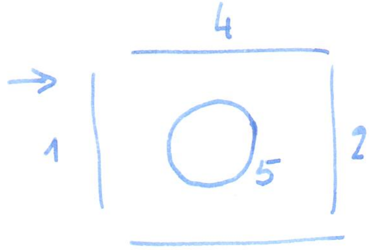
Formulations faibles - Dernière version

•) Densité



I) $-i\omega\rho_0\vec{v} = -\vec{\nabla}p + \eta\nabla^2\vec{v} + (\varsigma + \frac{1}{3}\eta)\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + \underbrace{f_0 e^{ikx}\hat{x}}_{\vec{f}}$

$\vec{v} = \tilde{\vec{v}}\, e^{ikx}$

$$\begin{cases} \vec{f} = f_0\hat{x}\, e^{ikx} \\ \Phi = \Phi_0\, e^{ikx} \\ \vec{f} = -\vec{\nabla}\Phi \Rightarrow f_0 = -ik\Phi_0 \end{cases}$$

$p = \tilde{p}\, e^{ikx}$

Formulation faible

1) $\vec{\nabla}p = (\vec{\nabla}\tilde{p})\, e^{ikx} + (\vec{\nabla}e^{ikx})\, p$

$$\Rightarrow \boxed{\vec{\nabla}p = (\vec{\nabla}\tilde{p})\, e^{ikx} + (ik\tilde{p})\, e^{ikx}\,\hat{x}}$$

2) $\nabla^2\vec{v} = \nabla^2 v_x\,\hat{x} + \nabla^2 v_y\,\hat{y}$

•) $\nabla^2 v_x = \nabla^2(\tilde{v}_x e^{ikx}) = \vec{\nabla}\cdot\vec{\nabla}(\tilde{v}_x e^{ikx})$

$= \vec{\nabla}\cdot\left[(\vec{\nabla}\tilde{v}_x)\, e^{ikx} + \tilde{v}_x(\vec{\nabla}e^{ikx})\right]$

$= \vec{\nabla}\cdot\left[(\vec{\nabla}\tilde{v}_x)\, e^{ikx} + ik\tilde{v}_x\, e^{ikx}\,\hat{x}\right]$

$= (\nabla^2\tilde{v}_x)\, e^{ikx} + (\vec{\nabla}\tilde{v}_x)\cdot ik e^{ikx}\,\hat{x} + ik\,\vec{\nabla}\cdot(\tilde{v}_x\hat{x})\, e^{ikx}$

$+ (ik)\,\tilde{v}_x\hat{x}\cdot\vec{\nabla}(e^{ikx})$

$= (\nabla^2\tilde{v}_x)\, e^{ikx} + ik(\partial_x\tilde{v}_x)\, e^{ikx} + ik(\partial_x\tilde{v}_x)\, e^{ikx} - k^2 v_x e^{ikx}$

$$\Rightarrow \boxed{\nabla^2 v_x = (\nabla^2\tilde{v}_x)\, e^{ikx} + 2ik(\partial_x\tilde{v}_x)\, e^{ikx} - k^2 v_x e^{ikx}}$$

$$\Rightarrow \boxed{\nabla^2 v_y = (\nabla^2\tilde{v}_y)\, e^{ikx} + 2ik(\partial_x\tilde{v}_y)\, e^{ikx} - k^2 v_y e^{ikx}}$$

②

$$\Rightarrow \boxed{\nabla^2\vec{v} = (\nabla^2\vec{\tilde{v}})e^{ikx} + 2ik(\partial_x\vec{\tilde{v}})e^{ikx} - k^2\vec{\tilde{v}}e^{ikx}}$$

3) $\vec{\nabla}(\vec{\nabla}.\vec{v}) = ?$

$$\vec{\nabla}.\vec{v} = \vec{\nabla}.(\vec{\tilde{v}}e^{ikx}) = (\vec{\nabla}.\vec{\tilde{v}})e^{ikx} + \vec{\tilde{v}}.(\vec{\nabla}e^{ikx})$$

$$= (\vec{\nabla}.\vec{\tilde{v}})e^{ikx} + \vec{\tilde{v}}.(ik\,e^{ikx}\,\hat{x})$$

$$= (\vec{\nabla}.\vec{\tilde{v}})e^{ikx} + ik\tilde{v}_x\,e^{ikx}$$

$\Rightarrow$

$$\vec{\nabla}\left[(\vec{\nabla}.\vec{\tilde{v}})e^{ikx} + ik\tilde{v}_x e^{ikx}\right] = \vec{\nabla}\left[(\vec{\nabla}.\vec{\tilde{v}})e^{ikx}\right] + \vec{\nabla}\left[ik\tilde{v}_x e^{ikx}\right]$$

$$= \vec{\nabla}(\vec{\nabla}.\vec{\tilde{v}})e^{ikx} + ik(\vec{\nabla}.\vec{\tilde{v}})e^{ikx}\hat{x} + ik(\vec{\nabla}\tilde{v}_x)e^{ikx} - k^2 v_x e^{ikx}\hat{x}$$

$$\Rightarrow \boxed{\vec{\nabla}(\vec{\nabla}.\vec{v}) = \vec{\nabla}(\vec{\nabla}.\vec{\tilde{v}})e^{ikx} + ik(\vec{\nabla}.\vec{\tilde{v}})e^{ikx}\hat{x} + ik(\vec{\nabla}v_x)e^{ikx} - k^2 v_x e^{ikx}\hat{x}}$$

Donc après suppression des tildes (I) devient :

$$\boxed{\begin{aligned} -i\omega\rho_0\vec{v} = {} & -\vec{\nabla}p - ikp\hat{x} + \eta\nabla^2\vec{v} + 2i\eta k(\partial_x\vec{v}) - \eta k^2\vec{v} \\ & + (\xi + \tfrac{1}{3}\eta)\vec{\nabla}(\vec{\nabla}.\vec{v}) + (\xi + \tfrac{1}{3}\eta)(ik)(\vec{\nabla}.\vec{v})\hat{x} \\ & + (\xi + \tfrac{1}{3}\eta)(ik)(\vec{\nabla}v_x) - (\xi + \tfrac{1}{3}\eta)k^2 v_x\hat{x} + f_0\hat{x} \end{aligned}}$$

(I*)

• Les termes en k^2 sont négligés

Formulation faible de l'éq. (I*)

On multiplie les 2 côtés de l'éq par $\cdot\vec{u}$, fonc teste de $\vec{v}$ et on intègre dans V

Shape functions :

$\vec{v} \to \vec{u}$

$v_x \to u_x$

$v_y \to u_y$

$p \to r$

1) $-i\omega\rho_0 \vec{v}\cdot\vec{u}$

$$\Rightarrow \boxed{-i\omega\rho_0 \int_V \vec{v}\cdot\vec{u}\, dV}$$

2) $-(\vec{\nabla}p\cdot\vec{u})$

$$\vec{\nabla}\cdot(p\vec{u}) = \vec{u}\cdot\vec{\nabla}p + p(\vec{\nabla}\cdot\vec{u})$$

$$\Rightarrow -\int_V \vec{\nabla}p\cdot\vec{u}\, dV = -\int_V \vec{\nabla}\cdot(p\vec{u})\, dV + \int_V p(\vec{\nabla}\cdot\vec{u})\, dV$$

$$= \int_{S=1,2,3,4,5} p(\vec{u}\cdot\hat{n})\, dS + \int_V p(\vec{\nabla}\cdot\vec{u})\, dV$$

i) par symmetrie :

$$\int_{S=1} p(\vec{u}\cdot\hat{n})\, dS = -\int_{S=2} p(u\cdot\hat{n})\, dS$$

ii) par sym. :

$$\int_{S=3} p(\vec{u}\cdot\hat{n})\, dS = -\int_{S=4} p(\vec{u}\cdot\hat{n})\, dS$$

iii) par C.L. du problème ($\vec{v}_{\partial V} = 0 \Rightarrow \vec{u}_{\partial V} = 0$) :

$$\int_{S=5} p(\vec{u}\cdot\hat{n})\, dS = 0$$

$$\Rightarrow \boxed{-\int_V \vec{\nabla}p\cdot\vec{u}\, dV = +\int_V p(\vec{\nabla}\cdot\vec{u})\, dV}$$

④

3) $-ik p \hat{x} \cdot \vec{u} = -ik p u_x$

$$\Rightarrow \boxed{-ik \int_V p u_x \, dV}$$

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4) $\eta (\nabla^2 \vec{v}) \cdot \vec{u} = \eta (\nabla^2 v_x) u_x + \eta (\nabla^2 v_y) u_y$

$$\begin{cases} \vec{\nabla} \cdot (u_x \vec{\nabla} v_x) = u_x \nabla^2 v_x + \vec{\nabla} v_x \cdot \vec{\nabla} u_x \\ \vec{\nabla} \cdot (u_y \vec{\nabla} v_y) = u_y \nabla^2 v_y + \vec{\nabla} v_y \cdot \vec{\nabla} u_y \end{cases}$$

$$\eta \int_V (\nabla^2 \vec{v}) \cdot \vec{u} \, dV = \int_V \vec{\nabla} \cdot (u_x \vec{\nabla} v_x) \, dV + \int_V \vec{\nabla} \cdot (u_y \vec{\nabla} v_y) \, dV$$

$$- \int_V \vec{\nabla} v_x \cdot \vec{\nabla} u_x \, dV - \int \vec{\nabla} v_y \cdot \vec{\nabla} u_y \, dV$$

$$= \int_{S=1,2,3,4,5} u_x (\vec{\nabla} v_x \cdot \hat{n}) \, dS + \int_{S=1,2,3,4,5} u_y (\vec{\nabla} v_y \cdot \hat{n}) \, dS$$

$$- \int_V (\vec{\nabla} v_x \cdot \vec{\nabla} u_x + \vec{\nabla} v_y \cdot \vec{\nabla} u_y) \, dV$$

i) par sym : $\int_{S=3} u_x (\vec{\nabla} v_x \cdot \hat{n}) \, dS = - \int_{S=4} u_x (\vec{\nabla} v_x \cdot \hat{n}) \, dS \; \cancel{\neq 0}$

ii) par sym : $\int_{S=1} u_x (\vec{\nabla} v_x \cdot \hat{n}) \, dS = - \int_{S=2} u_x (\vec{\nabla} v_x \cdot \hat{n}) \, dS$

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iii) par C.L. $\vec{v}_{S=5} = 0 \Rightarrow \vec{u}_{S=5} = 0$, $\vec{\nabla} v_x{}_{S=5} = 0$

$$\Rightarrow \int_{S=5} u_x (\vec{\nabla} v_x \cdot \hat{n})\, dS = 0$$

iv) par sym : $\left(v_y \, \Big|_{S=3,4,1,2} = 0 \right) \Rightarrow u_y \, \Big|_{S=3,4,1,2} = 0$

$$\Rightarrow \boxed{\eta \int_V \nabla^2 \vec{v} \cdot \vec{u}\, dV = -\eta \int_V \left(\vec{\nabla} v_x \cdot \vec{\nabla} u_x + \vec{\nabla} v_y \cdot \vec{\nabla} u_y \right) dV}$$

5) $2i\eta k (\partial_x \vec{v}) \cdot \vec{u}$

$$\partial_x (\vec{v} \cdot \vec{u}) = (\partial_x \vec{v}) \cdot \vec{u} + (\partial_x \vec{u}) \cdot \vec{v}$$

$$\Rightarrow \int_V (\partial_x \vec{v}) \cdot \vec{u}\, dV = \int_V \partial_x (\vec{v} \cdot \vec{u})\, dV - \int_V (\partial_x \vec{u}) \cdot \vec{v}\, dV$$

$$= \int_{S=1,2,3,4,5} (\vec{v} \cdot \vec{u})\, n_x\, dS - \int_V (\partial_x \vec{u}) \cdot \vec{v}\, dV$$

i) $\displaystyle\int_3 (\vec{v} \cdot \vec{u})\, n_x\, dS = - \int_4 (\vec{v} \cdot \vec{u})\, n_x\, dS$

iii) $\displaystyle\int_{1,2,5} (\vec{v} \cdot \vec{u})\, dS = 0$

$$\Rightarrow \boxed{2i\eta k \int_V (\partial_x \vec{v}) \cdot \vec{u}\, dV = -2i\eta k \int_V (\partial_x \vec{u}) \cdot \vec{v}\, dV}$$

6) $-\eta k^2 \vec{v} \cdot \vec{u}$

$$\Rightarrow \boxed{-\eta k^2 \int_V \vec{v} \cdot \vec{u}\, dV} =$$

7) $(\varsigma + \frac{1}{3}\eta)\left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v}) \cdot \vec{u}\right]$

$$\vec{\nabla} \cdot \left[(\vec{\nabla} \cdot \vec{v})\vec{u}\right] = \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v})\right] \cdot \vec{u} + (\vec{\nabla} \cdot \vec{v})(\vec{\nabla} \cdot \vec{u})$$

$$\Rightarrow \int_V \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v})\right] \cdot \vec{u}\, dV = \int_V \vec{\nabla} \cdot \left[(\vec{\nabla} \cdot \vec{v})\vec{u}\right] dV - \int_V (\vec{\nabla} \cdot \vec{v})(\vec{\nabla} \cdot \vec{u})\, dV$$

$$= \int_S (\vec{\nabla} \cdot \vec{v})(\vec{u} \cdot \hat{n})\, dS - \int_V (\vec{\nabla} \cdot \vec{v})(\vec{\nabla} \cdot \vec{u})\, dV$$

i) $\int_{S=1} (\vec{\nabla} \cdot \vec{v})(\vec{u} \cdot \hat{n})\, dS = -\int_{S=2} (\vec{\nabla} \cdot \vec{v})(\vec{u} \cdot \hat{n})\, dS$

ii) $\int_{S=3,4} (\vec{\nabla} \cdot \vec{v})\underbrace{(\vec{u} \cdot \hat{n})}_{=0}\, dS = 0$

$$\Rightarrow \boxed{(\varsigma + \frac{1}{3}\eta)\int_V \left[\vec{\nabla}(\vec{\nabla} \cdot \vec{v})\right] \cdot \vec{u}\, dV = -(\varsigma + \frac{1}{3}\eta)\int_V (\vec{\nabla} \cdot \vec{v})(\vec{\nabla} \cdot \vec{u})\, dV}$$

8) $(\varsigma + \frac{1}{3}\eta)(ik)(\vec{\nabla}\cdot\vec{v})\hat{x}\cdot\vec{u} = i(\varsigma+\frac{1}{3}\eta)k(\vec{\nabla}\cdot\vec{v})u_x$

$$\Rightarrow \boxed{i(\varsigma + \frac{1}{3}\eta)k\int_V (\vec{\nabla}\cdot\vec{v})\,u_x\,dV}$$

9) $(\varsigma + \frac{1}{3}\eta)(ik)(\vec{\nabla}v_x\cdot\vec{u})$

$$\vec{\nabla}\cdot(v_x\vec{u}) = \vec{\nabla}v_x\cdot\vec{u} + (\vec{\nabla}\cdot\vec{u})v_x$$

$$\Rightarrow \int_V \vec{\nabla}v_x\cdot\vec{u}\,dV = \int_V \vec{\nabla}\cdot(v_x\vec{u})\,dV - \int_V (\vec{\nabla}\cdot\vec{u})v_x\,dV$$

$$= \int_S v_x(\vec{u}\cdot\hat{n}) - \int_V (\vec{\nabla}\cdot\vec{u})v_x\,dV$$

i) $\int_{S=1} v_x(\vec{u}\cdot\hat{n})\,dS = -\int_{S=2} v_x(\vec{u}\cdot\hat{n})\,dS$

ii) $\int_{S=3,4} v_x(\vec{u}\cdot\hat{n})\,dS = 0$

$$\Rightarrow \boxed{i(\varsigma + \frac{1}{3}\eta)k\int_V(\vec{\nabla}v_x\cdot\vec{u})\,dV = -i(\varsigma+\frac{1}{3}\eta)k\int_V(\vec{\nabla}\cdot\vec{u})v_x\,dV}$$

10) $-(\varsigma + \frac{1}{3}\eta)k^2 v_x\hat{x}\cdot\vec{u} = -(\varsigma+\frac{1}{3}\eta)k^2 v_x u_x$

$$\Rightarrow \boxed{-(\varsigma + \frac{1}{3}\eta)k^2\int_V v_x u_x\,dV}$$

11) $f_0 \hat{x} \cdot \vec{u} = f_0 u_x$

$$\Rightarrow \boxed{f_0 \int_V u_x \, dV}$$

Donc,

$$\boxed{\begin{aligned} &-i\omega \rho_0 \int_V \vec{v}\cdot\vec{u}\, dV - \int_V p(\vec{\nabla}\cdot\vec{u})\, dV + ik \int_V p\, u_x \, dV \\ &+ \eta \int_V (\vec{\nabla} v_x \cdot \vec{\nabla} u_x + \vec{\nabla} v_y \cdot \vec{\nabla} u_y)\, dV + 2i\eta k \int_V (\partial_x \vec{u})\cdot\vec{v}\, dV \\ &+ \eta k^2 \int_V \vec{v}\cdot\vec{u} + (\varsigma + \tfrac{1}{3}\eta) \int_V (\vec{\nabla}\cdot\vec{v})(\vec{\nabla}\cdot\vec{u})\, dV \\ &- i(\varsigma + \tfrac{1}{3}\eta) k \int_V (\vec{\nabla}\cdot\vec{v}) u_x \, dV + i(\varsigma + \tfrac{1}{3}\eta) k \int_V (\vec{\nabla}\cdot\vec{u}) v_x \, dV \\ &+ (\varsigma + \tfrac{1}{3}\eta) k^2 \int_V v_x u_x \, dV - f_0 \int_V u_x \, dV = 0 \end{aligned}}$$

(I - faible)

II) $-i\omega\rho_0 c_p \tau = -\beta_0 T_0 i\omega p + \kappa\nabla^2\tau$

$$\begin{cases} \tau = \tilde{\tau}\, e^{ikx} \\ p = \tilde{p}\, e^{ikx} \end{cases}$$

$$\nabla^2\tau = \nabla^2\left(\tilde{\tau}\, e^{ikx}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(\tilde{\tau}\, e^{ikx}\right)$$

$$= \vec{\nabla}\cdot\left[\left(\vec{\nabla}\tilde{\tau}\right) e^{ikx} + ik\,\tilde{\tau}\, e^{ikx}\,\hat{x}\right]$$

$$= \left(\nabla^2\tilde{\tau}\right) e^{ikx} + ik\left(\vec{\nabla}\tilde{\tau}\right)\cdot\hat{x}\; e^{ikx} + ik\left(\vec{\nabla}\cdot\hat{x}\right)\left(\tilde{\tau}\, e^{ikx}\right)$$

$$= \left(\nabla^2\tilde{\tau}\right) e^{ikx} + ik\left(\partial_x\tilde{\tau}\right) e^{ikx} + ik\,\partial_x\left(\tilde{\tau}\, e^{ikx}\right)$$

$$= \left(\nabla^2\tilde{\tau}\right) e^{ikx} + ik\left(\partial_x\tilde{\tau}\right) e^{ikx} + ik\left(\partial_x\tilde{\tau}\right) e^{ikx} - k^2\tilde{\tau}\, e^{ikx}$$

$$\Rightarrow \boxed{\nabla^2\tau = \left(\nabla^2\tilde{\tau}\right) e^{ikx} + 2ik\left(\partial_x\tilde{\tau}\right) e^{ikx} - k^2\tilde{\tau}\, e^{ikx}}$$

Après suppression des tildes (II) devient :

$$\boxed{-i\omega\rho_0 c_p \tau = -i\omega\beta_0 T_0 p + \kappa\nabla^2\tau + 2ik\kappa\left(\partial_x\tau\right) - k^2\kappa\tau}$$

(II*)

Formulation faible de l'éq II*

Multiplier les deux côtés de l'éq par s et on intègre :

1) $-i\omega \rho_0 C_p \tau s$

Shape functions

$\tau \longrightarrow s$

$p \longrightarrow r$

$$\Rightarrow \boxed{-i\omega \rho_0 C_p \int_V \tau s \, dV}$$

2) $-i\omega \beta_0 T_0 p s$

$$\Rightarrow \boxed{-i\omega \beta_0 T_0 \int_V p s \, dV}$$

3) $K \left(\nabla^2 \tau \right) s$

$$\vec{\nabla} \cdot \left(s \vec{\nabla} \tau \right) = s \nabla^2 \tau + \vec{\nabla} \tau \cdot \vec{\nabla} s$$

$$\Rightarrow \int_V s \nabla^2 \tau \, dV = \int_V \vec{\nabla} \cdot \left(s \vec{\nabla} \tau \right) dV - \int_V \vec{\nabla} \tau \cdot \vec{\nabla} s \, dV$$

$$= \int_S s \left(\vec{\nabla} \tau \cdot \hat{n} \right) dS - \int_V \vec{\nabla} \tau \cdot \vec{\nabla} s \, dV$$

i) $\displaystyle\int_{S=1} s \left(\vec{\nabla} \tau \cdot \hat{n} \right) dS = - \int_{S=2} s \left(\vec{\nabla} \tau \cdot \hat{n} \right) dS$

ii) $\displaystyle\int_{S=3,4} s \left(\vec{\nabla} \tau \cdot \hat{n} \right) dS = 0$

$$\Rightarrow \boxed{\kappa \int_V s\,(\nabla^2 \tau)\, dV = -\kappa \int_V \vec{\nabla}\tau \cdot \vec{\nabla} s \; dV}$$

4) $2ik\kappa\,(\partial_x \tau)\, s$

$$\partial_x(\tau s) = (\partial_x \tau)\, s + \tau\, \partial_x s$$

$$\Rightarrow \int_V s\,(\partial_x \tau)\, dV = \int_V \partial_x(\tau s)\, dV - \int_V \tau\, \partial_x s\, dV$$

$$= \underbrace{\int_S \tau s\, n_x\, dS}_{=0} - \int_V \tau\, \partial_x s\, dV$$

$$\Rightarrow \boxed{2ik\kappa \int_V (\partial_x \tau)\, s\, dV = -2ik\kappa \int_V \tau\, \partial_x s\, dV}$$

5) $-k^2 \kappa\, \tau s$

$$\Rightarrow \boxed{-k^2 \kappa \int_V \tau s\, dV}$$

Donc

$$\boxed{-i\omega\rho_0 c_p \int_V \tau s \, dV + i\omega\beta_0 T_0 \int_V p s \, dV + K \int_V \vec{\nabla}\tau \cdot \vec{\nabla} s \, dV + 2ikK \int_V \tau \, \partial_x s \, dV + k^2 K \int_V \tau s \, dV = 0}$$

(II - faible)

III) $\quad -i\omega b + \vec{\nabla} \cdot \vec{v} = 0$

$$\begin{cases} \vec{v} = \tilde{\vec{v}}\, e^{ikx} \\ b = \tilde{b}\, e^{ikx} \end{cases}$$

$$\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \left(\tilde{\vec{v}}\, e^{ikx}\right) = (\vec{\nabla} \cdot \tilde{\vec{v}})\, e^{ikx} + ik\hat{x} \cdot \tilde{\vec{v}}\, e^{ikx}$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{v} = (\vec{\nabla} \cdot \tilde{\vec{v}})\, e^{ikx} + ik v_x \, e^{ikx}}$$

Après supression des tildes III devient :

$$\Rightarrow \boxed{-i\omega b + \vec{\nabla} \cdot \vec{v} + ik v_x = 0}$$

(III*)

Shape function | $b \to \ell$

multiplier par ℓ et intégration.

Forme faible de III* :

$$\boxed{-i\omega \int_V b\ell + \int_V (\vec{\nabla} \cdot \vec{v})\ell \, dV + ik \int_V v_x \ell = 0}$$

(III - faible)

IV) $\gamma \chi_0 p = b + \beta_0 \tau$: multiplier par r fonction de test de p et intégration

$$\boxed{+ \gamma \chi_0 \int_V p\, r\, dV - \int_V b\, r\, dV - \int_V \beta_0 \tau\, r\, dV = 0}$$

(IV - faible)