Chapter 6. Determinants

Jae Seok Jang

In This Chapter…

- Determinant
  - A Scalar value
    - Numbers or functions
  - Only square matrix
  - Rule for determinant
    - Similar to the Wronskian of two functions in Chapter 2

행렬 변환에서 scalar

$$\left|\begin{pmatrix} t & \cos(t) \\ 4 & e^t \end{pmatrix}\right| = te^t - 4\cos(t)$$

function

There is a simple test to determine whether two solutions of equation (2.2) are linearly independent or dependent on an open interval I. Define the *Wronskian* $W(y_1, y_2)$ of two solutions y_1 and y_2 to be the 2×2 determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

Often we denote this Wronskian as just $W(x)$.

  - Develop some properties of determinants
  - Evaluate and make usage of determinants

Permutation

1,2,3,4,5,6의 permutation.
했을 때의 하나의 결과.

- Rearrangement of some integers

$$1, 2, 3, 4, 5, 6 \to 3, 1, 4, 5, 2, 6,$$

6×5×4×3×2×1 경우의 수

then $p(1) = 3$, $p(2) = 1$, $p(3) = 4$, $p(4) = 5$, $p(5) = 2$ and $p(6) = 6$.

A permutation is characterized as even or odd according to a rule we will illustrate. Consider the permutation

p(1) = 3
p(2) = 1.

$$p: 1, 2, 3, 4, 5 \to 2, 5, 1, 4, 3$$

of the integers 1, 2, 3, 4, 5. For each k in the permuted list on the right, count the number of integers to the right of k that are smaller than k. There is one number to the right of 2 smaller than 2, three numbers to the right of 5 smaller than 5, no numbers to the right of 1 smaller than 1, one number to the right of 4 smaller than 4, and no numbers to the right of 3 smaller than 3. Since $1 + 3 + 0 + 1 + 0 = 5$ is odd, p is an *odd permutation*. When this sum is even, p is an *even permutation*.

If p is a permutation on $1, 2, \cdots, n$, define

$$\sigma(p) = \begin{cases} 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is an odd permutation.} \end{cases}$$

Determinant

- Definition

The *determinant* of an $n \times n$ matrix $\mathbf{A}$ is defined to be

+냐 -냐 결정

$$\det \mathbf{A} = \sum_p \sigma(p) a_{1p(1)} a_{2p(2)} \cdots a_{np(n)} \tag{8.1}$$

with this sum extending over all permutations p of $1, 2, \cdots, n$. Note that $\det \mathbf{A}$ is a sum of terms, each of which is plus or minus a product containing one element from each row and each column of $\mathbf{A}$.

- Notation
  - $\det A$ as $|A|$

$$A_{3\times3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

행렬의 행의 수

1,2,3 → 1,2,3 even
1,3,2 odd
2,1,3 odd
2,3,1 even
3,1,2 even
3,2,1 odd.

$p_1(1) = 1, \ p_1(2) = 2, \ p_1(3) = 3$

$\sigma(p_1) a_{1p_1(1)} a_{2p_1(2)} a_{3p_1(3)} = (1) a_{11} a_{22} a_{33}$

$\sigma(p_2) a_{1p_2(1)} a_{2p_2(2)} a_{3p_2(3)} = (-1) a_{11} a_{23} a_{32}.$

Determinant

- Example) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
  - Only two permutations
    - $p_1: 1,2 \to 1,2$ and $p_2: 1,2 \to 2,1$

$$|\mathbf{A}| = \sigma(p_1) a_{1 p_1(1)} a_{2 p_1(2)} + \sigma(p_2) a_{1 p_2(1)} a_{2 p_2(2)}$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$

Determinant

- Example) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$
  - Six permutations

$$p_1 : 1, 2, 3 \to 1, 2, 3, \text{(even)};\ p_2 : 1, 2, 3, \to 1, 3, 2, \text{(odd)};$$

$$p_3 : 1, 2, 3 \to 2, 3, 1, \text{(even)};\ p_4 : 1, 2, 3, \to 2, 1, 3, \text{(odd)};$$

$$p_5 : 1, 2, 3, \to 3, 1, 2, \text{(even)};\ p_6 : 1, 2, 3, \to 3, 2, 1, \text{(odd)}.$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Determinant

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $A^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

A

column 중 하나만 가져옴
→ 하나가 다 0이게 되면
→ 값 0

- Some fundamental properties of determinants
  - $|A^t| = |A|$
  - $|A| = 0$, if A has a zero row or column
  - If B is formed from A by type I operation, $|B| = -|A|$

$$b_{11} = a_{31},\ b_{12} = a_{32},\ b_{13} = a_{33},$$
$$b_{21} = a_{21},\ b_{22} = a_{22},\ b_{23} = a_{23},$$
$$b_{31} = a_{11},\ b_{32} = a_{12},\ b_{33} = a_{13}.$$

$$\begin{aligned} |\mathbf{B}| &= b_{11}b_{22}b_{33} - b_{11}b_{23}b_{32} + b_{12}b_{23}b_{31} \\ &= -b_{12}b_{21}b_{33} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} \\ &= a_{31}a_{22}a_{13} - a_{31}a_{23}a_{12} + a_{32}a_{23}a_{11} \\ &= -a_{32}a_{21}a_{13} + a_{33}a_{21}a_{12} - a_{33}a_{22}a_{11} \\ &= -|\mathbf{A}|. \end{aligned}$$

$R_1 \sim R_n$ (linearly dependent) → determinant = 0

  - If two rows or two columns are same, $|A| = 0$
  - If B is formed from A by type II operation(α), $|B| = \alpha|A|$
  - If one row or column of A is a constant multiple of another row or column, $|A| = 0$ ← $\begin{pmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} \end{pmatrix}$

+type III → determinant는 동일.

Determinant

- Some fundamental properties of determinants
  - Each element of row k of A, $a_{kj} = b_{kj} + c_{kj}$
    - $B = \begin{matrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{matrix}$, $C = \begin{matrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{k1} & \cdots & c_{kj} & \cdots & c_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{matrix}$

      → $b_{k1}+c_{k1}$, $b_{kj}+c_{kj}$, $b_{kn}+c_{kn}$.
    - $|A| = |B| + |C|$
  - If D is formed from A by type III operation, $|D| = |A|$

$$\mathbf{D} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{i1} + a_{k1} & \alpha a_{i2} + a_{k2} & \cdots & \alpha a_{in} + a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix}}_{=0} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix}$$

linearly dependent.

same.

Determinant

- Some fundamental properties of determinants
  - If A is nonsingular, $|A| \neq 0$
  - If A, B are $n \times n$ matrices, $|AB| = |A||B|$

행렬 A의 역행렬의 존재.

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

$A^{-1} = \dfrac{1.}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} \text{"} \end{pmatrix}$

0 이 되면 X.

Evaluation of Determinants I

- Example) $A = \begin{array}{c} \begin{array}{ccc} \phantom{a_{11}} & 12 & 13 \end{array} \\ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{array}$

$1\ 2\ 3 \longrightarrow$

1 2 3	+	$a_{11}a_{22}a_{33}$
1 3 2	−	$a_{11}a_{23}a_{32}$
2 1 3	−	$\cancel{a_{12}a_{21}a_{33}}$
2 3 1	+	$\cancel{a_{12}a_{23}a_{31}}$
3 1 2	+	$\cancel{a_{13}a_{21}a_{32}}$
3 2 1	−	$\cancel{a_{13}a_{22}a_{31}}$

$$"a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}" = \boxed{a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$$

A_{11} 행렬 A에서 1행 1열을 제외한 행렬

$$|A| = a_{11}|A_{11}|$$

Evaluation of Determinants I

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $A_{21} = \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{33} \end{pmatrix}$

- Lemma 8.1

$|A| = (-1)^{2+1} \cdot a_{21} |A_{21}| = -a_{21} |A_{21}|$

Let $\mathbf{A}$ be $n \times n$, and suppose row k or column r has all zero elements, except perhaps for a_{kr}. Then

$$|\mathbf{A}| = (-1)^{k+r} a_{kr} |\mathbf{A}_{kr}|, \tag{8.3}$$

where $\mathbf{A}_{kr}$ is the $n-1 \times n-1$ matrix formed by deleting row k and column r of $\mathbf{A}$. ♦

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

a_{13}

$|A| = (-1)^{1+3} a_{13} |A_{13}| = a_{13} |A_{13}|$

Evaluation of Determinants I

- Example) $A = \begin{pmatrix} 4 & 2 & -3 \\ 3 & 4 & 6 \\ 2 & -6 & 8 \end{pmatrix} \rightarrow B = \begin{pmatrix} 4 & 2 & -3 \\ -5 & 0 & 10 \\ 14 & 0 & 2 \end{pmatrix}$
  - row2 of B: -2*(row1)+row2
  - row3 of B: 3*(row1)+row3
  - If B is formed from A by type III operation, $|B| = |A|$
  - $|A| = |B|$
    - $|B| = (-1)^{1+2}(2)|B_{12}| = -2\begin{vmatrix} -5 & 10 \\ 14 & 2 \end{vmatrix} = -2(-1 - 140) = 300$

$$-2 \cdot \begin{vmatrix} -5 & 10 \\ 14 & 2 \end{vmatrix} = -2(-10 - 140) = 300$$

Evaluation of Determinants I

- $$\mathbf{A} = \begin{pmatrix} -6 & 0 & 1 & 3 & 2 \\ -1 & 5 & 0 & 1 & 7 \\ 8 & 3 & 2 & 1 & 7 \\ 0 & 1 & 5 & -3 & 2 \\ 1 & 15 & -3 & 9 & 4 \end{pmatrix}. \Rightarrow \mathbf{B} = \begin{pmatrix} -6 & 0 & 1 & 3 & 2 \\ -1 & 5 & 0 & 1 & 7 \\ 20 & 3 & 0 & -5 & 3 \\ 30 & 1 & 0 & -18 & -8 \\ -17 & 15 & 0 & 18 & 10 \end{pmatrix}.$$

$$\mathbf{C} = \begin{pmatrix} -1 & 5 & 1 & 7 \\ 20 & 3 & -5 & 3 \\ 30 & 1 & -18 & -8 \\ -17 & 15 & 18 & 10 \end{pmatrix}. \Rightarrow \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 20 & 103 & 15 & 143 \\ 30 & 151 & 12 & 202 \\ -17 & -70 & 1 & -109 \end{pmatrix}.$$

$$\mathbf{E} = \begin{pmatrix} 103 & 15 & 143 \\ 151 & 12 & 202 \\ -70 & 1 & -109 \end{pmatrix}. \Rightarrow \mathbf{F} = \begin{pmatrix} 1153 & 0 & 1778 \\ 991 & 0 & 1510 \\ -70 & 1 & -109 \end{pmatrix}.$$

Evaluation of Determinants II

Cofactor expansion.

- Cofactor expansion

THEOREM 8.2 ***Cofactor Expansion by a Row***

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow \left((-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13} \right).$

For any k with $1 \le i \le n$.

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}. \qquad (8.4)$$

THEOREM 8.3 ***Cofactor Expansion by a Column***

For any j with $1 \le j \le n$,

$$|\mathbf{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}. \qquad (8.5)$$

determinant.

$\left((-1)^{1+1} a_{11} M_{11} + (-1)^{2+1} a_{21} M_{21} + (-1)^{3+1} a_{31} M_{31} \right).$

Evaluation of Determinants II

- Cofactor expansion

$$|\mathbf{A}| = |[a_{ij}]| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$\det\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = afkp - aflo - agjp + agln + ahjo - ahkn - bekp + belo + bgip - bglm - bhio + bhkm$$
$$+ cejp - celn - cfip + cflm + chin - chjm - dejo + dekn + dfio - dfkm - dgin + dgjm$$

$$= a\det\begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b\det\begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix} + c\det\begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d\det\begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$

$$= \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} k & l \\ o & p \end{vmatrix} - \begin{vmatrix} a & c \\ e & g \end{vmatrix} \cdot \begin{vmatrix} j & l \\ n & p \end{vmatrix} + \begin{vmatrix} a & d \\ e & h \end{vmatrix} \cdot \begin{vmatrix} j & k \\ n & o \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} \cdot \begin{vmatrix} i & l \\ m & p \end{vmatrix} - \begin{vmatrix} b & d \\ f & h \end{vmatrix} \cdot \begin{vmatrix} i & k \\ m & o \end{vmatrix} + \begin{vmatrix} c & d \\ g & h \end{vmatrix} \cdot \begin{vmatrix} i & j \\ m & n \end{vmatrix}$$

$$+ \begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{k2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

Evaluation of Determinants II

- Example) $A = \begin{pmatrix} -6 & 3 & 7 \\ 12 & -5 & -9 \\ 2 & 4 & -6 \end{pmatrix}$
  - $|A| = -6\begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} - 3\begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} + 7\begin{vmatrix} 12 & -5 \\ 2 & 4 \end{vmatrix} = 172$

$$-6\begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} - 3\begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} + 7\begin{vmatrix} 12 & -5 \\ 2 & 4 \end{vmatrix}$$

$$= -6(30+36) - 3(-72+18) + 7(48+10)$$

$$= -6 \cdot 66 - 3 \cdot -54 + 7 \cdot 58$$

$$= -396 + 162 + 406$$

$$= 10 + 162$$

$$= 172$$

↓ column에 대해 계산해도
똑같이 나옴.

A Determinant Formula for A^{-1}

$(A : I_n) \sim (A_R : A^{-1})$, $A_R \to I_n$

- Elements of a matrix inverse

$A x_1 = e_1$
$A x_2 = e_2$ → solution → $A^{-1} = (x_1\ x_2\ \cdots\ x_n)$

$e_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

THEOREM 8.4 ***Elements of a Matrix Inverse***

Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix and define an $n \times n$ matrix $\mathbf{B} = [b_{ij}]$ by

$$b_{ij} = \frac{1}{|\mathbf{A}|}(-1)^{i+j} M_{ji}.$$

Then $\mathbf{B} = \mathbf{A}^{-1}$. ◆

- ○ M_{ji}: determinant of $(n-1)\times(n-1)$ matrix from A removing row j and column i

$A^{-1} = B = [b_{ij}]$

$b_{ij} = \frac{1}{|A|}(-1)^{i+j} M_{ji}$ → 행렬 A에서 j번째 행과 i번째 열을 제외한 나머지 행렬의 (determinant)

$A_{3\times3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$A^{-1} = B = (b_{ij})$ → (역행렬의 element)

$b_{23} = \frac{1}{|A|}(-1)^{2+3} M_{32}$

$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

A Determinant Formula for A^{-1}

- Example) $A = \begin{pmatrix} -2 & 4 & 1 \\ 6 & 3 & -3 \\ 2 & 9 & -5 \end{pmatrix}$

$b_{23} = \frac{1}{|A|}(-1)^{2+3} M_{32}$

$= \frac{1}{-120} \cdot (-1) \cdot \begin{vmatrix} -2 & 1 \\ 6 & -3 \end{vmatrix}$

$= \frac{1}{120}(6-6) = 0$

$|A| = -2\begin{vmatrix} 3 & -3 \\ 9 & -5 \end{vmatrix} - 4\begin{vmatrix} 6 & -3 \\ 2 & 5 \end{vmatrix} + 1\begin{vmatrix} 6 & 3 \\ 2 & 9 \end{vmatrix}$

$= -2(-15+27) - 4(30+6) + 1(54-6)$

$= -2(12) - 4(36) + 48$

$= -24 - 144 + 48$

$= -168 + 48 = -120$

Cramer's Rule

- A determinant formula for the unique solution of a nonhomogeneous system $\boxed{AX = B}$, when A is nonsingular
  - $x_k = \frac{1}{|A|}|A(k;B)|$, for $k = 1,2,\ldots,n$,
    - $\boxed{A(k;B)}$ is the matrix obtained from A by replacing column k of A with B

Let **A** be a nonsingular $n \times n$ matrix of numbers, and **B** be an $n \times 1$ matrix of numbers. Then the unique solution of $\mathbf{AX} = \mathbf{B}$ is determined by

$$x_k = \frac{1}{|\mathbf{A}|}|\mathbf{A}(k;\,\mathbf{B})| \tag{8.7}$$

for $k = 1, 2, \cdots, n$, where $\mathbf{A}(k;\,\mathbf{B})$ is the matrix obtained from **A** by replacing column k of **A** with **B**. ♦

$AX=B$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \rightarrow x_k = \frac{1}{|A|}\left|[A_1, A_2, B, A_n]\right|$

$= \frac{1}{|A|}\left|[A_1, A_2, \ldots B \ldots, A_n]\right|$

Cramer's Rule

k.

$k|A|$

- A determinant formula for the unique solution of a nonhomogeneous system $AX = B$, when A is nonsingular
  - $x_k = \frac{1}{|A|}|A(k;B)|$, for $k = 1,2,\ldots,n$,

    $A(k;B)$

    - $A(k;B)$ is the matrix obtained from A by replacing column k of A with B

$b_1 =$

$$x_k|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k}x_k & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k}x_k & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk}x_k & \cdots & a_{nn} \end{vmatrix}. \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

$x_k|A|.$

$$= \begin{vmatrix} a_{1k}x_k + a_{11}x_1 + a_{12}x_2 + \cdots \\ a_{21}x_k + a_{21}x_1 + a_{22}x_2 + \cdots \\ a_{3k}x_k + a_{31}x_1 + a_{32}x_2 + \cdots \end{vmatrix}$$

Cramer's Rule

- A determinant formula for the unique solution of a nonhomogeneous system $AX = B$, when A is nonsingular
  - $x_k = \frac{1}{|A|}|A(k;B)|$, for $k = 1,2,\ldots,n$,
    - $A(k;B)$ is the matrix obtained from A by replacing column k of A with B

$$x_k|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k}x_k & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k}x_k & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk}x_k & \cdots & a_{nn} \end{vmatrix}. \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Cramer's Rule

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} \ \end{pmatrix}$$

- A determinant formula for the unique solution of a nonhomogeneous system $AX = B$, when A is nonsingular
  - $x_k = \frac{1}{|A|}|A(k;B)|$, for $k = 1,2,\ldots,n$,

    $AX = B.$

    - $A(k;B)$ is the matrix obtained from A by replacing column k of A with B

$$x_k|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \overbrace{a_{11}x_1 + \cdots + a_{1n}x_n}^{a_{1k}x_k \quad b_1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{21}x_1 + \cdots + a_{2n}x_n \ \ b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \underbrace{a_{n1}x_1 + \cdots + a_{nn}x_n}_{a_{nk}x_k} \ \ b_n & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} = |\mathbf{A}(k;\mathbf{B})| \ \frac{1}{|A|}$$

Cramer's Rule

- Example) $$\begin{aligned} x_1 - 3x_2 - 4x_3 &= 1 \\ -x_1 + x_2 - 3x_3 &= 14 \\ x_2 - 3x_3 &= 5 \end{aligned}$$

$$x_1 = \frac{1}{|A|}\begin{vmatrix} 1 & -3 & -4 \\ 14 & 1 & -3 \\ 5 & 1 & -3 \end{vmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & -3 & -4 \\ -1 & 1 & -3 \\ 0 & 1 & -3 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 14 \\ 5 \end{pmatrix}}_{B}.$$

$$x_2 = \frac{1}{|A|}\begin{vmatrix} 1 & 1 & -4 \\ -1 & 14 & -3 \\ 0 & 5 & -3 \end{vmatrix}.$$

$$x_3 = \frac{1}{|A|}\begin{vmatrix} 1 & -3 & 1 \\ -1 & 1 & 14 \\ 0 & 1 & 5 \end{vmatrix}.$$