PhD Research
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Macroscopic theory of sound propagation in rigid-framed porous materials allowing for spatial dispersion: principle and validation

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Outline

INTRODUCTION

THEORY

- Averaging
- Acoustic Macroscopic equations: local and nonlocal
- Viscothermal fluid
  - Procedure for computing effective density
  - Procedure for computing effective bulk modulus
- Procedure to determine effective density and bulk modulus in porous media

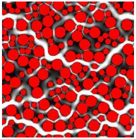
VALIDATION

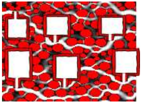
- Circular tube
- Arrays of rigid cylinders
- Arrays of Helmholtz resonators

CONCLUSION/PERSPECTIVE

A macroscopic theory describing sound propagation through porous media

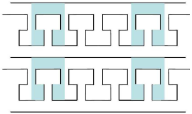
- Unbounded saturated porous media: fluid-solid
  - Solid is rigid
  - Fluid is viscothermal
- Local theory (Classical Equivalent-Fluid = order "0" of the Homogenization Theory = temporal dispersion)
  - Wavelength $\lambda \gg L$
  - Microscopic scale: the fluid is considered to be incompressible $\rightsquigarrow$ $\nabla.\mathbf{v} = 0$
  - Local theory is not complete...





Generalization

- Nonlocal theory
  - Temporal dispersion
    AND
  - Nonlocal effects due to spatial dispersion
  - Microscopic scale: the fluid is considered to be compressible $\rightsquigarrow$ $\nabla.\mathbf{v} \neq 0$
  - It describes also the short wavelength propagation: no constraint for the wavelength
  - Analogy with Maxwell's theory of electromagnetics
  - Upscaling procedure leads to determine two acoustic permittivities, using a thermodynamic postulate
  - Periodic media: propagation along a symmetry axis



Introduction: upscaling procedure

How to determine the nonlocal acoustic permittivities from microstructure?

- Solving two independent action-response problems at microscale:
  - Response of the fluid subjected to a bulk force
  - Response of the fluid subjected to a heating source
- Volume-averaging the fields
  in particular, using the thermodynamic postulate:

  Poynting-Schoch concept of acoustic part of energy current density

$\Longrightarrow$ frequency and wave number dependent effective density
effective bulk modulus

Viscothermal fluid equations for a small perturbation + interface conditions

- In the viscothermal fluid

Mass balance: $$\frac{\partial b}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

Momentum balance: $$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \mathbf{v} + \left(\zeta + \tfrac{1}{3}\eta\right) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{v}\right)$$

Energy balance: $$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \boldsymbol{\nabla}^2 \tau$$

State: $$\gamma \chi_0 p = b + \beta_0 \tau$$

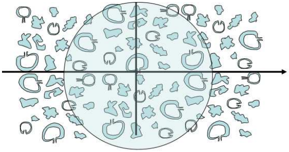
- On the fluid-solid interface

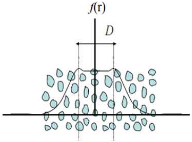
$$\mathbf{v} = 0, \quad \tau = 0$$

Theory: macroscopic fields

- Microscopic scale: Navier-Stokes-Fourier
- D: averaging scale = REV periodic media: REV = 1 period
- Indicator function: $I(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in \mathcal{V}^f \text{ fluid region} \\ 0, & \mathbf{r} \in \mathcal{V}^s \text{ solid region} \end{cases}$
- Definition $\mathbf{V} = \langle \mathbf{v}(t,\mathbf{r}) \rangle = \int d\mathbf{r}'\, f(\mathbf{r}'-\mathbf{r}) I(\mathbf{r}') \mathbf{v}(t,\mathbf{r}')$

  $\mathrm{B} = \langle b(t,\mathbf{r}) \rangle = \int d\mathbf{r}'\, f(\mathbf{r}'-\mathbf{r}) I(\mathbf{r}') b(t,\mathbf{r}')$





$$f(\mathbf{r}) = \frac{1}{(\pi L^2)^{3/2}} e^{-r^2/L^2}$$

Similarly, in electromagnetics:

$$\mathbf{E} = \langle \mathbf{e}(t,\mathbf{r}) \rangle$$
$$\mathbf{B} = \langle \mathbf{b}(t,\mathbf{r}) \rangle$$

Theory: macroscopic electromagnetic equations

Maxwell equations

Field equations

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = 0$$
$$\frac{\partial \mathbf{D}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{H}$$

Constitutive relations

$$\mathbf{D} = \hat{\epsilon}\,\mathbf{E}$$

$$\mathbf{H} = \hat{\mu}^{-1}\mathbf{B}$$

- Temporal dispersion (Local)

  $\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\ \epsilon(t-t')\mathbf{E}(t',\mathbf{r}) \rightsquigarrow \mathbf{D}(\omega,\mathbf{r}) = \epsilon(\omega)\mathbf{E}(\omega,\mathbf{r})$
  $\mathbf{H}(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\ \mu^{-1}(t-t')\mathbf{B}(t',\mathbf{r}) \rightsquigarrow \mathbf{H}(\omega,\mathbf{r}) = \mu^{-1}(\omega)\mathbf{B}(\omega,\mathbf{r})$

- Temporal dispersion + Spatial dispersion (Nonlocal)

  $\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}'\epsilon(t-t',\mathbf{r}-\mathbf{r}')\mathbf{E}(t',\mathbf{r}')$
  $\mathbf{H}(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}'\mu^{-1}(t-t',\mathbf{r}-\mathbf{r}')\mathbf{B}(t',\mathbf{r}')$
  $\rightsquigarrow$
  $\mathbf{D}(\omega,\mathbf{k}) = \epsilon(\omega,\mathbf{k})\mathbf{E}(\omega,\mathbf{k})$
  $\mathbf{H}(\omega,\mathbf{k}) = \mu^{-1}(\omega,\mathbf{k})\mathbf{B}(\omega,\mathbf{k})$

Maxwellian acoustic equations

Field equations

$$\frac{\partial B}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{V} = 0$$
$$\frac{\partial \mathbf{D}}{\partial t} = -\boldsymbol{\nabla} H$$

Constitutive relations

$$\mathbf{D} = \hat{\rho}\, \mathbf{V}$$
$$H = \hat{\chi}^{-1} B$$

- Temporal dispersion (Local)

  $\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \; \rho(t-t')\mathbf{V}(t',\mathbf{r}) \quad \rightsquigarrow \mathbf{D}(\omega,\mathbf{r}) = \rho(\omega)\mathbf{V}(\omega,\mathbf{r})$

  $H(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \; \chi^{-1}(t-t')B(t',\mathbf{r}) \rightsquigarrow H(\omega,\mathbf{r}) = \chi^{-1}(\omega)B(\omega,\mathbf{r})$

- Temporal dispersion + Spatial dispersion (Nonlocal)

  $\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \rho(t-t',\mathbf{r}-\mathbf{r}')\mathbf{V}(t',\mathbf{r}')$

  $H(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \chi^{-1}(t-t',\mathbf{r}-\mathbf{r}')B(t',\mathbf{r}')$

  $\rightsquigarrow$

  $\mathbf{D}(\omega,\mathbf{k}) = \rho(\omega,\mathbf{k})\mathbf{V}(\omega,\mathbf{k})$

  $H(\omega,\mathbf{k}) = \chi^{-1}(\omega,\mathbf{k})B(\omega,\mathbf{k})$

Theory: definition of the field H

- Local theory

$$H = \langle p \rangle$$

- Nonlocal theory: Poynting-Schoch condition

$$S = H\mathbf{V} = \langle pv \rangle$$

$=$ Acoustic part of energy current density

- Electromagnetism

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$=$ Electromagnetic part of energy current density

Theory: viscothermal fluid - Kirchhoff

Maxwellian acoustics equations

$$\frac{\partial b}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$
$$\frac{\partial \mathbf{d}}{\partial t} = -\boldsymbol{\nabla} h$$

$$\mathbf{d} = \hat{\rho}\, \mathbf{v}$$
$$h = \hat{\chi}^{-1} b$$

- Microscopic Eqs for longitudinal motions

$$\frac{\partial b}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \qquad \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p + \left(\zeta + \tfrac{4}{3}\eta\right) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{v}\right)$$

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \boldsymbol{\nabla}^2 \tau, \qquad \gamma \chi_0 p = b + \beta_0 \tau$$

- Identification of the field h

$$s = h\mathbf{v} = p\mathbf{v} \rightsquigarrow \quad h = p$$

Theory: viscothermal fluid - Kirchhoff

- Nonlocal density and bulk modulus

$$\rho(\omega, k) = \rho_0 \left(1 + \frac{\frac{4\eta}{3}+\zeta}{\rho_0} \frac{k^2}{-i\omega}\right) \rightsquigarrow \text{inertial and viscous effects}$$

$$\chi^{-1}(\omega, k) = \chi_0^{-1} \left[1 - \frac{\gamma-1}{\gamma} \left(1 - \frac{i\omega}{-i\omega + \frac{\kappa}{\rho_0 C_V} k^2}\right)\right] \rightsquigarrow \text{elastic and thermal effects}$$

- Nonlocal dispersion equation

$$\rho(\omega, k)\chi(\omega, k)\omega^2 = k^2 \qquad \Longrightarrow \text{Kirchhoff equation}$$

$$-\omega^2 + \left[c_a^2 - i\omega \left(\frac{\kappa}{\rho_0 C_V} + \frac{\frac{4\eta}{3}+\zeta}{\rho_0}\right)\right] k^2 - \frac{\kappa}{\rho_0 C_V i\omega} \left[c_i^2 - i\omega \frac{\frac{4\eta}{3}+\zeta}{\rho_0}\right] k^4 = 0$$

Theory: viscothermal fluid - nonlocal action/response problems

- Nonlocal density

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p + \left(\zeta + \tfrac{4}{3}\eta\right) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{v}\right) + \mathbf{F} e^{-i\omega t + ikx}$$

where $\mathbf{F} e^{-i\omega t+ikx} = -\boldsymbol{\nabla} \left(\mathcal{P} e^{-i\omega t+ikx}\right)$

Response fields: $\mathbf{v}(t,x) = v_x \mathbf{e}_x e^{-i\omega t+ikx}$, $p(t,x) = p e^{-i\omega t+ikx}$, b, τ

Maxwell: $$-i\omega \underbrace{\rho(\omega,k) v_x}_{d} = -ik \underbrace{\left(p + \mathcal{P}\right)}_{h}$$

- Nonlocal bulk modulus

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \nabla^2 \tau + \dot{Q} e^{-i\omega t+ikx}$$

where $\dot{Q} e^{-i\omega t+ikx} = \beta_0 T_0 \dfrac{\partial}{\partial t} \left(\mathcal{P} e^{-i\omega t+ikx}\right)$

Response fields: $\mathbf{v}$, p, b', τ

Maxwell: $$\underbrace{p + \mathcal{P}}_{h} = \chi^{-1}(\omega,k) \underbrace{\left(b' + \gamma \chi_0 \mathcal{P}\right)}_{b}$$

Theory: procedure to determine effective density

- In the visco-thermal fluid

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0$$
$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\zeta + \tfrac{1}{3}\eta\right) \nabla \left(\nabla . \mathbf{v}\right) + \mathbf{F} e^{-i\omega t + ikx}$$
$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \nabla^2 \tau$$
$$\gamma \chi_0 p = b + \beta_0 \tau$$

- On the fluid-solid interface

$$\mathbf{v} = 0, \quad \tau = 0$$

where $\mathbf{F} e^{-i\omega t + ikx} = -\boldsymbol{\nabla} \left(\mathcal{P} e^{-i\omega t + ikx}\right)$

$$\rho(\omega, k) = \frac{k \left(\mathcal{P} + P(\omega, k)\right)}{\omega \left\langle v(\omega, k, \mathbf{r})\right\rangle}$$

where $P\langle \mathbf{v} \rangle = \langle p \mathbf{v} \rangle$

Theory: procedure to determine effective bulk modulus

- In the visco-thermal fluid

$$\frac{\partial b'}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \mathbf{v} + \left(\zeta + \tfrac{1}{3}\eta\right) \boldsymbol{\nabla} \left(\boldsymbol{\nabla} . \mathbf{v}\right)$$

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \boldsymbol{\nabla}^2 \tau + \dot{Q} e^{-i\omega t + ikx}$$

$$\gamma \chi_0 p = b' + \beta_0 \tau$$

- On the fluid-solid interface

$$\mathbf{v} = 0, \quad \tau = 0$$

where $\dot{Q} e^{-i\omega t + ikx} = \beta_0 T_0 \dfrac{\partial}{\partial t} \left(\mathcal{P} e^{-i\omega t + ikx}\right)$

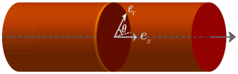
$$\chi^{-1}(\omega, k) = \frac{P(\omega, k) + \mathcal{P}}{\langle b'(\omega, k, \mathbf{r}) \rangle + \phi \gamma \chi_0 \mathcal{P}}$$

where $P\langle \mathbf{v} \rangle = \langle p \mathbf{v} \rangle$

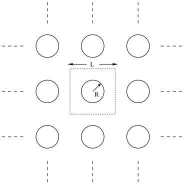
Validation

New theory is validated in three periodic microstructure:

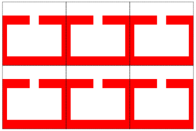
- Circular tube $\rightsquigarrow$ wavenumbers, impedances
  - Kirchhoff dispersion equation
  - Zwikker-Kosten model (local theory)
  - Nonlocal theory



- 2D arrays of rigid cylinders $\rightsquigarrow$ phase velocities
  $\lambda \sim L$
  - Quasi-exact multiple scattering method
  - Local theory
  - Nonlocal theory

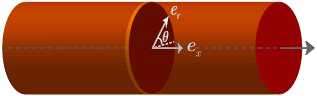


- 2D array of Helmhotz resonators $\rightsquigarrow$ wavenumbers
  $\lambda \gg L$
  - Bloch-wave modelling
  - Nonlocal theory



Circular tube

- Computing the wavenumbers and impedances obtained by the local theory, nonlocal theory and Kirchhoff's equation, considering axisymmetrical modes



$$\left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1}\right)\frac{1}{\varphi_{1w}}\frac{\partial\varphi_1}{\partial r_w} - \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2}\right)\frac{1}{\varphi_{2w}}\frac{\partial\varphi_2}{\partial r_w} - \frac{k^2}{\frac{-i\omega}{\nu}+k^2}\left(\frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2}\right)\frac{1}{\varphi_w}\frac{\partial\varphi}{\partial r_w} = 0$$

  - Kirchhoff's (1868): exact discrete wavenumbers
  - Zwikker-Kosten's model (1949): a unique wavenumber $k(\omega) = \omega\sqrt{\rho(\omega)\chi(\omega)}$
  - Nonlocal theory: discrete wavenumbers solutions to the dispersion equation $\rho(\omega, k)\chi(\omega, k)\omega^2 = k^2$

Nonlocal dispersion equation and Kirchhoff's give exactly the same results?

Kirchhoff

- Microscopic viscothermal fluid Eqs give the solution fields

$$\begin{aligned} &v_x(t,\mathbf{r}) = v_x(\omega,k,r)e^{-i\omega t+ikx}, && v_r(t,\mathbf{r}) = v_r(\omega,k,r)e^{-i\omega t+ikx} \\ &p(t,\mathbf{r}) = p(\omega,k,r)e^{-i\omega t+ikx}, && b(t,\mathbf{r}) = b(\omega,k,r)e^{-i\omega t+ikx} \\ &\tau(t,\mathbf{r}) = \tau(\omega,k,r)e^{-i\omega t+ikx} \end{aligned}$$

- Kirchhoff dispersion equation $\overset{\text{Newton}}{\Longrightarrow}$ $k_K(\omega)$
- Microscopic fields p and v_x can be then obtained explicitly
- Impedances are determined through

$$Z_K(\omega) = \frac{H}{\langle v_x\rangle} = \frac{\langle pv_x\rangle}{\langle v_x\rangle^2}, \text{ provided that } \langle pv_x\rangle = H\langle v_x\rangle$$

- Frequency-dependent densities and bulk modulii

$$\rho_K(\omega) = \frac{k_K Z_K}{\omega}, \qquad \chi_K^{-1}(\omega) = \frac{\omega Z_K}{k_K}$$

Circular tube: wavenumbers and impedances

- Viscothermal fluid Eqs with stirring force

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \mathbf{v} + \left(\zeta + \tfrac{1}{3}\eta\right) \boldsymbol{\nabla}\left(\boldsymbol{\nabla}.\mathbf{v}\right) + \mathbf{F} e^{-i\omega t + ikx}$$

$+$ Other equations...

where $\mathbf{F} e^{-i\omega t+ikx} = -\boldsymbol{\nabla}\left(\mathcal{P} e^{-i\omega t+ikx}\right) = -ik\mathbf{e}_x \mathcal{P} e^{-i\omega t+ikx}$

- Nonlocal density $\rho(\omega, k) = \dfrac{k\left(P(\omega,k) + \mathcal{P}\right)}{\omega\langle v_x(\omega,k,r)\rangle}$, with $\langle p v_x\rangle = P\langle v_x\rangle$

- Viscothermal fluid Eqs with stirring heating

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \boldsymbol{\nabla}^2 \tau + \dot{Q} e^{-i\omega t+ikx}$$

$+$ Other equations...

where $\dot{Q} e^{-i\omega t+ikx} = \beta_0 T_0 \dfrac{\partial}{\partial t}\left(\mathcal{P} e^{-i\omega t+ikx}\right) = -i\omega\beta_0 T_0 \mathcal{P} e^{-i\omega t+ikx}$

- Nonlocal bulk modulus $\chi^{-1}(\omega, k) = \dfrac{P(\omega,k) + \mathcal{P}}{\langle b'(\omega,k,r)\rangle + \gamma\chi_0\mathcal{P}}$ with $\langle p v_x\rangle = P\langle v_x\rangle$

Nonlocal

- For each ω and k, we can compute the nonlocal density $\rho(\omega, k)$ and bulk modulus $\chi^{-1}(\omega, k)$ by the two action-response problems
- Nonlocal dispersion equation

$$\rho(\omega, k)\chi(\omega, k)\omega^2 = k^2 \overset{\text{Newton}}{\Longrightarrow} k_{NL}(\omega)$$

- We obtain $\rho(\omega, k_{NL})$ and $\chi^{-1}(\omega, k_{NL})$
- Nonlocal impedances $Z_{NL}(\omega) = \sqrt{\rho(\omega, k_{NL})\chi^{-1}(\omega, k_{NL})}$

Zwikker-Kosten (Local)

- We have explicit expression of $k_L(\omega)$, $\rho_L(\omega)$, $\chi_L^{-1}(\omega)$ and $Z_L(\omega) = \sqrt{\rho_L(\omega)\chi_L^{-1}(\omega)}$

Circular tube: checkings

- Narrow tubes: $R = 10^{-4}$ m, $f = 100$ Hz
  - Wavenumbers

k_L	$7.01099685499484 + 6.61504658906530i$
k_K	$7.01099585405403 + 6.61504764250774i$
k_{NL}	$7.01099585405408 + 6.61504764250779i$

  - Impedances

Z_L	$1.122582910810147 \times 10^3 + 1.037174340699598 \times 10^3 i$
Z_K	$1.122582790953336 \times 10^3 + 1.037174463077600 \times 10^3 i$
Z_{NL}	$1.122582790953338 \times 10^3 + 1.037174463077570 \times 10^3 i$

Circular tube: checkings

- Wide tubes: $R = 10^{-3}$ m, $f = 10$ kHz
  - Wavenumbers

k_L	$1.877218171102030 \times 10^2 + 3.047328259173055i$
k_K	$1.877217761268940 \times 10^2 + 3.050105788888088i$
k_{NL}	$1.877217761268940 \times 10^2 + 3.050105788888080i$

  - Impedances

Z_L	$4.122429513133025 \times 10^2 + 2.490000257701453i$
Z_K	$4.122428467151478 \times 10^2 + 2.490594992301288i$
Z_{NL}	$4.122428467151478 \times 10^2 + 2.490594992301361i$

Circular tube: checkings

- Very wide tubes: $R = 10^{-2}m$, $f = 500kHz$
  - Wavenumbers

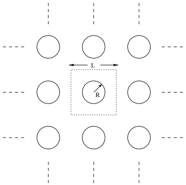
k_L	$9.238319493530025 \times 10^3 + 2.120643432935189i$
k_K	$9.230724176891270 \times 10^3 + 6.352252888390387i$
k_{NL}	$9.230724176891270 \times 10^3 + 6.352252888390393i$

  - Impedances

Z_L	$4.099012309061263 \times 10^2 + 3.362191197362033 \times 10^{-2}i$
Z_K	$2.443313123663708 \times 10^2 - 1.548257724791978 \times 10^3 i$
Z_{NL}	$2.443313123674136 \times 10^2 - 1.548257724791633 \times 10^3 i$

2D arrays of rigid cylinders

- Solving the microscopic equations with the geometry: 2D arrays of rigid cylinders- Finite Element FreeFem++



- Local theory: phase velocity of the unic wave
  $$c(\omega) = \frac{1}{\sqrt{\rho(\omega)\chi(\omega)}}$$
- Nonlocal theory: may be more than one wave solutions of the dispersion equation $\rho(k,\omega)\chi(k,\omega)\omega^2/k^2 = 1$
  phase velocity of the least attenuated wave $c(\omega) = \dfrac{\omega}{k}$
- Multiple scatering (quasi-exact): phase velocity of the least attenuated wave (Duclos et al., 2009)

Rigid cylinders: nonlocal effective density

- Equations to find the amplitudes of $\mathbf{v}(t,\mathbf{r}) = \mathbf{v}(\omega,k,\mathbf{r})e^{-i\omega t+ikx}$ and $p(t,\mathbf{r}) = p(\omega,k,\mathbf{r})e^{-i\omega t+ikx}$

  In $\mathcal{V}_f$:

$$-i\omega b + \nabla.\mathbf{v} + ikv_x = 0$$

$$-i\omega\rho_0\mathbf{v} = -\nabla p - ikp\mathbf{e}_x + \eta\nabla^2\mathbf{v} + 2ik\eta\frac{\partial\mathbf{v}}{\partial x} - \eta k^2\mathbf{v} + \left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla.\mathbf{v})$$

$$+ik\left(\zeta+\frac{1}{3}\eta\right)(\nabla.\mathbf{v})\mathbf{e}_x + ik\left(\zeta+\frac{1}{3}\eta\right)\nabla v_x - \left(\zeta+\frac{1}{3}\eta\right)k^2v_x\mathbf{e}_x - ik\mathbf{e}_x\mathcal{P}$$

$$-i\omega\rho_0c_p\tau = -i\omega\beta_0T_0p + \kappa\nabla^2\tau + 2ik\kappa\frac{\partial\tau}{\partial x} - k^2\kappa\tau$$

$$\gamma\chi_0 p = b + \beta_0\tau$$

  On $\partial\mathcal{V}$: $\mathbf{v} = 0, \qquad \tau = 0$

- Periodic conditions for field amplitudes on the border of the cell
- Poynting-Schoch: $P(\omega,k)\langle\mathbf{v}(\omega,k,\mathbf{r})\rangle = \langle p(\omega,k,\mathbf{r})\mathbf{v}(\omega,k,\mathbf{r})\rangle$
- Maxwell: $\rho(\omega,k) = \dfrac{k(P(\omega,k)+\mathcal{P})}{\omega\langle\mathbf{v}(\omega,k,\mathbf{r})\rangle\cdot\mathbf{e}_x}$

Rigid cylinders: nonlocal effective bulk modulus

- Equations to find the amplitudes of $\mathbf{v}(t,\mathbf{r}) = \mathbf{v}(\omega,k,\mathbf{r})e^{-i\omega t+ikx}$, $p(t,\mathbf{r}) = p(\omega,k,\mathbf{r})e^{-i\omega t+ikx}$ and $b'(t,\mathbf{r}) = b'(\omega,k,\mathbf{r})e^{-i\omega t+ikx}$

  In $\mathcal{V}_f$:

$$
\begin{aligned}
&-i\omega b' + \boldsymbol{\nabla}.\mathbf{v} + ikv_x = 0\\
&-i\omega\rho_0\mathbf{v} = -\boldsymbol{\nabla}p - ikp\mathbf{e}_x + \eta\boldsymbol{\nabla}^2\mathbf{v} + 2ik\eta\frac{\partial\mathbf{v}}{\partial x} - \eta k^2\mathbf{v} + \left(\zeta+\frac{1}{3}\eta\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{v})\\
&+ik\left(\zeta+\frac{1}{3}\eta\right)(\boldsymbol{\nabla}.\mathbf{v})\mathbf{e}_x + ik\left(\zeta+\frac{1}{3}\eta\right)\boldsymbol{\nabla}v_x - \left(\zeta+\frac{1}{3}\eta\right)k^2v_x\mathbf{e}_x\\
&-i\omega\rho_0c_p\tau = -i\omega\beta_0T_0p + \kappa\boldsymbol{\nabla}^2\tau + 2ik\kappa\frac{\partial\tau}{\partial x} - k^2\kappa\tau - i\omega\beta_0T_0\mathcal{P}\\
&\gamma\chi_0p = b' + \beta_0\tau
\end{aligned}
$$

  On $\partial\mathcal{V}$: $\mathbf{v}=0, \qquad \tau = 0$

- Periodic conditions for field amplitudes on the border of the cell
- Poynting-Schoch: $P(\omega,k)\langle\mathbf{v}(\omega,k,\mathbf{r})\rangle = \langle p(\omega,k,\mathbf{r})\mathbf{v}(\omega,k,\mathbf{r})\rangle$
- Maxwell: $\chi^{-1}(\omega,k) = \dfrac{P(\omega,k)+\mathcal{P}}{\langle b'(\omega,k,\mathbf{r})\rangle + \phi\gamma\chi_0\mathcal{P}}$

Rigid cylinders: local effective density and bulk modulus

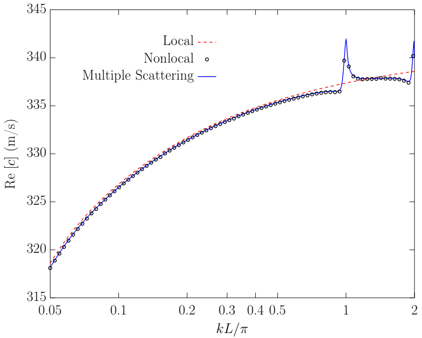
- A harmonic bulk force $\mathbf{f}(t) = f_0 e^{-i\omega t}\mathbf{e}_x$, with constant f_0, is applied
  In $\mathcal{V}_f$:
  $$\nabla.\mathbf{v} = 0$$
  $$-i\omega\rho_0\mathbf{v} = -\nabla p + \eta\nabla^2\mathbf{v} + f_0\mathbf{e}_x$$
  On $\partial\mathcal{V}$: $\mathbf{v} = 0$
  Gives the amplitudes of the fields $\mathbf{v}(t,\mathbf{r}) = \mathbf{v}(\omega,\mathbf{r})e^{-i\omega t}$ and $p(t,\mathbf{r}) = p(\omega,\mathbf{r})e^{-i\omega t}$
  Effective density: $\rho(\omega) = -\dfrac{f_0}{i\omega\langle\mathbf{v}(\omega,\mathbf{r})\rangle}$
- Applying a stirring heating $\dot{Q}(t) = \dot{Q}_0 e^{-i\omega t}$, gives the amplitude of the field $\tau(t,\mathbf{r}) = \tau(\omega,\mathbf{r})e^{-i\omega t}$ through
  In $\mathcal{V}_f$: $-i\omega\rho_0 c_p\tau = \kappa\nabla^2\tau + \dot{Q}_0$
  On $\partial\mathcal{V}$: $\tau = 0$
  Effective bulk modulus: $\chi^{-1}(\omega) = \chi_0^{-1}\left[\gamma + (\gamma-1)\frac{i\omega\rho_0 c_p\langle\tau(\omega,\mathbf{r})\rangle}{\dot{Q}_0}\right]^{-1}$

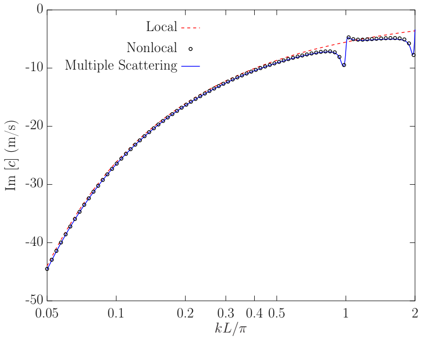
Rigid cylinders: phase velocities

- The medium is filled with air
- Local phase velocity of the single wave $c(\omega) = \dfrac{1}{\sqrt{\rho(\omega)\chi(\omega)}}$
- Nonlocal: For each ω solving nonlocal dispersion equation $\rho(\omega, k)\chi(\omega, k)\omega^2 = k^2 \overset{\text{Newton}}{\Longrightarrow}$ the least attenuated wave number $k(\omega)$
- Initial values of k are taken with 20% of discrepancy with respect to the least attenuated Bloch wavenumber obtained by multiple scattering method
- Phase velocity of the least attenuated wave $c(\omega) = \dfrac{\omega}{k}$

Rigid cylinders: results

$\phi = 0.99$

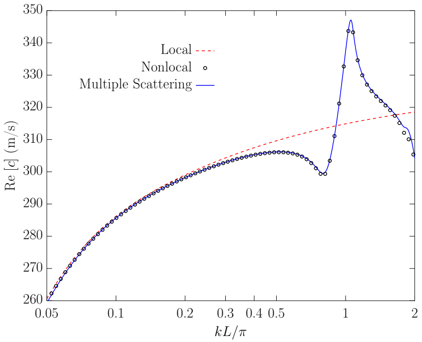


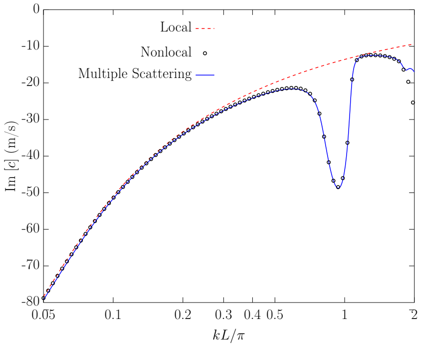


$L = 10\ \mu m,\ R = 1.8\ \mu m$

Rigid cylinders: results

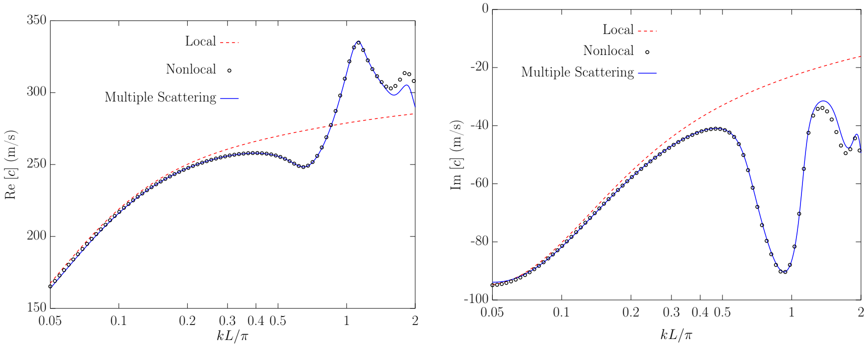
$\phi = 0.9$





$L = 10\ \mu m,\ R = 1.8\ \mu m$

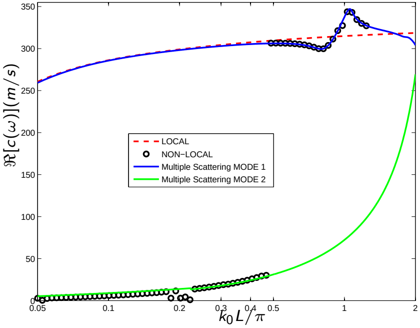
$\phi = 0.7$

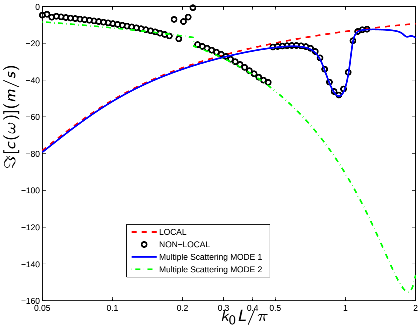


$L = 10\ \mu m,\ R = 1.8\ \mu m$

2nd mode

$$\phi = 0.9$$

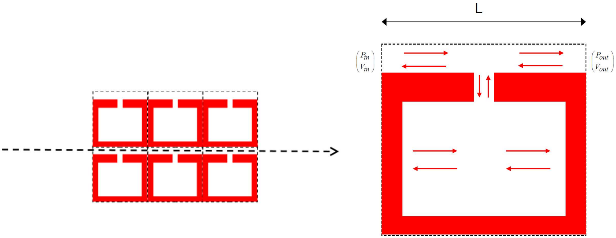




$L = 10\ \mu m,\ R = 1.8\ \mu m$

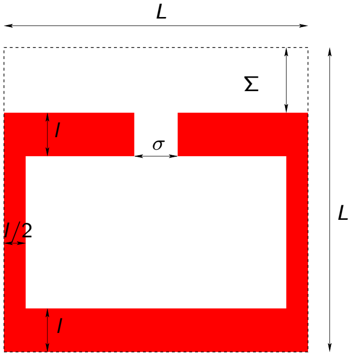
Array of Helmholtz resonators

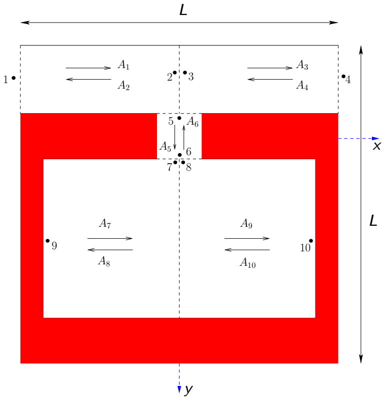
- Bloch-wave modelling vs nonlocal theory: looking for the wavenumbers



  - Bloch-wave modelling: computing the least attenuated Bloch-wavenumber k_B
    $$\begin{pmatrix} P_{out} \\ V_{out} \end{pmatrix} = \begin{pmatrix} P_{in} \\ V_{in} \end{pmatrix} e^{ik_B(\omega)L}$$
  - Nonlocal theory: wavenumber of the least attenuated wave among solutions to the dispersion equation
    $\rho(k,\omega)\chi(k,\omega)\omega^2 = k^2$

Array of Helmholtz resonators: continuity relations





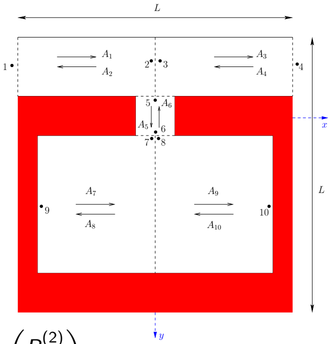
- Continuity relations

$$P_t^{(4)} = e^{ikL} P_t^{(1)} \qquad V_t^{(4)} = e^{ikL} V_t^{(1)}$$
$$P_t^{(3)} = P_t^{(2)} \qquad P_n^{(5)} = P_t^{(2)}$$
$$V_t^{(2)} - V_t^{(3)} = V_n^{(5)} \qquad P_n^{(6)} = P_c^{(7)}$$
$$V_n^{(6)} + V_c^{(7)} = V_c^{(8)} \qquad P_c^{(7)} = P_c^{(8)}$$
$$V_c^{(9)} = 0 \qquad V_c^{(10)} = 0$$

t: tube
n: neck
c: cavity

- Zwikker and Kosten's equations in the cavity

$$-i\omega\frac{\rho_c(\omega)}{S_c}V_c = -\frac{\partial P_c}{\partial x}$$
$$i\omega S_c\chi_c(\omega)P_c = \frac{\partial V_c}{\partial x}$$



Solution: $\begin{pmatrix} P_c \\ V_c \end{pmatrix} = \begin{pmatrix} 1 \\ Y_c \end{pmatrix} A^+ e^{ik_c x} + \begin{pmatrix} 1 \\ -Y_c \end{pmatrix} A^- e^{-ik_c x}$

$$\implies Y_6 = V_n^{(6)}/P_n^{(6)} \implies Y_r = V_n^{(5)}/P_n^{(5)}$$

$$\begin{pmatrix} P_t^{(3)} \\ V_t^{(3)} \end{pmatrix} = e^{ik_BL}\begin{pmatrix} \cos k_tL & -\frac{i}{Y_t}\sin k_tL \\ -iY_t\sin k_tL & \cos k_tL \end{pmatrix}\begin{pmatrix} P_t^{(2)} \\ V_t^{(2)} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{Y_r} - e^{ik_BL}\left(\frac{1}{Y_r}\cos k_tL - \frac{i}{Y_t}\sin k_tL\right) & -\frac{1}{Y_r}\left(1 + e^{ik_BL}\cos k_tL\right) \\ e^{ik_BL}\left(i\frac{Y_t}{Y_r}\sin k_tL - \cos k_tL\right) & 1 - e^{ik_BL}\frac{iY_t}{Y_r}\sin k_tL \end{pmatrix}\begin{pmatrix} V_t^{(2)} \\ V_t^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies e^{2ik_BL} - De^{ik_BL} + 1 = 0, \qquad \text{with} \qquad D = \left(2\cos k_tL - i\frac{Y_r}{Y_t}\sin k_tL\right)$$

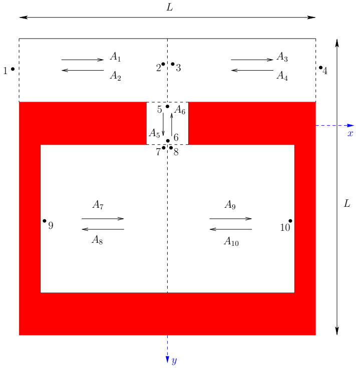
$$k_B = -\frac{i}{L}\ln\left(\frac{D}{2} \pm \sqrt{\frac{D^2}{4} - 1}\right)$$

Array of Helmholtz resonators: effective density

- Zwikker and Kosten's equations

  A bulk force $f(t,x) = f_0 e^{-i\omega t + ikx}$ is applied in the $+x$ direction

$$\rho(\omega, k) = \frac{f_0 - ikH}{-i\omega\langle v\rangle}, \qquad H(\omega, k) = \langle pv\rangle \,/\, \langle v\rangle$$



$$-i\omega \frac{\rho_\alpha(\omega)}{S_\alpha} V_\alpha = -\frac{\partial P_\alpha}{\partial x} + \underbrace{f}_{=\,0,\ \text{for the neck}}$$

$$i\omega S_\alpha \chi_\alpha(\omega) P_\alpha = \frac{\partial V_\alpha}{\partial x}$$

For the neck: $x \to y$

Solutions: $$\begin{pmatrix} P_\alpha \\ V_\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ Y_\alpha \end{pmatrix} A_m^+ e^{ik_\alpha x} + \begin{pmatrix} 1 \\ -Y_\alpha \end{pmatrix} A_m^- e^{-ik_\alpha x} + \underbrace{\begin{pmatrix} B_\alpha \\ C_\alpha \end{pmatrix} f_0 e^{ik_\alpha x}}_{=\,0,\ \text{for the neck}}$$

10 continuity relations $\Longrightarrow$ 10 equations yielding 10 amplitudes A_m

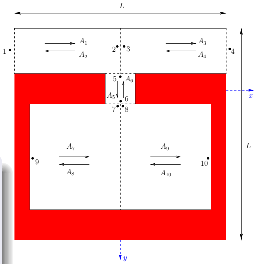
$$\langle v\rangle = \frac{1}{L^2}\left(\int_{-L/2}^{0} V_t\, dx + \int_0^{L/2} V_t\, dx + \int_{-(L-l)/2}^{0} V_c\, dx + \int_0^{(L-l)/2} V_c\, dx\right)$$

$$\langle pv\rangle = \frac{1}{L^2}\left(\int_{-L/2}^{0} P_t V_t\, dx + \int_0^{L/2} P_t V_t\, dx + \int_{-(L-l)/2}^{0} P_c V_c\, dx + \int_0^{(L-l)/2} P_c V_c\, dx\right)$$

Array of Helmholtz resonators: effective bulk modulus

- Zwikker and Kosten's equations

  A stirring heating $\dot{Q}(t,x) = -i\omega\beta_0 T_0 \mathcal{P} e^{-i\omega t + ikx}$ is applied $\implies \chi^{-1}(\omega,k) = \dfrac{H(\omega,k)+\mathcal{P}}{\langle b'(\omega,k,\mathbf{x})\rangle + \phi\gamma\chi_0\mathcal{P}}$



$$-i\omega\frac{\rho_\alpha(\omega)}{S_\alpha}V_\alpha = -\frac{\partial P_\alpha}{\partial x}$$

$$i\omega S_\alpha\chi_\alpha(\omega)P_\alpha + i\omega(S_\alpha\chi_\alpha(\omega) - \gamma S_\alpha\chi_0)\mathcal{P}\underbrace{\langle e^{ikx}\rangle}_{\text{for the neck}} = \frac{\partial V_\alpha}{\partial x}$$

For the neck: $x \to y$

Solutions: $$\begin{pmatrix} P_\alpha \\ V_\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ Y_\alpha \end{pmatrix} A_m^+ e^{ik_\alpha x} + \begin{pmatrix} 1 \\ -Y_\alpha \end{pmatrix} A_m^- e^{-ik_\alpha x} + \begin{pmatrix} B_\alpha \\ C_\alpha \end{pmatrix}\mathcal{P}e^{ik_\alpha x}$$

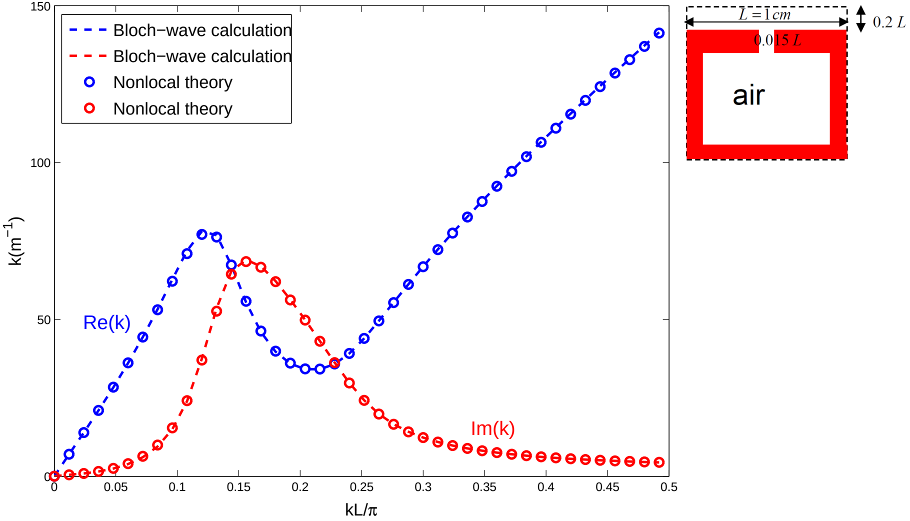
10 continuity relations $\implies$ 10 equations yielding 10 amplitudes A_m

$$\langle v\rangle = \tfrac{1}{L^2}\left(\int_{-L/2}^{0} V_t\,dx + \int_0^{L/2} V_t\,dx + \int_{-(L-l)/2}^{0} V_c\,dx + \int_0^{(L-l)/2} V_c\,dx\right)$$

$$\langle pv\rangle = \tfrac{1}{L^2}\left(\int_{-L/2}^{0} P_tV_t\,dx + \int_0^{L/2} P_tV_t\,dx + \int_{-(L-l)/2}^{0} P_cV_c\,dx + \int_0^{(L-l)/2} P_cV_c\,dx\right)$$

$$-i\omega\langle b'\rangle = -\tfrac{1}{L^2}\int \boldsymbol{\nabla}\cdot\mathbf{v}\,dxdy = -\tfrac{1}{L^2}\oint \mathbf{v}\cdot\mathbf{n}\,dS = -\tfrac{1}{L^2}\left(-V_t^{(1)} + V_t^{(4)}\right)$$

Array of Helmhotz resonators: results



Conclusions and perspectives

- Inspired by the electromagnetic theory and a thermodynamic concept, a new nonlocal macroscopic theory of sound propagation in rigid-framed porous media saturated with a viscothermal fluid has been successfully established
- An upscaling procedure to coarse-grain the dissipative fluid dynamics has been proposed
- No constraint for the wavelength is required
- Maxwellian formulation of the sound propagation in viscothermal fluids leads to the Kirchhoff equation
- The new theory and upscaling procedure has been validated with three geometries of the porous structure
  - Circular tube
  - Arrays of rigid cylinders
  - Array of Helmholtz resonators

Conclusions and perspectives

- FEM numerical simulations to compute the wavenumbers of the higher order modes for the case of lattice of rigid cylinders are in progress
- FEM numerical simulations to compute the wavenumbers for the case of Helmholtz resonators are in progress
- Geometries leading to spatial dispersion for developing more efficient sound absorbing materials
- Comparison with the higher order of classical homogenization theory
- How to generalize the present nonlocal theory when the medium is bounded, and when the medium is poroelastic?