Chapter 7. Eigenvalues and Diagonalization and Special Matrices

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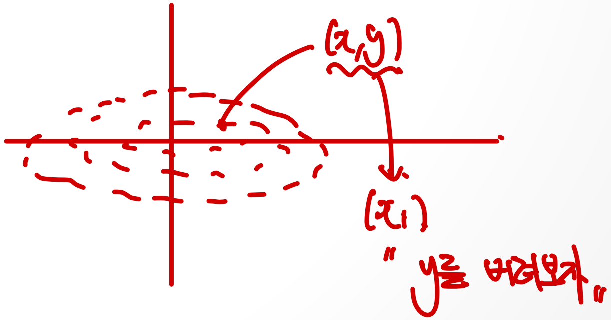
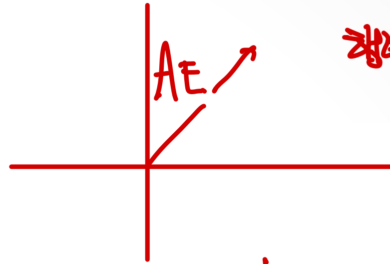


Eigenvalues and Eigenvectors

- Definition

$$\mathbf{A}\mathrm{E} = \lambda \mathrm{E}. \tag{9.1}$$



행렬 A를 곱했더니
λ만큼 커짐.

행렬 A와 관계가 있는 수 있다

We call E an *eigenvector* associated with the eigenvalue λ.

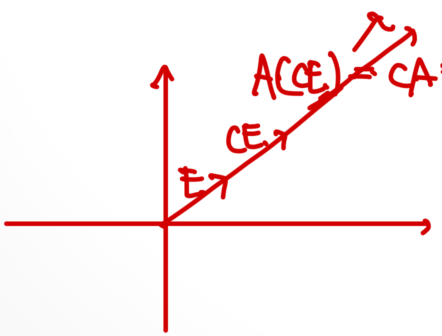
E는 λ와 관계가 있다 E가 바뀌면 λ도 다르고
λ가 다르면 E가 다르다.

If c is a nonzero number and $\mathbf{A}\mathrm{E} = \lambda \mathrm{E}$, then

$$\mathbf{A}(c\mathrm{E}) = c\mathbf{A}\mathrm{E} = c\lambda \mathrm{E} = \lambda(c\mathrm{E}).$$

E

This means that nonzero constant multiples of eigenvectors are eigenvectors (with the same eigenvalue).



$A(cE) = cAE = c\lambda E = \lambda(cE).$

행렬 A의 eigenvector야

$AE' = \lambda E'$

Eigenvalues and Eigenvectors

- Example) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
  - $A\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 4 \end{pmatrix}$
    - $\lambda = 0,\ E = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$
    - $\begin{pmatrix} 0 \\ 4\alpha \end{pmatrix}$ is also an eigenvector, zero can be an eigenvalue
    - An eigenvector has to be a nonzero vector

$A \cdot \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$\lambda = 0,\ E = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ $\quad \alpha E = \alpha\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4\alpha \end{pmatrix}$

0에 대응하는 eigenvector

$(A - \lambda I) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \left| \begin{pmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right| = (1-\lambda)\lambda = 0$

① $\lambda = 0$

$E_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

② $\lambda = 1$

$(A - \lambda I)E_2 = \begin{pmatrix} 1-1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$E_2 = \begin{pmatrix} e_1 \\ 0 \end{pmatrix} = e_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

linearly independent

Eigenvalues and Eigenvectors

$\underset{\text{matrix}}{A}E = \lambda E$

eigen value 2개
eigen vector 2개

- Example) $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

행렬 A의 eigen value eigen vector 1개 이상일 수 있음 존재.

  - $A\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$
    - $\lambda = 1, E = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$
  - $A\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$
    - $\lambda = -1, E = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

linearly independent

꼭 E가 1개 아닐 수도 있음

$(A-\lambda I) = \begin{pmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix}$

$(1-\lambda)(1-\lambda)(-1-\lambda)$

$= (1-\lambda)^2(-1-\lambda) = 0$

$\lambda = 1, \quad \lambda = -1$

$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-e_2 = 0$
$e_3 = 1$

$E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eigenvalues and Eigenvectors

- Computation of eigenvalues and eigenvectors
  - From definition
    - $\lambda E - AE = 0$
    - $(\lambda I_n - A)E = 0$
      - Equivalent to $(\lambda I_n - A)X = 0$
      - The equation has a nontrivial solution E
        - Eigenvectors are nonzero vectors
      - $|\lambda I_n - A| = 0$
        - Corresponding to each eigenvalue λ, a nontrivial solution of $(\lambda I_n - A)X = 0$

If expanded, the determinant on the left is a polynomial of degree n in the unknown λ, and is called the *characteristic polynomial* of $\mathbf{A}$. Thus

$$p_{\mathbf{A}}(\lambda) = |\lambda \mathbf{I}_n - \mathbf{A}|.$$

$AE = \lambda E$

$AE - \lambda E = 0$

$(A - \lambda I)E = 0$ ⇒ linear equation

B → coefficient matrix.

E → non-zero vector가 되어야 함

$X = 0$ → 구하는 값. parameter.

$BX = 0$ 의 non-trivial solution.

B가 역행렬이 존재?

~~B~~ $BX = 0$ ∴ 역행렬 B는 singular.

$X = 0$ vector.

$B = A - \lambda I$ 역행렬 존재X

characteristic equation

polynomial

$|A - \lambda I| = 0$

① 방정식 A의 eigenvalue λ

② $(A - \lambda I)E = 0$ → eigen vector

Eigenvalues and Eigenvectors

- Theorem 9.1 eigenvalues an eigenvectors of A
  - Let A be an $n{\times}n$ matrix of numbers
    - λ is an eigenvalue of A iff λ is a root of the characteristic polynomial of A
      - $p_A(\lambda) = |\lambda I_n - A| = 0$
        - p_A has degree n, A has n eigenvalues and its corresponding eigenvectors
    - If λ is an eigenvalue of A. Its corresponding eigenvector is a nontrivial solution of $(\lambda I_n - A)X = 0$
    - If E is an eigenvector associated with the eigenvalues λ, then so is cE for any nonzero number c.

Eigenvalues and Eigenvectors

- Example) $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

Eigenvalues and Eigenvectors

- Example) $A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$

$(A-\lambda I) = \left| \begin{pmatrix} 1-\lambda & -2 \\ 2 & -\lambda \end{pmatrix} \right| = (1-\lambda)\cdot(-\lambda)+4=0$

$\lambda^2-\lambda+4=0$

$\lambda = \frac{1 \pm \sqrt{15}i}{2}$: complex eigenvalue.

$(A-\lambda I) = \begin{pmatrix} 1-\frac{1-\sqrt{15}i}{2} & -2 \\ 2 & -\frac{1+\sqrt{15}i}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

첫번째행 $\times \frac{2}{1-\sqrt{15}i}$ $\quad -2 \times \frac{2}{1-\sqrt{15}i} = \frac{-4(1+\sqrt{15}i)}{1+15}$

$\begin{pmatrix} 1 & -\frac{1+\sqrt{15}i}{4} \\ 2 & -\frac{1+\sqrt{15}i}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$e_1 = \frac{1+\sqrt{15}i}{4} e_2$ $\quad e_2 = e_2$

① $\lambda_1 = \frac{1+\sqrt{15}i}{2}$ $\quad E_1 = \begin{pmatrix} \frac{1+\sqrt{15}i}{4} \\ 1 \end{pmatrix}$

② $\lambda_2 = \frac{1-\sqrt{15}i}{2}$ $\quad E_2 = \begin{pmatrix} \frac{1-\sqrt{15}i}{4} \\ 1 \end{pmatrix}$

linearly independent!

$\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{15}i}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{15}i}{2} \times \frac{1+\sqrt{15}i}{4} \\ \frac{1+\sqrt{15}i}{2} \end{pmatrix}$

$\frac{1+\sqrt{15}i}{2} \times \frac{1+\sqrt{15}i}{4} = \frac{1-15+2\sqrt{15}i}{8} = \frac{-14+2\sqrt{15}i}{8} = \frac{-7+\sqrt{15}i}{4}$

$\frac{1+\sqrt{15}i}{4} - 2 = \frac{-7+\sqrt{15}i}{4}$

$= \frac{-7+\sqrt{15}i}{4} = \frac{-7+\sqrt{15}i}{4}$

Eigenvalues and Eigenvectors

A: real number matrix.
$(\lambda = \alpha + i\beta)$

- The conjugate of λ
  - Let A be a real number matrix, $\lambda = \alpha + i\beta$
  - The conjugate($\bar{\lambda}$) of λ is also an eigenvalue
  - E is an eigenvector corresponding to λ
    - $\bar{E}$ is an eigenvector corresponding to $\bar{\lambda}$
    - $\overline{AE} = \bar{\lambda}\bar{E}$
      - $\bar{A} = A$, A is a real number matrix

$\overline{AE} = \overline{\lambda E}$

$= A\bar{E} = \bar{\lambda}\bar{E}$

A: real number matrix.

Eigenvalues and Eigenvectors

eigen vector E를 알고있다.

→ eigen value 어떻게 구하지!

$AE = \lambda E$

vector vector

λ → 1x1 matrix

- Lemma 9.1

$\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$ $(1+\sqrt{2} \;\; 1)\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$

  - Let A be a $n \times n$ matrix, λ be an eigenvalue of A, with eigenvector E
    - $\lambda = \dfrac{\bar{E}^t AE}{\bar{E}^t E}$ $\qquad \lambda = \dfrac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \overline{e_i} e_j}{\sum_{j=1}^{n} |e_j|^2}.$
    - $AE = \lambda E \rightarrow \bar{E}^t AE = \lambda \bar{E}^t E \rightarrow \lambda = \dfrac{\bar{E}^t AE}{\bar{E}^t E}$

$\bar{E}^t(AE) = \bar{E}^t \lambda E.$

$\dfrac{\text{vector} \cdot \text{vector}}{\text{scalar}} = \lambda \bar{E}^t E.$

$\bar{E} \cdot E = 1\times1$

= scalar.

$$\overline{\mathrm{E}}^t \mathrm{AE} = (e_1 \quad e_2 \quad \cdots \quad e_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} \quad \overline{\mathrm{E}}^t \mathrm{AE} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \overline{e_i} e_j.$$

$$\overline{\mathrm{E}}^t \mathrm{E} = (\overline{e_1} \quad \overline{e_2} \quad \cdots \quad \overline{e_n}) \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \sum_{j=1}^{n} \overline{e_j} e_j = \sum_{j=1}^{n} |e_j|^2.$$

$\therefore \lambda = \dfrac{\bar{E}^t AE}{\bar{E}^t E}$

Eigenvectors and Eigenvalues

$EA = \lambda A$

- Theorem 9.2

linearly independent → subspace

subspace의 basis가 될 수 있음

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

- Proof

  $\cdots \mathbf{v}_p, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_r$ → $\mathbf{v}_{p+1}$이 $\mathbf{v}_1 \sim \mathbf{v}_p$ linear combination으로 표현가능

  - Assume that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependence set with $\mathbf{v}_1 \neq 0$

    $\mathbf{v}_1 \sim \mathbf{v}_p$ → linearly independent

  - Let p be the least index such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding vectors(linearly independent)
    - $\lambda_{p+1} \times A\left(\boxed{c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1}}\right)$ (1) $\times \lambda_{p+1}$

    $\lambda_1 - \lambda_{p+1} = 0$, $\lambda_p - \lambda_{p+1} = 0$ distinct 하지 않다.

  - Multiplication A to both side of (1), eq(2)
    - $c_1 A\mathbf{v}_1 + \cdots + c_p A\mathbf{v}_p = A\mathbf{v}_{p+1} \rightarrow c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p = \lambda_{p+1}\mathbf{v}_{p+1}$
  - Multiplication λ_{p+1} to both side of (1), eq(3)
    - $c_1\lambda_{p+1}\mathbf{v}_1 + \cdots + c_p\lambda_{p+1}\mathbf{v}_p = \lambda_{p+1}\mathbf{v}_{p+1}$
  - ep(2)-eq(3)
    - $c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = 0$

  * $\mathbf{v}_1 \sim \mathbf{v}_p$까지 linearly independent

  $c_1 \sim c_p$ nonzero

Eigenvalues and Eigenvectors

$1 \cdot I_n - A$

$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}5&-4&4\\12&-11&12\\4&-4&5\end{pmatrix}$

$\begin{pmatrix}-4&4&-4\\-12&12&-12\\-4&4&-4\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix} \sim \begin{pmatrix}4&-4&4\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$

- Example) $A = \begin{pmatrix}5 & -4 & 4\\ 12 & -11 & 12\\ 4 & -4 & 5\end{pmatrix}$
  - Eigenvalues of A are -3, 1, 1 with 1 a $\boxed{\text{repeated root}}$ of the characteristic polynomial

    $E = \begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix} = \begin{pmatrix}e_2-e_3\\e_2\\e_3\end{pmatrix} = e_2\begin{pmatrix}1\\1\\0\end{pmatrix} + e_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$

  - $\lambda_1 = -3$, its corresponding $E_1 = \begin{pmatrix}1\\3\\1\end{pmatrix}$
  - $((1)I_n - A)X = \begin{pmatrix}-4 & 4 & -4\\ -12 & 12 & -12\\ -4 & 4 & -4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$
    - General solution of the equation, $\alpha\begin{pmatrix}1\\0\\-1\end{pmatrix} + \beta\begin{pmatrix}\cancel{0}\ 1\\1\\\cancel{1}\ 0\end{pmatrix}$

$1 \to \begin{pmatrix}-1\\0\\1\end{pmatrix},\ \begin{pmatrix}1\\1\\0\end{pmatrix}$

$-3 \to \begin{pmatrix}1\\3\\1\end{pmatrix}$

linearly independent [illegible]
[illegible] $\neq 0$

$\left|\begin{matrix}1&1&1\\0&1&3\\-1&0&1\end{matrix}\right| \neq 0$ linearly independent

$1\cdot 1 - 1(3-1) \neq 0$

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Eigenvalues and Eigenvectors

$A \rightarrow$ symmetric matrix $A = A^t$

- Theorem 9.3. eigenvalues of real symmetric matrices
  - By Lemma 9.1, for any eigenvalue λ of A, with eigenvector $E = (e_1, \dots, e_n)^t$
    - $\lambda = \frac{\bar{E}^t A E}{\bar{E}^t E}$
      - $\bar{E}^t E = \sum_{j=1}^{n} \left| e_j \right|^2 \rightarrow$ real number
      - $\overline{\bar{E}^t A E} = \overline{\bar{E}^t}\,\overline{AE} = E^t A \bar{E} \rightarrow 1{\times}1$ matrix
      - $(E^t A \bar{E})^t = \bar{E}^t A E$

Eigenvalues and Eigenvectors

A: symmetric

$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

- Theorem 9.4. orthogonality of eigenvectors
  - Let A be a real symmetric matrix, then eigenvectors associated with distinct eigenvalues are orthogonal
    - Let λ and μ be distinct eigenvalues of A
      - $E = (e_1, e_2, \dots, e_n)^t$, $G = (g_1, g_2, \dots, g_n)^t$ (eigen vector, eigen vector)
      - $E \cdot G = E^t G$ (λ_1, eigen value λ_2)
      - $\lambda E^t G = (AE)^t G = E^t A G = \mu E^t G$
      - $(\lambda - \mu) E^t G = 0$
      - $E^t G = 0$

$\lambda_1 \to E_1$

$\lambda_2 \to E_2$

orthogonal

$E \cdot G = E^t G$

$\lambda_1 E^t G = (AE)^t G$

$= E^t A^t G$

$= E^t A G = E^t \lambda_2 G = \lambda_2 E^t G$

$A = A^t$

$(\lambda_1 - \lambda_2) E^t G = 0$

$\lambda_1 \neq \lambda_2$

$E^t G = 0 = E \cdot G = 0$

E와 G는 orthogonal

Eigenvalues and Eigenvectors

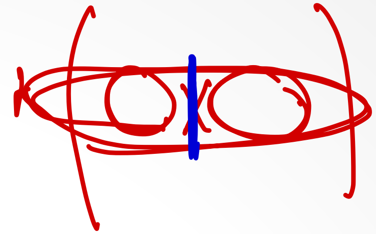
A: symmetric matrix

$$\begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

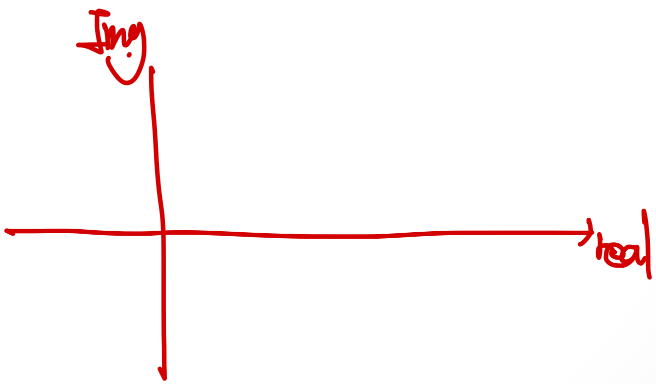
$$= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

- Example) $A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$
  - $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 4$
  - $E_1 = (0,1,0)^t, E_2 = (1,0,2)^t, E_3 = (2,0,-1)^t$ →
  - They are mutually orthogonal

Eigenvalues and Eigenvectors



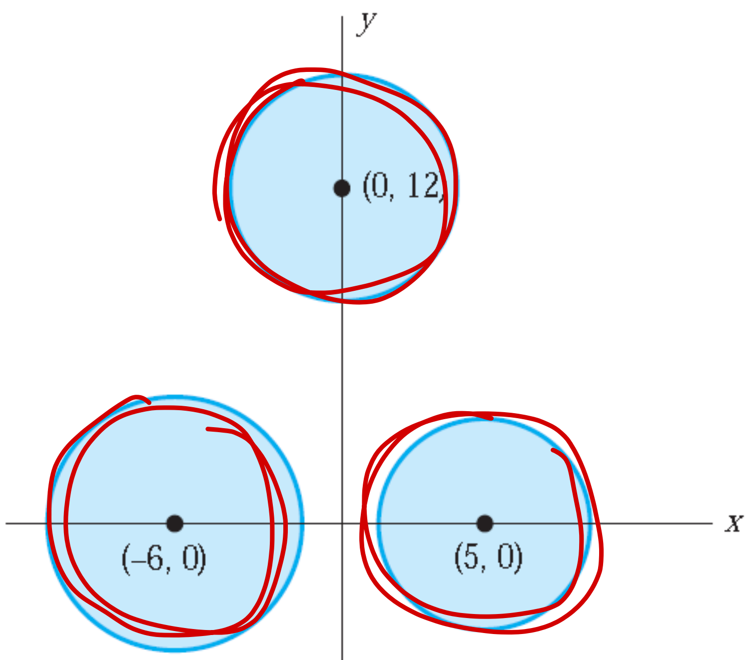
- Theorem 9.5. Gershgorin's theorem
  - Let A be a $n \times n$ matrix of complex or real numbers
    - $r_k = \sum_{j=1, j \neq k}^{n} |a_{kj}|$ → k번째 행에 대해서
    - Let C_k be the circle of radius r_k centered at (α_k, β_k), where $a_{kk} = \alpha_k + \beta_k i$
    - Then each eigenvalue of A, when plotted as a point in the complex plane, lies on or within one of the circles $C_1, \dots, C_n$



Eigenvalues and Eigenvectors

$r_1 = 1+3=4$

- Example) $A = \begin{pmatrix} 12i & 1 & 3 \\ 2 & -6 & 2+i \\ 3 & 1 & 5 \end{pmatrix}$ $r_2 = 4+i$
  - $p_A(\lambda) = \lambda^3 + (1-12i)\lambda^2 - (43+13i)\lambda - 68 + 381i$
    - $C_1 = (0, 12), r_1 = 1 + 3 = 4$
    - $C_2 = (-6, 0), r_2 = 2 + \sqrt{5}$
    - $C_3 = (5, 0), r_1 = 1 + 3 = 4$



→이 원에 eigenvalue가 있을 수 있음

Diagonalization

대각행렬

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$

- Properties of diagonal matrix

1. $\mathbf{A}+\mathbf{B}$ is diagonal with diagonal elements $a_{ii}+b_{ii}$.
2. $\mathbf{AB}$ is diagonal with diagonal elements $a_{ii}b_{ii}$.
3. $$|\mathbf{A}| = a_{11}a_{22}\cdots a_{nn},$$
   the product of the diagonal elements.
4. From (3), $\mathbf{A}$ is nonsingular exactly when each diagonal element is nonzero (so $\mathbf{A}$ has nonzero determinant). In this event, $\mathbf{A}^{-1}$ is the diagonal matrix having diagonal elements $1/a_{ii}$.
5. The eigenvalues of $\mathbf{A}$ are its diagonal elements.
6. $$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
   with all zero elements except for 1 in the $i, 1$ place, is an eigenvector corresponding to the eigenvalue a_{ii}.

$\lambda_1 = a_{11}$ $\quad \lambda_2 = a_{22}$ $\quad \lambda_3 = a_{33}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_1 I_n - A$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22}-a_{11} & 0 \\ 0 & 0 & a_{33}-a_{11} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$e_1 = e_1$

$(a_{22}-a_{11})e_2 = 0$

$(a_{33}-a_{11})e_3 = 0$

$e_2 = 0$, $e_3 = 0$

$(a_{22}-a_{11})e_2 = 0$

$e_1 = e_1$, $e_2 = 0$, $e_3 = 0$

eigen vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Diagonalization

- Problem definition

A square matrix is called a *diagonal matrix* if all the off-diagonal elements are zero. A diagonal matrix has the appearance

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & d_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & d_n \end{pmatrix}.$$

eigen value.

$P^{-1}AP = D$ "eigen value"

"eigen vector"

"$P^{-1}AP$"

(만들 수 있다!

An $n \times n$ matrix $\mathbf{A}$ is *diagonalizable* if there is an $n \times n$ matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. In this case we say that $\mathbf{P}$ *diagonalizes* $\mathbf{A}$.

Formed from eigenvectors

nxn matrix A is diagonalizable.
there is an nxn matrix P such that:

Diagonalization

- Example) $A = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}$

대각화.

$$\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

right

① eigen value

$$\begin{vmatrix} -1-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda) = 0 \quad \begin{pmatrix} \lambda_1 = -1 \\ \lambda_2 = 3 \end{pmatrix}$$

$$\begin{pmatrix} -1+1 & 4 \\ 0 & 3+1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \sim \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad \therefore \begin{cases} e_1 = e_1 \\ e_2 = 0 \end{cases}$$

$$\boxed{\lambda_1 = -1,\quad E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad e_1 = e_2,\ e_2 = e_2$$

$$\boxed{\lambda_2 = 3,\quad E_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

<대각화>

eigen value 순서대로 적어주어야 한다.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 0 & 3 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}}$$

$$A=\begin{pmatrix}-1 & 4\\ 0 & 3\end{pmatrix}$$

$$\begin{vmatrix}-1-\lambda & 4\\ 0 & 3-\lambda\end{vmatrix} = (-1-\lambda)(3-\lambda)=0$$

$\lambda=-1$ or $\lambda=3$

$$\begin{pmatrix}0 & 4\\ 0 & 4\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$4e_2=0$ $\quad$ $e_2=0$, $e_1=e_1$ $\quad$ $e_1\begin{pmatrix}1\\ 0\end{pmatrix}$

$=$

$\lambda=-1$

$E_1=\begin{pmatrix}1\\ 0\end{pmatrix}$

대각화 $\Leftarrow$

$\lambda_2=3$, $E_2=\begin{pmatrix}1\\ 1\end{pmatrix}$

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}-1 & 4\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}$$

eigen value.

$$\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 4\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 1\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}$$

Diagonalization

- Example) $A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$
  - Eigenvalues, -3, 1, 1
  - Eigenvectors, $(1,3,1)^t, (1,0,1)^t, (0,1,1)^t$
  - $P = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, P^{-1}AP = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

eigen vector

eigen value

Some Special Types of Matrices

- Orthogonal matrices

An $n \times n$ matrix is *orthogonal* if its transpose is its inverse:

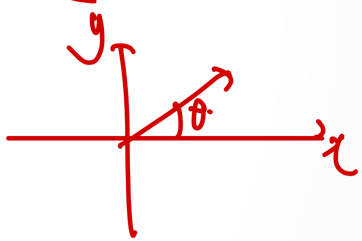
$$\mathbf{A}^{-1} = \mathbf{A}^t. \longrightarrow A^t A = I_n$$

In this event,

$$\mathbf{A}\mathbf{A}^t = \mathbf{A}^t\mathbf{A} = \mathbf{I}_n.$$

회전 행렬

$$\mathbf{A} = \begin{pmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2\sqrt{5} & -1\sqrt{5} \end{pmatrix}$$



$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

orthonormal matrix.

THEOREM 9.7

If $\mathbf{A}$ is orthogonal, then $|\mathbf{A}| = \pm 1$. ♦

Proof Because a matrix and its transpose have the same determinant,

$$|\mathbf{I}_n| = 1 = |\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{A}\mathbf{A}^t| = |\mathbf{A}||\mathbf{A}^t| = |\mathbf{A}|^2. \quad ♦$$

Some Special Types of Matrices

- Theorem 9.8
  - Let A be a $n \times n$ matrix of real numbers
    - A is orthogonal iff the row vectors are mutually orthogonal unit vectors in R^n
    - A is orthogonal iff the column vectors are mutually orthogonal unit vectors in R^n
  - Example of orthogonal matrices
    - Rotation matrix on 2D xy-plane

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \text{ or } \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$

Some Special Types of Matrices

- Theorem 9.9
  - An $n \times n$ real symmetric matrix with distinct eigenvalues can be diagonalized by an orthogonal matrix
  - Example) $S = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$
    - Eigenvalues, 2, -1, 4
    - Eigenvectors, $(0,1,0)^t$, $(1,0,2)^t$, $(2,0,-1)^t$
  - $Q = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$

길이를 1로 만들어서

놓으면

→ orthogonal matrix.

Some Special Types of Matrices

- Unitary matrices

We say that $\mathbf{U}$ is *unitary* if the inverse is the conjugate of the transpose (which is the same as the transpose of the conjugate):

$$\mathbf{U}^{-1} = \overline{\mathbf{U}}^{t}.$$

$U^{-1} = U^{t}$ → orthogonal

This means that

$$(\overline{\mathbf{U}})^{t}\mathbf{U} = \mathbf{U}(\overline{\mathbf{U}})^{t} = \mathbf{I}_n.$$

$$\mathbf{U} = \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

$U^{-1} = \overline{U}^{t}$

(Unitary → complex number 로 확장.)

Some Special Types of Matrices

- Unitary matrices
  - Theorem 9.11
    - Let λ be an eigenvalue of a unitary matrix U, then $|\lambda| = 1$
      - Let λ be an eigenvalue of U with eigenvector E
      - $UE = \lambda E$, $\overline{UE} = \overline{\lambda E}$
      - $(\overline{UE})^t = \bar{E}^t \bar{U}^t = \bar{E}^t U^{-1} = \overline{\lambda E}^t$
      - $\bar{E}^t E = \bar{\lambda} \bar{E}^t U E = \lambda \bar{E}^t \lambda E = \lambda^2 \bar{E}^t E$
      - $|\lambda| = 1$

Some Special Types of Matrices

- Hermitian and skew-Hermitian matrices

An $n \times n$ complex matrix $\mathbf{H}$ is *hermitian* if $\overline{\mathbf{H}} = \mathbf{H}^t$.

$(\overline{H} = H^t)$

An $n \times n$ complex matrix $\mathbf{S}$ is *skew-hermitian* if $\overline{\mathbf{S}} = -\mathbf{S}^t$.

$(\overline{S} = -S^t)$

$$\mathbf{H} = \begin{pmatrix} 15 & 8i & 6-2i \\ -8i & 0 & -4+i \\ 6+2i & -4-i & -3 \end{pmatrix} \quad \overline{\mathbf{H}} = \begin{pmatrix} 15 & -8i & 6+2i \\ 8i & 0 & -4-i \\ 6-2i & -4+i & -3 \end{pmatrix} = \mathbf{H}^t.$$

$$\mathbf{S} = \begin{pmatrix} 0 & 8i & 2i \\ 8i & 0 & 4i \\ 2i & 4i & 0 \end{pmatrix} \quad \overline{\mathbf{S}} = \begin{pmatrix} 0 & -8i & -2i \\ -8i & 0 & -4i \\ -2i & -4i & 0 \end{pmatrix} = -\mathbf{S}^t.$$

Some Special Types of Matrices

- Lemma 9.2
  - $Z = (z_1, z_2, \dots, z_n)^t$, complex $n \times 1$ matrix
    - If H is $n \times n$ Hermitian, then $\bar{Z}^t HZ$ is real
      - $\overline{\bar{Z}^t HZ} = Z^t \bar{H} \bar{Z}$
      - $(Z^t \bar{H} \bar{Z})^t = \bar{Z}^t \bar{H}^t Z = \bar{Z}^t HZ$
      - Its conjugate is equal to itself.
    - If H is $n \times n$ skew-Hermitian, then $\bar{Z}^t HZ$ is pure imaginary
      - $\overline{\bar{Z}^t HZ} = -\bar{Z}^t \bar{H} Z$
      - $\bar{Z}^t HZ = a + ib$
      - $a - ib = -a - ib$, a = 0

Some Special Types of Matrices

- Theorem 9.12
  - The eigenvalues of a Hermitian matrix are real
  - The eigenvalues of a skew-Hermitian matrix are pure imaginary

Proof By Lemma 9.1, an eigenvalue λ of any $n \times n$ matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{E}$, satisfies

$$\lambda = \frac{\overline{\mathbf{E}}^t \mathbf{A} \mathbf{E}}{\overline{\mathbf{E}}^t \mathbf{E}}.$$

We know that the denominator of this quotient is a positive number. Now use Lemma 9.2. If $\mathbf{A}$ is hermitian, the numerator is real, so λ is real. If $\mathbf{A}$ is skew-hermitian then the numerator is pure imaginary, so λ is pure imaginary. ♦

Some Special Types of Matrices

- Quadratic forms

A *quadratic form* is an expression

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\overline{z_j}z_k$$

in which the $a_{jk}'s$ and the $z_j's$ are complex numbers. If these quantities are all real, we say that we have a *real quadratic form*.

  - Complex number case

For $n = 2$, the quadratic form is

$$\sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}\overline{z_j}z_k = a_{11}\overline{z_1}z_1 + a_{12}\overline{z_1}z_2 + a_{21}z_1\overline{z_2} + a_{22}z_2\overline{z_2}.$$

- Real number case

$$\sum_{j=1}^{2}\sum_{k=1}^{2} a_{jk}x_jx_k = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2x_2$$

$$= a_1x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2.$$

Some Special Types of Matrices

- Example) $(x_1 \quad x_2)\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + 7x_1x_2 + 2x_2^2$

$$(x_1 \quad x_2)\begin{pmatrix} 1 & 7/2 \\ 7/2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + 7x_1x_2 + 2x_2^2$$

x^tAx → Quadratix form.

$x^tAX.$ →

Some Special Types of Matrices

- Theorem 9.13. principal axis theorem

Let $\mathbf{A}$ be a real symmetric matrix with distinct eigenvalues $\lambda_1, \cdots \lambda_n$. Then there is an orthogonal matrix $\mathbf{Q}$ such that the $\boxed{\text{change of variables}}$ $\mathbf{X} = \mathbf{QY}$ transforms the quadratic form $\sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} x_i x_j$ to

$$\sum_{j=1}^{n} \lambda_j y_j^2.$$

- Let Q be an orthogonal matrix that diagonalizes A
  - $\sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} x_i x_j = X^t A X = (QY)^t AQY = (Y^t Q^t) AQY = Y^t (Q^t AQ) Y = Y^t (Q^{-1}AQ) Y$

eigen value

D

eigen value

$$= \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2. \quad \blacklozenge$$

Some Special Types of Matrices

$x^tAx \rightarrow y^tDy$

선형 대수 시한

(대각성분 행렬)

- Example) $x_1^2 - 7x_1x_2 + x_2^2 \rightarrow X^t \overbrace{\begin{pmatrix} 1 & -\frac{7}{2} \\ -\frac{7}{2} & 1 \end{pmatrix}}^{A} X$ → singular value decomposition

  → 대각화 해서 편안한 형태로 표현하기.

  - Eigenvalues of A, $-5/2,\ 9/2$
  - Eigenvectors of A, $(1,1)^t,\ (-1,1)^t$
    - $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \rightarrow X = QY = \frac{1}{\sqrt{2}}\begin{pmatrix} y_1 - y_2 \\ y_1 + y_2 \end{pmatrix}$

$Y = Q^{-1}X$

선형대수

PCA

→ principal axis

This transforms the given quadratic form to its standard form

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = -\frac{5}{2}y_1^2 + \frac{9}{2}y_2^2,$$

in which there are no cross product y_1y_2 terms. ♦