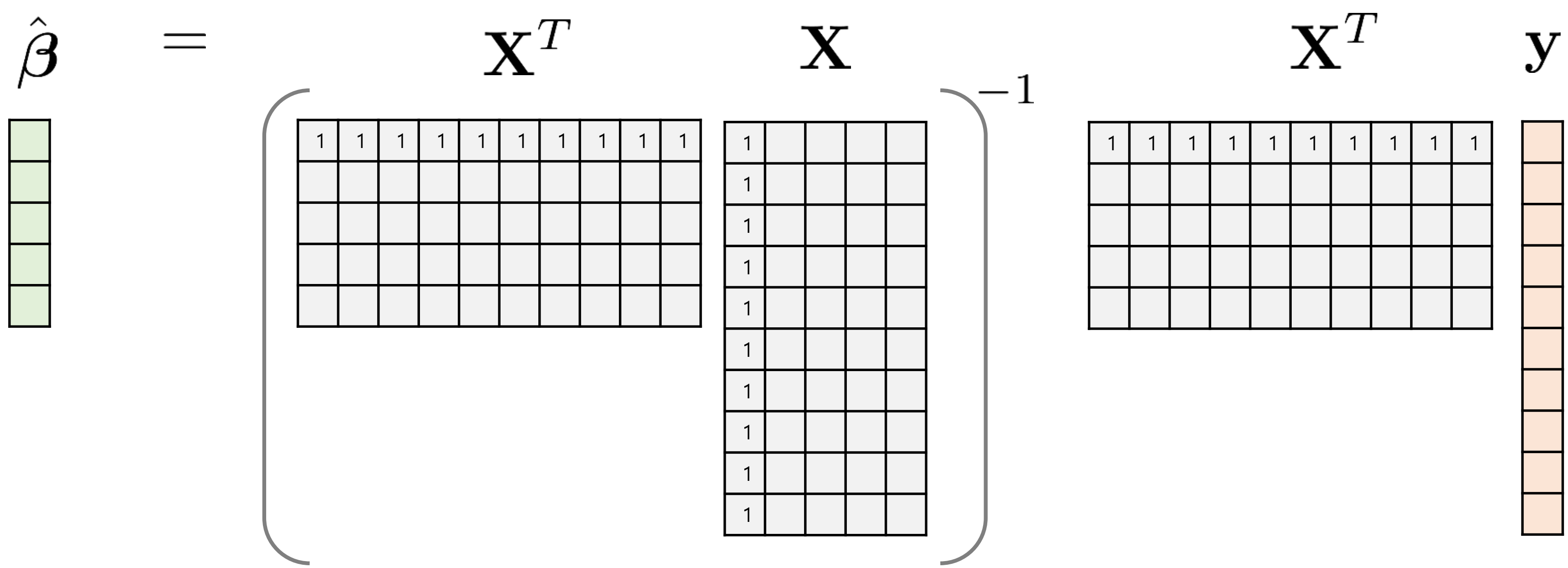
$$\Rightarrow \mathbf{X}^T\mathbf{y} + \mathbf{X}^T\mathbf{X}\hat{\boldsymbol{\beta}} = 0$$

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y} \longrightarrow$$ Unique and explicit solution exists!

Multiple Linear Regression

- Ordinary least square: Matrix solution

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$



Closed form solution for the regression coefficient

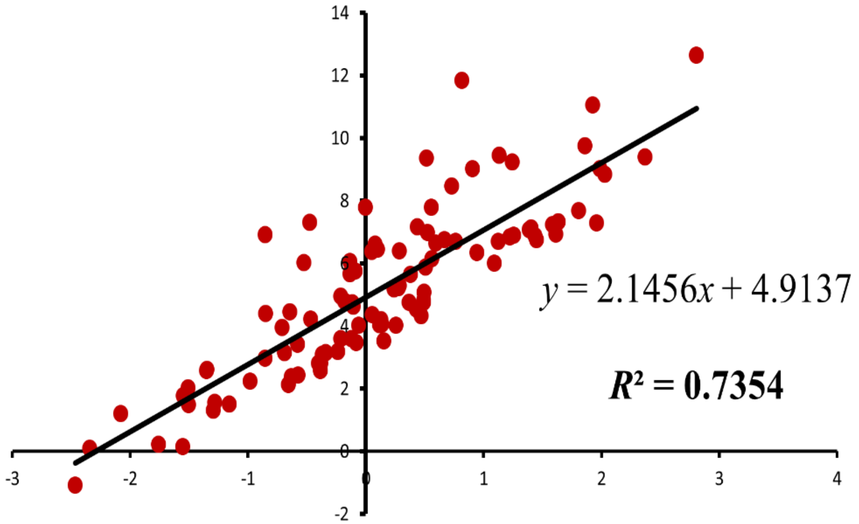
Multiple Linear Regression

- Ordinary least square
  - ✓ Finds the best estimates **β** when the following conditions are satisfied:
    - The noise ε follows a normal distribution.
    - The linear relationship is correct.
    - The cases are independent of each other.
    - The variability in Y values for a given set of predictors is the same regardless of the values of the predictors (homoskedasticity).

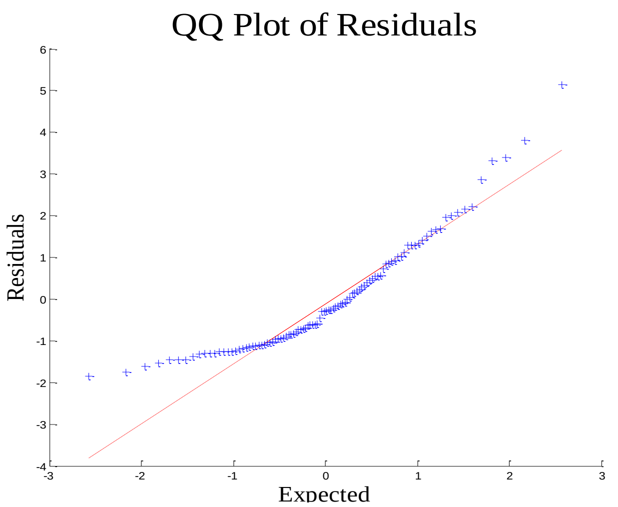
Multiple Linear Regression

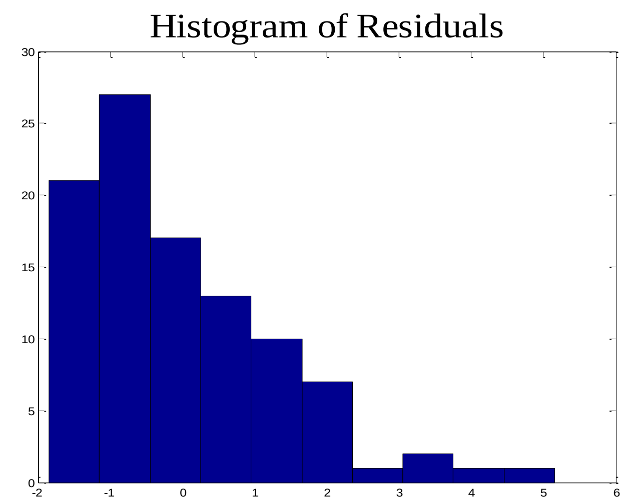
- Model checking

$$y = 2x + \varepsilon, \qquad \varepsilon \sim Gamma(2,1)$$



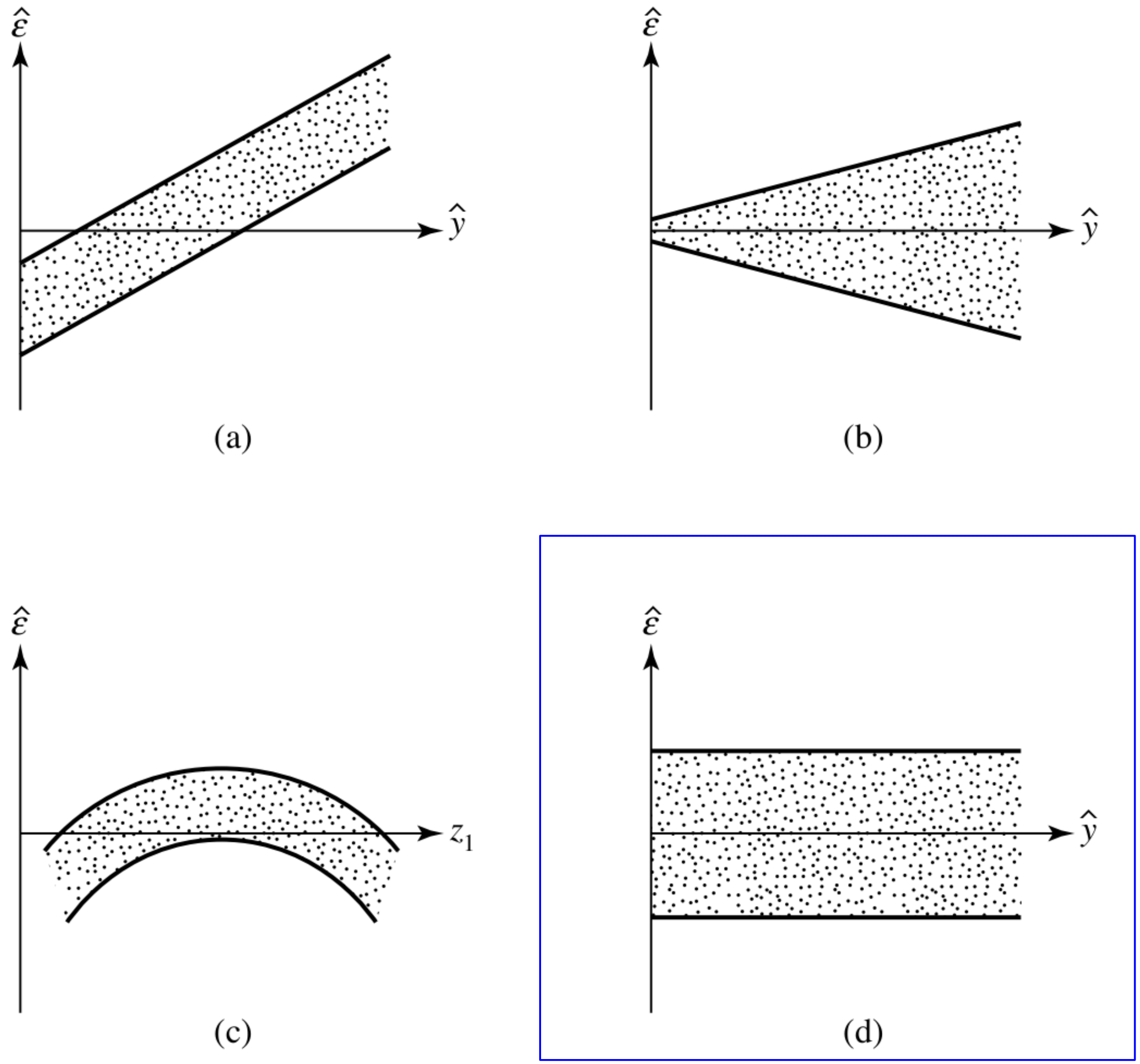
Regression model





Multiple Linear Regression

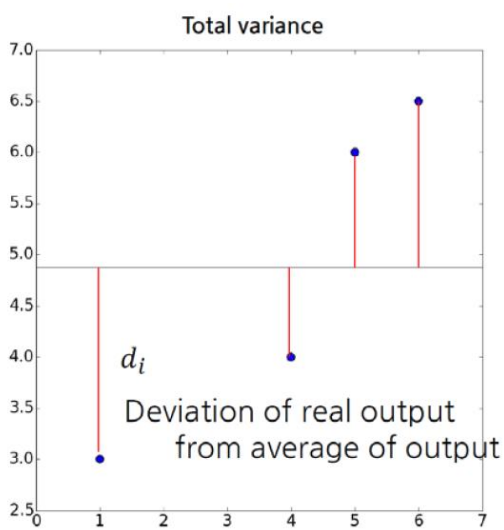
- Residual plots

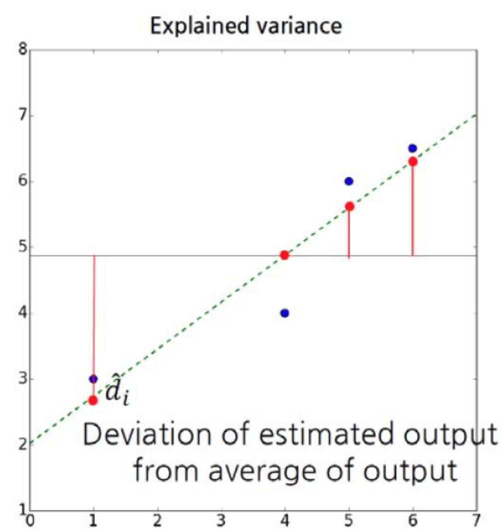


Multiple Linear Regression

- Goodness of fit
  - ✓ Sum-of-Squares Decomposition

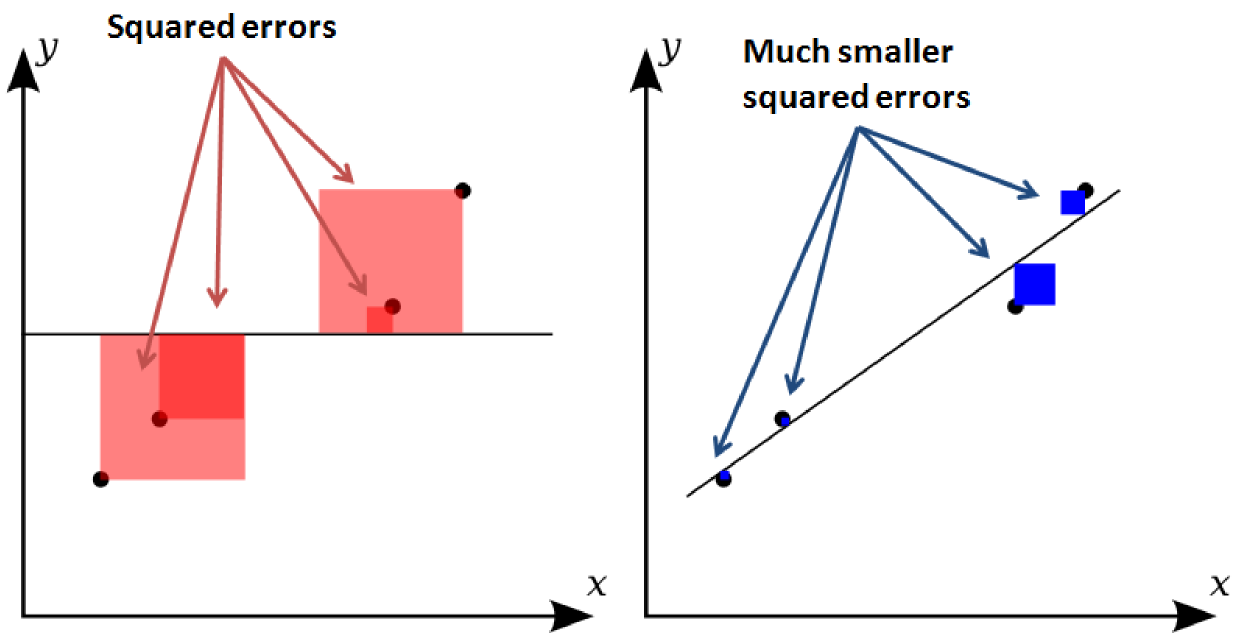
$$\underbrace{\boxed{\sum_{j=1}^{n}\left(y_j-\bar{y}\right)^2 \begin{pmatrix}\text{total sum of squares}\\ \text{about mean}\end{pmatrix}}}_{\text{SST}} = \underbrace{\boxed{\sum_{j=1}^{n}\left(\hat{y}_j-\bar{y}\right)^2 \begin{pmatrix}\text{regression}\\ \text{sum of squares}\end{pmatrix}}}_{\text{SSR}} + \underbrace{\boxed{\sum_{j=1}^{n}\hat{\varepsilon}_j^2 \begin{pmatrix}\text{residual (error)}\\ \text{sum of squares}\end{pmatrix}}}_{\text{SSE}}.$$





Multiple Linear Regression

- Goodness-of-fit: (Adjusted) R^2
  - ✓ Graphical interpretation



Computationally:

$$\text{R-squared} = 1 - \left[\frac{SS_{error}}{SS_{total}}\right]$$

Conceptually:
Force x and y to be independent, calculate **the squared error.**
Allow for a relationship between x and y, does this reduce your **error?**

http://compare2what.blogspot.kr/2013_01_01_archive.html

Multiple Linear Regression

- Goodness-of-fit: (Adjusted) R^2

$$R^2 = 1 - \frac{\sum_{j=1}^{n} \hat{\varepsilon}_j^2}{\sum_{j=1}^{n} (y_j - \bar{y})^2} = \frac{\sum_{j=1}^{n} (\hat{y}_j - \bar{y})^2}{\sum_{j=1}^{n} (y_j - \bar{y})^2}$$

$$R^2 = 1 - \frac{SSE}{SST} = \frac{SSR}{SST}$$

✓ Gives the proportion of the total variation in the y_i's explained by the predictor variables

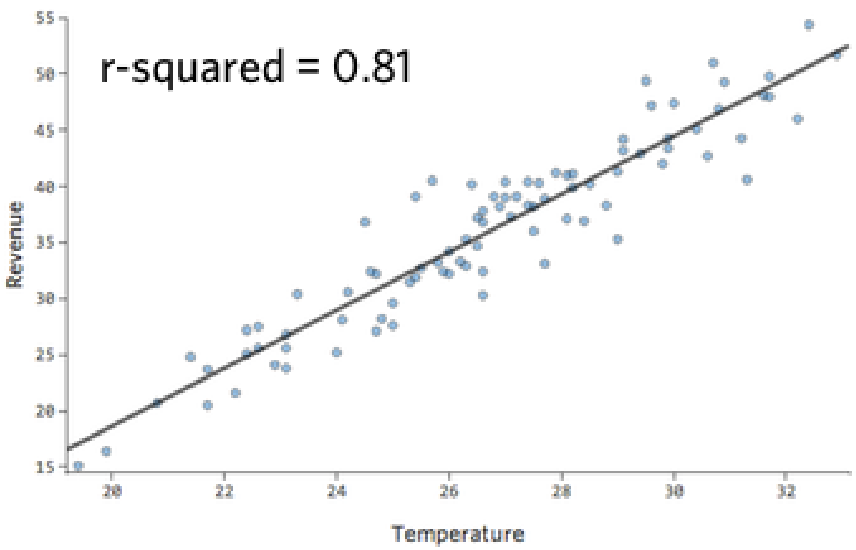
✓ $0 \leq R^2 \leq 1$

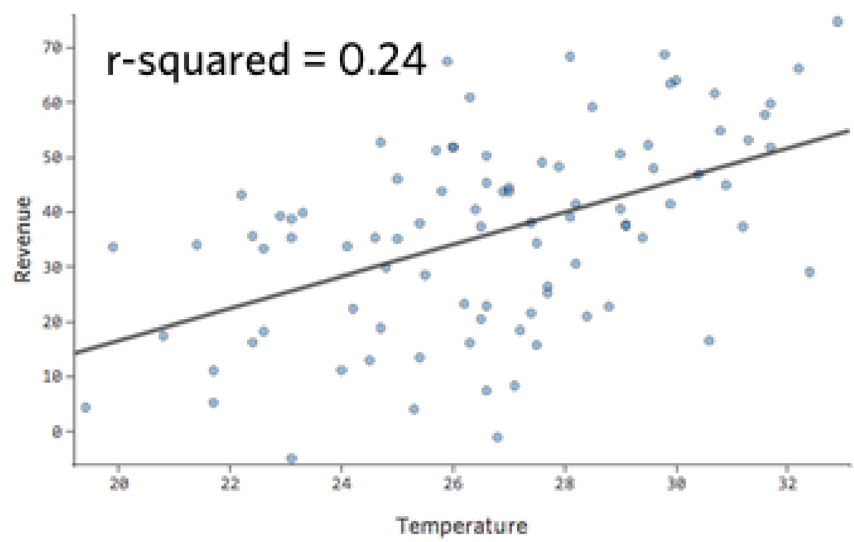
✓ $R^2 = 1$ → The fitted equation passes through all the data points

✓ $R^2 = 0$ → There is no linear relationship between the predictor variables and the target variable

Multiple Linear Regression

- Goodness-of-fit: (Adjusted) R^2
  - ✓ The proportionate reduction of total variation associated with the use of the predictor variable Z.





  - ▪ R^2 of your model is very high
    - I did a good job! (No!)
    - This dataset has a strong linear relationship between the X and y
    - Because everyone can have the same solution

Multiple Linear Regression

- Goodness-of-fit: (Adjusted) R^2
  - ✓ Adjusted R^2

$$R^2_{adj} = 1 - \left[\frac{n-1}{n-(p+1)} \right] \frac{SSE}{SST} \leq 1 - \frac{SSE}{SST} = R^2$$

  - ✓ R^2 increases monotonically when a (possibly not significant) new variable is added
  - ✓ Adjusted R^2 fix this problem
  - ✓ If an insignificant variable is added, the adjusted R^2 does not increase
- Model verification
  - ✓ Check whether the model satisfies the following assumptions
    - Residuals are independent
    - Residuals have zero mean and a constant variance

Multiple Linear Regression: Example

- Example: predict the selling price of Toyota corolla

Y

X

Price	Age_08_04	KM	Fuel_Type	HP	Met_Color	Automatic	cc	Doors	Quarterly_Tax	Weight
13500	23	46986	Diesel	90	1	0	2000	3	210	1165
13750	23	72937	Diesel	90	1	0	2000	3	210	1165
13950	24	41711	Diesel	90	1	0	2000	3	210	1165
14950	26	48000	Diesel	90	0	0	2000	3	210	1165
13750	30	38500	Diesel	90	0	0	2000	3	210	1170
12950	32	61000	Diesel	90	0	0	2000	3	210	1170
16900	27	94612	Diesel	90	1	0	2000	3	210	1245
18600	30	75889	Diesel	90	1	0	2000	3	210	1245
21500	27	19700	Petrol	192	0	0	1800	3	100	1185
12950	23	71138	Diesel	69	0	0	1900	3	185	1105
20950	25	31461	Petrol	192	0	0	1800	3	100	1185
19950	22	43610	Petrol	192	0	0	1800	3	100	1185
19600	25	32189	Petrol	192	0	0	1800	3	100	1185
21500	31	23000	Petrol	192	1	0	1800	3	100	1185
22500	32	34131	Petrol	192	1	0	1800	3	100	1185
22000	28	18739	Petrol	192	0	0	1800	3	100	1185
22750	30	34000	Petrol	192	1	0	1800	3	100	1185
17950	24	21716	Petrol	110	1	0	1600	3	85	1105
16750	24	25563	Petrol	110	0	0	1600	3	19	1065

Multiple Linear Regression: Example

- Data preprocessing
  - ✓ Create dummy variables for fuel types

	Fuel_type = Disel	**Fuel_type = Petrol**	**Fuel_type = CNG**
Diesel	1	0	0
Petrol	0	1	0
CNG	0	0	1

- Data partitioning
  - ✓ 60% training data / 40% validation data

Id	Model	Price	Age_08_04	Mfg_Month	Mfg_Year	KM	Fuel_Type_Diesel	Fuel_Type_Petrol
1	RRA 2/3-Doors	13500	23	10	2002	46986	1	0
4	RRA 2/3-Doors	14950	26	7	2002	48000	1	0
5	SOL 2/3-Doors	13750	30	3	2002	38500	1	0
6	SOL 2/3-Doors	12950	32	1	2002	61000	1	0
9	/VT I 2/3-Doors	21500	27	6	2002	19700	0	1
10	RRA 2/3-Doors	12950	23	10	2002	71138	1	0
12	BNS 2/3-Doors	19950	22	11	2002	43610	0	1
17	ORT 2/3-Doors	22750	30	3	2002	34000	0	1

Multiple Linear Regression: Example

- Fitted linear regression model

Input variables	Coefficient	Std. Error	p-value	SS
Constant term	-3608.418457	1458.620728	0.0137	97276410000
Age_08_04	-123.8319168	3.367589	0	8033339000
KM	-0.017482	0.00175105	0	251574500
Fuel_Type_Diesel	210.9862518	474.9978333	0.6571036	6212673
Fuel_Type_Petrol	2522.066895	463.6594238	0.00000008	4594.9375
HP	20.71352959	4.67398977	0.00001152	330138600
Met_Color	-50.48505402	97.85591125	0.60614568	596053.75
Automatic	178.1519013	212.0528565	0.40124047	19223190
cc	0.01385481	0.09319961	0.88188446	1272449
Doors	20.02487946	51.0899086	0.69526076	39265060
Quarterly_Tax	16.7742424	2.09381151	0	160667200
Weight	15.41666317	1.40446579	0	214696000

β

Significance Probability

Multiple Linear Regression: Example

- Interpret the result
  - ✓ Regression coefficient
    - Beta value for the corresponding predictor variable
    - The amount of change when the predictor variable increases by 1
    - If it is positive/negative, then the predictor variable and the target variable are positively/negatively correlated

Input variables	Coefficient	Std. Error	p-value	SS
Constant term	-3608.418457	1458.620728	0.0137	97276410000
Age_08_04	-123.8319168	3.367589	0	8033339000
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Weight	15.41666317	1.40446579	0	214696000

Multiple Linear Regression: Example

- Interpret the result
  - ✓ p-value
    - Indicate whether the regression coefficient is statistically significant or not
    - A predictor variable is important for modeling when its p-value is close to 0
    - Can be used to select significant variables (e.g., use the variables with p-value less than 0.05)

Input variables	Coefficient	Std. Error	p-value	SS
Constant term	-3608.418457	1458.620728	0.0137	97276410000
Age_08_04	-123.8319168	3.367589	0	8033339000
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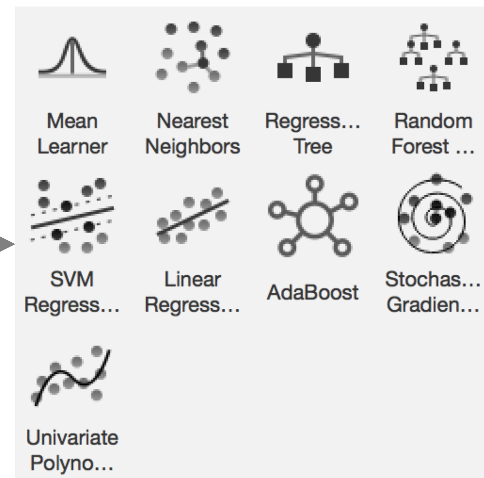
Weight	15.41666317	1.40446579	0	214696000

AGENDA

Evaluating Regression Models

- Example: predict a yield based on process sensors

	X1	X2	X3	…	Xp	Y
G1						95
G2						90
G3						92
G4						88
G5						75
G6						97
G7						91
G8						84
G9						83
G10						79



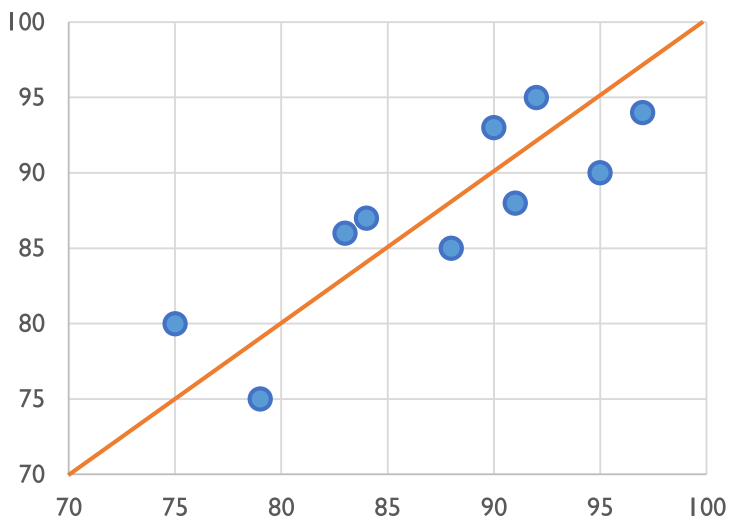
Y'
90
93
95
85
80
94
88
87
86
75

How accurate is this model?

Evaluating Regression Models

- Performance measure 1: Average Error
  - ✓ Compare the average difference between the actual and predicted y
  - ✓ Mislead to an inappropriate conclusion based on the sign effect

$$\text{Average Error} = \frac{1}{n}\sum_{i=1}^{n}(y_i - y_i')$$

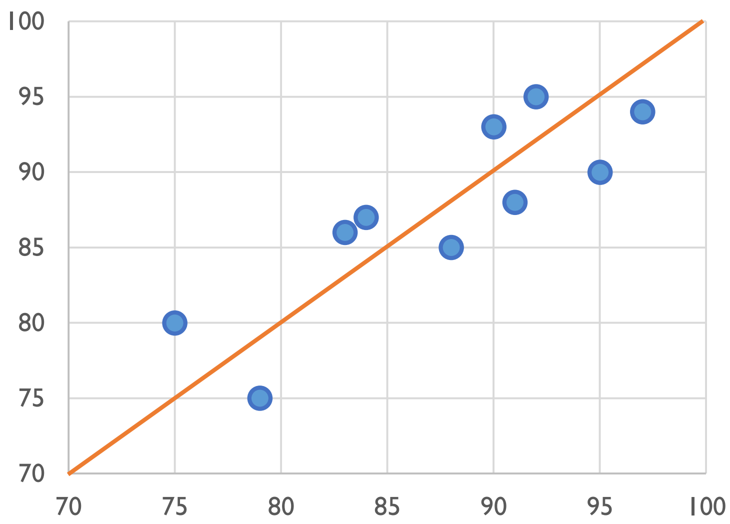


Y	Y'	Y-Y'
95	90	5
90	93	-3
92	95	-3
88	85	3
75	80	-5
97	94	3
91	88	3
84	87	-3
83	86	-3
79	75	4
Average Error		**0.1**

Evaluating Regression Models

- Performance measure 2: Mean Absolute Error (MAE)
  - ✓ Compute the average of absolute value of differences between the predicted and actual y

$$\mathrm{MAE} = \frac{1}{n}\sum_{i=1}^{n} |y_i - y_i'|$$



Y	Y'	|Y-Y'|
95	90	5
90	93	3
92	95	3
88	85	3
75	80	5
97	94	3
91	88	3
84	87	3
83	86	3
79	75	4
MAE		**3.5**

Evaluating Regression Models

- Performance measure 3: Mean Absolute Percentage Error (MAPE)
  - ✓ MAE can only provide the degree of absolute difference between the predicted and actual y but cannot provide the relative difference between them
  - ✓ Ex) The MAEs of the two models below are the same (MAE = 1)

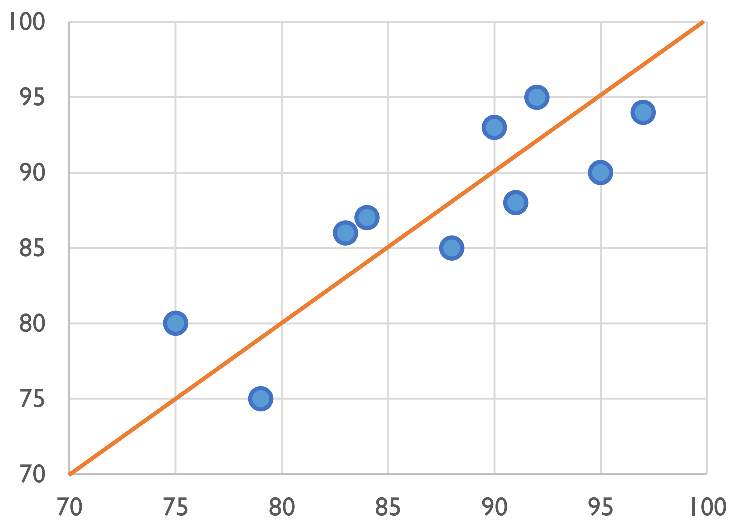
Y	Y'	Y-Y'
1	0	1
1	2	1
1	0	1
1	2	1
1	0	1
1	2	1
1	0	1
1	2	1
1	0	1
1	2	1
MAE		1

Y	Y'	Y-Y'
100	99	1
100	101	1
100	99	1
100	101	1
100	99	1
100	101	1
100	99	1
100	101	1
100	99	1
100	101	1
MAE		1

Evaluating Regression Models

- Performance measure 3: Mean Absolute Percentage Error (MAPE)
  - ✓ Provide the relative absolute difference in terms of %
  - ✓ Commonly adopted by domains in which relative differences are more important than the absolute difference (ex: quality control in manufacturing process)

$$\text{MAPE} = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{y_i - y_i'}{y_i}\right|$$

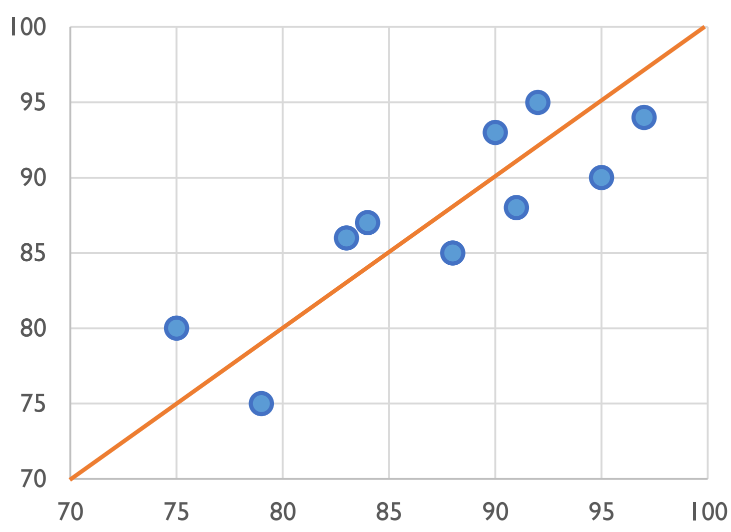


Y	Y'	|Y-Y'|	|Y-Y'|/|Y|
95	90	5	5.26%
90	93	3	3.33%
92	95	3	3.26%
88	85	3	3.41%
75	80	5	6.67%
97	94	3	3.09%
91	88	3	3.30%
84	87	3	3.57%
83	86	3	3.61%
79	75	4	5.06%
MAE		**3.5**	**4.06%**

Evaluating Regression Models

- Performance measure 4 & 5: (Root) Mean Squared Error ((R)MSE)
  - ✓ Use the square instead of absolute value to resolve the effect of sign

$$\mathrm{MSE} = \frac{1}{n}\sum_{i=1}^{n}(y_i - y_i')^2 \quad , \quad \mathrm{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - y_i')^2}$$



Y	Y'	$(Y-Y')^2$
95	90	25
90	93	9
92	95	9
88	85	9
75	80	25
97	94	9
91	88	9
84	87	9
83	86	9
79	75	16
MSE		**12.9**

$$\mathrm{RMSE} = \sqrt{12.9} = 3.59$$

AGENDA

R Exercise 1

- Data Set: Toyota Corolla Selling Price

Variable	Description	Variable	Description
		Guarantee_Period	Guarantee period in months
		ABS	Anti-Lock Brake System (Yes=1, No=0)
Price	Offer Price in EUROs	Airbag_1	Driver_Airbag (Yes=1, No=0)
Age_08_04	Age in months as in August 2004	Airbag_2	Passenger Airbag (Yes=1, No=0)
Mfg_Month	Manufacturing month (1-12)	Airco	Airconditioning (Yes=1, No=0)
Mfg_Year	Manufacturing Year	Automatic_airco	Automatic Airconditioning (Yes=1, No=0)
KM	Accumulated Kilometers on odometer	Boardcomputer	Boardcomputer (Yes=1, No=0)
Fuel_Type	Fuel Type (Petrol, Diesel, CNG)	CD_Player	CD Player (Yes=1, No=0)
HP	Horse Power	Central_Lock	Central Lock (Yes=1, No=0)
Met_Color	Metallic Color? (Yes=1, No=0)	Powered_Windows	Powered Windows (Yes=1, No=0)
Automatic	Automatic ((Yes=1, No=0)	Power_Steering	Power Steering (Yes=1, No=0)
CC	Cylinder Volume in cubic centimeters	Radio	Radio (Yes=1, No=0)
Doors	Number of doors	Mistlamps	Mistlamps (Yes=1, No=0)
Cylinders	Number of cylinders	Sport_Model	Sport Model (Yes=1, No=0)
Gears	Number of gear positions	Backseat_Divider	Backseat Divider (Yes=1, No=0)
Quarterly_Tax	Quarterly road tax in EUROs	Metallic_Rim	Metallic Rim (Yes=1, No=0)
Weight	Weight in Kilograms	Radio_cassette	Radio Cassette (Yes=1, No=0)
Mfr_Guarantee	Within Manufacturer's Guarantee period (Yes=1, No=0)	Parking_Assistant	Parking assistance system (Yes=1, No=0)
BOVAG_Guarantee	BOVAG (Dutch dealer network) Guarantee (Yes=1, No=0)	Tow_Bar	Tow Bar (Yes=1, No=0)

R Exercise I

- Define the performance evaluation function

```r
# Performance evaluation function for regression --------------------------
perf_eval_reg <- function(tgt_y, pre_y){
    # RMSE
    rmse <- sqrt(mean((tgt_y - pre_y)^2))
    # MAE
    mae <- mean(abs(tgt_y - pre_y))
    # MAPE
    mape <- 100*mean(abs((tgt_y - pre_y)/tgt_y))
    return(c(rmse, mae, mape))
}

# Initialize a performance summary table
perf_mat <- matrix(0, nrow = 2, ncol = 3)
rownames(perf_mat) <- c("Toyota Corolla", "Boston Housing")
colnames(perf_mat) <- c("RMSE", "MAE", "MAPE")
perf_mat
```

✓ perf_eval_reg( ) function

- Arguments: target values & predicted values
- Outputs: RMSE, MAE, MAPE

R Exercise I

- Load the data

```
# Dataset 1: Toyota Corolla
corolla <- read.csv("ToyotaCorolla.csv")

# Indices for the activated input variables
nCar <- nrow(corolla)
nVar <- ncol(corolla)

id_idx <- c(1,2)
category_idx <- 8
```

  - ✓ read.csv( ): a function that can read a csv file
  - ✓ nrow( ) & ncol( ): return the number of rows/columns in the dataframe
  - ✓ id_idx: id-related variables, irrelevant variable for analysis, will be removed
  - ✓ category_idx: catorical variable, will be transformed by 1-of-C coding

R Exercise I

- Data preprocessing: 1-of-C coding

Price	Age_08_04	Mfg_Month	Mfg_Year	KM	Fuel_Type	HP	Met_Color	Automatic	cc
13500	23	10	2002	46986	Diesel	90	1	0	2000
13750	23	10	2002	72937	Diesel	90	1	0	2000
13950	24	9	2002	41711	Diesel	90	1	0	2000
14950	26	7	2002	48000	Diesel	90	0	0	2000
13750	30	3	2002	38500	Diesel	90	0	0	2000
12950	32	1	2002	61000	Diesel	90	0	0	2000
16900	27	6	2002	94612	Diesel	90	1	0	2000
18600	30	3	2002	75889	Diesel	90	1	0	2000
21500	27	6	2002	19700	Petrol	192	0	0	1800
12950	23	10	2002	71138	Diesel	69	0	0	1900
20950	25	8	2002	31461	Petrol	192	0	0	1800
19950	22	11	2002	43610	Petrol	192	0	0	1800
19600	25	8	2002	32189	Petrol	192	0	0	1800
21500	31	2	2002	23000	Petrol	192	1	0	1800
22500	32	1	2002	34131	Petrol	192	1	0	1800

KM	HP	Met_Color
46986	90	1
72937	90	1
41711	90	1
48000	90	0
38500	90	0
61000	90	0
94612	90	1
75889	90	1
19700	192	0
71138	69	0
31461	192	0
43610	192	0
32189	192	0
23000	192	1

…

Petrol	Diesel	CNG
0	1	0
0	1	0
0	1	0
0	1	0
0	1	0
0	1	0
0	1	0
0	1	0
1	0	0
0	1	0
1	0	0
1	0	0
1	0	0
1	0	0

✓ Transform one categorical variable to C binary variables

- C is the number of categories

R Exercise 1

- 1-of-C Coding process

```
# Transform a categorical variable into a set of binary variables
dummy_p <- rep(0,nCar)
dummy_d <- rep(0,nCar)
dummy_c <- rep(0,nCar)

p_idx <- which(corolla$Fuel_Type == "Petrol")
d_idx <- which(corolla$Fuel_Type == "Diesel")
c_idx <- which(corolla$Fuel_Type == "CNG")

dummy_p[p_idx] <- 1
dummy_d[d_idx] <- 1
dummy_c[c_idx] <- 1
```

✓ dummy_p(c/d): initialize a zero vector with length nCar

✓ p_idx: Store the row index with Fuel_Type == "Petrol" (do the same job for d_idx and c_idx)

✓ dummy_p[p_idx] <- 1: replace 0 by 1 for the rows in the p_idx

R Exercise I

- Combine the dataset and split the data

```
Fuel <- data.frame(dummy_p, dummy_d, dummy_c)
names(Fuel) <- c("Petrol","Diesel","CNG")

# Prepare the data for MLR
corolla_mlr_data <- cbind(corolla[,-c(id_idx, category_idx)], Fuel)

# Split the data into the training/validation sets
set.seed(12345)
corolla_trn_idx <- sample(1:nCar, round(0.7*nCar))
corolla_trn_data <- corolla_mlr_data[corolla_trn_idx,]
corolla_val_data <- corolla_mlr_data[-corolla_trn_idx,]
```

✓ Create a new data frame "Fuel" by combining three dummy variables

✓ Combine the dataset with the original corolla dataset and Fuel dataset (use cbind( ) function)

✓ Split the data: 70% for training and 30% for validation

R Exercise I

- Training the model

```
# Train the MLR
mlr_corolla <- lm(Price ~ ., data = corolla_trn_data)
mlr_corolla
summary(mlr_corolla)
plot(mlr_corolla)
```

✓ lm( ): linear regression

- Price ~ : Formula
  - The left side of ~ is the target variable
  - The right side of ~ are the predictor variables (. means all variables except the target variable)
  - data = corolla_trn_data: data used to estimate the regression coefficients

✓ Summary( ): print the result of the regression model

✓ plot( ): draw four plots for the regression model

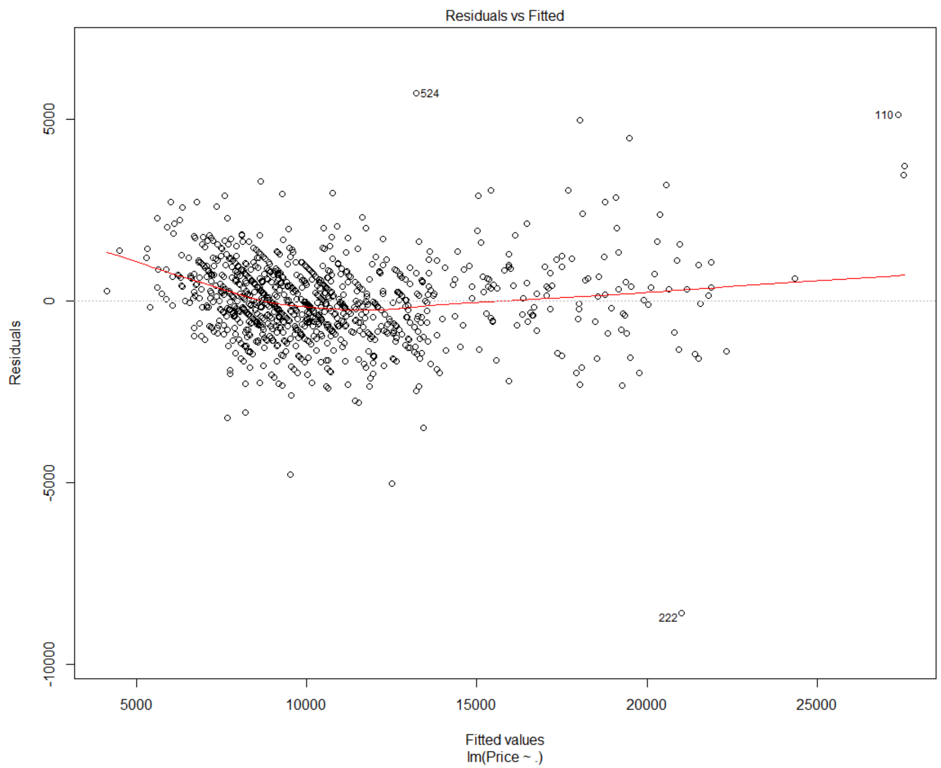
R Exercise I

- Interpret the results
  - ✓ Estimate: estimated regression coefficients
  - ✓ Std.Error: standard error of the estimated coefficients
  - ✓ t value: t-statistic for the hypothesis test
  - ✓ Pr(>|t|): p-value for the regression coefficient, the smaller the p-value, the more significant the variable
  - ✓ Adjusted R-squared
  - ✓ NA: variable is removed because of multicollinearity problem

```
> summary(full_model)

Call:
lm(formula = Price ~ ., data = trn_data)

Residuals:
    Min      1Q  Median      3Q     Max 
-8569.6  -637.3   -42.9   650.5  5720.8 

Coefficients: (3 not defined because of singularities)
               Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -752.65641 1775.96778  -0.424 0.671805    
Age_08_04    -118.43999    4.29358 -27.585  < 2e-16 ***
Mfg_Month     -95.78658   11.11831  -8.615  < 2e-16 ***
Mfg_Year             NA         NA      NA       NA    
KM             -0.01727    0.00136 -12.702  < 2e-16 ***
HP             20.46048    3.59449   5.692 1.66e-08 ***
Met_Color     -64.93066   81.74804  -0.794 0.427228    
Automatic     338.02084  156.47529   2.160 0.031000 *  
cc             -0.10770    0.07958  -1.353 0.176246    
Doors          10.46213   43.60079   0.240 0.810418    
Cylinders            NA         NA      NA       NA    
Gears         183.36459  196.28703   0.934 0.350451    
                              :
Tow_Bar      -217.67837   85.11397  -2.557 0.010694 *  
Petrol       2280.57458  387.15749   5.891 5.30e-09 ***
Diesel       1004.02078  377.75296   2.658 0.007993 ** 
CNG                  NA         NA      NA       NA    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1128 on 971 degrees of freedom
Multiple R-squared:  0.9046,	Adjusted R-squared:  0.9014 
F-statistic: 279.1 on 33 and 971 DF,  p-value: < 2.2e-16
```

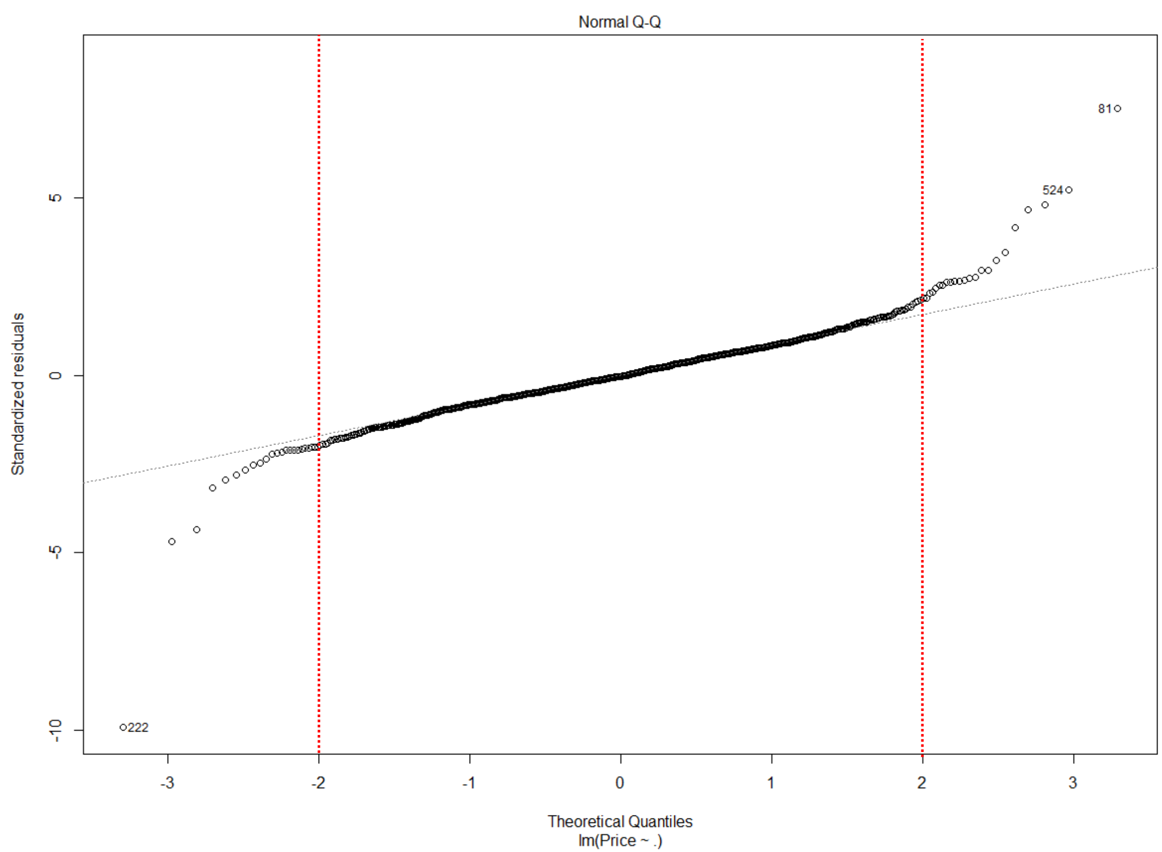
R Exercise I

- Interpret the result
  - ✓ Figure 1: used to check the following assumption
    - ▪ The variability in Y values for a given set of predictors is the same regardless of the values of the predictors (homoskedasticity)



R Exercise I

- Interpret the result
  - ✓ Figure 2: used to check the following assumption
    - The noise ε follows a normal distribution

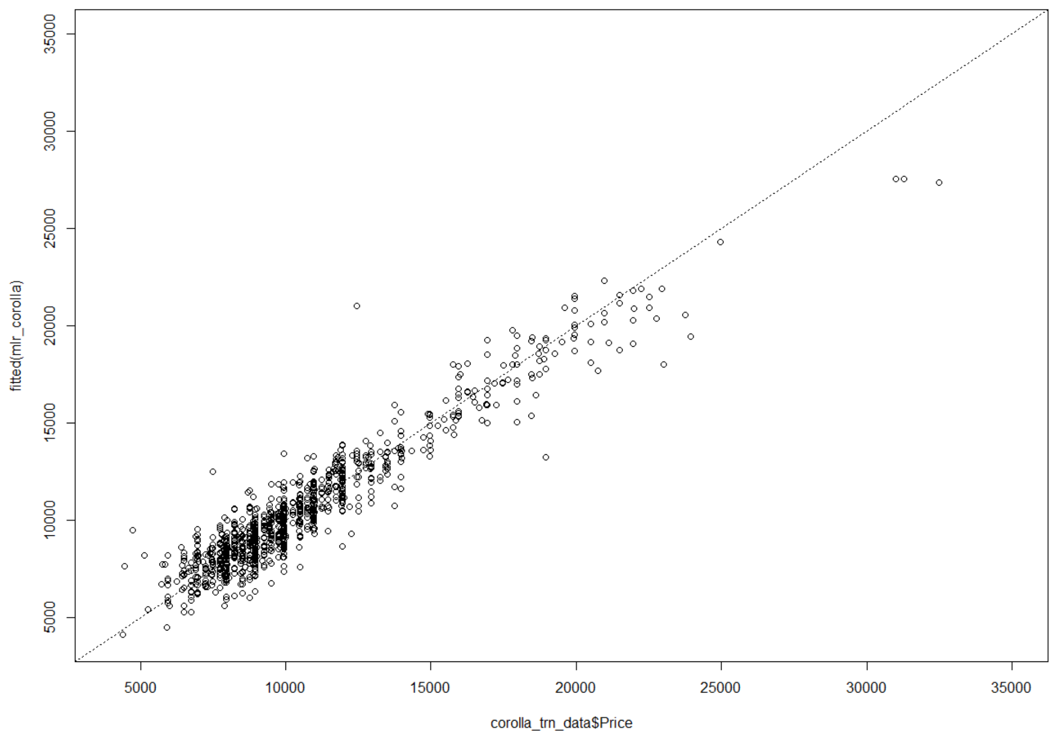


R Exercise I

- Interpret the result

```
# Plot the result
plot(corolla_trn_data$Price, fitted(mlr_corolla), xlim = c(4000,35000),
     ylim = c(4000,35000))
abline(0,1,lty=3)
```

✓ Plot the relationship between the actual target values (x-axis) and the predicted values (y-axis)



R Exercise I

- Normality check for the residuals

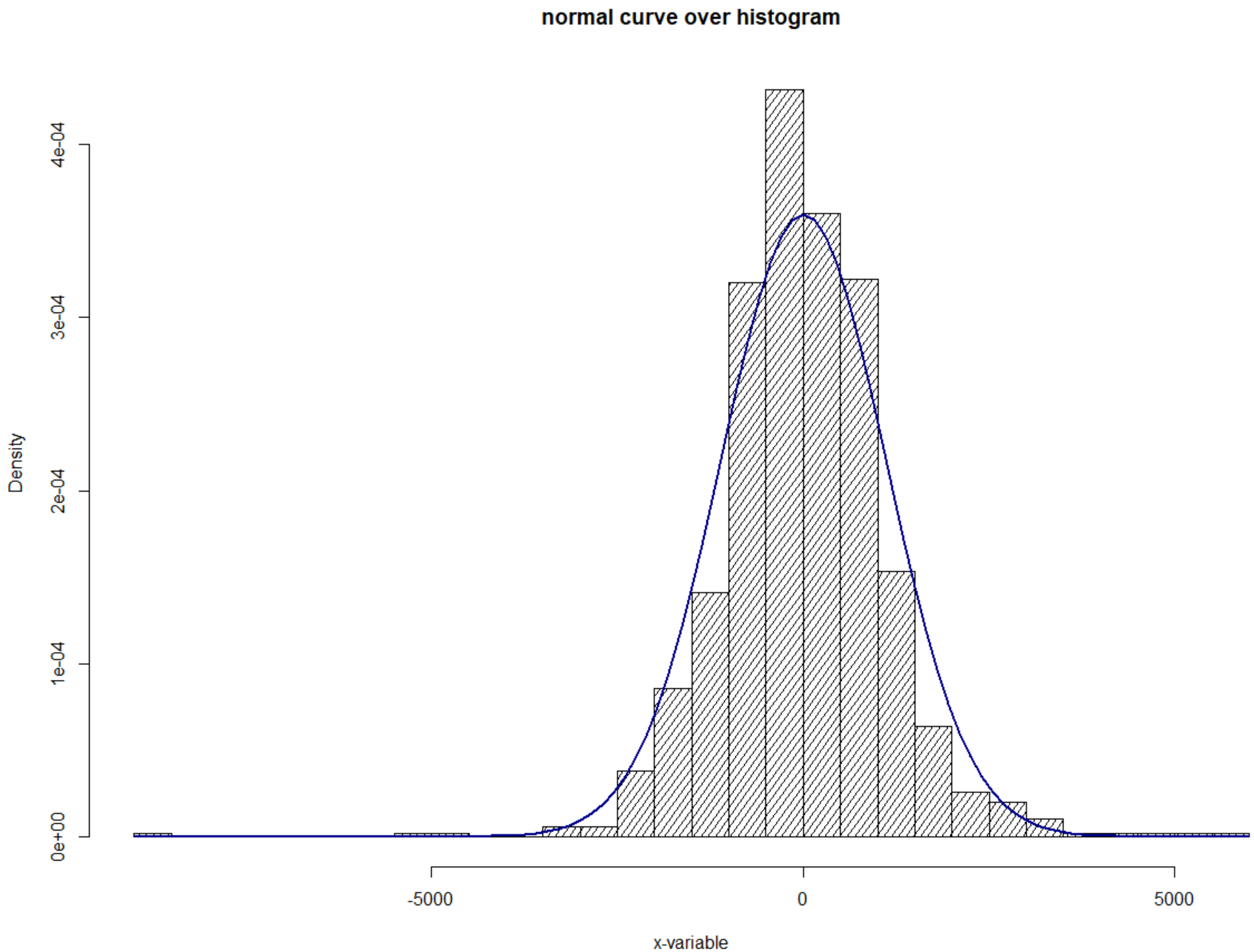
```
# normality test of residuals
corolla_resid <- resid(mlr_corolla)
m <- mean(corolla_resid)
std <- sqrt(var(corolla_resid))

hist(corolla_resid, density=20, breaks=50, prob=TRUE, xlab="x-variable",
     main="normal curve over histogram")
curve(dnorm(x, mean=m, sd=std), col="darkblue", lwd=2, add=TRUE, yaxt="n")

skewness(corolla_resid)
kurtosis(corolla_resid)
```

- ✓ hist( ): draw a histogram
- ✓ curve( ): draw a curve for a certain probability density function
- ✓ skewness: 0 if the dataset follows a normal distribution
- ✓ kurtosis: 3 if the dataset follows a normal distribution

```
> skewness(corolla_resid)
[1] -0.1355675
> kurtosis(corolla_resid)
[1] 8.630819
```

R Exercise I

- Histogram & Normal Density Curve



R Exercise I

- Prediction performance of the regression model

```r
# Performance Measure
mlr_corolla_haty <- predict(mlr_corolla, newdata = corolla_val_data)
perf_mat[1,] <- perf_eval_reg(corolla_val_data$Price, mlr_corolla_haty)
perf_mat
```

```
> perf_mat
                    RMSE      MAE     MAPE
Toyota Corolla  1085.124 814.0598 8.007297
Boston Housing     0.000   0.0000 0.000000
```

R Exercise 2

- Boston Housing Price
  - ✓ Predict the median price of houses in a unit district

Variable	Description
CRIM	per capita crime rate by town.
ZN	proportion of residential land zoned for lots over 25,000 sq.ft.
INDUS	proportion of non-retail business acres per town.
NOX	nitrogen oxides concentration (parts per 10 million).
RM	average number of rooms per dwelling.
AGE	proportion of owner-occupied units built prior to 1940.
DIS	weighted mean of distances to five Boston employment centres.
TAX	full-value property-tax rate per \$10,000.
PTRATIO	pupil-teacher ratio by town.
Black	1000(Bk - 0.63)^2 where Bk is the proportion of blacks by town.
LSTAT	lower status of the population (percent).
MEDV	median value of owner-occupied homes in \$1000s.

R Exercise 2

- Load the dataset and preprocess the data

```
# Dataset 2: Boston Housing
boston_housing <- read.csv("BostonHousing.csv")
nHome <- nrow(boston_housing)
nVar <- ncol(boston_housing)

# Split the data into the training/validation sets
boston_trn_idx <- sample(1:nHome, round(0.7*nHome))
boston_trn_data <- boston_housing[boston_trn_idx,]
boston_val_data <- boston_housing[-boston_trn_idx,]
```

  ✓ Unlike "corolla" dataset, all variables are numerical variables

  ✓ No special data preprocessing is requred

R Exercise 2

- Training the model and plot the results

```
# Train the MLR
mlr_boston <- lm(MEDV ~ ., data = boston_trn_data)
mlr_boston

summary(mlr_boston)
plot(mlr_boston)

# Plot the result
plot(boston_trn_data$MEDV, fitted(mlr_boston), xlim = c(-5,50), ylim = c(-5,50))
abline(0,1,lty=3)
```

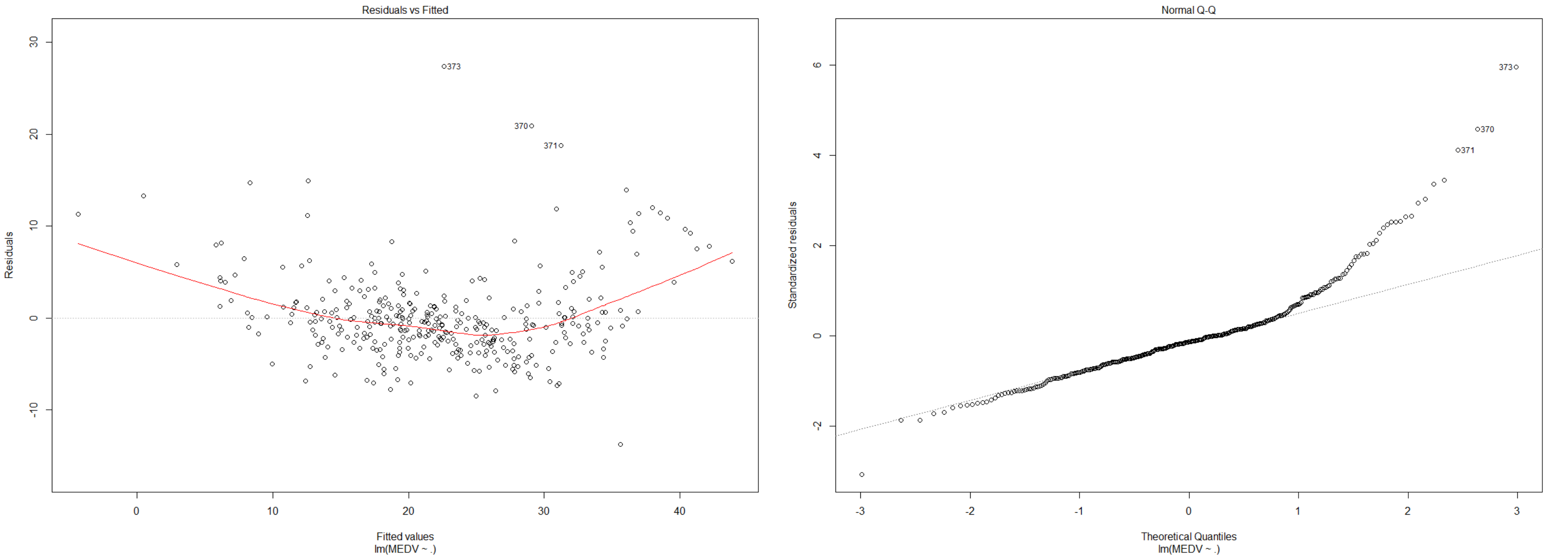
R Exercise 2

- Fitted model
  - ✓ Adjusted R^2: 0.7347 (smaller than that of corolla model)
  - ✓ All variables except INDUS, AGE, and TAX are statistically significant (alpha = 0.05)

```
> summary(mlr_boston)

Call:
lm(formula = MEDV ~ ., data = boston_trn_data)

Residuals:
     Min       1Q   Median       3Q      Max 
-13.7098  -2.6401  -0.5686   1.3348  27.3746 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  2.779e+01  5.744e+00   4.839 1.98e-06 ***
CRIM        -7.268e-02  3.435e-02  -2.115   0.0351 *  
ZN           3.485e-02  1.710e-02   2.038   0.0423 *  
INDUS       -8.059e-03  6.755e-02  -0.119   0.9051    
NOX         -1.218e+01  4.643e+00  -2.623   0.0091 ** 
RM           4.287e+00  4.920e-01   8.714  < 2e-16 ***
AGE         -2.460e-02  1.492e-02  -1.649   0.1000    
DIS         -1.573e+00  2.456e-01  -6.404 5.00e-10 ***
TAX          7.515e-04  2.755e-03   0.273   0.7852    
PTRATIO     -8.879e-01  1.495e-01  -5.941 6.98e-09 ***
B            1.237e-02  3.067e-03   4.034 6.78e-05 ***
LSTAT       -4.910e-01  6.139e-02  -7.999 1.97e-14 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.658 on 342 degrees of freedom
Multiple R-squared:  0.743,	Adjusted R-squared:  0.7347 
F-statistic: 89.89 on 11 and 342 DF,  p-value: < 2.2e-16
```

R Exercise 2

- Residual plot and normal QQ plot



✓ Residuals might not follow a normal distribution

R Exercise 2

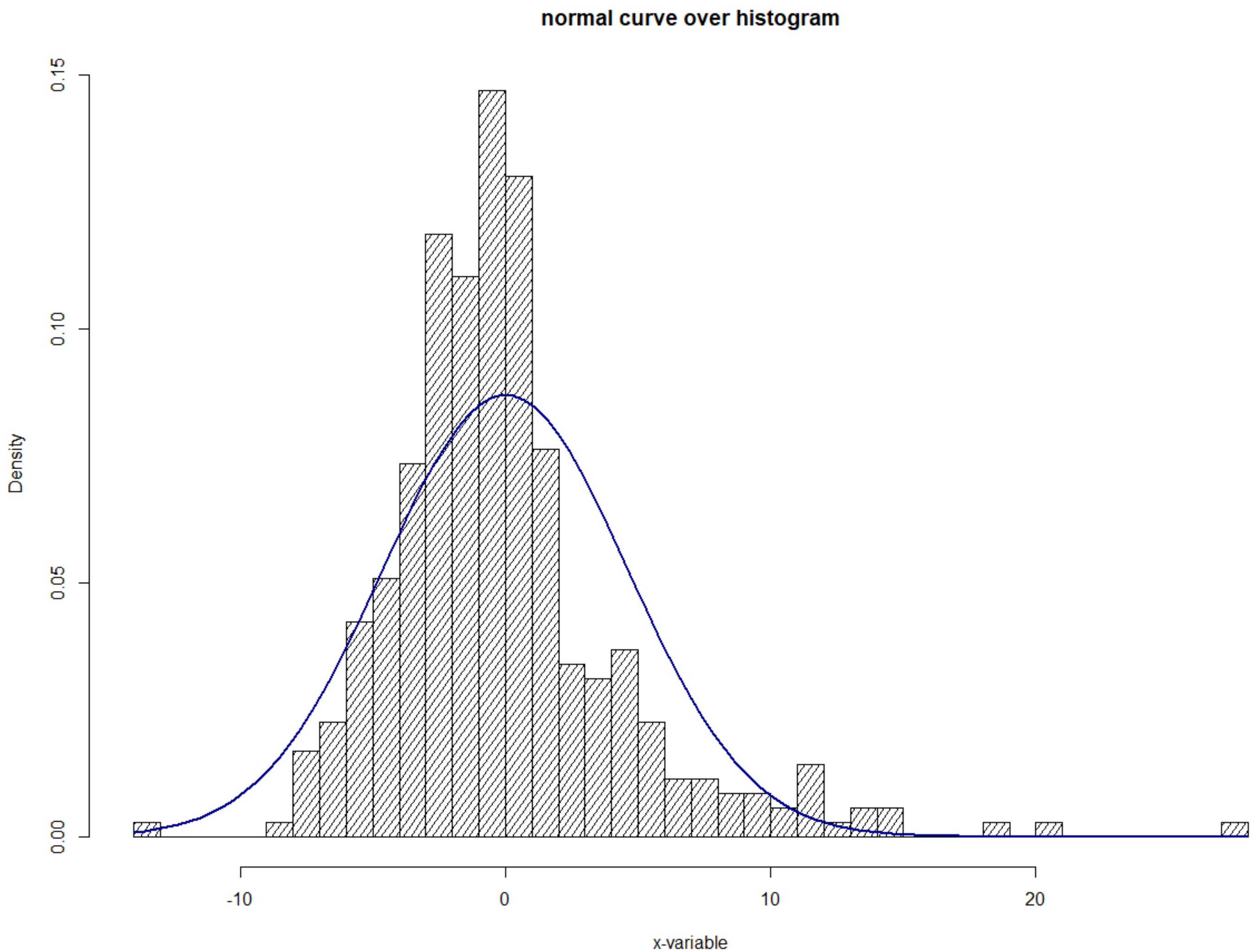
- Normality check for the residuals

```
# normality test of residuals
boston_resid <- resid(mlr_boston)
m <- mean(boston_resid)
std <- sqrt(var(boston_resid))

hist(boston_resid, density=20, breaks=50, prob=TRUE, xlab="x-variable",
     main="normal curve over histogram")
curve(dnorm(x, mean=m, sd=std), col="darkblue", lwd=2, add=TRUE, yaxt="n")

skewness(boston_resid)
kurtosis(boston_resid)
```

```
> skewness(boston_resid)
[1] 1.656679
> kurtosis(boston_resid)
[1] 8.669806
```

R Exercise 2

- Normality check for the residuals



R Exercise 2

- Prediction performance of the regression model

```
# Performance Measure
mlr_boston_haty <- predict(mlr_boston, newdata = boston_val_data)
perf_mat[2,] <- perf_eval_reg(boston_val_data$MEDV, mlr_boston_haty)
perf_mat
```

```
> perf_mat
                       RMSE        MAE      MAPE
Toyota Corolla 1085.124496 814.059765  8.007297
Boston Housing    5.476226   3.834064 19.276196
```


ANY
questions?