Chapter 3. Determinant

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In This Chapter…

- Determinant
  - A Scalar value
    - Numbers or functions
  - Only square matrix
  - Rule for determinant
    - Similar to the 2×2 matrix in Chapter 2
  - Develop some properties of determinants
  - Evaluate and make usage of determinants

Introduction

- In Chapter 2
  - A 2×2 invertible matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
    - $ad - bc \neq 0$
  - For a 3×3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix} \quad (1)$$

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \quad (2)$$

$$\Delta = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Permutation

- Rearrangement of some integers

$$1, 2, 3, 4, 5, 6 \rightarrow 3, 1, 4, 5, 2, 6,$$

then $p(1) = 3$, $p(2) = 1$, $p(3) = 4$, $p(4) = 5$, $p(5) = 2$ and $p(6) = 6$.

A permutation is characterized as even or odd according to a rule we will illustrate. Consider the permutation

$$p: 1, 2, 3, 4, 5 \rightarrow 2, 5, 1, 4, 3$$

of the integers 1, 2, 3, 4, 5. For each k in the permuted list on the right, count the number of integers to the right of k that are smaller than k. There is one number to the right of 2 smaller than 2, three numbers to the right of 5 smaller than 5, no numbers to the right of 1 smaller than 1, one number to the right of 4 smaller than 4, and no numbers to the right of 3 smaller than 3. Since $1 + 3 + 0 + 1 + 0 = 5$ is odd, p is an *odd permutation*. When this sum is even, p is an *even permutation*.

If p is a permutation on $1, 2, \cdots, n$, define

$$\sigma(p) = \begin{cases} 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is an odd permutation.} \end{cases}$$

Determinant

$\det A = \sum_{p}$.

- Definition

The *determinant* of an $n \times n$ matrix $\mathbf{A}$ is defined to be

$$\det \mathbf{A} = \sum_{p} \sigma(p) a_{1p(1)} a_{2p(2)} \cdots a_{np(n)} \tag{8.1}$$

with this sum extending over all permutations p of $1, 2, \cdots, n$. Note that $\det \mathbf{A}$ is a sum of terms, each of which is plus or minus a product containing one element from each row and each column of $\mathbf{A}$.

- Notation
  - $\det A$ as $|A|$

Determinant

- Example) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
  - Only two permutations
    - p_1: ~~1,2 → 1,2~~ and p_2: 1,2 → 2,1

$$|\mathbf{A}| = \sigma(p_1) a_{1p_1(1)} a_{2p_1(2)} + \sigma(p_2) a_{1p_2(1)} a_{2p_2(2)}$$

$$= a_{11}a_{22} - a_{12}a_{21}.$$

$\begin{pmatrix} p_1 : 1,2 \longrightarrow 1,2 \\ p_2 : 1,2 \longrightarrow 2,1 \end{pmatrix}$

$\sigma(p_1) = 0$ (even) $+1$

$\sigma(p_2) = 1$ (odd) -1

$|A| = \overset{+1}{\sigma(p_1)}\, a_{1\,\overset{1}{p_1(1)}}\, a_{2\,\overset{2}{p_1(2)}} + \overset{-1}{\sigma(p_2)}\, a_{1\,\overset{2}{p_2(1)}}\, a_{2\,\overset{1}{p_2(2)}}$

$= [a_{11}a_{22} - a_{12}a_{21}]$

Determinant

$\sigma(p_1)\, a_{1p_1(1)}\, a_{2p_1(2)}\, a_{3p_1(3)}$

$+\ \sigma(p_2)\, a_{1p_2(1)}\, a_{2p_2(2)}\, a_{3p_2(3)}$

$+\ \sigma(p_3)\, a_{1p_3(1)}\, a_{2p_3(2)}\, a_{3p_3(3)}.$

6개가 나오게 된다.

- Example) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
  - Six permutations

$p_1 : 1, 2, 3 \to 1, 2, 3, \text{(even)};\ p_2 : 1, 2, 3, \to 1, 3, 2, \text{(odd)};$

$p_3 : 1, 2, 3 \to 2, 3, 1, \text{(even)};\ p_4 : 1, 2, 3, \to 2, 1, 3, \text{(odd)};$

$p_5 : 1, 2, 3, \to 3, 1, 2, \text{(even)};\ p_6 : 1, 2, 3, \to 3, 2, 1, \text{(odd)}.$

Determinant

- Some fundamental properties of determinants
  - $|A^t| = |A|$
  - $|A| = 0$, if A has a zero row or column
  - If B is formed from A by type I operation, $|B| = -|A|$ → type I

$$
\begin{aligned}
b_{11} &= a_{31},\ b_{12} = a_{32},\ b_{13} = a_{33},\\
b_{21} &= a_{21},\ b_{22} = a_{22},\ b_{23} = a_{23},\\
b_{31} &= a_{11},\ b_{32} = a_{12},\ b_{33} = a_{13}.
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{B}| &= b_{11}b_{22}b_{33} - b_{11}b_{23}b_{32} + b_{12}b_{23}b_{31}\\
&= -b_{12}b_{21}b_{33} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31}\\
&= a_{31}a_{22}a_{13} - a_{31}a_{23}a_{12} + a_{32}a_{23}a_{11}\\
&= -a_{32}a_{21}a_{13} + a_{33}a_{21}a_{12} - a_{33}a_{22}a_{11}\\
&= -|\mathbf{A}|.
\end{aligned}
$$

해가 없는 경우

  - If two rows or two columns are same, $|A| = 0$
  - If B is formed from A by type II operation(α), $|B| = \alpha|A|$
  - If one row or column of A is a constant multiple of another row or column, $|A| = 0$

$2x_1 + 3x_2 = 0$
$4x_1 + 6x_2 = 0$

trivial solution

$2x_1 + 3x_2 = 0$

Determinant

$2x_1 = -3x_2$

trivial

- Some fundamental properties of determinants
  - Each element of row k of A, $a_{kj} = b_{kj} + c_{kj}$

    fundamental rule.

    - B = $\begin{matrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{matrix}$, C= $\begin{matrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{k1} & \cdots & c_{kj} & \cdots & c_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{matrix}$

    - $|A| = |B| + |C|$

  - If D is formed from A by type III operation, $|D| = |A|$

    linearly dependent $\rightarrow$ 0

$$\mathbf{D} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{i1} + a_{k1} & \alpha a_{i2} + a_{k2} & \cdots & \alpha a_{in} + a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix}$$

|D|

linearly dependent. det = 0

det = |A|

Determinant

- Some fundamental properties of determinants
  - If A is nonsingular, $|A| \neq 0$
  - If A, B are $n \times n$ matrices, $|AB| = |A||B|$

$|AB| = |A|\,|B|$

Evaluation of Determinants I

$(-1)^{1+1} a_{11} |A_{11}|$

- Example) $A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= a_{11} |A_{11}|$

$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$= a_{11} (a_{22}a_{33} - a_{23}a_{32})$

Determinant

- Definition

co-factor expansion

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Determinant

- Theorem 1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

- Theorem 2

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$$

Properties of Determinants

- Theorem 3

Row Operations

Let A be a square matrix.

type III operation.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

type I operation.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

type II operation.

Evaluation of Determinants I

- Lemma 8.1

Let $\mathbf{A}$ be $n \times n$, and suppose row k or column r has all zero elements, except perhaps for a_{kr}. Then

$$|\mathbf{A}| = (-1)^{k+r} a_{kr} |\mathbf{A}_{kr}|, \tag{8.3}$$

where $\mathbf{A}_{kr}$ is the $n-1 \times n-1$ matrix formed by deleting row k and column r of $\mathbf{A}$. ♦

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Evaluation of Determinants I

- Example) $A = \begin{pmatrix} 4 & 2 & -3 \\ 3 & 4 & 6 \\ 2 & -6 & 8 \end{pmatrix} \rightarrow B = \begin{pmatrix} 4 & 2 & -3 \\ -5 & 0 & 12 \\ 14 & 0 & -1 \end{pmatrix}$
  - row2 of B: -2*(row1)+row2
  - row3 of B: 3*(row1)+row3
  - If B is formed from A by type III operation, $|B| = |A|$
  - $|A| = |B|$
    - $|B| = (-1)^{1+2}(2)|B_{12}| = -2\begin{vmatrix} -5 & 12 \\ 14 & -1 \end{vmatrix} = -2(5-168) = 326$

$= -2\begin{vmatrix} -5 & 12 \\ 14 & -1 \end{vmatrix} = -2(5-168)$

$= -2 \times -163$

$= 326$

Evaluation of Determinants I

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}$$ 6.

- $$\mathbf{A} = \begin{pmatrix} -6 & 0 & 1 & 3 & 2 \\ -1 & 5 & 0 & 1 & 7 \\ 8 & 3 & 2 & 1 & 7 \\ 0 & 1 & 5 & -3 & 2 \\ 1 & 15 & -3 & 9 & 4 \end{pmatrix}. \Rightarrow \mathbf{B} = \begin{pmatrix} -6 & 0 & 1 & 3 & 2 \\ -1 & 5 & 0 & 1 & 7 \\ 20 & 3 & 0 & -5 & 3 \\ 30 & 1 & 0 & -18 & -8 \\ -17 & 15 & 0 & 18 & 10 \end{pmatrix}.$$

$$\mathbf{C} = \begin{pmatrix} -1 & 5 & 1 & 7 \\ 20 & 3 & -5 & 3 \\ 30 & 1 & -18 & -8 \\ -17 & 15 & 18 & 10 \end{pmatrix}. \Rightarrow \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 20 & 103 & 15 & 143 \\ 30 & 151 & 12 & 202 \\ -17 & 70 & 1 & -109 \end{pmatrix}.$$

$$\mathbf{E} = \begin{pmatrix} 103 & 15 & 143 \\ 151 & 12 & 202 \\ -70 & 1 & -109 \end{pmatrix}. \Rightarrow \mathbf{F} = \begin{pmatrix} 1153 & 0 & 1778 \\ 991 & 0 & 1510 \\ -70 & 1 & -109 \end{pmatrix}.$$

Evaluation of Determinants II

- Cofactor expansion

THEOREM 8.2 *Cofactor Expansion by a Row*

For any k with $1 \leq i \leq n$.

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}. \quad \blacklozenge \tag{8.4}$$

THEOREM 8.3 *Cofactor Expansion by a Column*

For any j with $1 \leq j \leq n$,

$$|\mathbf{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}. \quad \blacklozenge \tag{8.5}$$

KNU VIRTUAL REALITY LABORATORY

Evaluation of Determinants II

- Cofactor expansion

$$|\mathbf{A}| = |[a_{ij}]| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$\det\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = \begin{aligned} &afkp - aflo - agjp + agln + ahjo - ahkn - bekp + belo + bgip - bglm - bhio + bhkm \\ &+ cejp - celn - cfip + cflm + chin - chjm - dejo + dekn + dfio - dfkm - dgin + dgjm \end{aligned}$$

$$= a\det\begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b\det\begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix} + c\det\begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d\det\begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$

$$= \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} k & l \\ o & p \end{vmatrix} - \begin{vmatrix} a & c \\ e & g \end{vmatrix} \cdot \begin{vmatrix} j & l \\ n & p \end{vmatrix} + \begin{vmatrix} a & d \\ e & h \end{vmatrix} \cdot \begin{vmatrix} j & k \\ n & o \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} \cdot \begin{vmatrix} i & l \\ m & p \end{vmatrix} - \begin{vmatrix} b & d \\ f & h \end{vmatrix} \cdot \begin{vmatrix} i & k \\ m & o \end{vmatrix} + \begin{vmatrix} c & d \\ g & h \end{vmatrix} \cdot \begin{vmatrix} i & j \\ m & n \end{vmatrix}$$

$$+ \begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{k2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

Evaluation of Determinants II

- Example) $A = \begin{pmatrix} -6 & 3 & 7 \\ 12 & -5 & -9 \\ 2 & 4 & -6 \end{pmatrix}$
  - $|A| = -6\begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} - 3\begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} + 7\begin{vmatrix} 12 & -5 \\ 2 & 4 \end{vmatrix} = 172$

$= -6\,(30+36) - 3\,(-72+18) + 7\cdot(48+10)$

$= -6\cdot 66 - 3\cdot -54 + 7\cdot 58$

$= -396 + 162 + 406$

$= 172$

A Determinant Formula for A^{-1}

$b_{ij} = \frac{1}{|A|}(-1)^{i+j} M_{ji}$

- Elements of a matrix inverse

THEOREM 8.4 ***Elements of a Matrix Inverse***

Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix and define an $n \times n$ matrix $\mathbf{B} = [b_{ij}]$ by

$$b_{ij} = \frac{1}{|\mathbf{A}|}(-1)^{i+j} M_{ji}.$$

Then $\mathbf{B} = \mathbf{A}^{-1}$. ♦

→ determinant를 이용해 역행렬 계산가능

  - ○ M_{ji}: determinant of $(n-1)\times(n-1)$ matrix from A removing row j and column i

$b_{ij} = \frac{1}{|A|}(-1)^{i+j} M_{ji}$

A Determinant Formula for A^{-1}

- Example) $A = \begin{pmatrix} -2 & 4 & 1 \\ 6 & 3 & -3 \\ 2 & 9 & -5 \end{pmatrix}$

$A^{-1} = B$

$B_{11} = \begin{vmatrix} 3 & -3 \\ 9 & -5 \end{vmatrix} = -15-(-27) = 12$

$B_{12} = -\begin{vmatrix} 6 & -3 \\ 2 & -5 \end{vmatrix} = -(-30-(-6)) = -(-24) = 24$

$B_{13} = \begin{vmatrix} 6 & 3 \\ 2 & 9 \end{vmatrix} = 54-6 = 48$

$B_{21} = -\begin{vmatrix} 4 & 1 \\ 9 & -5 \end{vmatrix} = -(-20-9) = 29$

$B_{22} = \begin{vmatrix} -2 & 1 \\ 2 & -5 \end{vmatrix} = (10-2) = 8$

$B_{23} = -\begin{vmatrix} -2 & 4 \\ 2 & 9 \end{vmatrix} = -(-18-8) = 26$

$B_{31} = \begin{vmatrix} 4 & 1 \\ 3 & -3 \end{vmatrix} = -12-3 = -15$

$B_{32} = -\begin{vmatrix} -2 & 1 \\ 6 & -3 \end{vmatrix} = -(6-6) = 0$

$B_{33} = \begin{vmatrix} -2 & 4 \\ 6 & 3 \end{vmatrix} = (-6-24) = -30$

$B = \begin{pmatrix} 12 & 29 & -15 \\ 24 & 8 & 0 \\ 48 & 26 & -30 \end{pmatrix} \times \frac{1}{\det(A)}$

$\begin{pmatrix} -2 & 4 & 1 \\ 2 & 1 & -1 \\ 2 & 9 & -5 \end{pmatrix}$

$\begin{pmatrix} -2 & 4 & 1 \\ 0 & 5 & 0 \\ 0 & 13 & -4 \end{pmatrix}$

$-2 \cdot (-1)^{1+1} \begin{vmatrix} 5 & 0 \\ 13 & -4 \end{vmatrix}$

$= -2\,(-20) = 120$

$\times 3$

$\therefore \begin{bmatrix} 0.1 & 0.24 & -0.125 \\ 0.2 & 0.07 & 0 \\ 0.4 & 0.22 & -0.25 \end{bmatrix}$

Cramer's Rule

역행렬이 존재

- A determinant formula for the unique solution of a nonhomogeneous system $AX = B$, when A is nonsingular
  - $x_k = \frac{1}{|A|}|A(k;B)|$, for $k = 1,2,\dots,n$,

    해를 구함

    - $A(k;B)$ is the matrix obtained from A by replacing column k of A with B

Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix of numbers, and $\mathbf{B}$ be an $n \times 1$ matrix of numbers. Then the unique solution of $\mathbf{AX} = \mathbf{B}$ is determined by

$$x_k = \frac{1}{|\mathbf{A}|}|\mathbf{A}(k;\mathbf{B})| \tag{8.7}$$

for $k = 1, 2, \cdots, n$, where $\mathbf{A}(k;\mathbf{B})$ is the matrix obtained from $\mathbf{A}$ by replacing column k of $\mathbf{A}$ with $\mathbf{B}$. ◆

Cramer's Rule

- A determinant formula for the unique solution of a nonhomogeneous system $AX = B$, when A is nonsingular.
  - ○ $x_k = \frac{1}{|A|}|A(k;B)|$, for $k = 1,2,\dots,n$,
    - $A(k;B)$ is the matrix obtained from A by replacing column k of A with B

replacing column k of A with B.

$$x_k|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k}x_k & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k}x_k & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk}x_k & \cdots & a_{nn} \end{vmatrix}. \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

B.

$x_k|A| = |A(k,B)|$

$x_k = \frac{1}{|A|}|A(k,B)|$

Cramer's Rule

- A determinant formula for the unique solution of a nonhomogeneous system $AX = B$, when A is nonsingular
  - $x_k = \frac{1}{|A|}|A(k;B)|$, for $k = 1,2,\ldots,n$,
    - $A(k;B)$ is the matrix obtained from A by replacing column k of A with B

type II operation

type III operation → determinant는 변하지 X

$$
x_k|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{11}x_1 + \cdots + a_{1n}x_n & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{21}x_1 + \cdots + a_{2n}x_n & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n1}x_1 + \cdots + a_{nn}x_n & \cdots & a_{nn} \end{vmatrix}
$$

$$
= \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} = |\mathbf{A}(k;\mathbf{B})|
$$

Cramer's Rule

- Example) $$\begin{aligned} x_1 - 3x_2 - 4x_3 &= 1 \\ -x_1 + x_2 - 3x_3 &= 14 \\ x_2 - 3x_3 &= 5 \end{aligned}$$

"계산하면 답 도출 가능"

$X_1 = \frac{1}{|A|}\left|A(1,B)\right|$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ -1 & 1 & -3 & 14 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -3 & -4 & 1 \\ 0 & -2 & -7 & 15 \\ 0 & 1 & -3 & 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -4 & 1 \\ 0 & -2 & -7 & 15 \\ 0 & 0 & -\frac{13}{2} & \frac{25}{2} \end{array}\right)$$

non-singular

$$X_3 = \frac{1}{|A|}\begin{vmatrix} 1 & -3 & 1 \\ -1 & 1 & 14 \\ 0 & 1 & 5 \end{vmatrix}$$

$$X_1 = \frac{1}{|A|}\left|A(1,B)\right| = \frac{1}{|A|}\begin{vmatrix} 1 & -3 & -4 \\ 14 & 1 & -3 \\ 5 & 1 & -3 \end{vmatrix}$$

$$X_2 = \frac{1}{|A|}\left|A(2,B)\right| = \frac{1}{|A|}\begin{vmatrix} 1 & 1 & -4 \\ -1 & 14 & -3 \\ 0 & 5 & -3 \end{vmatrix}$$

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

$$\overset{A}{\left(\begin{matrix} 3 & -2 \\ -5 & 4 \end{matrix}\right.}\left|\begin{matrix} 6 \\ 8 \end{matrix}\right).$$

$$12-(10)$$

$$x_1 = \frac{1}{|A|}\,|A(1,b)|$$
$$= \frac{1}{2}\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}$$
$$= \frac{1}{2}(24+16)$$
$$= 20$$

$$x_2 = \frac{1}{2}\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}$$
$$= \frac{1}{2}(24+30)$$
$$= \frac{1}{2}\cdot 54$$
$$= 27$$

#1.

$$A=\begin{bmatrix} 3 & -1 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$|A| = 3\left|\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}\right|$$

$$= 3\cdot 2\cdot \begin{vmatrix} 1 & 5 & 0 \\ \cancel{2} & \cancel{4} & \cancel{-1} \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 3\cdot 2\cdot 1 \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix}$$

$$= 3\cdot 2\cdot(-2)$$

$$= -12$$

#

type III $\rightarrow$ determinant isn't change

$$A = \begin{bmatrix} 5 & -1 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ 0 & -15 & 2 & 5 \\ 0 & 5 & 0 & -6 \end{bmatrix}$$

$$5\begin{vmatrix} 3 & 0 & -4 \\ \cancel{-15} & \cancel{2} & \cancel{5} \\ 5 & 0 & -6 \end{vmatrix} = 10\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 10\left(-18-(-20)\right)$$

$$= 10\cdot 2 = 20$$

$$A = \left| \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \right|$$

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 3 & -4 & -2 \\ -12 & 10 & 10 \\ 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \cdot 2 \begin{vmatrix} 3 & -4 & -2 \\ -6 & 5 & 5 \\ 0 & -3 & 2 \end{vmatrix} \qquad -4+\frac{5}{2}$$

$$-2+\frac{5}{2}$$

$$= 2 \cdot 2 \begin{vmatrix} 0 & -\frac{3}{2} & \frac{1}{2} \\ -6 & 5 & 5 \\ 0 & -3 & 2 \end{vmatrix} \qquad -3-\left(-\frac{3}{2}\right)$$

$$-\frac{3}{2}$$

$$= 4 \cdot (-1) \cdot (-6) \begin{vmatrix} -\frac{3}{2} & \frac{1}{2} \\ -3 & 2 \end{vmatrix} \qquad \overset{12}{\cancel{24}} \cdot -\frac{3}{\cancel{2}}$$

$$= 24 \left| -3 - \left(-\frac{3}{2}\right) \right|$$

$$= 24\left(-3+\frac{3}{2}\right) = -36$$

Properties of Determinant

- $|A| = |A^T|$
  - a_{1i}: each element of first row vector of A
  - a_{i1}: each element of first column vector of A
  - C_{1i}: cofactor expansion of A(except first row and ith column)
  - C_{i1}: cofactor expansion of A(except ith row and first column)
  - $|A| = \sum_{i=1}^{n} a_{1i}C_{1i}$
  - $|A^T| = \sum_{i=1}^{n} a_{i1}C_{i1}$
  - By definition of determinant
    - $\sum_{i=1}^{n} a_{1i}C_{1i} = \sum_{i=1}^{n} a_{i1}C_{i1}$
    - $|A| = |A^T|$

Properties of Determinant

- $\det AB = (\det A)(\det B)$
  - Verification by example
    - $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$
    - $\det A = 9$, $\det B = 5$
    - $AB = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$
      - $\det AB = 25\times13 - 20\times14 = 325 - 280 = 45 = 9\times5$
  - $\det EA = (\det E)(\det A)$
    - $\det E = \begin{cases} 1 & E\ is\ a\ row\ replacement \\ -1 & E\ is\ an\ interchange \\ r & E\ is\ a\ sclar\ mutiplication \end{cases}$

Warning: A common misconception is that Theorem 6 has an analogue for *sums* of matrices. However, det$(A + B)$ is *not* equal to $\det A + \det B$, in general.

$|A+B| \neq |A| + |B|$

Properties of Determinant

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 2 & 8 \end{bmatrix}$$

-3·2   4-6   -10   4-12   -2 -8   = -10   8-18

- Linearity property of the determinant function
  - $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{j-1} \quad \mathbf{x} \quad \mathbf{a}_{j+1} \quad \cdots \quad \mathbf{a}_2]$ → 이 행렬의 determinant 를 T → linear 하다.
  - Define a transformation T from $\mathbb{R}^n$ to $\mathbb{R}$
    - $T = \det[\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{j-1} \quad \mathbf{x} \quad \mathbf{a}_{j+1} \quad \cdots \quad \mathbf{a}_2]$
      - $T(c\mathbf{x}) = cT(\mathbf{x})$ → TypeⅡ operation.
      - $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

$T(u) = \det([a_1, \cdots, u, \cdots, a_m])$

$T(v) = \det([a_1, \cdots, v, \cdots, a_m])$

$T(u+v) = \det([a_1, \cdots, u+v, \cdots, a_m])$

↓ co-factor expansion

Define a transformation T from

$$\left|\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\right| + \left|\begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}\right| = \left|\begin{bmatrix} 1 & 6 & 3 \\ 1 & 6 & 3 \\ 1 & 6 & 3 \end{bmatrix}\right|$$

$[a_1, \cdots u \cdots, a_m]$ → det = □

$[a_1, \cdots v \cdots, a_m]$ → det = △

Determinants as Area or Volume

- Theorem 9

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

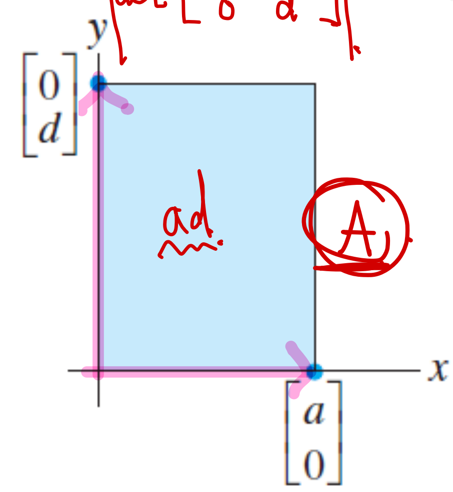
→ 부호에 민감해 지니까. → 절댓값

PROOF The theorem is obviously true for any 2×2 diagonal matrix:

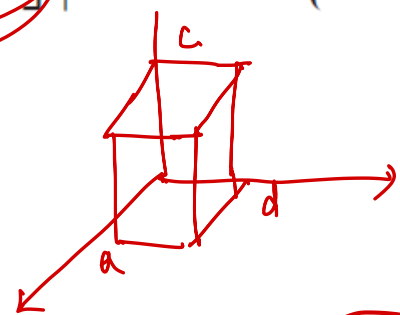
$$\left|\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right| = |ad| = \left\{\begin{matrix}\text{area of} \\ \text{rectangle}\end{matrix}\right\}$$

A

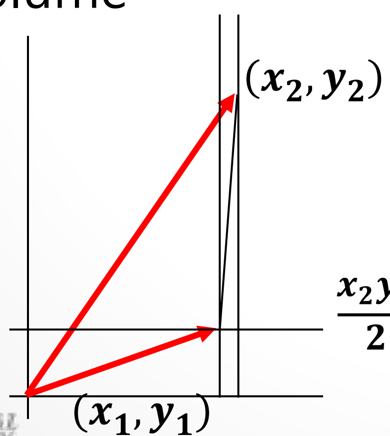
$$\left|\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right|$$



- In $\mathbb{R}^3$
  - Volume



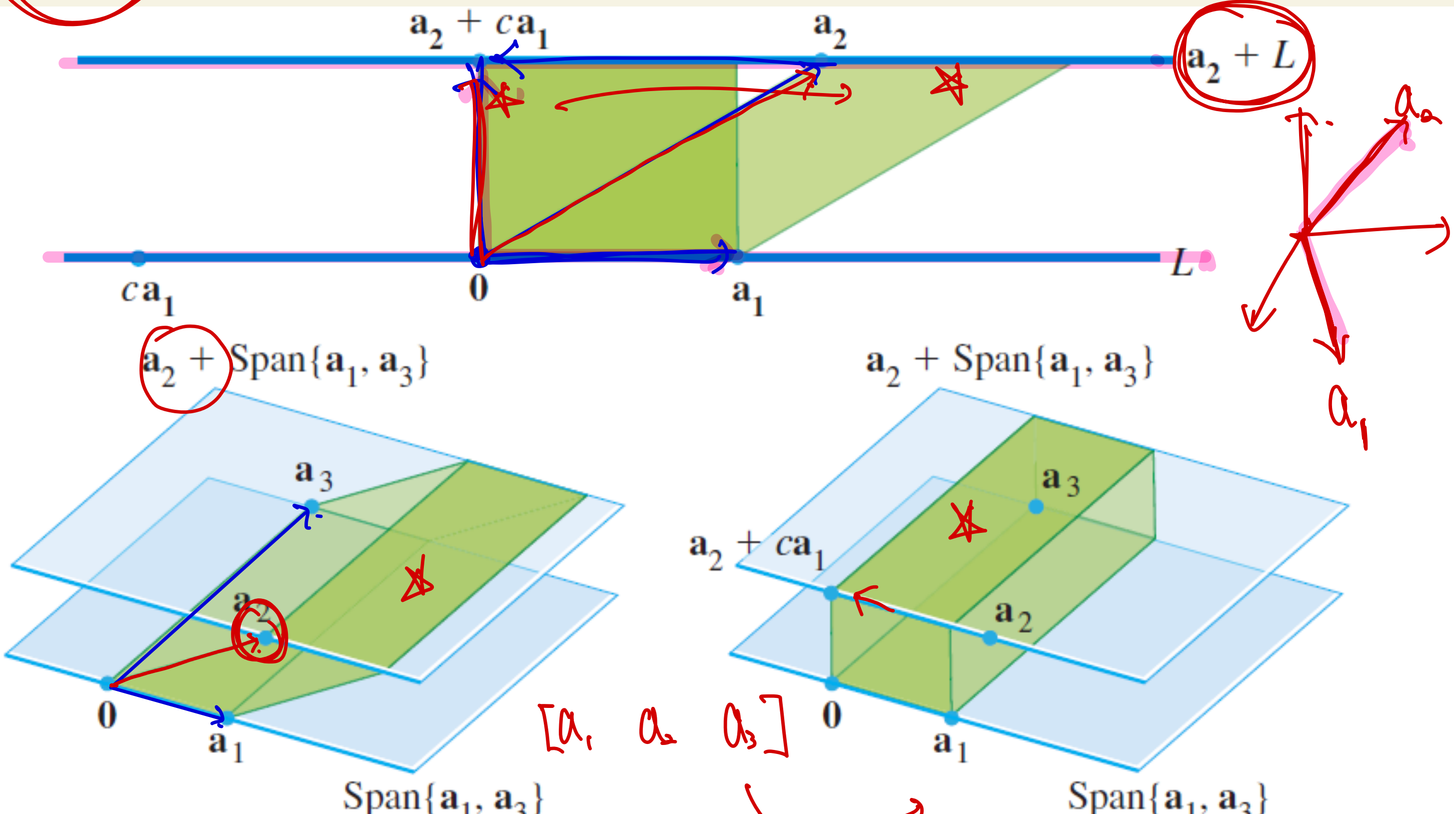
det $|A|$



$$\frac{x_2 y_2}{2} - \frac{x_1 y_1}{2} - (x_2 - x_1) y_1 - \frac{(x_2 - x_1)(y_2 - y_1)}{2} = \frac{1}{2}(x_1 y_2 - x_2 y_1)$$

Determinants as Area or Volume

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$.



$[a_1 \quad a_2 \quad a_3]$

FIGURE 4 Two parallelepipeds of equal volume.

$[a_1 \quad a_2+ca_1 \quad a_3]$

Linear Transformations

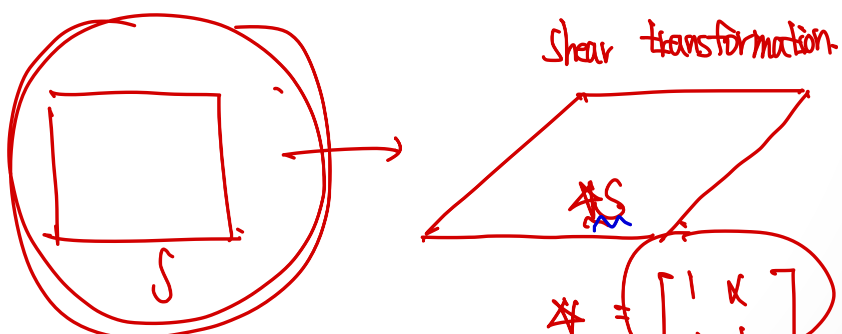
- Theorem 10

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the "linear transformation determined" by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \tag{6}$$



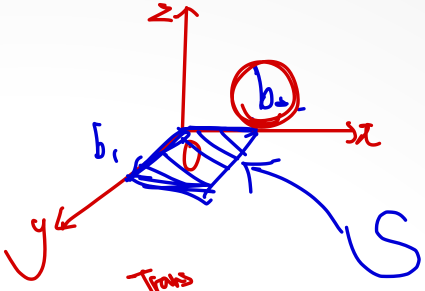
$* = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

1 → 넓이 변화 X

$|\det| \cdot \{\text{area of } S\}$

Linear Transformations

- Theorem 10



$T(s_1b_1+s_2b_2) = s_1T(b_1)+s_2T(b_2)$

$= s_1Ab_1 + s_2Ab_2$

$[Ab_1 \quad Ab_2]\ B$

PROOF Consider the 2×2 case, with $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2\,]$. A parallelogram at the origin in $\mathbb{R}^2$ determined by vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ has the form

$s_1Ab_1 + s_2Ab_2$

$$S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2 : 0 \le s_1 \le 1,\ 0 \le s_2 \le 1\}$$

The image of S under T consists of points of the form

$B = [b_1 \quad b_2]$

$$T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) = s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2)$$
$$= s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2$$

$T(s_1b_1+s_2b_2)$

$= s_1T(b_1) + s_2T(b_2)$

where $0 \le s_1 \le 1$, $0 \le s_2 \le 1$. It follows that $T(S)$ is the parallelogram determined by the columns of the matrix $[\,A\mathbf{b}_1 \quad A\mathbf{b}_2\,]$. This matrix can be written as AB, where $B = [\,\mathbf{b}_1 \quad \mathbf{b}_2\,]$. By Theorem 9 and the product theorem for determinants,

$|\det AB| = |\det A|\,|\det B|$

$s_1Ab_1 + s_2Ab_2$

$$\{\text{area of } T(S)\} = |\det AB| = |\det A| \cdot |\det B|$$
$$= |\det A| \cdot \{\text{area of } S\} \qquad (7)$$

$A\,[b_1 \quad b_2]$

area of S

$A = [a_1, a_2]$

$[Ab_1 \quad Ab_2]$

$S = [s_1b_1 + s_2b_2]$

AB

Linear Transformations

- Theorem 10

An arbitrary parallelogram has the form $\mathbf{p} + S$, where $\mathbf{p}$ is a vector and S is a parallelogram at the origin, as above. It is easy to see that T transforms $\mathbf{p} + S$ into $T(\mathbf{p}) + T(S)$. (See Exercise 26.) Since translation does not affect the area of a set,

$$\begin{aligned}
\{\text{area of } T(\mathbf{p} + S)\} &= \{\text{area of } T(\mathbf{p}) + T(S)\} \\
&= \{\text{area of } T(S)\} && \text{Translation} \\
&= |\det A| \cdot \{\text{area of } S\} && \text{By equation (7)} \\
&= |\det A| \cdot \{\text{area of } \mathbf{p} + S\} && \text{Translation}
\end{aligned}$$

This shows that (5) holds for all parallelograms in $\mathbb{R}^2$. The proof of (6) for the 3×3 case is analogous. ■

Linear Transformations

The conclusions of Theorem 10 hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

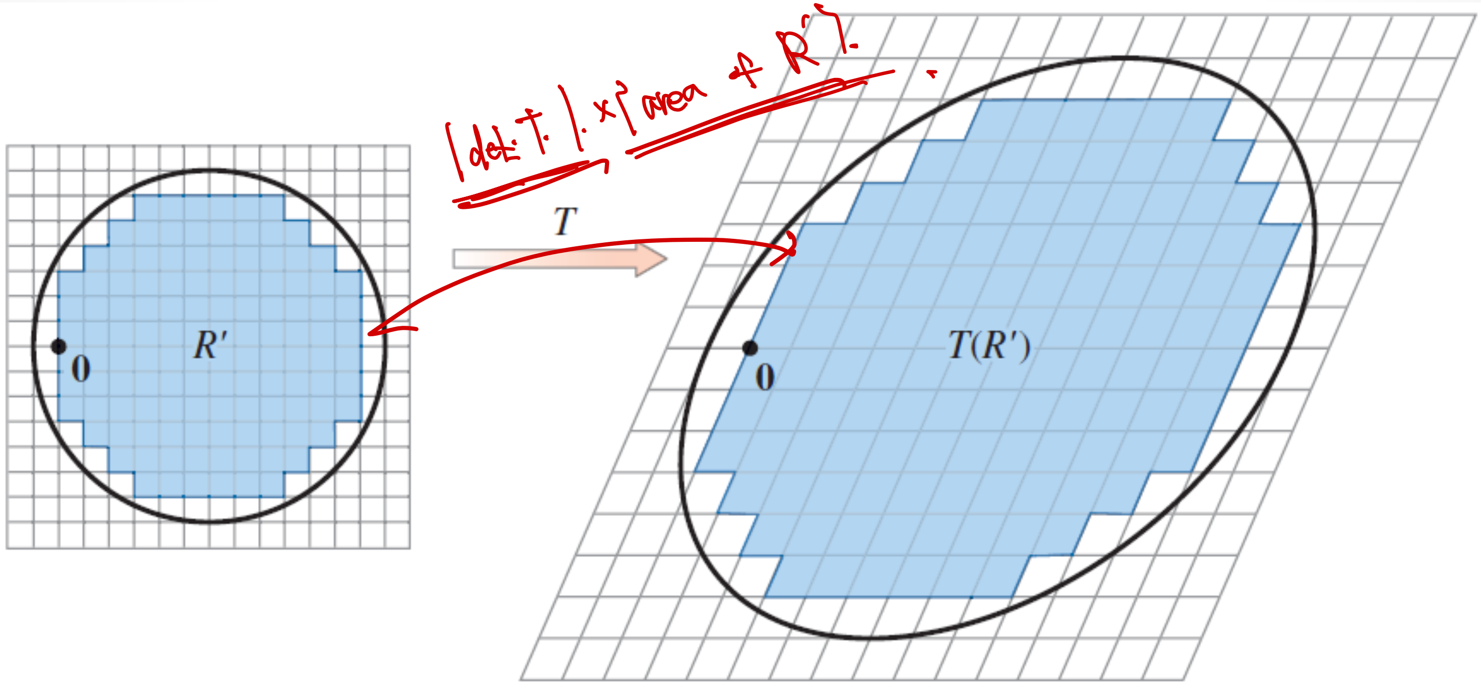


FIGURE 7 Approximating $T(R)$ by a union of parallelograms.

Linear Transformations

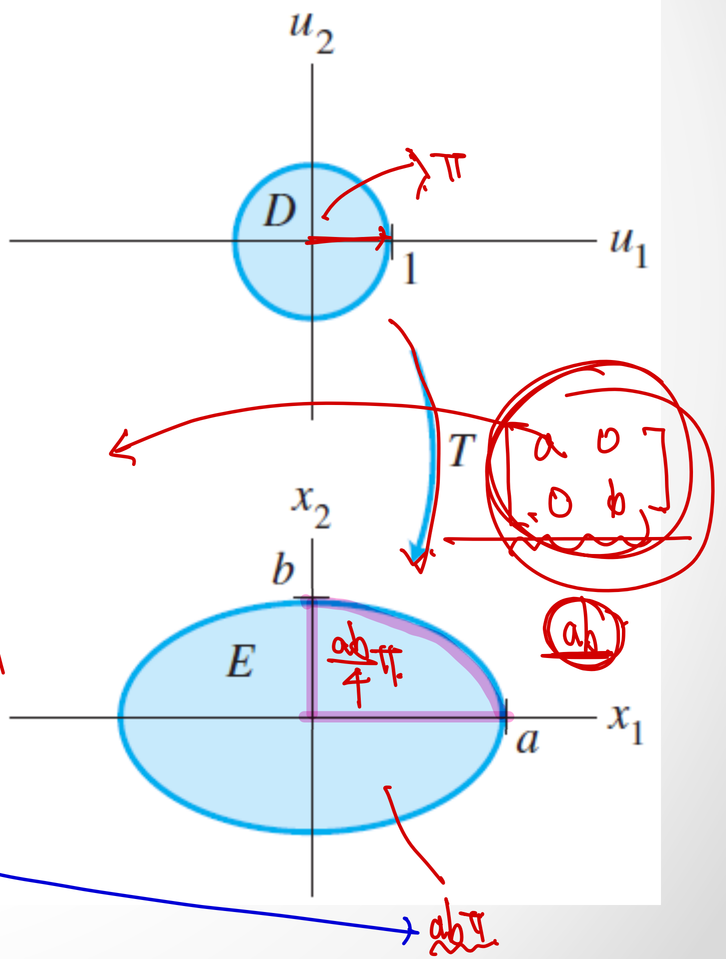
EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$ab\pi$

$$\left| \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right| \times \pi$$

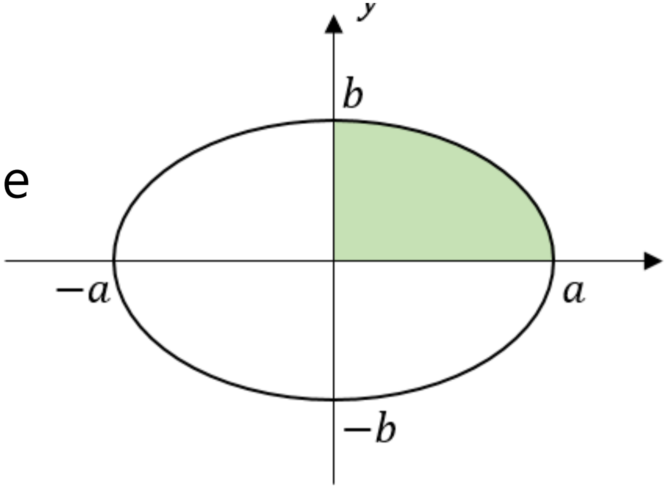
$= ab\pi$



Linear Transformations

EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$



- $x_2 = \pm b\sqrt{1 - \frac{x_1^2}{a^2}}$
- The quarter-area of the given ellipse
  - $\int_0^a b\sqrt{1 - \frac{x_1^2}{a^2}}\, dx_1$: $x_1 = a\sin\theta$
  - $\int_0^{\frac{\pi}{2}} ab\sqrt{1 - \sin^2\theta}\cos\theta\, d\theta$
  - $ab\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = ab\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} d\theta = \frac{ab}{4}[\sin 2\theta + 2\theta]_0^{\frac{\pi}{2}}$
  - $\frac{ab}{4}[0 + \pi - 0 - 0] = \frac{ab}{4}\pi$

KNU VIRTUAL REALITY LABORATORY

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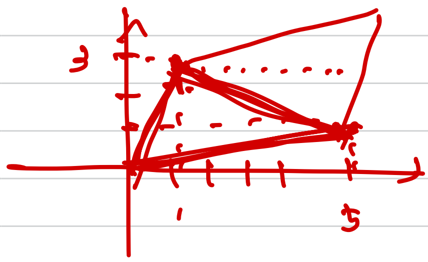
$b_1 = \begin{bmatrix}1\\3\end{bmatrix}$ $b_2 = \begin{bmatrix}5\\1\end{bmatrix}$



let $A = \begin{bmatrix}1 & -0.1\\ 0 & 2\end{bmatrix}$

$x \mid Ax$

2

$\det A = 2$

$\det\begin{bmatrix}1 & 5\\ 3 & 1\end{bmatrix}$

$\therefore |1-15| = 14$

$2 \cdot 14 = 28$