Chapter ~~10~~ 8. Systems of Linear Differential Equations

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VIRTUAL
REALITY
LABORATORY

Linear Systems

$$\begin{cases} a_1x_1+b_1x_2=c_1 \\ a_2x_1+b_2x_2=c_2 \end{cases}$$

$$\begin{bmatrix} a_{11}(t) & & \\ & \ddots & \\ & & a_{nn}(t) \end{bmatrix}\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

- Definition $\quad \mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{G}(t)$
  - Apply matrices to the solutions of a system of n linear differential equations in n unknown functions

n개의 식

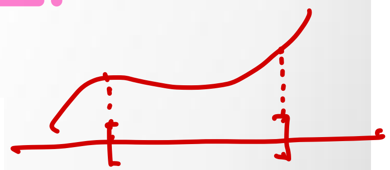
n개의 미분

"n개"

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + g_1(t) \\ x_2'(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + g_2(t) \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + g_n(t). \end{aligned}$$

상수 or function

  - $a_{ij}(t)$: continuous, $g_j(t)$: piecewise continuous on some interval



$$\mathbf{A}(t) = [a_{ij}(t)],\ \mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \text{ and } \mathbf{G}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

Linear Systems

- Theorem 10.1

Let I be an open interval containing t_0. Suppose $\mathbf{A}(t) = [a_{ij}(t)]$ is an $n \times n$ matrix of functions that are continuous on I, and let

$$\mathbf{G}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

be an $n \times 1$ matrix of functions that are continuous on I. Let $\mathbf{X}^0$ be a given $n \times 1$ matrix of real numbers. Then the initial value problem:

미분 방정식.

$$\mathbf{X}' = \mathbf{AX} + \mathbf{G};\ \boxed{\mathbf{X}(t_0) = \mathbf{X}^0} \to \text{initial value.}$$

has a unique solution that is defined for all t in I. ♦

The Homogeneous System $X' = AX$

$y'' + p(x)y' + Q(x)y = 0$

$x_1'(t) = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots$

$x_2'(t) = a_{21}(t)x_1 + a_{22}(t)x_2 \cdots$

$X' - AX = 0$

- Solution of $X' = AX$
  - Φ_1 and Φ_2 $\to$ $c_1\Phi_1 + c_2\Phi_2$

A set of k solutions $\mathbf{X}_1, \cdots, \mathbf{X}_k$ is *linearly dependent* on an open interval I (which can be the entire real line) if one of these solutions is a linear combination of the others, for all t in I. This is equivalent to the assertion that there is a linear combination

(Vector 형태의 solution)

$$c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \cdots + c_k\mathbf{X}_k(t) = 0$$

for all t in I, with at least one of the coefficients $c_1, \cdots, c_k$ nonzero.

We call these solutions *linearly independent* on I if they are not linearly dependent on I. This means that no one of the solutions is a linear combination of the others. Alternatively, these solutions are linearly independent if and only if the only way an equation

$$c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + \cdots + c_k\mathbf{X}_k(t) = 0$$

can hold for all t in I is for each coefficient to be zero: $c_1 = c_2 = \cdots = c_k = 0$.

The Homogeneous System $X' = AX$

- Example) $X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X$
  - Linearly independent solutions
    - $\Phi_1(t) = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix}$, $\Phi_2(t) = \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}$
  - Linearly dependent solutions
    - Φ_1, Φ_2, Φ_3
    - $\Phi_3(t) = \begin{pmatrix} (-5-6t)e^{3t} \\ (4+3t)e^{3t} \end{pmatrix} = 4\Phi_1(t) + 3\Phi_2(t)$

x_1 $\qquad$ x_2

linearly dependent

$x_1' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} x_1$

$\boxed{\begin{pmatrix} -6 \\ 3 \end{pmatrix} e^{3t}} = \begin{pmatrix} -6e^{3t} \\ 3e^{3t} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{3t} = \boxed{\begin{pmatrix} -6 \\ 3 \end{pmatrix} e^{3t}}$

$x_2(t) = \begin{pmatrix} 1-2t \\ t \end{pmatrix} e^{3t}$

(linearly independent) $x_2'(t) = \boxed{\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 3-6t \\ 3t \end{pmatrix} e^{3t}}$

$= \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1-2t \\ t \end{pmatrix} e^{3t}$

$= \begin{pmatrix} 1-2t-4t \\ 1-2t+5t \end{pmatrix} = \boxed{\begin{pmatrix} 1-6t \\ 1+3t \end{pmatrix} e^{3t}}$

The Homogeneous System $X' = AX$

- Theorem 10.2. test for independence of solutions

Suppose that

n개의 solution vector

$$\Phi_1(t) = \begin{pmatrix} \varphi_{11}(t) \\ \varphi_{21}(t) \\ \vdots \\ \varphi_{n1}(t) \end{pmatrix}, \Phi_2(t) = \begin{pmatrix} \varphi_{12}(t) \\ \varphi_{22}(t) \\ \vdots \\ \varphi_{n2}(t) \end{pmatrix}, \cdots, \Phi_n(t) = \begin{pmatrix} \varphi_{1n}(t) \\ \varphi_{2n}(t) \\ \vdots \\ \varphi_{nn}(t) \end{pmatrix}$$

are n solutions of $\mathbf{X}' = \mathbf{AX}$ on an open interval I. Let t_0 be any number in I. Then

1. $\Phi_1, \Phi_2, \cdots, \Phi_n$ are linearly independent on I if and only if $\Phi_1(t_0), \Phi_2(t_0), \cdots, \Phi_n(t_0)$ are linearly independent, when considered as vectors in R^n.
2. $\Phi_1, \Phi_2, \cdots, \Phi_n$ are linearly independent on I if and only if

"Wronskian Test"

$$\begin{vmatrix} \varphi_{11}(t_0) & \varphi_{12}(t_0) & \cdots & \varphi_{1n}(t_0) \\ \varphi_{21}(t_0) & \varphi_{22}(t_0) & \cdots & \varphi_{2n}(t_0) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{n1}(t_0) & \varphi_{n2}(t_0) & \cdots & \varphi_{nn}(t_0) \end{vmatrix} \neq 0.$$

The Homogeneous System $X' = AX$

$t_0 = 0$

- Example) Example) $X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X$
  - $\Phi_1(t) = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix}, \Phi_2(t) = \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}$
  - $\Phi_1(0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \Phi_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - $\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

linearly independent 하다고 말할 수 있다.

$\Phi_1(t_0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. $\Phi_2(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

The Homogeneous System $X' = AX$

- Theorem 10.3

Let $\mathbf{A}(t) = [a_{ij}(t)]$ be an $n \times n$ matrix of functions that are continuous on an open interval I. Then

1. The system $\mathbf{X}' = \mathbf{AX}$ has n linearly independent solutions on I.
2. Given n linearly independent solutions $\Phi_1(t), , \cdots , \Phi_n(t)$ defined on I, every solution on I is a linear combination of $\Phi_1(t), , \cdots , \Phi_n(t)$.

We call

general solution

$$c_1\Psi_1(t) + \cdots + c_n\Psi_n(t)$$

the *general solution* of $\mathbf{X}' = \mathbf{AX}$ when these solutions are linearly independent. Every solution is contained in this expression by varying the choices of the constants. In the language of linear algebra, the set of all solutions of $\mathbf{X}' = \mathbf{AX}$ is a vector space of dimension n, hence any n linearly independent solutions form a basis.

The Homogeneous System $X' = AX$

$$c_1\begin{pmatrix}-2e^{3t}\\ e^{3t}\end{pmatrix} + c_2\begin{pmatrix}(1-2t)e^{3t}\\ te^{3t}\end{pmatrix}$$
→ general solution

- Example) $X' = \begin{pmatrix}1 & -4\\ 1 & 5\end{pmatrix}X$
  - $\Omega(t) = \begin{pmatrix}-2e^{3t} & (1-2t)e^{3t}\\ e^{3t} & te^{3t}\end{pmatrix}$ → fundamental matrix
  - $\Omega C = \begin{pmatrix}-2e^{3t} & (1-2t)e^{3t}\\ e^{3t} & te^{3t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$

$$\begin{pmatrix}-2e^{3t} & (1-2t)e^{3t}\\ e^{3t} & te^{3t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

$$= c_1\begin{pmatrix}-2e^{3t}\\ e^{3t}\end{pmatrix} + c_2\begin{pmatrix}(1-2t)e^{3t}\\ te^{3t}\end{pmatrix} = c_1\Phi_1(t) + c_2\Phi_2(t)$$

The Homogeneous System $X' = AX$

$$\begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

대입.

$X(t_0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

- Example) $X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X, X(0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$
  - $X(0) = \Omega(0)C = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \to \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} C = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$
  - $C = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
  - $X(t) = \Omega(t) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2e^{3t} - 8te^{3t} \\ 3e^{3t} + 4te^{3t} \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -1\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= -\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

general solution

$$x(t) = \Omega(t)C$$

$$= \begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} e^{3t}(-2-8t) \\ e^{3t}(3+4t) \end{pmatrix}}$$

particular solution.

The Nonhomogeneous System

$y'' + p(x)y' + Q(x)y = f(x)$

$(Y_1 - Y_2) \rightarrow$ associated homogeneous solution

- Nonhomogeneous linear system $X' = AX + G$
  - Any two solutions Ψ_1, Ψ_2 of $X' = AX + G$
    - $\Psi_1 - \Psi_2 \rightarrow$ a solution of the homogeneous system $X' = AX$
    - $\Psi_1 - \Psi_2 = \Omega K$ with $n \times 1$ constant matrix K
    - $\Psi_1 = \Psi_2 + \Omega K$

The Nonhomogeneous System

- Theorem 10.4

Let Ω be a fundamental matrix for the homogeneous system $\boxed{\mathbf{X}' = \mathbf{A}\mathbf{X}}$. Let Ψ_p be any particular solution of the nonhomogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{G}$. Then every solution of the nonhomogeneous system has the form

$$\mathbf{X} = \underbrace{\Omega\mathbf{C}}_{\text{homo's}} + \underbrace{\Psi_p}_{\text{solution.}}.$$

any particular solution

For the homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, form a fundamental matrix Ω whose columns are n linearly independent solutions. The general solution is $\mathbf{X} = \Omega\mathbf{C}$.

For the nonhomogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{G}$, first find the general solution $\Omega\mathbf{C}$ of the associated homogeneous system $\mathbf{X}' = \mathbf{A}\mathbf{X}$. Then find any particular solution Ψ_p of the nonhomogeneous system. The general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{G}$ is $\mathbf{X} = \Omega\mathbf{C} + \Psi_p$.

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Solution of $X' = AX$ for Constant A

$y'' + p(x)y' + q(x)y = 0$
constant

- To find n linearly independent solutions for a linear system $X' = AX$

To carry out this strategy we will focus on the special case that $\mathbf{A}$ is a real, constant matrix. Taking a cue from the constant coefficient, second order differential equation, attempt solutions of the form $\mathbf{X} = \mathbf{E}e^{\lambda t}$, with $\mathbf{E}$ an $n \times 1$ matrix of numbers and λ a number. For this to be a solution, we need

$$(\mathbf{E}e^{\lambda t})' = \mathbf{E}\lambda e^{\lambda t} = \mathbf{A}\mathbf{E}e^{\lambda t}.$$

This will be true if

$$\mathbf{A}\mathbf{E} = \lambda\mathbf{E},$$

which holds if λ is an eigenvalue of $\mathbf{A}$ with associated eigenvector $\mathbf{E}$.

$X = Ee^{\lambda t}$

$(Ee^{\lambda t})' = E\lambda e^{\lambda t} = AX$
$= AEe^{\lambda t}$
$(AE = \lambda E)$

행렬 A의 eigen value와 eigen vector.

Solution of $X' = AX$ for Constant A

THEOREM 10.5

Let $\mathbf{A}$ be an $n \times n$ matrix of real numbers. If λ is an eigenvalue with associated eigenvector $\mathbf{E}$, then $\mathbf{E}e^{\lambda t}$ is a solution of $\mathbf{X}' = \mathbf{AX}$. ◆

We need n linearly independent solutions to write the general solution of $\mathbf{X}' = \mathbf{AX}$. The next theorem addresses this.

THEOREM 10.6

Let $\mathbf{A}$ be an $n \times n$ matrix of real numbers. Suppose $\mathbf{A}$ has eigenvalues $\lambda_1, \cdots, \lambda_n$, and suppose there are n corresponding eigenvectors $\mathbf{E}_1, \cdots, \mathbf{E}_n$ that are linearly independent. Then $\mathbf{E}_1 e^{\lambda_1 t}, \cdots, \mathbf{E}_n e^{\lambda_n t}$ are linearly independent solutions. ◆

① λ = distinct real
② λ = repeated
③ λ = complex

Solution of $X' = AX$ for Constant A

- Example) $X' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} X$

$$\begin{pmatrix} \lambda-4 & -2 \\ -3 & \lambda-3 \end{pmatrix} \rightarrow (\lambda-4)(\lambda-3)-6 = \lambda^2-7\lambda+6$$

① $\lambda_1 = 1$.

$$\begin{pmatrix} -3 & -2 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\xi_1 = \alpha \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} = \frac{\alpha}{3}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{t}$$

② $\lambda_2 = 6$

$$\begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = x_2$.

x_2 is free.

$$\xi_2 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}$$

$$X = \begin{pmatrix} -2e^{t} & e^{6t} \\ 3e^{t} & e^{6t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Solution of $X' = AX$ for Constant A

- Example) $X' = \begin{pmatrix} 5 & 14 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix} X$ → 한번 해보자.

Solution of $X' = AX$ for Constant A

$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$

$y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$

- Solutions with complex eigenvalue

Let $\mathbf{A}$ be an $n \times n$ matrix of real numbers. Let $\alpha + i\beta$ be a complex eigenvalue with corresponding eigenvector $\mathbf{U} + i\mathbf{V}$, in which $\mathbf{U}$ and $\mathbf{V}$ are real $n \times 1$ matrices. Then

$e^{(\alpha+\beta i)t} \rightarrow e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$

$$e^{\alpha t}[\cos(\beta t)\mathbf{U} - \sin(\beta t)\mathbf{V}]$$

$$\frac{1}{2}(\Phi_1(t) + \Phi_2(t))$$

and $\begin{pmatrix} 2+4i \\ 5 \\ 1-2i \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} i$

$$e^{\alpha t}[\sin(\beta t)\mathbf{U} + \cos(\beta t)\mathbf{V}]$$

are linearly independent solutions of $\mathbf{X}' = \mathbf{AX}$. ♦

$$\frac{1}{2i}(\Phi_1(t) - \Phi_2(t)),$$

$$\Phi_1(t) = e^{(\alpha+i\beta)t}(\mathbf{U} + i\mathbf{V}) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(U + iV)$$

$$= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\mathbf{U} + i\mathbf{V}) \quad e^{\alpha t}(\cos(\beta t)U - \sin(\beta t)V) + ie^{\alpha t}(\sin(\beta t)U + \cos(\beta t)V)$$

$$= e^{\alpha t}(\cos(\beta t)\mathbf{U} - \sin(\beta t)\mathbf{V}) + ie^{\alpha t}(\sin(\beta t)\mathbf{U} + \cos(\beta t)\mathbf{V})$$

$$\Phi_2(t) = e^{(\alpha-i\beta)t}(\mathbf{U} - i\mathbf{V})$$

Φ_3: $\frac{1}{2}(\Phi_1 + \Phi_2) = e^{\alpha t}(\cos(\beta t)U - \sin(\beta t)V)$

Φ_4: $\frac{1}{2i}(\Phi_1 - \Phi_2) = ie^{\alpha t}(\sin(\beta t)U + \cos(\beta t)V)$

$$= e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))(\mathbf{U} - i\mathbf{V})$$

$$= e^{\alpha t}(\cos(\beta t)\mathbf{U} - \sin(\beta t)\mathbf{V}) + ie^{\alpha t}(-\cos(\beta t)\mathbf{V} - \sin(\beta t)\mathbf{U}),$$

Solution of $X' = AX$ for Constant A

$\Phi_3 = e^{\alpha t}(\cos(\beta t)U - \sin(\beta t)V)$

$\Phi_4 = e^{\alpha t}(\sin(\beta t)U + \cos(\beta t)V)$

$\Phi_3 = \left(\cos(\sqrt{3}t)\begin{pmatrix}1\\0\\-3\end{pmatrix} - \sin(\sqrt{3}t)\begin{pmatrix}0\\-2\sqrt{3}\\\sqrt{3}\end{pmatrix}\right)e^{-t}$

$= \begin{pmatrix} e^{-t}\cos(\sqrt{3}t) \\ e^{-t}2\sqrt{3}\cdot\sin(\sqrt{3}t) \\ e^{-t}(-3\cos(\sqrt{3}t) - \sqrt{3}\sin(\sqrt{3}t)) \end{pmatrix}$

- Example) $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{pmatrix}$
  - Eigenvalues: $2, -1+\sqrt{3}i, -1-\sqrt{3}i$
  - Eigenvectors: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-2\sqrt{3}i\\-3+\sqrt{3}i\end{pmatrix}, \begin{pmatrix}1\\2\sqrt{3}i\\-3-\sqrt{3}i\end{pmatrix}$
  - $\begin{pmatrix}1\\-2\sqrt{3}i\\-3+\sqrt{3}i\end{pmatrix} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \begin{pmatrix}0\\-2\sqrt{3}\\\sqrt{3}\end{pmatrix} i = U + V$
  - Two complex solutions
    - $e^{-t}[\cos(\sqrt{3}t)\,U - \sin(\sqrt{3}t)\,V], e^{-t}[\cos(\sqrt{3}t)\,U + \sin(\sqrt{3}t)\,V]$

$$\Omega(t) = \begin{pmatrix} e^{2t} & e^{-t}\cos(\sqrt{3}t) & e^{-t}\sin(\sqrt{3}t) \\ 0 & 2\sqrt{3}e^{-t}\sin(\sqrt{3}t) & -2\sqrt{3}e^{-t}\cos(\sqrt{3}t) \\ 0 & e^{-t}[-3\cos(\sqrt{3}t) - \sqrt{3}\sin(\sqrt{3}t)] & e^{-t}[\sqrt{3}\cos(\sqrt{3}t) - 3\sin(\sqrt{3}t)] \end{pmatrix}$$

Solution of $X' = AX$ for Constant A ③ repeated.

- Solutions with n linearly dependent eigenvectors
  - Example) $A = \begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix}$ → $\left|\begin{pmatrix} \lambda-1 & -3 \\ 3 & \lambda-7 \end{pmatrix}\right| = \lambda^2 - 8\lambda + 7 + 9 = (\lambda-4)^2.$

    $\lambda = 4$
    - Eigenvalue: 4
    - Eigenvector: $\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
    - One solution: $\Phi_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$\begin{cases} y_1 = e^{rx} \\ y_2 = xe^{rx}. \end{cases}$

Solution of $X' = AX$ for Constant A

$X' = AX$

- Solutions with n linearly dependent eigenvectors
  - Example) $A = \begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix}$
    - Let $E_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
    - Attempt a second solution of the form
      - $\Phi_2 = E_1 t e^{4t} + E_2 e^{4t}$ with 2×1 constant matrix E_2
      - $\Phi_2' = A\Phi_2$
      - $E_1[e^{4t} + 4te^{4t}] + 4E_2e^{4t} = AE_1te^{4t} + AE_2e^{4t}$
      - $E_1 + 4tE_1 + 4E_2 = AE_1t + AE_2$, $AE_1 = 4E_1$
      - $AE_2 - 4E_2 = E_1 \rightarrow (A - 4I_2)E_2 = E_1$
        - $E_2 = \begin{pmatrix} \alpha \\ \frac{1+3\alpha}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix}$
        - $\Phi_2 = \begin{pmatrix} 1 + t \\ \frac{4}{3} + t \end{pmatrix} e^{4t}$

$\Phi_2 = E_1 t e^{\lambda t} + E_2 e^{\lambda t}$

$\Phi_2' = E_1 e^{\lambda t} + \lambda E_1 t e^{\lambda t} + \lambda E_2 e^{\lambda t}$

$= E_1 e^{\lambda t} + \lambda(E_1 t + E_2) e^{\lambda t}$

$= A(E_1 t e^{\lambda t} + E_2 e^{\lambda t})$

$E_1 + \lambda(E_1 t + E_2) = AE_1 t + AE_2$

$AE_1 = \lambda E_1$

$E_1 + \lambda E_2 = AE_2$

$(A - \lambda)E_2 = E_1$

$(A-\lambda)E_2 = E_1$

$-3x_1 + 3x_2 = 1$

$x_1 = x_2 - \frac{1}{3}$

x_2

$\begin{pmatrix} x_2 - \frac{1}{3} \\ x_2 \end{pmatrix}$

$\begin{pmatrix} \alpha - \frac{1}{3} \\ \alpha \end{pmatrix} \rightarrow E_2 \rightarrow \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$

$\lambda = 4$, $E_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A - 4I)E_2 = E_1$

$\begin{pmatrix} 1-4 & 3 \\ -3 & 7-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Phi_2 = E_1 t e^{\lambda t} + E_2 e^{\lambda t}$

$= t\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} e^{4t} = \begin{pmatrix} t + \frac{2}{3} \\ t + 1 \end{pmatrix} e^{4t}$

Solution of $X' = AX$ for Constant A

$X' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} X$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 4t + \frac{8}{3} \\ 4t + 4 \end{pmatrix} e^{4t} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} t + \frac{2}{3} \\ t + 1 \end{pmatrix} e^{4t}$

$= \begin{pmatrix} t + \frac{2}{3} + 3t + 3 \\ \end{pmatrix} e^{4t}$

Thus, suppose an eigenvalue λ has multiplicity $k > 1$. If there are k linearly independent solutions associated with λ, then we can produce k linearly independent solutions corresponding to λ.

If λ only has r linearly independent associated eigenvectors and $r < k$, we need from λ a total of $r - k$ more solutions linearly independent from the others.

If $r - k = 1$, we need one more solution, which can be obtained as in Example 10.10.

If $r - k = 2$, proceed as in Example 10.11 to find another linearly independent solution.

If $r - k = 3$, follow the pattern of the previous cases, trying

$$\Phi_4(t) = \frac{1}{3!}\mathbf{E}_1 t^3 e^{\lambda t} + \frac{1}{2}\mathbf{E}_2 t^2 e^{\lambda t} + \mathbf{E}_3 t e^{\lambda t} + \mathbf{E}_4 e^{\lambda t}$$

distinct

$\lambda_1 \cdots \lambda_n$

$\downarrow$

$E_1 \cdots E_n$

where $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ were found in generating preceding solutions.

If $r - k = 4$, try

$$\Phi_5(t) = \frac{1}{4!}\mathbf{E}_1 t^4 e^{\lambda t} + \frac{1}{3!}\mathbf{E}_2 t^3 e^{\lambda t} + \frac{1}{2}\mathbf{E}_3 t^2 e^{\lambda t} + \mathbf{E}_4 t e^{\lambda t} + \mathbf{E}_5 e^{\lambda t}.$$

$\lambda_1, \cdots, \lambda_j$ $\quad$ $\lambda_{j+1}, \cdots \lambda_n$

* multiplicity 2 $\quad$ λ_1, λ_4

$\Phi_1 = E_1 e^{\lambda t}$

$\Phi_2 = E_1 t e^{\lambda t} + E_2 e^{\lambda t}$

$\Phi_3 = \frac{1}{2!} E_1 t^2 e^{\lambda t} + E_2 t e^{\lambda t} + E_3 e^{\lambda t}$ …

→ 한개씩 증가하면 된다

Solution of $X' = AX + G$

- Variation of parameters
  - Fundamental matrix $\Omega(t)$ for homogeneous system $X' = AX$
    - The general solution of the homogeneous system ΩC
  - A particular solution of the nonhomogeneous system
    - $\Psi_p(t) = \Omega(t)U(t)$, with $n \times 1$ matrix $U(t)$
    - $(\Omega U)' = \Omega' U + \Omega U' = A(\Omega U) + G = (A\Omega)U + G$
    - $\Omega' = A\Omega$, $\Omega' U = (A\Omega)U$
    - $\Omega U' = G$, Ω is nonsingular
      - $U' = \Omega^{-1}G$
      - $U(t) = \int \Omega^{-1}(t)G(t)dt$
  - General solution of $X' = AX$
    - $X(t) = \Omega(t)C + \Omega(t)U(t)$

$y'' + p(x)y' + Q(x)y = f(x)$

$y_h = c_1y_1 + c_2y_2$

$y_p = v_1(x)y_1 + v_2(x)y_2$

$\Omega C = x_h$

$\Omega V(t) = x_p$

$X' = AX + G$

에 대입

$\cancel{\Omega' V(t)} + \Omega V'(t)$
$= \cancel{A\Omega V(t)} + G$

$\Omega' = A\Omega$

$\Omega V'(t) = G$

$V'(t) = \Omega^{-1}G$

$V(t) = \int U^{-1}G$

$\boxed{X = \Omega C + \Omega \int U^{-1}G}$

Solution of $X' = AX + G$

- Example) $X' = \begin{pmatrix} 1 & -10 \\ -1 & 4 \end{pmatrix} X + \begin{pmatrix} t \\ 1 \end{pmatrix}$
  - Eigenvalues: -1, 6
  - Eigenvectors: $\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
  - $\Omega(t) = \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix}, \Omega^{-1}(t) = \frac{1}{7}\begin{pmatrix} e^{t} & 2e^{t} \\ -e^{-6t} & 5e^{-6t} \end{pmatrix}$
  - $U'(t) = \Omega^{-1}(t)G(t) = \frac{1}{7}\begin{pmatrix} e^{t} & 2e^{t} \\ -e^{-6t} & 5e^{-6t} \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2e^{t} + te^{t} \\ 5e^{-6t} - te^{-6t} \end{pmatrix}$
  - $U(t) = \int \Omega^{-1}(t)G(t)dt = \begin{pmatrix} \frac{(t+1)e^{t}}{7} \\ (-29.252)e^{-6t} + \left(\frac{1}{42}\right)te^{-6t} \end{pmatrix}$

$$\mathbf{X}(t) = \Omega(t)\mathbf{C} + \Omega(t)\mathbf{U}(t) = \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix}\mathbf{C}$$

$$+ \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix}\begin{pmatrix} (t+1)e^{t}/7 \\ (-29/252)e^{-6t} + (1/42)te^{-6t} \end{pmatrix}$$

$$= \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix}\mathbf{C} + \frac{1}{3}\begin{pmatrix} 17/6 + (49/7)t \\ 1/12 + t/2 \end{pmatrix}. \quad \blacklozenge$$

$\begin{pmatrix} 1 & -10 \\ -1 & 4 \end{pmatrix} \to \begin{pmatrix} \lambda-1 & 10 \\ 1 & \lambda-4 \end{pmatrix} = (\lambda-1)(\lambda-4) - 10$
$= \lambda^2 - 5\lambda - 6$
$= (\lambda-6)(\lambda+1)$

$\lambda_1 = 6 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\quad \lambda_2 = -1 \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 5 & 10 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x_1 = -2x_2$

$\begin{pmatrix} -2 & 10 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad x_1 = 5x_2$

$\Omega = \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix}$ $\quad \Omega^{-1} = \frac{1}{5e^{5t} + 2e^{5t}}\begin{pmatrix} e^{6t} & 2e^{6t} \\ -e^{-t} & 5e^{-t} \end{pmatrix}$

$= \frac{1}{7}\begin{pmatrix} e^{t} & 2e^{t} \\ -e^{-6t} & 5e^{-6t} \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix}$

$= \frac{1}{7}\begin{pmatrix} te^{t} + 2e^{t} \\ -te^{-6t} + 5e^{-6t} \end{pmatrix}$

$(t-1)e^{t} + 2e^{t} = (t+1)e^{t}$

$= \begin{pmatrix} \frac{(t+1)e^{t}}{7} \\ (-29.252)e^{-6t} + \left(\frac{1}{42}\right)te^{-6t} \end{pmatrix}.$

Solution of $X' = AX + G$

$P = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$ $P^{-1} = -\frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$.

$X = PZ$ $X' = PZ' = AX + G$
$= A(PZ) + G = (AP)Z + G$.
$Z' = (P^{-1}AP)Z + P^{-1}G$

- Solution by diagonalizing A
  - Example) $X' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} X + \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$
    - Eigenvalues and eigenvectors: $2 \rightarrow \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $6 \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
    - $P = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$, $P^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}$
    - $X = PZ, X' = PZ'$
      - $PZ' = A(PZ) + G$, $PZ' = (AP)Z + G$, $Z' = (P^{-1}AP)Z + P^{-1}G$
      - $Z' = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} Z + \frac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$, $z_1 = c_1 e^{2t} + e^{3t} + 1$, $z_2 = c_2 e^{6t} - e^{3t} - \frac{1}{3}$

$Z' = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} Z + \frac{1}{4}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} Z + \frac{1}{4}\begin{pmatrix} -8+4e^{3t} \\ 8+12e^{3t} \end{pmatrix}$

$Z' = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 2z_1 + \frac{1}{4}(-8+4e^{3t}) \end{pmatrix}$

linear equation

$z_1' = 2z_1 - 2 + e^{3t}$

$z_1' - 2z_1 = -2 + e^{3t}$

$(e^{-2x} z_1)' = -2e^{-2x}$

$e^{\int -2 dx}$

e^{-2x}

$$\mathbf{X}(t) = \mathbf{PZ}(t) = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{2t} + e^{3t} + 1 \\ c_2 e^{6t} - e^{3t} - 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} -3c_1 e^{2t} + c_2 e^{6t} - 4e^{3t} - 10/3 \\ c_1 e^{2t} + c_2 e^{6t} + 2/3 \end{pmatrix} = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}\mathbf{C} + \begin{pmatrix} -4e^{3t} - 10/3 \\ 2/3 \end{pmatrix}$$

$$X' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} X.$$

$$\begin{vmatrix} \lambda-3 & -3 \\ -1 & \lambda-5 \end{vmatrix} = \lambda^2 - 8\lambda + 15 - 3 = \lambda^2 - 8\lambda + 12$$

$$X = PZ = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} + e^{3t} + 1 \\ c_2 e^{6t} - e^{3t} - \frac{1}{3} \end{pmatrix}$$

① $\lambda = 2$ $\quad E_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -1 & -3 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

② $\lambda = 6$ $\quad E_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$X = PZ$ $\qquad X' = AX.$

$$X' = PZ' = (AP)Z$$

$$Z' = \underbrace{P^{-1}AP}_{D} Z.$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{pmatrix} z_1' = 2z_1 \\ z_2' = 6z_2. \end{pmatrix} \qquad z_1 = e^{2t}, \quad z_2 = e^{6t}.$$

$$\Omega_D = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{6t} \end{pmatrix}.$$

$$\Omega = P\Omega_D = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{6t} \end{pmatrix} = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}$$