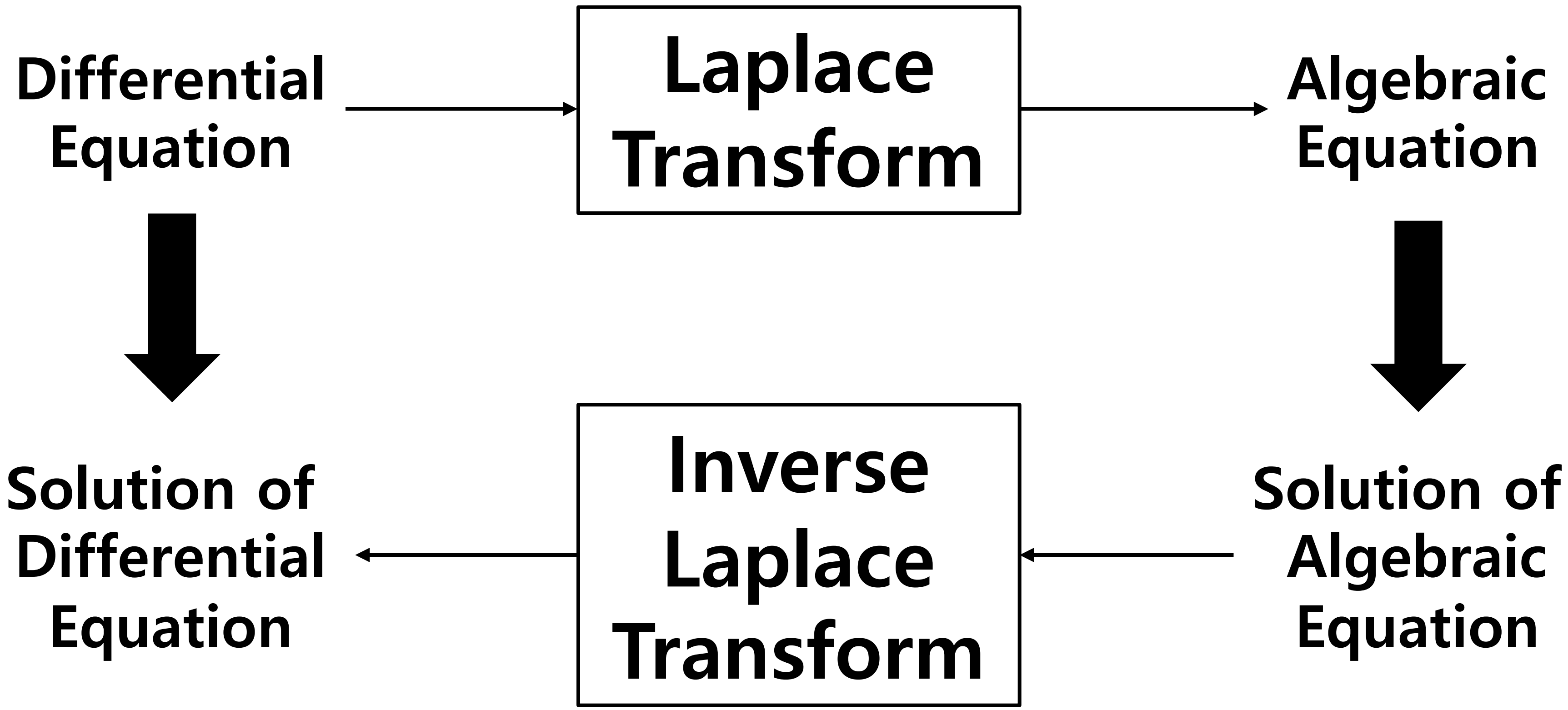
Laplace Transform

Objective…

- Definition and notation for Laplace transform
- Solution of initial value problem
- Shifting and the Heaviside function
- Convolution
- Impulse and the Delta function
- Solutions of System
- Polynomial Coefficients

Why Laplace Transform??



Definition and Notation

- Some algebra problem
  - Easy to solve than initial value problems

- Laplace transform
  - Initial value problem to
    - Algebra problem
    - Solution of the algebra problem
    - Solution of the initial value problem
  - $\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt$
    - Integration with respect to t
    - A function of new variable s for all s such that this integral converges

Laplace Transform

- Ex) $f(t) = e^{at}$
  - $\mathcal{L}[f](s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s}\left[e^{(a-s)t}\right]_{t=0}^{t=\infty} = \frac{1}{(s-a)}$, for $s > a$
  - The Laplace transform of $f(t) = e^{at}$
    - $F(s) = \frac{1}{s-a}$, for $s > a$
- Ex) $f(t) = c$
  - $\mathcal{L}[f](s) = \int_0^\infty c e^{-st}\, dt = -\frac{c}{s}\left[e^{-st}\right]_{t=0}^{t=\infty} = \frac{c}{s}, F(s) = \frac{c}{s}$

By l'Hôpital's rule, $\lim_{t \to a} \frac{f(x)}{g(x)} = \lim_{t \to a} \frac{f'(x)}{g'(x)}$

- Ex) $f(t) = t$
  - $\mathcal{L}[f](s) = \int_0^\infty t e^{-st}\, dt = \left[-\frac{1}{s} t e^{-st}\right]_{t=0}^{t=\infty} - \int_0^\infty \left(-\frac{1}{s} e^{-st}\right) dt = \boxed{\lim_{t \to \infty}\left(-\frac{t}{s e^{st}}\right)} - \left[\frac{1}{s^2} e^{-st}\right]_{t=0}^{t=\infty} = \frac{1}{s^2}, F(s) = \frac{1}{s^2}$

Laplace Transform

By l'Hôpital's rule, $\lim_{t \to a} \frac{f(x)}{g(x)} = \lim_{t \to a} \frac{f'(x)}{g'(x)}$

- Ex) $f(t) = t^2$
  - $\mathcal{L}[f](s) = \int_0^\infty t^2 e^{-st}\,dt = \left[-\frac{1}{s} t^2 e^{-st}\right]_{t=0}^{t=\infty} - \int_0^\infty \left(-\frac{2}{s} e^{-st}\right) dt = \boxed{\lim_{t \to \infty}\left(-\frac{t^2}{se^{st}}\right)} + \frac{2}{s}\boxed{\int_0^\infty te^{-st}\,dt} = \frac{2}{s^3}, F(s) = \frac{2}{s^3}$

    $\frac{1}{s^2}$

- Ex) $f(t) = t^n$
  - $f(t) = c \to F(s) = \frac{c}{s}$
  - $f(t) = t \to F(s) = \frac{1}{s^2}$
  - $f(t) = t^2 \to F(s) = \frac{2}{s^3}$
  - $f(t) = t^n \to F(s) = \frac{n!}{s^{n+1}}$

Laplace Transform

- Ex) $f(t) = \sin(at)$
  - $\mathcal{L}[f](s) = \boxed{\int_0^\infty \sin(at)\, e^{-st}\, dt} = \left[-\frac{1}{s}\sin(at)\, e^{-st}\right]_{t=0}^{t=\infty} - \int_0^\infty -\frac{a}{s}\cos(at)\, e^{-st}\, dt =$ $\left[-\frac{a}{s^2}\cos(at)\, e^{-st}\right]_{t=0}^{t=\infty} - \frac{a^2}{s^2}\int_0^\infty \sin(at)\, e^{-st}\, dt = \frac{a}{s^2} - \frac{a^2}{s^2}\boxed{\int_0^\infty \sin(at)\, e^{-st}\, dt}$
  - $F(s) = \frac{a}{s^2+a^2}$
- Ex) $f(t) = \cos(at)$
  - $\mathcal{L}[f](s) = \int_0^\infty \cos(at)\, e^{-st}\, dt = \left[-\frac{1}{s}\cos(at)\, e^{-st}\right]_{t=0}^{t=\infty} - \frac{a}{s}\int_0^\infty \sin(at) e^{-st}\, dt = \frac{1}{s} - \frac{a^2}{s(s^2+a^2)} =$ $\frac{s}{s^2+a^2}$
  - $F(s) = \frac{s}{s^2+a^2}$

Laplace Transform

- Ex) e^{iat}
  - $\mathcal{L}[f](s) = \int_0^\infty e^{iat} e^{-st}\, dt = \int_0^\infty e^{-(s-ai)t}\, dt = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2} = \mathcal{L}[\cos(at)] + i\mathcal{L}[\sin(at)]$
- Ex) $e^{iat} = \cos(at) + i\sin(at)$
  - Linear property of Laplace transform
    - $\mathcal{L}\left[e^{iat}\right](s) = \mathcal{L}[\cos(at)](s) + \mathcal{L}[\sin(at)](s) = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$

Laplace Transform

- Ex) $f(s) = e^{at}\sin(bt)$

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st}e^{at}\sin(bt)\,dt = \int_0^\infty e^{(a-s)t}\sin(bt)\,dt$$

$$= -\frac{1}{b}\left[e^{(a-s)t}\cos(bt)\right]_{t=0}^{t=\infty} + \frac{a-s}{b}\int_0^\infty e^{(a-s)t}\cos(bt)\,dt$$

$$= \frac{1}{b} + \frac{a-s}{b}\left(\left[\frac{1}{b}e^{(a-s)t}\sin(bt)\right]_{t=0}^{t=\infty} - \frac{a-s}{b}\int_0^\infty e^{(a-s)t}\cos(bt)\,dt\right) = \frac{1}{b} - \left(\frac{a-s}{b}\right)^2\int_0^\infty e^{(a-s)t}\sin(bt)\,dt$$

– $\frac{b^2+(a-s)^2}{b^2}\int_0^\infty e^{(a-s)t}\sin(bt)\,dt = \frac{1}{b}$

– $\int_0^\infty e^{(a-s)t}\sin(bt)\,dt = \frac{b}{(s-a)^2+b^2}$, for $s > a$

Laplace Transform

- Ex) $f(s) = e^{at}\cos(bt)$
  - Find out the Laplace transform of the given function

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st}e^{at}\cos(bt)\,dt = \int_0^\infty e^{(a-s)t}\cos(bt)\,dt = \frac{1}{b}\left[e^{(a-s)t}\sin(bt)\right]_{t=0}^{t=\infty} - \frac{a-s}{b}\int_0^\infty e^{(a-s)t}\sin(bt)\,dt$$

$$= -\frac{a-s}{b}\left(-\frac{1}{b}\left[e^{(a-s)t}\cos(bt)\right]_{t=0}^{t=\infty} + \frac{a-s}{b}\int_0^\infty e^{(a-s)t}\cos(bt)\,dt\right) = -\frac{a-s}{b^2} - \left(\frac{a-s}{b}\right)^2\int_0^\infty e^{(a-s)t}\cos(bt)\,dt$$

  - $-\frac{a-s}{b^2} - \left(\frac{a-s}{b}\right)^2\int_0^\infty e^{(a-s)t}\cos(bt)\,dt = \int_0^\infty e^{(a-s)t}\cos(bt)\,dt$
  - $\int_0^\infty e^{(a-s)t}\cos(bt)\,dt = \frac{s-a}{(s-a)^2+b^2}$, for $s > a$

Laplace Transform

- Properties
  - Linearity
    - $\mathcal{L}[f+g](s) = \mathcal{L}[f](s) + \mathcal{L}[g](s) = F + G$
    - $\mathcal{L}[cf](s) = c\mathcal{L}[f](s) = cF$
  - Inverse of Laplace transform
    - $\mathcal{L}^{-1}[F] = f$ exactly, when $\mathcal{L}[f] = F$
    - Linearity
      - $\mathcal{L}^{-1}[F+G] = \mathcal{L}^{-1}[F] + \mathcal{L}^{-1}[G] = f + g$
      - $\mathcal{L}^{-1}[cF] = c\mathcal{L}^{-1}[F] = cf$

TABLE 3.1 Laplace Transforms of Selected Functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
(1) 1	$\frac{1}{s}$	(8) $t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
(2) t^n	$\frac{n!}{s^{n+1}}$	(9) $t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
(3) e^{at}	$\frac{1}{s-a}$	(10) $e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
(4) $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	(11) $e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
(5) $e^{at}-e^{bt}$	$\frac{a-b}{(s-a)(s-b)}$	(12) $\sinh(at)$	$\frac{a}{s^2-a^2}$
(6) $\sin(at)$	$\frac{a}{s^2+a^2}$	(13) $\cosh(at)$	$\frac{s}{s^2-a^2}$
(7) $\cos(at)$	$\frac{s}{s^2+a^2}$	(14) $\delta(t-a)$	e^{-as}

In each of Problems 1 through 5, use Table 3.1 to determine the Laplace transform of the function.

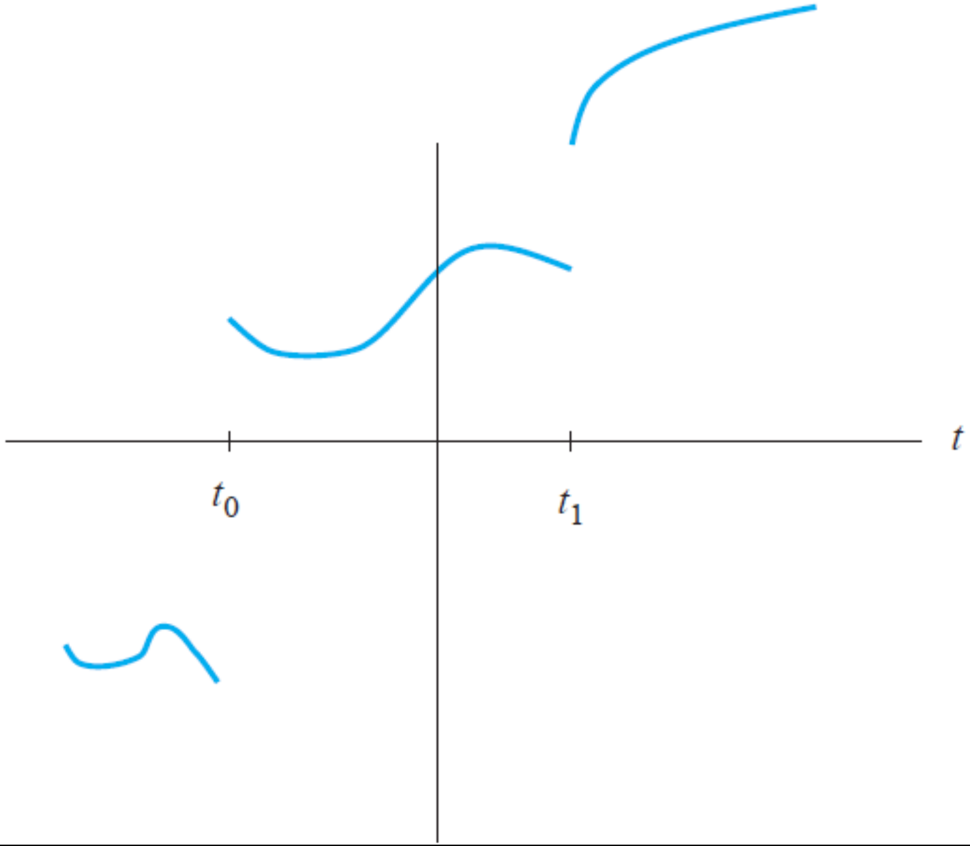
1. $f(t) = 3t\cos(2t)$
2. $g(t) = e^{-4t}\sin(8t)$
3. $h(t) = 14t - \sin(7t)$
4. $w(t) = \cos(3t) - \cos(7t)$
5. $k(t) = -5t^2 e^{-4t} + \sin(3t)$

In each of Problems 6 through 10, use Table 3.1 to determine the inverse Laplace transform of the function.

6. $R(s) = \frac{7}{s^2-9}$
7. $Q(s) = \frac{s}{s^2+64}$
8. $G(s) = \frac{5}{s^2+12} - \frac{4s}{s^2+8}$
9. $P(s) = \frac{1}{s+42} - \frac{1}{(s+3)^4}$
10. $F(s) = \frac{-5s}{(s^2+1)^2}$

A Piecewise Continuous Function

- $f(t)$ is defined at least on $[a, b]$. Then f is piecewise function if
  - f is contiguous at all but perhaps finitely many points of $[a, b]$
  - If f is not continuous at t_0 in (a, b), then $f(t)$ has finite limits from both sides at t_0
  - $f(t)$ has finite limits as t approaches a and as t approaches b from within the interval



Ex) a function g is defined as $g(t) = \begin{cases} \frac{1}{t} & for\ 0 < t \leq 1 \\ 0 & for\ t = 0 \end{cases}$

g is continuous on $(0, 1]$, but is not piecewise continuous on $[0, 1]$
, because $\lim_{t \to 0+} g(t) = \infty$

Solution of Initial Value Problem

- Transform of a derivative
  - Let $f(t)$ be a continuous function for $t \geq 0$
  - Suppose that $f'(t)$ is piecewise continuous on $[0, k]$ for every $k > 0$
  - Suppose that $\lim_{t \to \infty} \frac{f(t)}{e^{st}} = 0$ if $s > 0$
  - $\mathcal{L}[f'(t)](s) = \int_0^\infty e^{-st} f'(t)\, dt = [f(t)e^{-st}]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt$: integration by part
    - $s\int_0^\infty e^{-st} f(t)\, dt = s\mathcal{L}[f](s) = sF(s)$
    - $[f(t)e^{-st}]_0^\infty = \lim_{t \to \infty} \frac{f(t)}{e^{st}} - f(0) = -f(0)$
    - $f(0)$: evaluate from the original function at $y = 0$
    - $\mathcal{L}[f'(t)](s) = sF(s) - f(0)$
    - If f has a jump discontinuity at 0
      - $\mathcal{L}[f'](s) = sF(s) - f(0+)$, where $f(0+) = \lim_{t \to 0+} f(t)$

Solution of Initial Value Problem

- Transform of a derivative
  - Let $f(t)$ and $f'(t)$ be a continuous function for $t \geq 0$
  - Suppose that $f''(t)$ is piecewise continuous on $[0, k]$ for every $k > 0$, and $\lim_{t\to\infty} \frac{f'(t)}{e^{st}} = 0$ if $s > 0$
  - $\mathcal{L}[f''(t)](s) = \int_0^\infty e^{-st} f''(t)\, dt = [f'(t)e^{-st}]_0^\infty + s\int_0^\infty e^{-st} f'(t)\, dt$: integration by part
    - $s\int_0^\infty e^{-st} f'(t)\, dt = s\big(s\mathcal{L}[f](s) - f(0)\big) = s^2F(s) - sf(0)$
    - $[f'(t)e^{-st}]_0^\infty = \lim_{t\to\infty} \frac{f'(t)}{e^{st}} - f'(0) = -f'(0)$
    - $f'(0)$: evaluate from the original function at $y = 0$
    - $\mathcal{L}[f''(t)](s) = s^2F(s) - sf(0) - f'(0)$
- Transform of a higher derivative
  - Let $f, f', f^{(n-1)}$ be continuous for $t > 0$
  - Suppose $f^{(n)}$ is piecewise continuous on $[0, k]$ for every $k > 0$
  - Suppose $\lim_{k\to\infty} e^{-sk} f^{(j)}(k) = 0$, for $s > 0$ and $j = 1, 2, \dots, n-1$
    - $\mathcal{L}\left[f^{(n)}\right](s) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Solution of Initial Value Problem

- Ex) $y' - 4y = 1; y(0) = 1$ →a linear 1st-order differential equation
  - According to process in Chapter 1
    - Integrating factor: $e^{\int -4dx} = e^{-4x}$
    - $e^{-4x}y' - 4e^{-4x}y = e^{-4x}$
    - $(e^{-4x}y')' = e^{-4x}$
    - $e^{-4x}y = -\frac{1}{4}e^{-4x} + c$
    - $y = -\frac{1}{4} + ce^{4x}$
    - $y(0) = -\frac{1}{4} + c = 1 \rightarrow c = \frac{5}{4}$
    - $y = \frac{5}{4}e^{4x} - \frac{1}{4} = \frac{1}{4}(5e^{4x} - 1)$

Solution of Initial Value Problem

- Ex) $y' - 4y = 1; y(0) = 1$
  - $\mathcal{L}[y' - 4y](s) = \mathcal{L}[y'](s) - 4\mathcal{L}[y](s) = sY(s) - y(0) - 4Y(s) = (s-4)Y(s) - 1$
  - $\mathcal{L}[1](s) = \frac{1}{s}$
  - $(s-4)Y(s) - 1 = \frac{1}{s}$
  - $Y(s) = \frac{1}{s-4} + \frac{1}{s(s-4)}$
  - $\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] = e^{4t}, \mathcal{L}^{-1}\left[\frac{1}{s(s-4)}\right] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s}\right] = \frac{1}{4}(e^{4t} - 1)$
  - $y(t) = \frac{5}{4}e^{4t} - \frac{1}{4}$

Solution of Initial Value Problem

- Ex) $y'' + 4y' + 3y = e^t; y(0) = 0, y'(0) = 2$
  - The method of undetermined coefficient in Chapter2
    - The associated homogeneous equation, $y'' + 4y' + 3y = 0$
    - $y_1 = e^{-t}, y_2 = e^{-3t}$
    - $y_p = ae^t$
    - $y_p' = ae^t$
    - $y_p'' = ae^t$
    - Substitute y_p'' and y_p' to the given equation
      - $8ae^t = e^t \rightarrow a = \frac{1}{8}$
    - $y(t) = c_1 e^{-t} + c_2 e^{-3t} + \frac{1}{8} e^t$
    - $y(0) = c_1 + c_2 + \frac{1}{8} = 0, y'(0) = -c_1 - 3c_2 + \frac{1}{8} = 2$
      - $c_1 = \frac{7}{8}, c_2 = -\frac{7}{8}$
    - $y(t) = \frac{3}{4} e^{-t} - \frac{7}{8} e^{-3t} + \frac{1}{8} e^t$

Solution of Initial Value Problem

- Ex) $y'' + 4y' + 3y = e^t; y(0) = 0, y'(0) = 2$
  - $\mathcal{L}[y''] + 4\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[e^t]$
    - $s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 3Y(s) = s^2Y(s) + 4sY(s) + 3Y - 2 = \frac{1}{s-1}$
    - $Y(s) = \frac{2s-1}{(s-1)(s^2+4s+3)} = \frac{2s-1}{(s-1)(s+1)(s+3)} \rightarrow \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3} = \frac{2s-1}{(s-1)(s+1)(s+3)}$
      - $A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1) = 2s - 1$
        - » $s = 1 \rightarrow A = \frac{1}{8}, s = -1 \rightarrow B = \frac{3}{4}, s = -3 \rightarrow C = -\frac{7}{8}$
      - $Y(s) = \frac{1}{8}\frac{1}{s-1} - \frac{3}{4}\frac{1}{s+1} - \frac{7}{8}\frac{1}{s+3}$
    - $y(t) = \frac{1}{8}e^t + \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}$

The Shifting Theorems

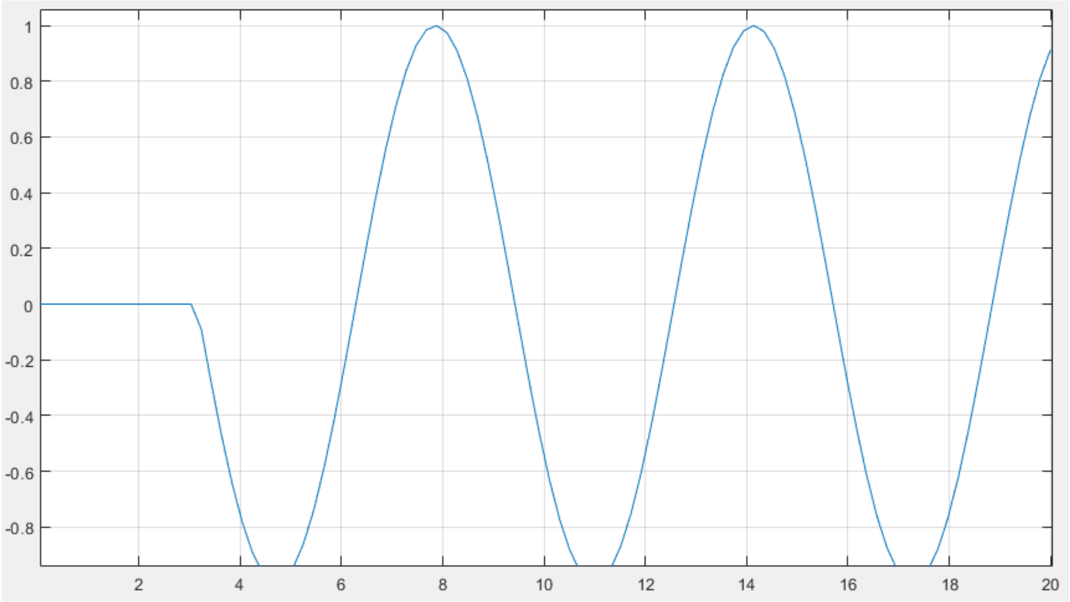
- Let a be a positive number of $f(t)$ a function
- Replace t with $t-a$, the result is the shifted function $f(t-a)$
  - The graph of $f(t)$ shifted a units to the positive direction of t
- $\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$
- $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{-st}\left(e^{at} f(t)\right) dt = \mathcal{L}[e^{at} f(t)](s)$
- Ex) $\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2} = F(s) \rightarrow \mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2} = F(s-a)$
- Ex) $\mathcal{L}[t^3] = \frac{6}{s^4} \rightarrow \mathcal{L}[t^3 e^{7t}] = \frac{6}{(s-7)^4}$

The Shifting Theorems

- $\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)$
- Ex) $\mathcal{L}^{-1}\left[\frac{4}{s^2+4s+20}\right] = \mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+16}\right] = F(s+2)$
  - $\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right] = \sin(4t)$
  - $\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+16}\right] = e^{-2t}\sin(4t)$
- Ex) $\mathcal{L}^{-1}\left[\frac{3s-1}{s^2-6s+2}\right] = \mathcal{L}^{-1}\left[\frac{3s-1}{(s-3)^2-7}\right] = \mathcal{L}^{-1}\left[\frac{3(s-3)+8}{(s-3)^2-7}\right] = \mathcal{L}^{-1}\left[\frac{3(s-3)}{(s-3)^2-7} + \frac{8}{(s-3)^2-7}\right] = G(s-3) + K(s-3)$
  - $G(s) = \frac{3s}{s^2-7}, K(s) = \frac{8}{s^2-7}$
  - $\mathcal{L}^{-1}[G(s)] = 3\cosh(\sqrt{7}\,t), \mathcal{L}^{-1}[K(s)] = \frac{8}{\sqrt{7}}\sinh\left(\sqrt{7}t\right)$
  - $\mathcal{L}^{-1}\left[\frac{3s-1}{s^2-6s+2}\right] = 3e^{3t}\cosh(\sqrt{7}\,t) + \frac{8}{\sqrt{7}}e^{3t}\sinh\left(\sqrt{7}t\right)$

The Heaviside Function

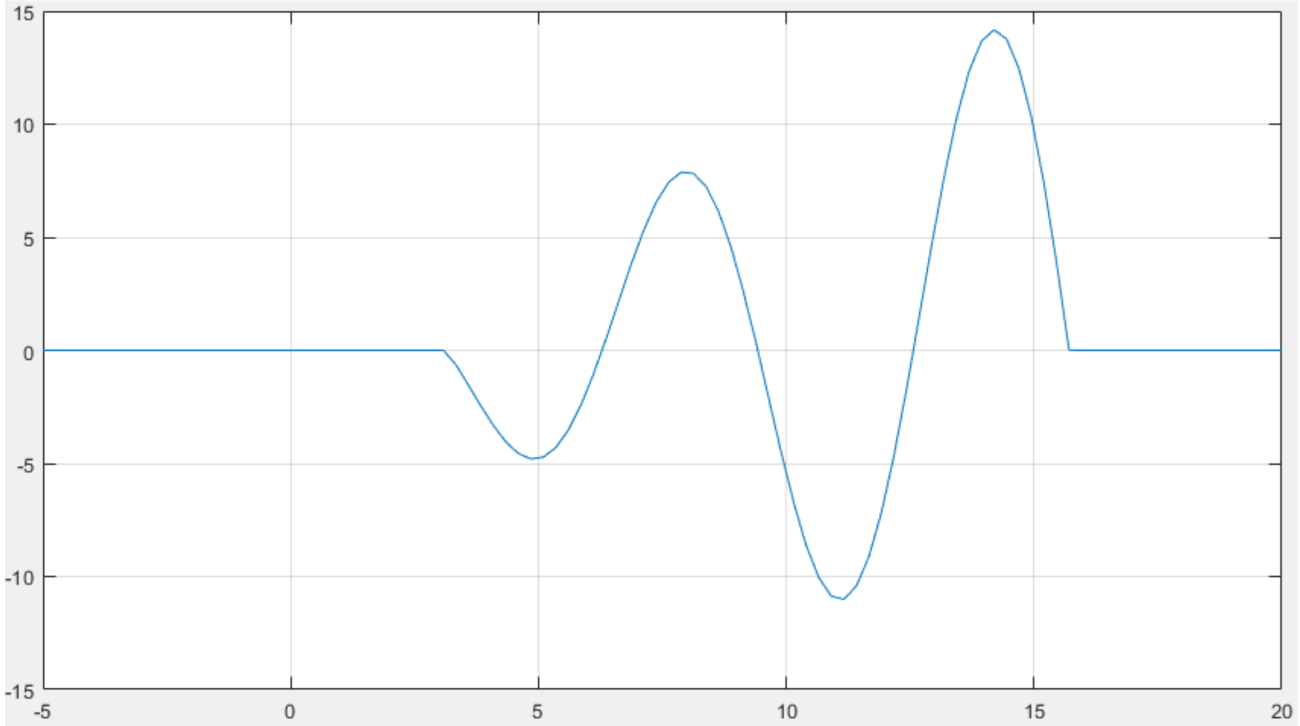
- The unit step function, or Heaviside function
  - $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$
- The shifted Heaviside function
  - $H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$
- A role of the shifted Heaviside function
  - Turn a signal function $f(t)$ off until time $t = a$
  - Turn a signal function $f(t)$ on for all later times
  - $H(t-a)g(t) = \begin{cases} 0 & t < a \\ g(t) & t \geq a \end{cases}$

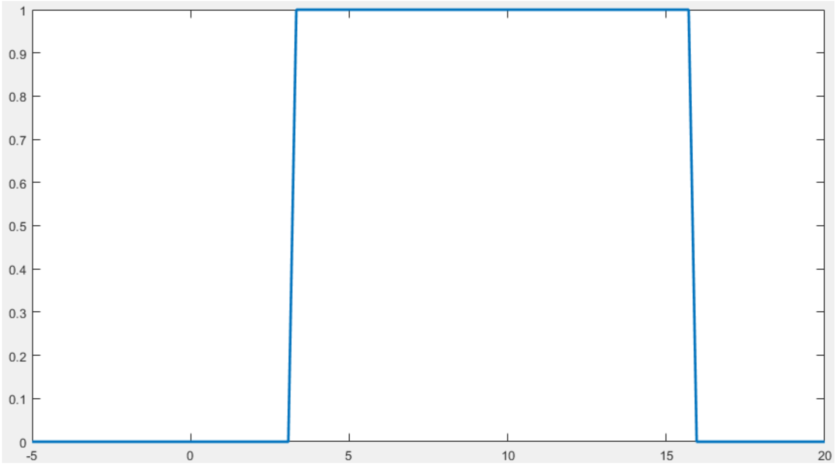


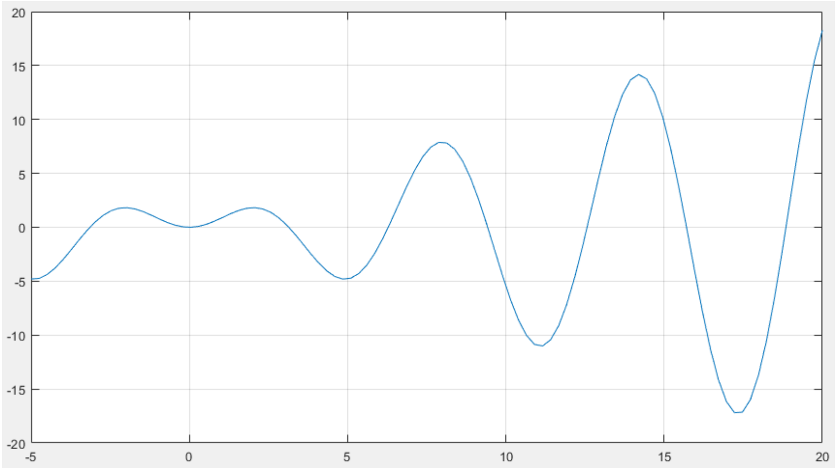
$H(t-\pi)\sin(t)$

The Pulse Function

- A pulse function as a difference of two shifted Heaviside functions for $0 < a < b$
  - $H(t-a) - H(t-b) = \begin{cases} 0 & t < a \\ 1 & a \le t < b \\ 0 & t \ge b \end{cases}$
  - Ex) $\big(H(t-\pi) - H(t-5\pi)\big)t\sin(t)$







The Laplace Transform of Pulse Function(Shifting Theorem)

- Ex) The Laplace transform of a Heaviside function with constant function
  - $\mathcal{L}[H(t-a)](s) = \int_0^\infty e^{-st} H(t-a)\, dt = \int_a^\infty e^{-st}\, dt = -\frac{1}{s}[e^{-st}]_a^\infty = \frac{e^{-as}}{s} = e^{-as}\mathcal{L}[1](s)$
- Ex) The Laplace transform of a Heaviside function with $f(t)$
  - $\mathcal{L}[H(t-a)f(t-a)](s) = \int_0^\infty e^{-st} H(t-a) f(t-a)\, dt = \int_a^\infty e^{-st} f(t-a)\, dt = \int_0^\infty e^{-s(T+a)} f(T)\, dT = e^{-as}F(s)$
- Finally, $\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$ and $\mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a)$

The Laplace Transform of Pulse Function(Shifting Theorem)

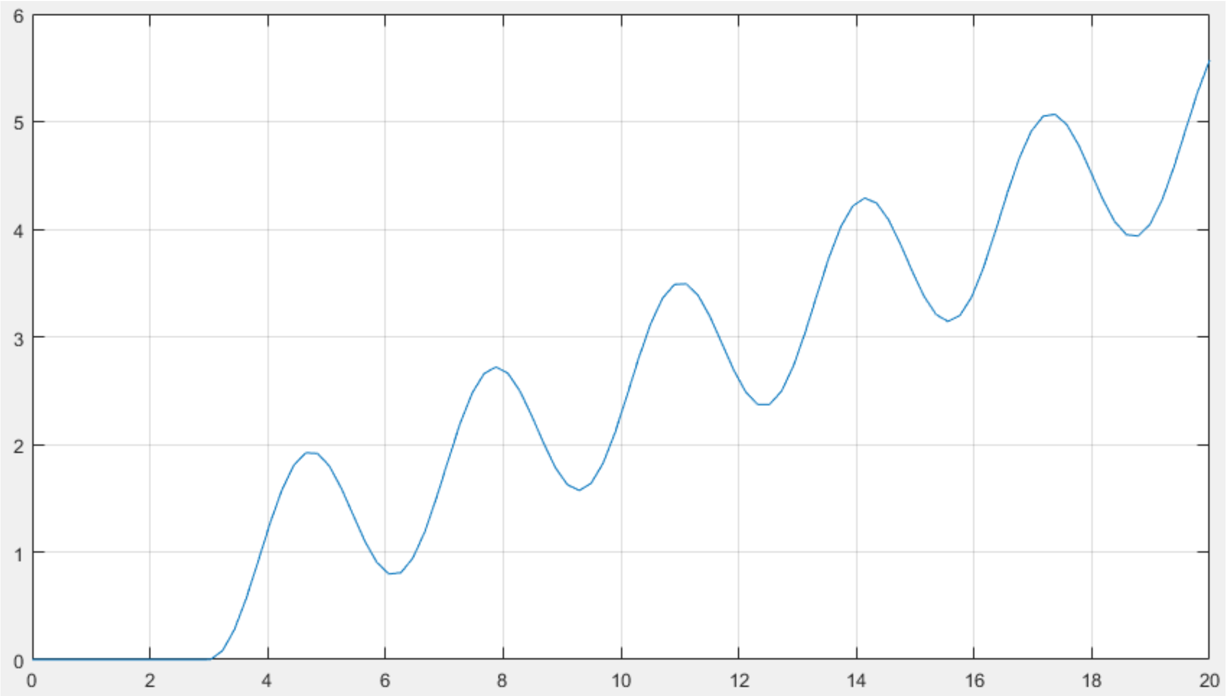
- Ex) Compute $\mathcal{L}[g(t)](s)$ for $g(t) = \begin{cases} 0 & t < 2 \\ t^2 + 1 & t \geq 2 \end{cases}$
  - $t^2 + 1 = (t-2)^2 + 4(t-2) + 5$
  - $\mathcal{L}[g(t)](s) = \mathcal{L}[H(t-2)(t^2+1)] = \mathcal{L}\left[H(t-2)\left((t-2)^2 + 4(t-2) + 5\right)\right]$
    - $\mathcal{L}[H(t-2)(t-2)^2] = e^{-2s}\mathcal{L}[t^2] = e^{-2s}\left(\frac{2}{s^3}\right)$
    - $\mathcal{L}\left[H(t-2)\left(4(t-2)\right)\right] = 4e^{-2s}\mathcal{L}[t] = e^{-2s}\left(\frac{4}{s^2}\right)$
    - $\mathcal{L}[H(t-2)5] = e^{-2s}\left(\frac{5}{s}\right)$
  - $\mathcal{L}[g(t)](s) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{5}{s}\right)$

The Laplace Transform of Pulse Function(Shifting Theorem)

- Ex) Compute $\mathcal{L}^{-1}\left[\frac{se^{-3s}}{s^2+4}\right]$
  - $\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right](t) = \cos(2t), \mathcal{L}^{-1}\left[\frac{se^{-3s}}{s^2+4}\right] = H(t-3)\cos\big(2(t-3)\big)$
- Solve the initial value problem $y'' + 4y = f(t); y(0) = y'(0) = 0$, where $f(t) = \begin{cases} 0 & t < 3 \\ t & t \geq 3 \end{cases}$
  - $\mathcal{L}[y'' + 4y] = s^2Y(s) - sy(0) - y'(0) + 4Y(s) = (s^2+4)Y(s)$
  - $\mathcal{L}[f(t)] = \mathcal{L}[H(t-3)t] = \mathcal{L}[H(t-3)(t-3) + 3H(t-3)] = \frac{e^{-3s}}{s^2} + \frac{3e^{-3s}}{s} = \frac{3s+1}{s^2}e^{-3s}$
  - $Y(s) = e^{-3s}\left(\frac{3s+1}{s^2(s^2+4)}\right) = e^{-3s}\left(\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right) + \frac{3}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right)$
    - $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{4s^2}\right](t) = \frac{1}{4}H(t-3)(t-3), \mathcal{L}^{-1}\left[\frac{1}{4}\frac{e^{-3s}}{s^2+4}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{8}\frac{2e^{-3s}}{s^2+4}\right](t) = \frac{1}{8}H(t-3)\sin\big(2(t-3)\big)$
    - $\mathcal{L}^{-1}\left[\frac{3}{4}\frac{e^{-3s}}{s}\right](t) = \frac{3}{4}H(t-3), \mathcal{L}^{-1}\left[\frac{3}{4}\frac{se^{-3s}}{s^2+4}\right](t) = \frac{3}{4}H(t-3)\cos\big(2(t-3)\big)$
  - $y(t) = \frac{3}{4}H(t-3) + \frac{1}{4}H(t-3)(t-3) - \frac{3}{4}H(t-3)\cos\big(2(t-3)\big) - \frac{1}{8}H(t-3)\sin\big(2(t-3)\big)$

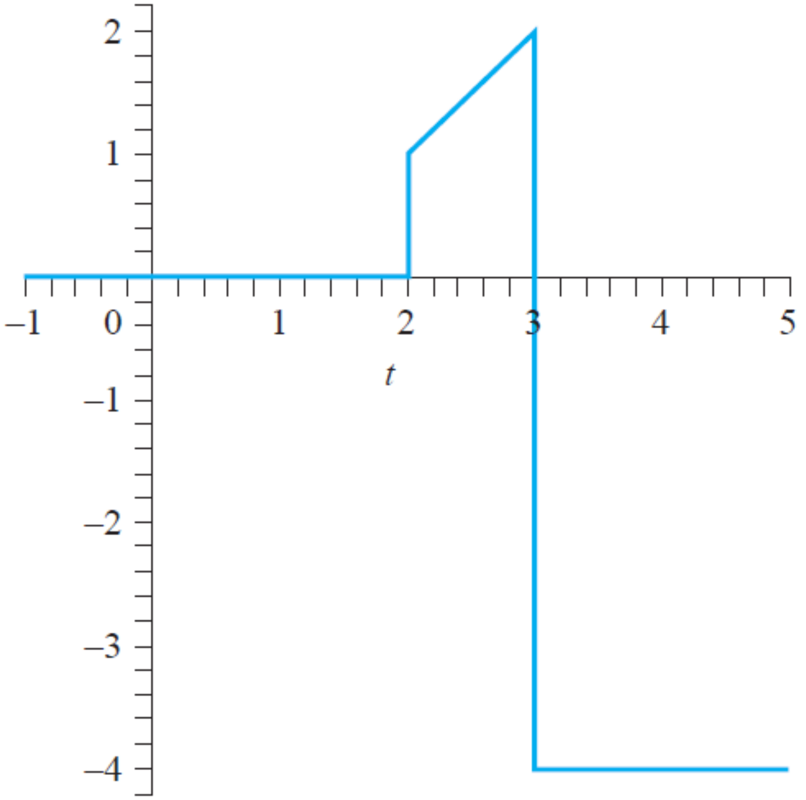
The Laplace Transform of Pulse Function(Shifting Theorem)

- Solve the initial value problem $y'' + 4y = f(t); y(0) = y'(0) = 0$, where $f(t) = \begin{cases} 0 & t < 3 \\ t & t \geq 3 \end{cases}$
  - $y(t) = \frac{3}{4}H(t-3) + \frac{1}{4}H(t-3)(t-3) - \frac{3}{4}H(t-3)\cos\big(2(t-3)\big) - \frac{1}{8}H(t-3)\sin\big(2(t-3)\big)$
  - Finally, $y(t) = \begin{cases} 0 & t < 3 \\ \frac{1}{8}\Big(2t - 6\cos\big(2(t-3)\big) - \sin\big(2(t-3)\big)\Big) & t \geq 3 \end{cases}$



The Laplace Transform of Pulse Function(Shifting Theorem)

- Ex) Write a function in terms of step functions of the following function
  - $f(t) = \begin{cases} 0 & t < 2 \\ t-1 & 2 \le t < 3 \\ -4 & t \ge 3 \end{cases}$
  - $f(t)$ as consisting of two nonzero parts
    - The part that is $t-1$ for $2 \le t < 3$ and part that is -4 for $t \ge 3$
    - Turn on $t-1$ at time 2 and turn it off at time 3
      - $\big(H(t-2) - H(t-3)\big)(t-1)$
    - Turn -4 on at time 3
      - $-4H(t-3)$
  - $f(t) = \big(H(t-2) - H(t-3)\big)(t-1) - 4H(t-3)$



Heaviside's Formula

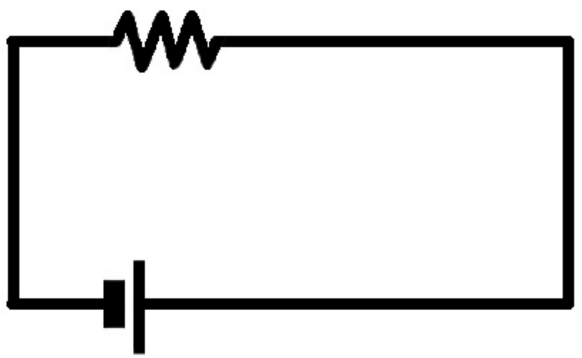
- Suppose $F(s) = \frac{p(s)}{q(s)}$ with p and q polynomial and q of higher degree than p
  - Assume that q can be factored into linear factors
    - $q(s) = c(s - a_1)(s - a_2) \dots (s - a_n)$ with a nonzero constant c and n distinct number a_j
      - None of the a_j's are roots of $p(s)$
    - Let $q_j(s)$ be the polynomial of degree $n - 1$ formed by omitting the factor $s - a_j$ from $q(s)$ for $j = 1, 2, \dots, n$
      - $q_1(s) = c(s - a_2) \dots (s - a_n)$
    - $\mathcal{L}^{-1}[F(s)](t) = \sum_{j=1}^{n} \frac{p(a_j)}{q_j(a_j)} e^{a_j t}$: Heaviside's formula
      - $\frac{p(s)}{q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}$
      - $\mathcal{L}^{-1}[F(s)](t) = A_1 e^{a_1 t} + \cdots + A_n e^{a_n t}$
      - $(s - a_1)\frac{p(s)}{q(s)} = A_1 + A_2 \frac{s-a_1}{s-a_2} + \cdots + A_n \frac{s-a_1}{s-a_n}$, $\lim_{s \to a_1} (s - a_1)\frac{p(s)}{q(s)} = A_1$

Heaviside's Formula

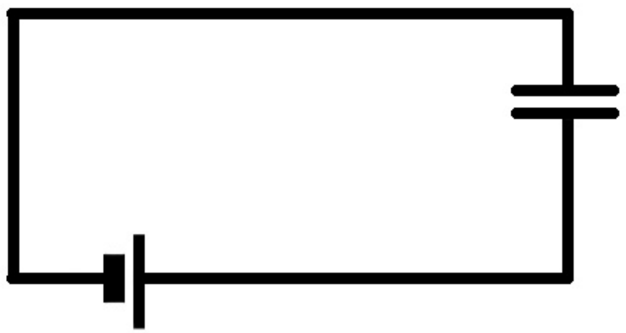
- Ex) $F(s) = \frac{s}{(s^2+4)(s-1)} = \frac{s}{(s-2i)(s+2i)(s-1)}$
  - Let $p(s) = s$ and $q(s) = (s-2i)(s+2i)(s-1)$
    - $a_1 = 2i, a_2 = -2i, a_3 = 1$
  - $q_1(2i) = 4i(2i-1), q_2(-2i) = -4i(-2i-1), q_3(1) = (1-2i)(1+2i)$
  - $p(2i) = 2i, p(-2i) = -2i, p(1) = 1$
  - $\mathcal{L}^{-1}[F(s)](t) = \frac{2i}{4i(2i-1)}e^{2it} + \frac{2i}{4i(-2i-1)}e^{-2it} + \frac{1}{(1-2i)(1+2i)}e^{t} = \frac{-1-2i}{10}e^{2it} + \frac{-1+2i}{10}e^{-2it} + \frac{1}{5}e^{t} = -\frac{1}{10}\left(e^{2it} + e^{-2it}\right) - \frac{2i}{10}\left(e^{2it} - e^{-2it}\right) + \frac{1}{5}e^{t} = -\frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) + \frac{1}{5}e^{t}$

The Laplace Transform of Pulse Function(Shifting Theorem)

- Circuit connected a resistance
  - $I = \frac{V}{R}$
    - V: 전원장치의 기전력
    - R: 저항의 값
    - I: 전류

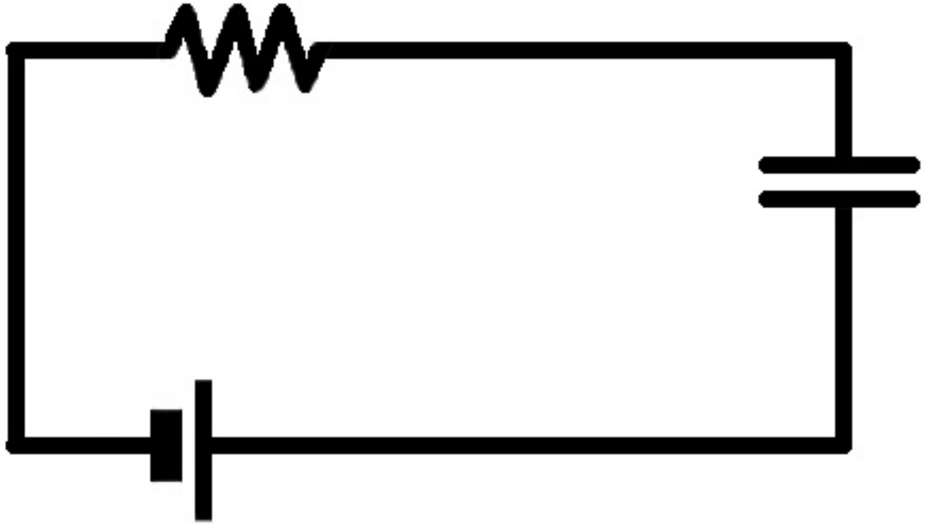


- Circuit connected a capacitor
  - $Q = CV$
    - V: 축전기 양단에 걸린 전압
    - C: 축전기의 전기용량
    - Q: 축전기에 저장되는 전하량



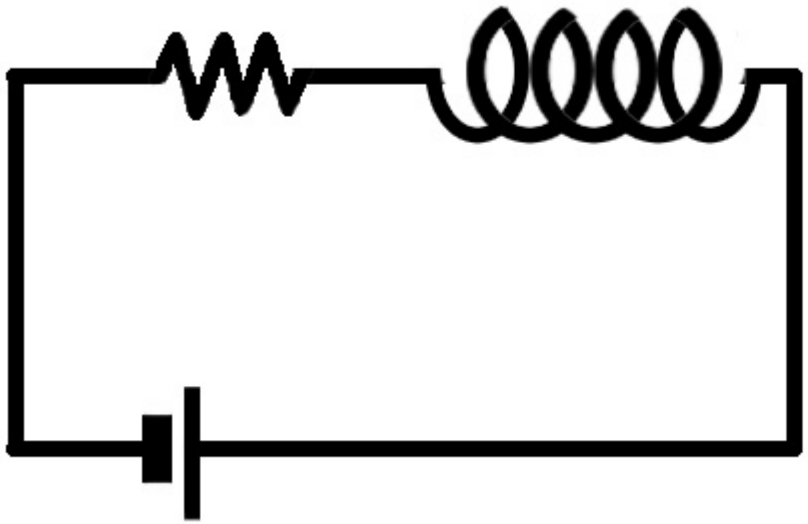
The Laplace Transform of Pulse Function(Shifting Theorem)

- RC-circuit(Circuit connected a resistance and capacitor)
  - $V = V_R + V_C$
    - V: 전원장치의 기전력
    - V_R: 저항에서의 전압강하
    - V_C: 축전기에서의 전압강하
  - $V = IR + \frac{Q}{C}$
    - $I = \frac{dQ}{dt}$
    - $V = R\frac{dQ}{dt} + \frac{Q}{C} = Rq'(t) + \frac{1}{C}q(t)$



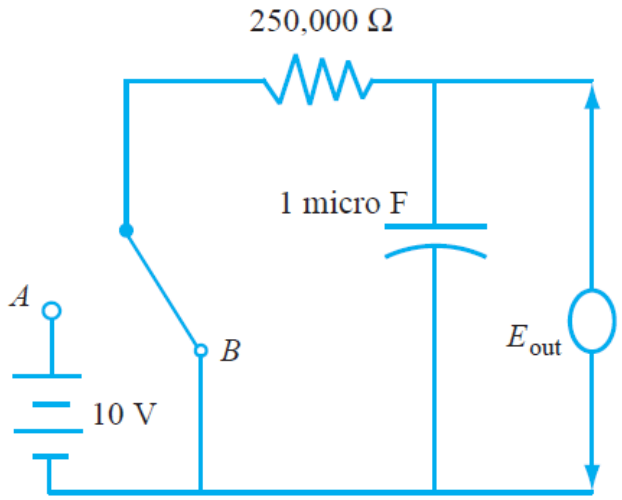
The Laplace Transform of Pulse Function(Shifting Theorem)

- RL-circuit(Circuit connected a resistance and capacitor)
  - $V = V_R + V_L$
    - V: 전원장치의 기전력
    - V_R: 저항에서의 전압강하
    - V_L: 코일에서의 전압강하
  - $V = IR + L\frac{dI}{dt}$
    - $I = \frac{dQ}{dt}$
    - $V = R\frac{dQ}{dt} + L\frac{dI}{dt} = Rq'(t) + Lq''(t)$

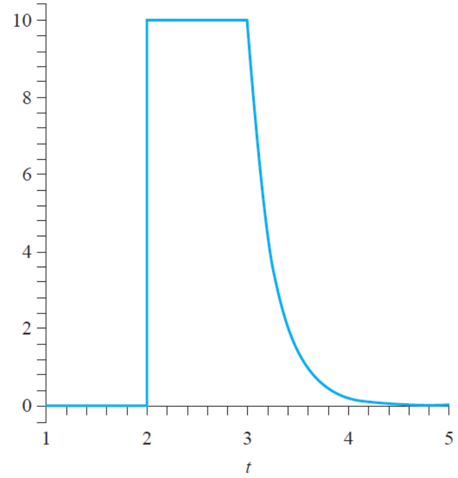


The Laplace Transform of Pulse Function(Shifting Theorem)

Suppose the capacitor in the circuit initially has a charge of zero and there is no initial current. At time $t = 2$ seconds, the switch is thrown from position B to A, held there for 1 second, and then switched back to B. We want the output voltage E_{out} on the capacitor. $E(t) = 10\big(H(t-2) - H(t-3)\big)$

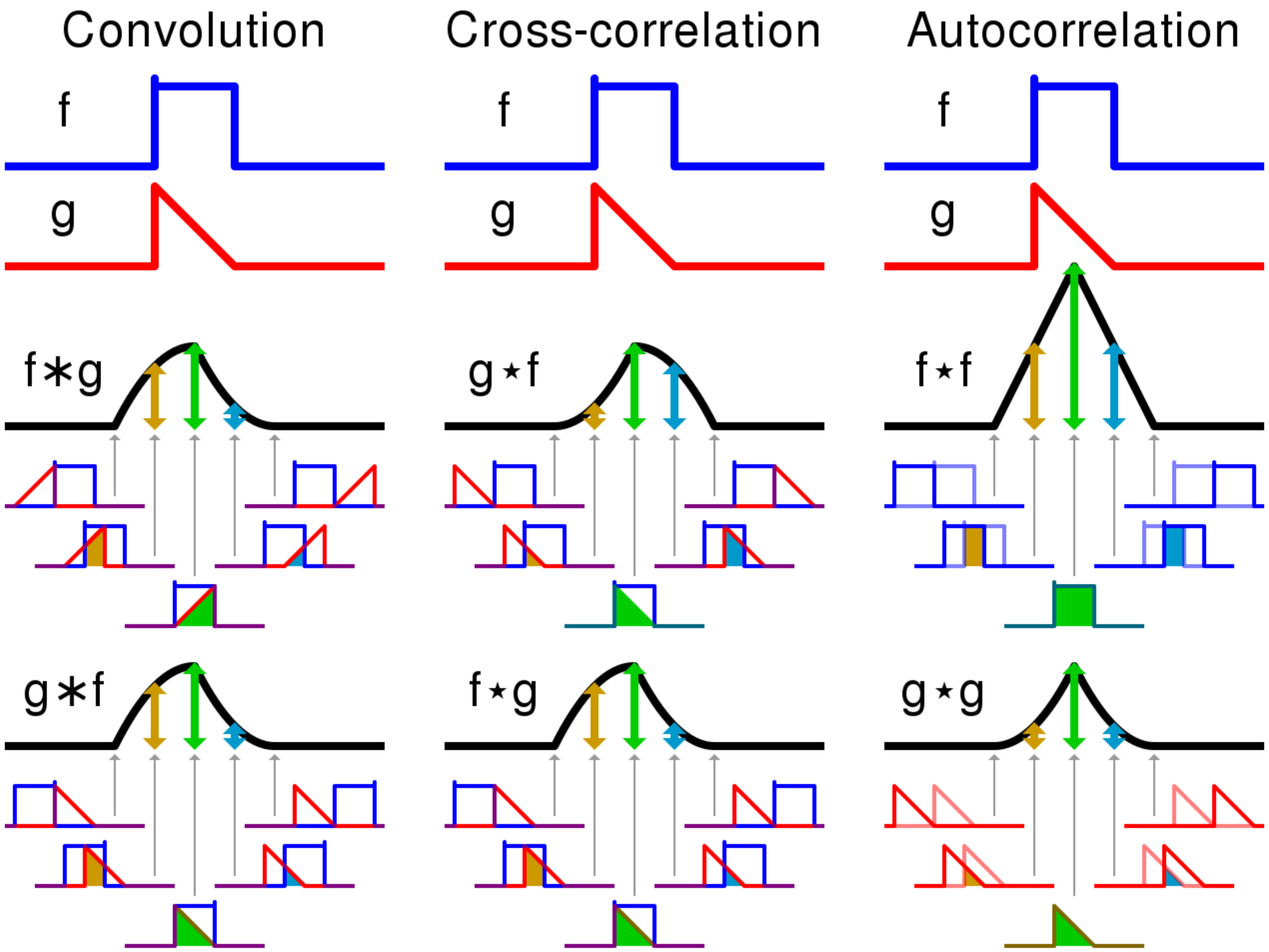


- $\boxed{\text{By Kirchhoff's voltage law}\left(Ri(t) + \frac{1}{C}q(t) = E(t)\right)}$
  - $250000q'(t) + 10^6 q(t) = E(t);\ q(0) = 0$
  - Solve the given equation for $q(t)$
    - $250000\big(sQ(s) - q(0)\big) + 10^6 Q(s) = \mathcal{L}[E(t)](s)$
    - $\mathcal{L}[E(t)](s) = \frac{10e^{-2s}}{s} - \frac{10e^{-3s}}{s}$
    - $(2.5)(10^5)sQ(s) + (10^6)Q(s) = \frac{10e^{-2s}}{s} - \frac{10e^{-3s}}{s} = \frac{10}{s}(e^{-2s} - e^{-3s})$
    - $Q(s) = \frac{10(e^{-2s} - e^{-3s})}{s\left((2.5)(10^5)s + 10^6\right)} = \frac{(4)(10^{-5})}{s(s+4)}e^{-2s} - \frac{(4)(10^{-5})}{s(s+4)}e^{-3s}$
    - $Q(s) = 10^{-5}\left(\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s+4}\right) - 10^{-5}\left(\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s+4}\right)$
  - $q(t) = 10^{-5}H(t-2)\left(1 - e^{-4(t-2)}\right) - 10^{-5}H(t-3)\left(1 - e^{-4(t-3)}\right)$



Convolution

- Convolution
  - A mathematical operation on two functions f and g to produce a third function that expresses how the shape of one is modified by the other
    - $f * g = \int_{-\infty}^{\infty} f(x)g(t-x)\,dx$
- Cross-correlation
  - A measure of similarity of two series as a function of displacement of one relative to the other
    - $f \star g = \int_{-\infty}^{\infty} f(x)g(t+x)\,dx$
- Actually, $-\infty$
  - The lower bound in the integral
  - Define at $t_0 = 0$



Convolution

- Properties
  - $f * g = g * f$(commutative property)
  - $(f * g) * h = f * (g * h)$(associative property)
  - $f * (g + h) = f * g + f * h$(distribute property)
  - $f * 0 = 0$
  - $f * 1 \neq f$
    - $t * 1 = \int_0^t \tau \, d\tau = \frac{t^2}{2} \neq t$
  - $f * f \geq 0$ is incorrect
    - $\sin(t) * \sin(t) = \frac{1}{2}(-t\cos(t) + \sin(t)) < 0$, for $t = 2\pi$
- **On the Laplace transform**
  - $\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]$ and $f * g = \mathcal{L}^{-1}\big[\mathcal{L}[f]\mathcal{L}[g]\big]$

Convolution

- Proof of $\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]$
  - $\mathcal{L}[f * g] = \int_0^\infty \left(\int_0^t f(\tau) g(t-\tau)\, d\tau \right) e^{-st}\, dt = \int_0^\infty \int_0^t f(\tau) g(t-\tau) e^{-st}\, d\tau\, dt$

Convolution

- Ex) Compute $\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right]$
  - From the previous method
    - $\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right] = \frac{1}{16}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s-4} + \frac{4}{(s-4)^2}\right] = \frac{1}{16}(1 - e^{4t} + 4t4^{4t})$
  - From convolution theory
    - $\frac{1}{s(s-4)^2} = \frac{1}{s}\frac{1}{(s-4)^2}$
    - $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1, \mathcal{L}^{-1}\left[\frac{1}{(s-4)^2}\right] = te^{4t}$
    - Finally, $\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right] = 1 * te^{4t} = \int_0^t \tau e^{4\tau}\, d\tau = \frac{1}{4}[\tau e^{4\tau}]_0^t - \frac{1}{16}[e^{4\tau}]_0^t = \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t} + \frac{1}{16}$

Convolution

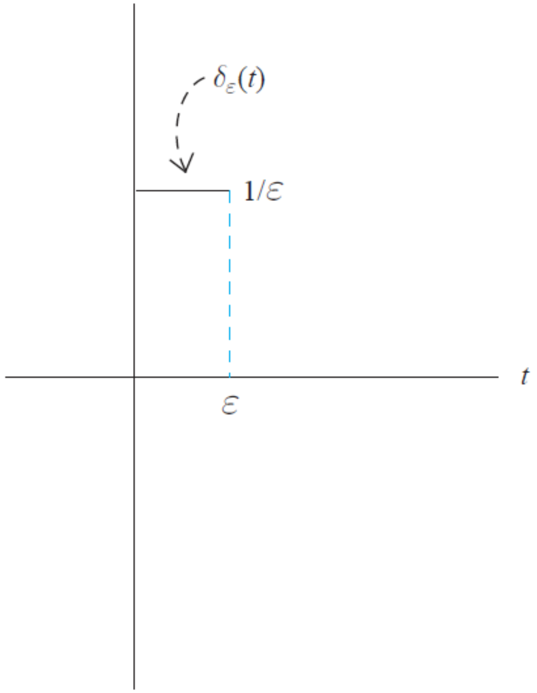
- Ex) Solve the initial value problem $y'' - 2y' - 8y = f(t); y(0) = 1, y'(0) = 0$
  - The left-hand side after the Laplace transform
    - $s^2Y(s) - sf(0) - f'(0) - 2sY(s) + 2f(0) - 8Y(s) = (s^2 - 2s - 8)Y(s) - s + 2$
  - The right-hand side after the Laplace transform, $F(s)$
  - The given equation after the Laplace transform
    - $Y(s) = \frac{s-2}{(s-4)(s+2)} + \frac{1}{(s-4)(s+2)}F(s) = \frac{1}{3}\left(\frac{1}{s-4} + \frac{2}{s+2}\right) + \frac{1}{6}\left(\frac{1}{s-4} - \frac{1}{s+2}\right)F(s)$
    - $y(t) = \frac{1}{3}e^{4t} + \frac{2}{3}e^{-2t} + \frac{1}{6}\left(e^{4t} * f(t)\right) - \frac{1}{6}\left(e^{-2t} * f(t)\right)$

Convolution

- Ex) Solve for $f(t)$ in the integral equation, $f(t) = 2t^2 + \int_0^t f(t-\tau)e^{-\tau}\, d\tau$
  - $f(t) = 2t^2 + f(t) * e^{-\tau}$
  - The Laplace transform to both side
    - $F(s) = \frac{4}{s^3} + \frac{1}{s+1}F(s) \rightarrow F(s) = \frac{4(s+1)}{s^4} = \frac{4}{s^3} + \frac{4}{s^4}$
  - $f(t) = 2t^2 + \frac{2}{3}t^3$

Impulses and the Dirac Delta Function

- Impulse
  - A force of extremely large magnitude applied over a very short period of time
    - $\delta_\epsilon(t) = \frac{1}{\epsilon}[H(t) - H(t-\epsilon)]$, for any positive number ϵ
- A delta function
  - As the limit of the pulse as $\epsilon \to 0$, as the duration shrinks to zero and the height goes to infinity
    - $\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)$

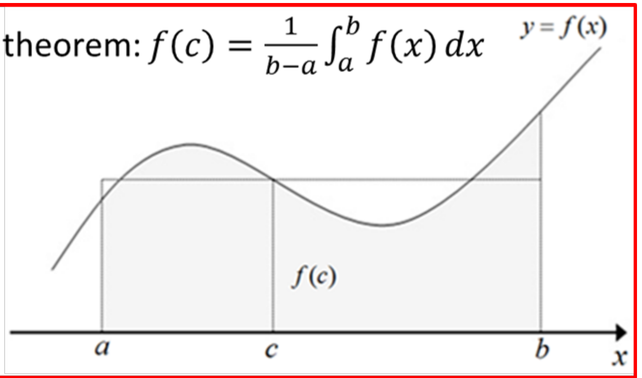


Impulses and the Dirac Delta Function

- The Laplace transform of a shifted pulse function
  - $\delta_\epsilon(t-a) = \frac{1}{\epsilon}[H(t-a) - H(t-a-\epsilon)]$
  - $\mathcal{L}[\delta_\epsilon(t-a)] = \mathcal{L}\left[\frac{1}{\epsilon}[H(t-a) - H(t-a-\epsilon)]\right] = \frac{1}{\epsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s}\right) = \frac{e^{-as}-e^{-(a+\epsilon)s}}{\epsilon s} = \frac{e^{-as}(1-e^{-\epsilon s})}{\epsilon s}$
- The Laplace transform of a delta function
  - $\mathcal{L}[\delta(t-a)] = \lim_{\epsilon\to 0}\mathcal{L}[\delta_\epsilon(t-a)] = e^{-as}\left(\because \lim_{\epsilon\to 0}\left(\frac{1-e^{-\epsilon s}}{\epsilon s}\right) = \lim_{\epsilon\to 0}\left(\frac{se^{-\epsilon s}}{s}\right) = e^0 = 1\right)$
- The Laplace transform of the delta function
  - In particular, with $a = 0$(no shifted)
    - $\mathcal{L}[\delta(t)] = 1$

Impulses and the Dirac Delta Function

★Mean value theorem: $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$



- Filtering property

Let $a > 0$ and let f be integrable on $[0, \infty)$ and continuous at a. Then

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$$

Proof:

$$\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt = \int_0^\infty \frac{1}{\epsilon}[H(t-a) - H(t-a-\epsilon)]f(t)\,dt = \frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\,dt = \frac{1}{\epsilon}\big(\epsilon f(t_\epsilon)\big), (a < t_\epsilon < a+\epsilon)$$

As $\epsilon \to 0+, a+\epsilon \to a$, so $t_\epsilon \to a$, and by continuity, $f(t_\epsilon) = f(a)$

$$\lim_{\epsilon\to 0}\int_0^\infty f(t)\delta_\epsilon(t-a)\,dt = \int_0^\infty f(t)\left(\lim_{\epsilon\to 0}\delta_\epsilon(t-a)\right)dt = \int_0^\infty f(t)\,\delta(t-a)dt = \lim_{\epsilon\to 0+} f(t_\epsilon) = f(a)$$

Ex) $f(t) = e^{-st}$, $\int_0^\infty f(t)\delta(t-a)\,dt = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$

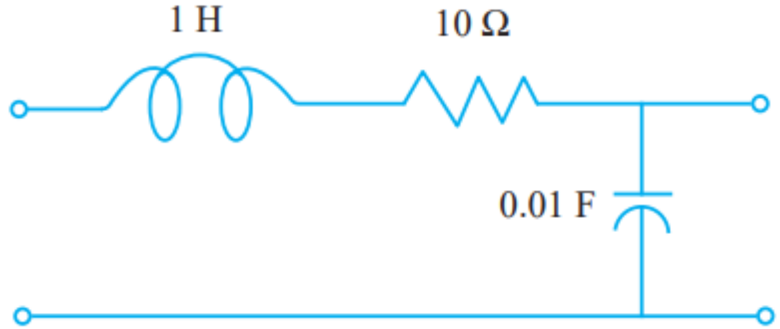
Finally, $f * \delta = f$. The delta function therefore acts as an identity for the “product” defined by the convolution of two functions

Impulses and the Dirac Delta Function

- Ex) Solve the initial value problem $y'' + 2y' + 2y = \delta(t-3); y(0) = y'(0) = 0$
  - The left-hand side after the Laplace transform
    - $s^2Y(s) + 2sY(s) + 2Y(s) = (s^2 + 2s + 2)Y(s)$
  - The right-hand side after the Laplace transform, e^{-3s}
  - $Y(s) = \frac{e^{-3s}}{(s+1)^2+1}$
  - $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin(t)$: the inverse of the shifting function, with $a = -1$
  - $y(t) = H(t-3)e^{-(t-3)}\sin(t-3)$

Impulses and the Dirac Delta Function

Ex) Suppose the current and charge on the capacitor in the circuit of the following figure are zero at time zero. We want to describe the output voltage response to a transient modeled by $\delta(t)$. The output voltage is $\frac{q(t)}{C}$, so we will determine $q(t)$. By Kirchhoff's voltage law, $Li' + Ri + \frac{1}{C}q$ with assumption $q(0) = q'(0) = 0$.



Systems of Linear Differential Equation

- A Linear system
  - a mathematical model of a system based on the use of a linear operator
  - Ex) find the intersection point of two lines
    - $l_1: a_1x + b_1y + c_1 = 0, l_2: a_2x + b_2y + c_2 = 0$
- Ex) Solve the initial value problem for the following linear differential system
  - $eq(1)\colon\ x'' - 2x' + 3y' + 2y = 4,\ \ eq(2)\colon\ 2y' - x' + 3y = 0; x(0) = x'(0) = y(0) = 0$
  - Eq(1) after the Laplace transform
    - $s^2X(s) - 2sX(s) + 3Y(s) + 2Y(s) = (s^2 - 2s)X(s) + 5Y(s) = \frac{4}{s}$
  - Eq(2) after the Laplace transform
    - $2sY(s) - sX(s) + 3Y(s) = -sX(s) + (2s + 3)Y(s) = 0$
  - After solving the transformed linear system
    - $X(s) = \frac{4s+6}{s^2(s+2)(s-1)}, Y(s) = \frac{2}{s(s+2)(s-1)}$
    - $x(t) = -\frac{7}{2} - 3t + \frac{1}{6}e^{-2t} + \frac{10}{3}e^t, y(t) = -1 + \frac{1}{3}e^{-2t} + \frac{2}{3}e^t$

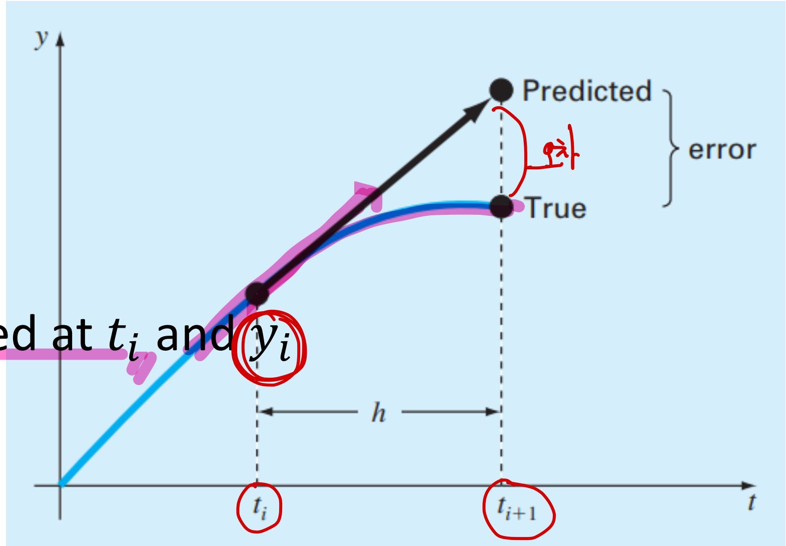
Approximation of Solutions

Euler's method

y_i

$y_{i+1} = y_i + f(t_i, y_i)h$

- The first derivative for a direct estimate of the slope at t_i
  - $\phi = f(t_i, y_i)$, where $f(t_i, y_i)$ is the differential equation evaluated at t_i and y_i
- $y_{i+1} = y_i + f(t_i, y_i)h$
  - Euler's method(the Euler-Cauchy or point-slope method)
  - Predicted a new value using the slope to extrapolate linearly over the step size h



에러가 쌓인다

Problem state: Use Euler's method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1. The initial condition at $t = 0$ is $y = 2$.
Note that the exact solution can be determined analytically as

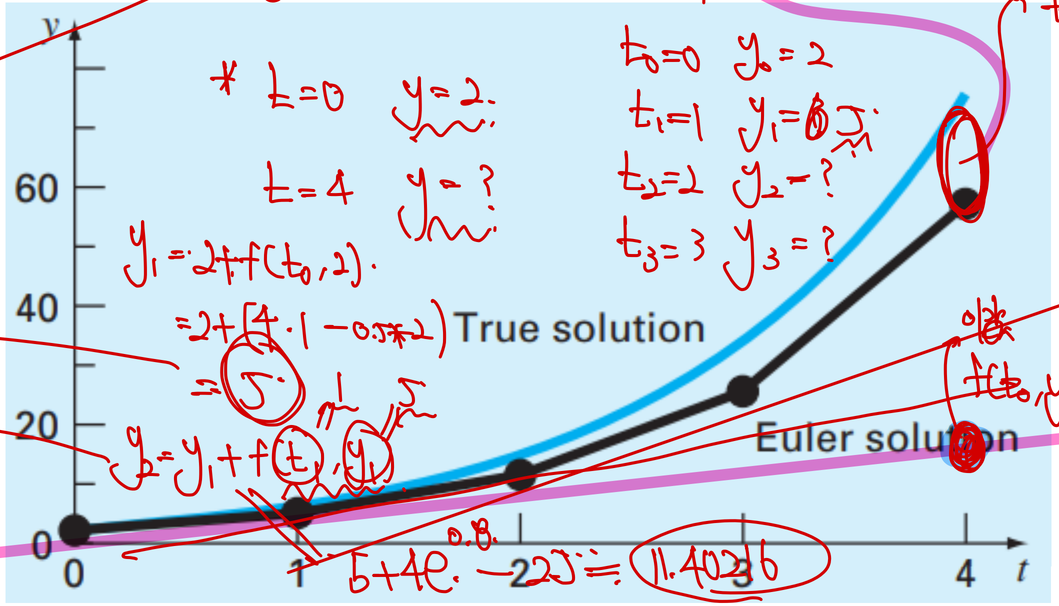
$$y = \frac{4}{1.3}\left(e^{0.8t} - e^{-0.5t}\right) + 2e^{-0.5t}$$

y_{true} : 참 값을 보기 위해서.

t	y_{true}	y_{Euler}	$\lvert\varepsilon_t\rvert$ (%)
0	2.00000	2.00000	
1	6.19463	5.00000	19.28
2	14.84392	11.40216	23.19
3	33.67717	25.51321	24.24
4	75.33896	56.84931	24.54

but 계산 결과

step 을 나누어서 오차이 한개의 값과 같을 것



* $t=0$ $y=2$

$t=4$ $y=?$

$t_0=0$ $y_0=2$

$t_1=1$ $y_1=5$

$t_2=2$ $y_2=?$

$t_3=3$ $y_3=?$

$y_1 = 2 + f(t_0, 2)$

$= 2 + (4 \cdot 1 - 0.5 \times 2)$

$= 5$

$y_2 = y_1 + f(t_1, y_1)$

$5 + 4e^{0.8} - 2.5 = 11.40216$

이전 $f(t_0, y_0)$

total error

$y' = 4e^{0.8t} - 0.5y$: linear equation.

$y' + 0.5y = 4e^{0.8t}$

Integrating factor

$\rightarrow e^{\int \frac{1}{2} dx} = e^{\frac{1}{2}x}$

$\int (y \cdot e^{0.5t})' dt = \int 4 \cdot e^{1.3t} dt$

$y = \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) - 2e^{-0.5t}$

$e^{0.5t} y = \frac{4}{1.3} e^{1.3t} + C$

$y = \frac{4}{1.3} e^{0.8t} + C \cdot e^{-0.5t}$

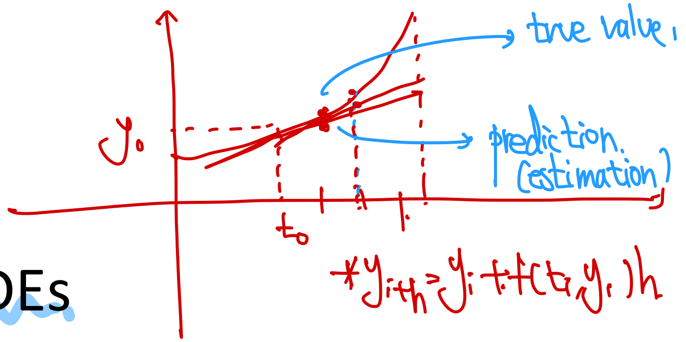
$y(0) = 2$. $\quad \frac{4}{1.3} + C = 2$

$C = 2 - \frac{40}{13} = -\frac{14}{13}$

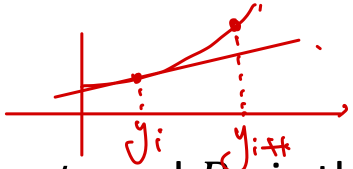
" $y = \frac{4}{13} e^{0.8t} - \frac{14}{13} e^{-0.5t}$ "

$= \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) + \frac{4}{1.3} e^{-0.5t} - \frac{1.4}{1.3} e^{-0.5t}$

Error analysis for Euler's method



- Two types of errors of the numerical solution of ODEs
  - Truncation, or discretization, errors by the nature of the techniques employed to approximate value of y
    - **Local truncation error**, results from an application of the method in questions over a single step
    - 전파.
    - **Propagated truncation error**, results from the approximations produced during the previous steps 해서 쌓이는 것.
  - Roundoff error by the limited numbers of significant digits that can be retained by a computer
    → 반올림오차.
- From a Taylor expansion to deriving Euler's method



  - $y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 + \cdots + \frac{y_i^n}{n!} h^n + R_n$, where $h = t_{i+1} - t_i$ and R_n is the remainder term
    ↳ 이 부분이 truncation error
    - $R_n = \frac{y^{n+1}(\xi)}{(n+1)!} h^{n+1}$, where ξ lies somewhere in the interval from t_i to t_{i+1}
      → 오차가 가장 많이 dependent 하는 값이다.
  - $y_{i+1} = y_i + f(t_i, y_i)h + \frac{f'(t_i,y_i)}{2!} h^2 + \cdots + \frac{f^{(n-1)}(t_i,y_i)}{n!} h^n + O(h^{n+1})$
    - $E_t = \frac{f'(t_i,y_i)}{2!} h^2 + \cdots + O(h^{n+1})$, where E_t is the true local truncation error
    - For sufficiently small h, higher-order term can be negligible, $E_a = \frac{f'(t_i,y_i)}{2!} h^2$ or $E_a = O(h^2)$
      h가 작다면.

Error analysis for Euler's method

- Global truncation error as step size

*그래프 그려봐.

$y' = y + e^t$

Problem state: Compare a numerical solution by Euler's method to ground truth for a differential equation $y' - y = e^t$ with initial value $y(0) = 0$ as $h_1 = 0.1$ and $h_2 = 0.01$ from $t = 0$ to $t = 1$

$t_0 = 0$ $y_0 = 0$
$t_1 = 0.1$ $y_1 = y_0 + (y_0 + e^{t_0})$
$t_2 = 0.2$ $y_2 = y_1 + (y_1 + e^{t_1})$
$\vdots$
$t_{10} = 1$ $y_{10} = y_9 + (y_9 + e^{t_9})$

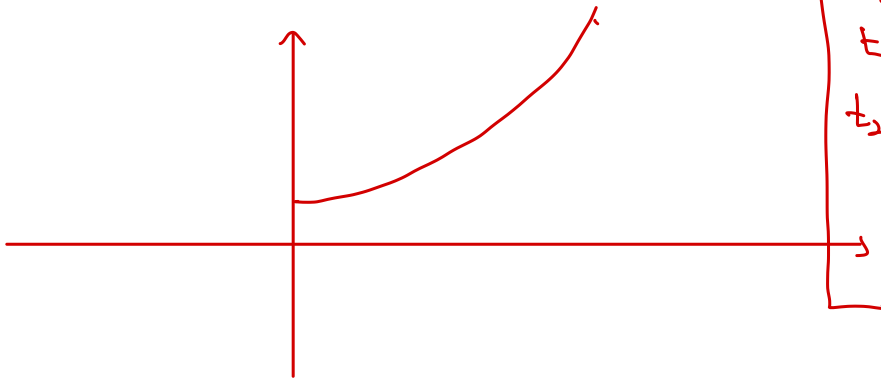
- Analytic solution
  - Actually, in this case, one of the well-known differential equation as linear equation $y' - P(t)y = Q(t)$
  - Multiplication non-zero value to both side, $e^{\int P(t)dt} = e^{\int -1 dt} = e^{-t}$
    - $e^{-t}y' - e^{-t}y = (e^{-t}y)' = e^t e^{-t} = 1$
    - $\int (e^{-t}y)'\, dt = \int 1\, dt \to e^{-t}y = t + C \to y = te^t + Ce^{-t}$
    - From the initial value, $y(0) = C = 0$
    - Finally, $y = te^t$

integrating factor. e^{-t}

① true value.

$e^{-t}y = t + C$

$y = te^t$ $y(0) = 0.$



10개

$t_0 = 0$ $y_0 = 0$
$t_1 = 0.01$ $y_1 = y_0 + (y_0 + e^{t_0})$
$t_2 = 0.02$ $y_2 = y_1 + (y_1 + e^{t_1})$
$\vdots$
$t_{100} = 1$ $y_{100} = y_{99} + (y_{99} + e^{t_{99}})$

Error analysis for Euler's method

Numerical solution, $h_1 = 0.1$

$y_1 = y_0 + 0.1y'(t_0, y_0) \rightarrow y_1 = 0 + 0.1(0 + e^0) = 0.1$

$y_2 = y_1 + 0.1y'(t_1, y_1) \rightarrow y_2 = 0.1 + 0.1(0.1 + e^{0.1}) \cong$ 0.220517

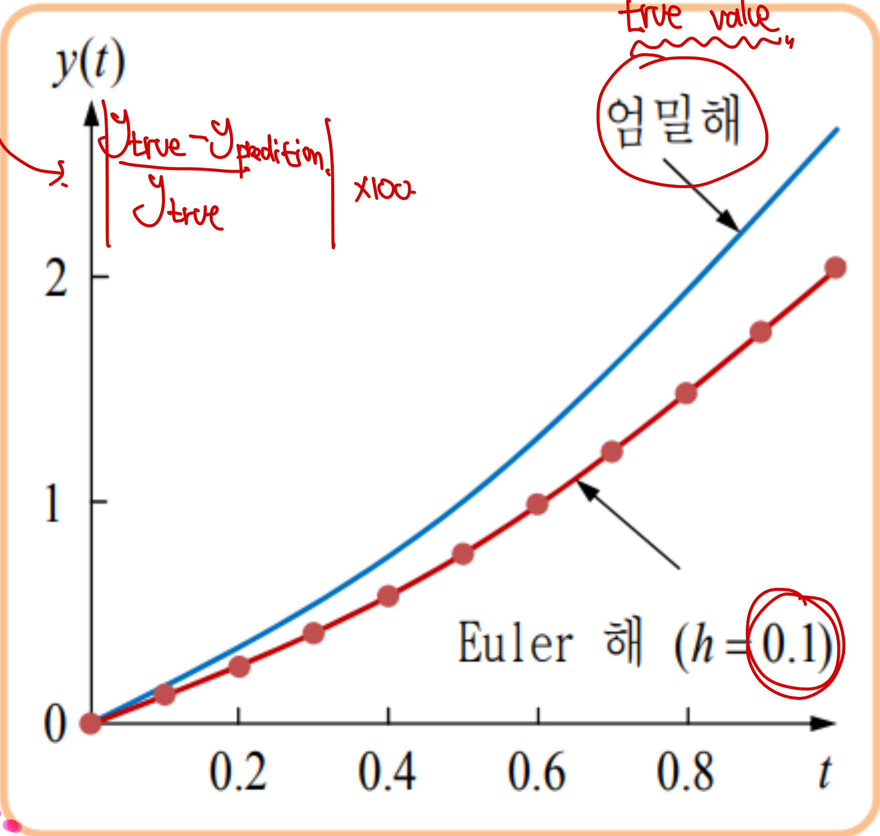
- Global truncation error as step size

Problem state: Compare a numerical solution by Euler's method to ground truth for a differential equation $y' - y = e^t$ with initial value $y(0) = 0$ as $h_1 = 0.1$ and $h_2 = 0.01$ from $t = 0$ to $t = 1$

$y' = y + e^t$

h	도함수 ($f(x)$)	수치해	엄밀해	상대오차 (%)
0.0	1.000000	0.000000	0.000000	-
0.1	1.205171	0.100000	0.110517	9.516
0.2	1.441920	0.220517	0.244281	9.728
0.3	1.714568	0.364709	0.404958	9.939
0.4	2.027991	0.536166	0.596730	10.149
0.5	2.387686	0.738965	0.824361	10.359
0.6	2.799852	0.977734	1.093271	10.568
0.7	3.271471	1.257719	1.409627	10.776
0.8	3.810407	1.584866	1.780433	10.984
0.9	4.425510	1.965907	2.213643	11.191
1.0	**5.126739**	**2.408458**	**2.718282**	$\boxed{\textbf{11.398}}$

$\left|\frac{y_{true} - y_{prediction}}{y_{true}}\right| \times 100$



Error analysis for Euler's method

Numerical solution, $h_2 = 0.01$

$$y_1 = y_0 + 0.01y'(t_0, y_0) \rightarrow y_1 = 0 + 0.01(0 + e^0) = 0.01$$

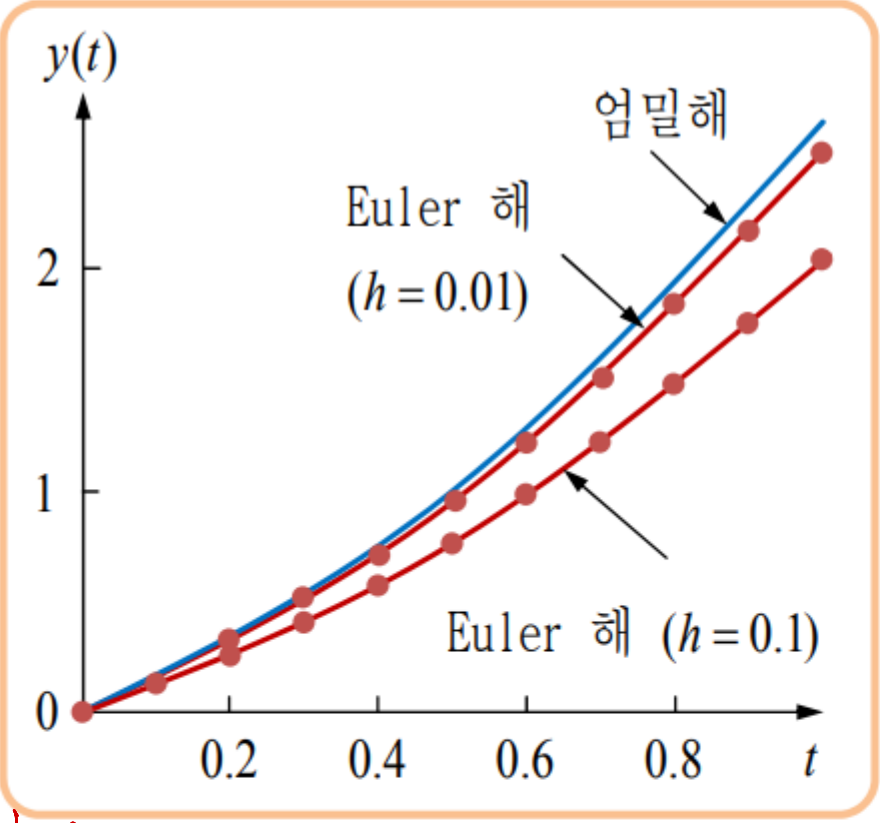
$$y_2 = y_1 + 0.01y'(t_1, y_1) \rightarrow y_2 = 0.01 + 0.01(0.01 + e^{0.01}) \cong 0.020201$$

- Global truncation error as step size

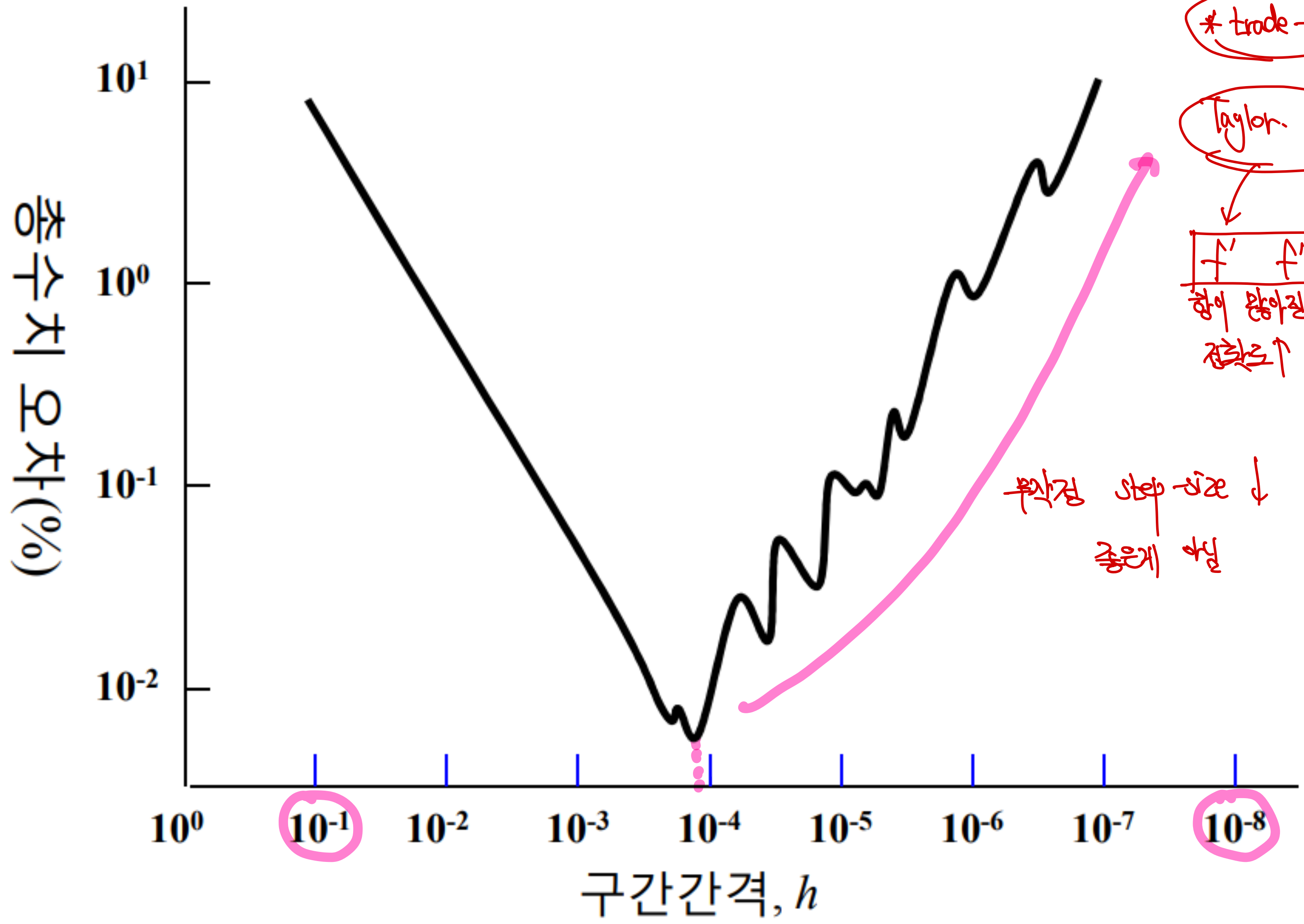
Problem state: Compare a numerical solution by Euler's method to ground truth for a differential equation $y' - y = e^t$ with initial value $y(0) = 0$ as $h_1 = 0.1$ and $h_2 = 0.01$ from $t = 0$ to $t = 1$

h	도함수 ($f(x)$)	수치해	엄밀해	상대오차 (%)
0.00	1.00000	0.000000	0.000000	-
0.01	1.02005	0.010000	0.010101	1.015
0.02	1.04040	0.020201	0.020404	1.018
0.03	1.06105	0.030605	0.030914	1.010
0.04	1.08202	0.041215	0.041632	1.013
0.05	1.10330	0.052035	0.052564	1.020
…	…	…	…	…
0.97	5.16526	2.527315	2.558806	1.244
0.98	5.24342	2.578968	2.611167	1.249
0.99	5.32263	2.631402	2.664322	1.251
1.00	**5.40291**	**2.684629**	**2.718282**	$\boxed{\textbf{1.246}}$

step-size를 ↓ 축.



Error analysis for Euler’s method



* trade-off.

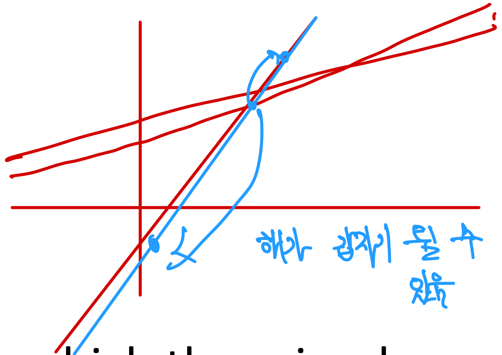
Taylor expansion

$\boxed{f' \quad f'' \quad f'''}$

항이 많아질 수록 정확도↑ but 계산량↑.

무작정 step-size ↓ 좋은게 아님

Stability of Euler's method



$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

- An unstable numerical solution of ODEs
  – If errors grow exponentially for a problem for which there is a bounded solution
- The stability of a particular application depending on…
  – The differential equation, the numerical method, and the step size

numerical solution

  – Ex) $\frac{dy}{dt} = -ay$ with $y(0) = y_0$

$\frac{1}{y}dy = -adt \rightarrow \ln|y| = -at + C \rightarrow y = y_0e^{-at}$

  - $\frac{dy}{dt} = -ay \rightarrow \frac{1}{y}dy = -adt \rightarrow \ln|y| = -at + C \rightarrow y = e^{-at+C} \rightarrow y = Ae^{-at} \rightarrow y(0) = A = y_0$
  - Finally, $y = y_0e^{-at}$
  - Numerically…

$-1 < 1-ah < 1$ $-1 < |1-ah| < 1$ 수렴 조건

$-2 < -ah$ $\frac{2}{a} > h$ $-ah < 0$ $ah > 0$ $h > 0$

특정 value에 수렴해야 한다. 발산하면 X.

  – $y_{i+1} = y_i + \frac{dy_i}{dt}h \rightarrow y_{i+1} = y_i - ay_ih \rightarrow y_{i+1} = y_i(1 - ah)$, where $1 - ah$ is called an amplification factor
    » If its absolute value is greater than unity, the solution will grow in an unbounded fashion
    » Depending on the step size h, $h > \frac{2}{a}$, $|y_i| \rightarrow \infty$ as $i \rightarrow \infty$

$* y_{i+1} = y_i + \frac{dy_i}{dt}h$

      - Euler's method is said to be conditionally stable

    » Ill-conditioned ODEs

$(1-ah) * y_i \rightarrow y_{i+1}$ $y_i - ay_ih = (1-ah)y_i$

      - Always error growing regardless of the method

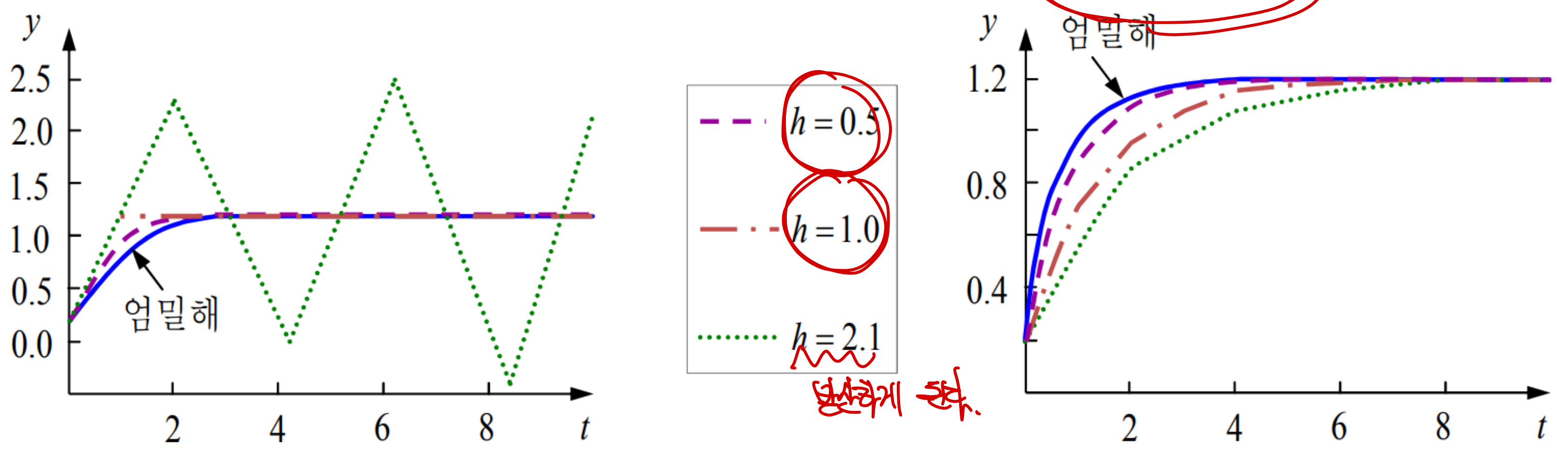
$*(1-ah)$ ∴ 등비 수열.

Stability of Euler's method

$$y_{i+1} = y_i + y_i' h.$$
$$= y_i + (1.2 - y_i) h.$$
$$= (1-h) y_i + 1.2h$$

- Implicit method to Euler's method
  - $y_{i+1} = y_i + hf(t_i, y_i)$, for $i = 0,1,\dots,n$
    - Substitute $f(t_i, y_i)$ to $f(t_{i+1}, y_{i+1})$ as $y_{i+1} = y_i + hf(t_{i+1}, y_{i+1})$
  - Ex) $y' + y = 1.2, y(0) = 0.2$
    - Explicit method, $y_{i+1} = 1.2h + (1-h)y_i$
    - Implicit method, $y_{i+1} = \frac{y_i + 1.2h}{1+h}$

$$y_{i+1} = y_i + (1.2 - y_{i+1}) h$$
$$= y_i + 1.2h - h y_{i+1}$$
$$y_{i+1} = \frac{y_i + 1.2h}{1+h}$$



(a) 양함수 해 (b) 음함수 해

Stability of Euler's method

- Implicit method
  - $y' = -\lambda y, y(0) = 1$

Improvements of Euler's method

- From Euler's method to Heun's method
  - Euler's method, $\boxed{y_{i+1} = y_i + f(t_i, y_i)h}$
  - Heun's method
    - Prediction equation, $y_{i+1}^0 = y_i + f(t_i, y_i)h$
      - Prediction of y'_{i+1}, $y'_{i+1} = f\left(t_{i+1}, y_{i+1}^0\right)$
    - Now, we have two slopes, $\boxed{y'_i}$ and $\boxed{y'_{i+1}}$
    - Correct equation
      - $y_{i+1} = y_i + \bar{y}'h = y_i + \dfrac{f(t_i,y_i)+f\left(t_{i+1},y_{i+1}^0\right)}{2}h$

기울기가 많으면 더 정확하지 않겠나.

- Predictor-corrector approach

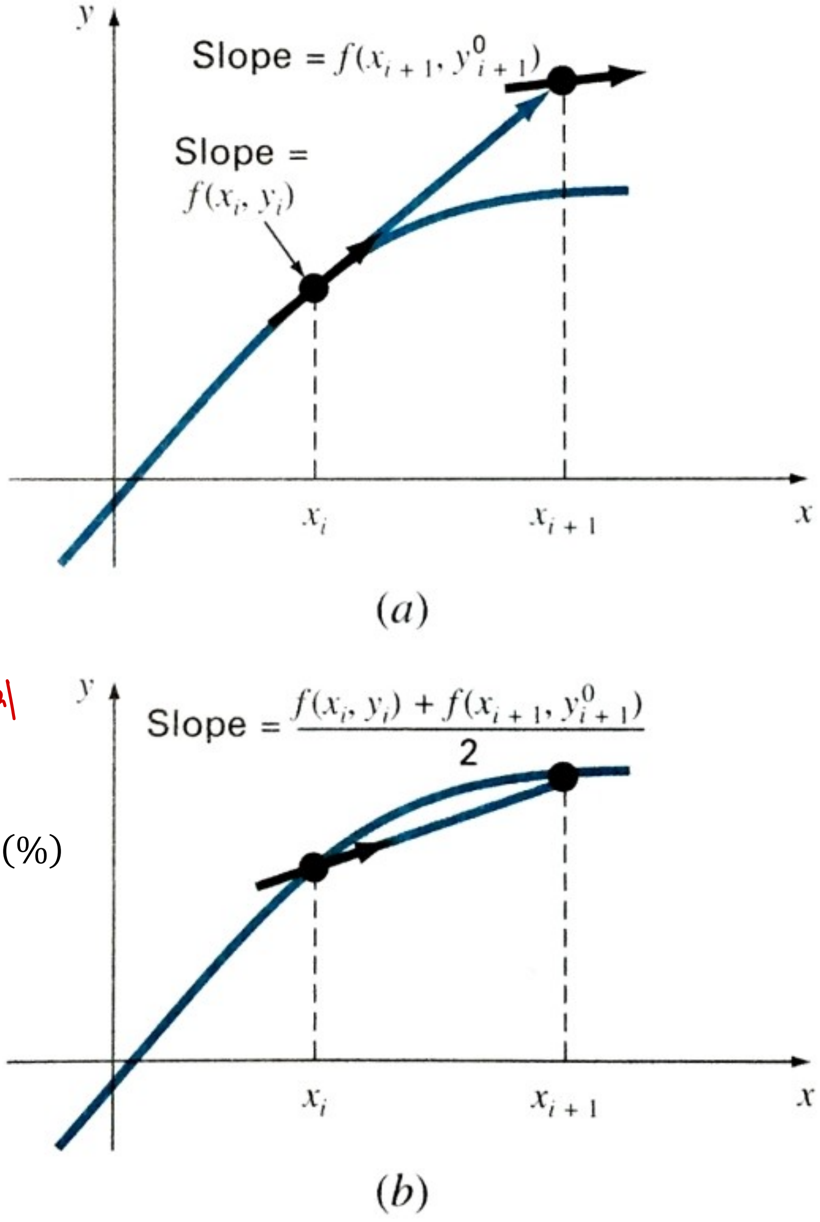
$$|\boldsymbol{\varepsilon_a}| = \left|\frac{\boldsymbol{y_{i+1}^j - y_{i+1}^{j-1}}}{\boldsymbol{y_{i+1}^j}}\right| \times \mathbf{100}(\%)$$

Predictor (Fig. 22.4*a*): $\quad y_{i+1}^0 = y_i^m + f(t_i, y_i)h$

Corrector (Fig. 22.4*b*): $\quad y_{i+1}^j = y_i^m + \dfrac{f(t_i, y_i^m) + f\left(t_{i+1}, y_{i+1}^{j-1}\right)}{2}h$

(for $j = 1, 2, \ldots, m$)

$$y_{i+1}^j \longleftarrow y_i^m + \frac{f(t_i, y_i^m) + f(t_{i+1}, y_{i+1}^{j-1})}{2}h$$



Improvements of Euler’s method

Problem statement: Use Heun’s method with iteration to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1. The initial condition at $t = 0$ is $y = 2$. Employ a stopping criterion of 0.00001% to terminate the corrector iterations.

- At first, $f(t_0, y_0) = f(0,2) = 4e^0 - 0.5(2) = 3$
  - Prediction step, $y_1^0 = y_0 + f(t_0, y_0)h = 2 + 3(1) = 5$
  - Correction step, 1st iteration
    - $y_1' = f(t_1, y_1^0) = 4e^{0.8(1)} - 0.5(5) = 6.402164$
    - $\bar{y}' = \frac{f(t_0,y_0)+f(t_1,y_1^0)}{2} = \frac{3+6.402164}{2} = 4.701082$
    - $y_1^1 = y_0 + \bar{y}'h = 2 + 4.701082(1) = 6.701082, |\varepsilon_a| = \left|\frac{6.701082-5}{6.701082}\right| \times 100(\%) = 25.39$
  - Correction step, 2nd iteration
    - $y_1' = f(t_1, y_1^1) = 4e^{0.8(1)} - 0.5(6.701082)$
    - $\bar{y}' = \frac{f(t_0,y_0)+f(t_1,y_1^1)}{2} = \frac{3+4e^{0.8(1)}-0.5(6.701082)}{2}$
    - $y_1^2 = y_0 + \bar{y}'h = 2 + \frac{3+4e^{0.8(1)}-0.5(6.701082)}{2}(1) \cong 6.275811, |\varepsilon_a| = \left|\frac{6.275811-6.701082}{6.275811}\right| \times 100(\%) = 6.776\%$

$f(t_0, y_0) = 4-1 = 3 \quad t_0 = 0 \quad y_0 = 2$

$y_1^0 = y_0 + f(t_0, y_0)h = 2+3\cdot 1 = 5$

$y_1' = f(t_1, y_1^0) = 4e^{0.8(1)} - 0.5\cdot 5 = 6.4xxx$

$\bar{y}_1 = \frac{3+6.4xx}{2} = 4.7xxx$

$y_1^1 = y_0 + 4.7xx \times 1 = 6.7xxx$

$y_1' = f(t_1, y_1^1) = 4e^{0.8\cdot 1} - 0.5\times(6.7xx) = ?$

$\bar{y}' = \frac{f(t_0,y_0)+f(t_1,y_1^1)}{2} = \frac{3+y_1'}{2}$

Improvements of Euler’s method

TABLE 22.2 Comparison of true and numerical values of the integral of $y' = 4e^{0.8t} - 0.5y$, with the initial condition that $y = 2$ at $t = 0$. The numerical values were computed using the Euler and Heun methods with a step size of 1. The Heun method was implemented both without and with iteration of the corrector.

				Without Iteration		With Iteration	
t	y_{true}	y_{Euler}	$\lvert\varepsilon_t\rvert$ (%)	y_{Heun}	$\lvert\varepsilon_t\rvert$ (%)	y_{Heun}	$\lvert\varepsilon_t\rvert$ (%)
0	2.00000	2.00000		2.00000		2.00000	
1	6.19463	5.00000	19.28	6.70108	8.18	6.36087	2.68
2	14.84392	11.40216	23.19	16.31978	9.94	15.30224	3.09
3	33.67717	25.51321	24.24	37.19925	10.46	34.74328	3.17
4	75.33896	56.84931	24.54	83.33777	10.62	77.73510	3.18

Improvements of Euler's method

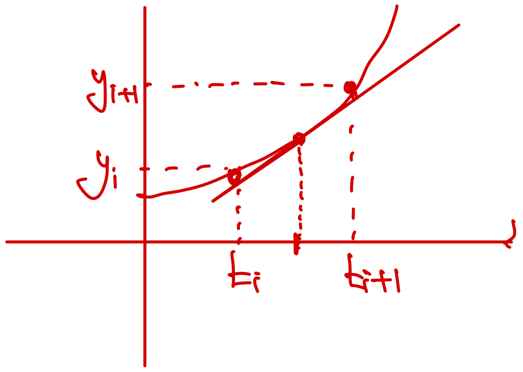
- Local and global truncation error of Heun's method
  - Local truncation error
    - $\frac{dy}{dt} = f(t) \rightarrow dy = f(t)dt$, integrating both side
    - $\int_{y_i}^{y_{i+1}} 1\, dy = \int_{y_i}^{y_{i+1}} f(t)\, dt \rightarrow y_{i+1} - y_i = \int_{y_i}^{y_{i+1}} f(t)\, dt$
      - By trapezoidal rule, $\int_{t_i}^{t_{i+1}} f(t)\, dt = \frac{f(t_{i+1})+f(t_i)}{2} h$
      - Finally, $y_{i+1} - y_i = \int_{y_i}^{y_{i+1}} f(t)\, dt \rightarrow y_{i+1} = y_i + \frac{f(t_{i+1})+f(t_i)}{2} h$

      수치 해석

    - Local truncation error relating to error of the trapezoidal rule, $E_t = -\frac{f''(\xi)}{12} h^3$, for $t_i \leq \xi \leq t_{i+1}$
  - Global truncation error, $O(h^2)$

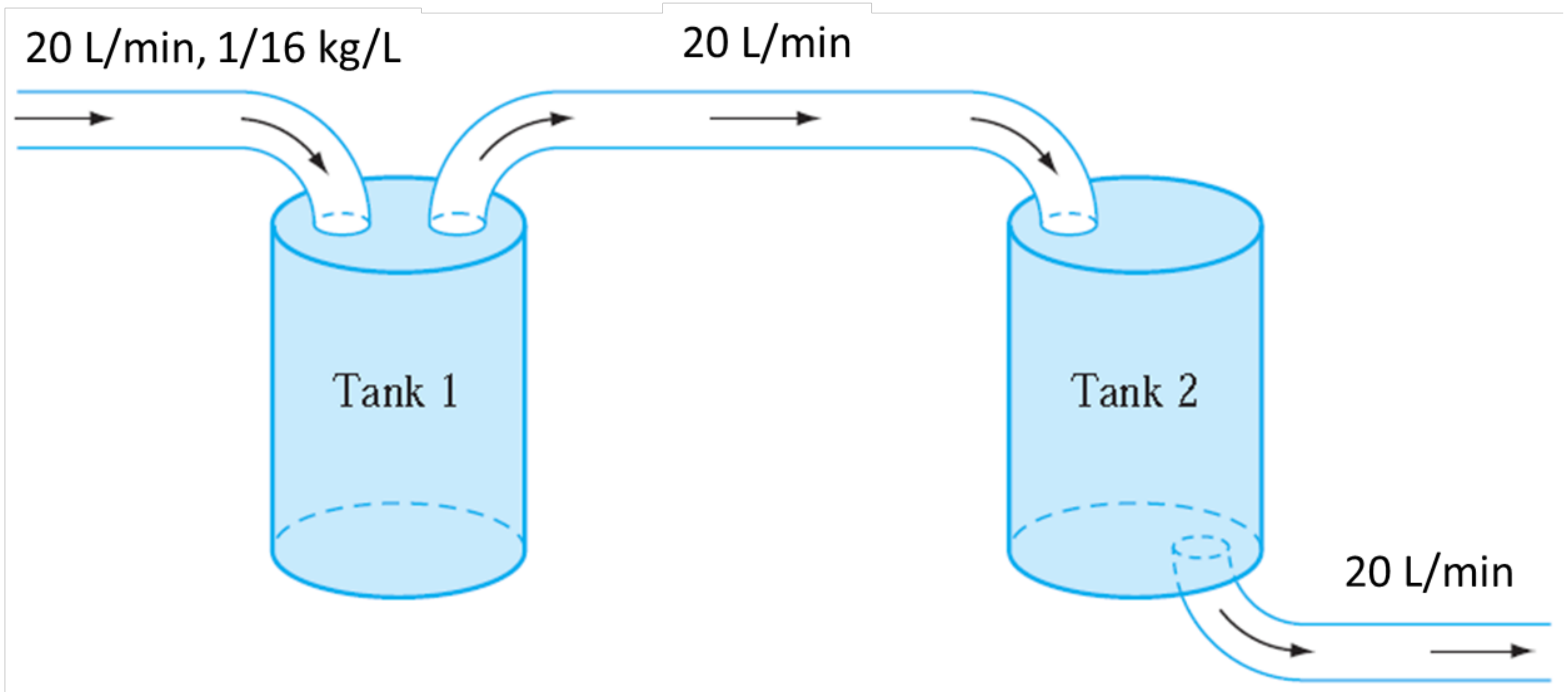
Improvements of Euler's method

- The midpoint method
  - Euler's method to predict a value of y at the midpoint of the interval
    - $y_{i+\frac{1}{2}} = y_i + f(t_i, y_i)\frac{h}{2}$
  - Calculate a slope at the midpoint with $y_{\frac{i+1}{2}}$
    - $y'_{i+\frac{1}{2}} = f\left(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\right)$
  - Predict new value with a slope at the midpoint
    - $y_{i+1} = y_i + f\left(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\right)h$
  - $\int_a^b f(x)\,dx \cong (b-a)f(x_1)$, where x_1 is the midpoint of interval (a, b)
    - $\int_{t_i}^{t_{i+1}} f(t)\,dt \cong hf\left(t_{i+\frac{1}{2}}\right)$
  - Local and global truncation error, $O(h^3)$ and $O(h^2)$

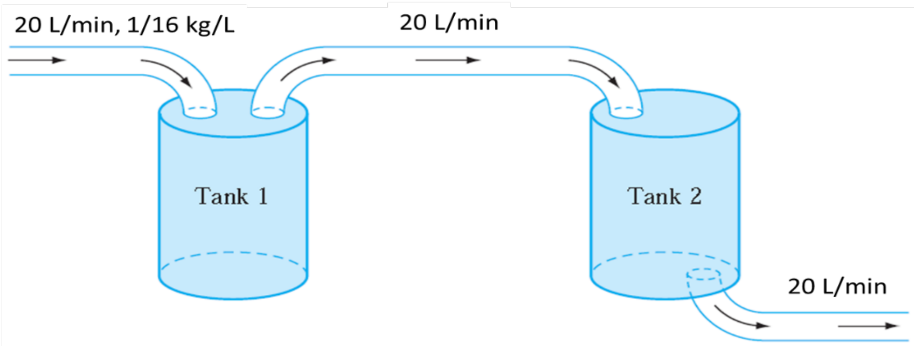


Recall Two Tank Problem…

Two tanks are connected as in Figure 1.6. Tank 1 initially contains 10 kilograms of salt dissolved in 400 liters of brine. Tank 2 initially contains 600 liters of brine in which 45 kilograms of salt have been dissolved. At time zero, a brine solution containing 1/16 kilogram of salt per liter is added to tank 1 at the rate of 20 liters per minute. Tank 1 has an output that discharges brine into tank 2 at the rate of 20 liters per minute, and tank 2 also has a discharge of 20 liters per minute. Determine that amount of salt in each tank at any time $t > 0$. Also determine when the concentration of salt in tank 2 is a minimum and how much salt is in this tank at that time.
Hint: solve for the amount of salt in tank 1 at time t and use this solution to determine the amount in tank 2.



Recall Two Tank Problem…



- $q_1(t)$: the number of kilograms of salt in tank 1
  - $\frac{\partial q_1}{\partial t} = \frac{1}{16}\left(\frac{kg}{L}\right)20\left(\frac{L}{m}\right) - \frac{q_1}{400}\left(\frac{kg}{L}\right)20\left(\frac{L}{m}\right) = \frac{5}{4} - \frac{1}{20}q_1, q_1(0) = 10$

$Q_1' = \frac{5}{4} - \frac{1}{20}Q_1$

- $q_2(t)$: the number of kilograms of salt in tank 2
  - $\frac{dq_2}{dt} = \frac{q_1}{400}\left(\frac{kg}{L}\right)20\left(\frac{L}{m}\right) - \frac{q_2}{600}\left(\frac{kg}{L}\right)20\left(\frac{L}{m}\right) = \frac{1}{20}q_1 - \frac{1}{30}q_2, q_2(0) = 45$

$Q_2' = \frac{1}{20}Q_1 - \frac{1}{30}Q_2$

- Solve by Laplace transform

$t_0 \quad Q_0$

$\vdots$

10개

- Solve by systems of ordinary differential equations

중요

- Find out approximation solutions from $t = 0$ to $t = 2$ with step size 0.2