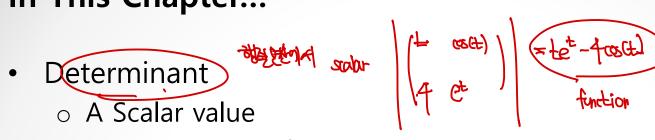
Chapter 6. Determinants

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In This Chapter...



- - Numbers or functions
- Only square matrix
- Rule for determinant
 - Similar to the Wronskian of two functions in Chapter 2

There is a simple test to determine whether two solutions of equation (2.2) are linearly independent or dependent on an open interval I. Define the Wronskian $W(y_1, y_2)$ of two solutions y_1 and y_2 to be the 2 × 2 determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \underbrace{y_1 y'_2 - y_2 y'_1}_{V}.$$

Often we denote this Wronskian as just W(x).

- Develop some properties of determinants
- Evaluate and make usage of determinants



Permutation

• Rearrangement of some integers
$$1, 2, 3, 4, 5, 6 \longrightarrow 3, 1, 4, 5, 2, 6,$$
 then $p(1) = 3$, $p(2) = 1$, $p(3) = 4$, $p(4) = 5$, $p(5) = 2$ and $p(6) = 6$.

A permutation is characterized as even or odd according to a rule we will illustrate. Consider the permutation

$$p: 1, 2, 3, 4, 5 \rightarrow 2, 5, 1, 4, 3$$

of the integers 1, 2, 3, 4, 5. For each k in the permuted list on the right, count the number of integers to the right of k that are smaller than k. There is one number to the right of 2 smaller than 2, three numbers to the right of 5 smaller than 5, no numbers to the right of 1 smaller than 1, one number to the right of 4 smaller than 4, and no numbers to the right of 3 smaller than 3. Since 1+3+0+1=5 is odd, p is an odd permutation. When this sum is even, p is an even permutation.

If p is a permutation on $1, 2, \dots, n$, define

$$\sigma(p) = \begin{cases} 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is an odd permutation.} \end{cases}$$



Definition

The *determinant* of an $n \times n$ matrix **A** is defined to be

$$\det \mathbf{A} = \sum_{p} \sigma(p) a_{1p(1)} a_{2p(2)} \cdots a_{np(n)}$$
(8.1)

with this sum extending over all permutations p of $1, 2, \dots, n$. Note that det A is a sum of terms, each of which is plus or minus a product containing one element from each row and each column of A.

- Notation
 - \circ det A as |A|

$$A_{3/3} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \begin{pmatrix} A_{21} & A_{22} & A_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \begin{pmatrix} A_{21} & A_{22} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{21} & A_{22} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{22} & A_{23} & A_{23} \\ A_{$$



- Example) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
 - Only two permutations

•
$$p_1: 1,2 \to 1,2$$
 and $p_2: 1,2 \to 2,1$

$$|\mathbf{A}| = \sigma(p_1) a_{1p_1(1)} a_{2p_1(2)} + \sigma(p_2) a_{1p_2(1)} a_{2p_2(2)}$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$



• Example)
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Six permutations



$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & & \\ \alpha_{21} & \alpha_{22} & & \\ \end{pmatrix} \qquad A^{\frac{1}{2}} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & & \\ \alpha_{12} & \alpha_{22} & & \\ \end{pmatrix}.$$

Some fundamental properties of determinants

$$\circ \boxed{A^t| = |A|}$$

- $\circ |A| = 0$, if A has a zero row or column
- o If B is formed from A by type I operation |B| = -|A|

$$b_{11}=a_{31},\,b_{12}=a_{32},\,b_{13}=a_{33},$$
 $b_{21}=a_{21},\,b_{22}=a_{22},\,b_{23}=a_{23},$ $b_{31}=a_{11},\,b_{32}=a_{12},\,b_{33}=a_{13}.$

$$|\mathbf{B}| = b_{11}b_{22}b_{33} - b_{11}b_{23}b_{32} + b_{12}b_{23}b_{31}$$

$$= -b_{12}b_{21}b_{33} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31}$$

$$= a_{31}a_{22}a_{13} - a_{31}a_{23}a_{12} + a_{32}a_{23}a_{11}$$

$$= -a_{32}a_{21}a_{13} + a_{33}a_{21}a_{12} - a_{33}a_{22}a_{11}$$

$$= -|\mathbf{A}|.$$

- o If two rows or two columns are same, |A| = 0
- o If B is formed from A by type II operation(α), $|B| = \alpha |A|$
- o If one row or column of A is a constant multiple of another row or column, A = 0



- Some fundamental properties of determinants
 - \circ Each element of row k of A a_{kj} + (b_{kj}) + (c_{kj})

$$B = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{kj} & \cdots & a_{nn} \end{bmatrix}$$

$$O[A] = \begin{bmatrix} B \\ C \end{bmatrix}$$

 a_{11}

o If D is formed from A by type III operation, $(D) \neq |A|$

$$\mathbf{D} = egin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ lpha a_{i1} + a_{k1} & lpha a_{i2} + a_{k2} & \cdots & lpha a_{in} + a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & \vdots & \ddots & \ddots & \ddots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \ddots \\ a_{i2} & \cdots & a_{in} \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{in} \end{pmatrix}$$

some.

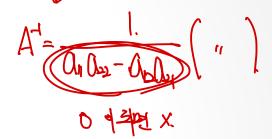


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A= (11 012).

Some fundamental properties of determinants

- If A is nonsingular, $|A| \neq 0$.
- o If A, B are $n \times n$ matrices, |AB| = AB





Evaluation of Determinants I

• Example)
$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
 $\begin{pmatrix} a_{23} & b_{23} \\ a_{24} & a_{32} \\ a_{33} & a_{33} \end{pmatrix}$ $\begin{pmatrix} a_{23} & b_{24} \\ a_{24} & a_{23} \\ a_{34} & a_{34} \end{pmatrix}$





Evaluation of Determinants $I_{A_{-}}$ $\left(\begin{array}{c} O_{-} & O_{-} \\ O_{-} & O_{-}$

Let **A** be $n \times n$, and suppose row k or column r has all zero elements, except perhaps for a_{kr} . Then

$$|\mathbf{A}| = (-1)^{k+r} a_{kr} |\mathbf{A}_{kr}|,$$
 (8.3)

where \mathbf{A}_{kr} is the $n-1 \times n-1$ matrix formed by deleting row k and column r of \mathbf{A} .

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

$$|A| = (A)^{H3} \log |A_B| = \log |A_B|.$$



Evaluation of Determinants I

• Example)
$$A = \begin{pmatrix} 4 & 2 & -3 \\ 3 & 4 & 6 \\ 2 & -6 & 8 \end{pmatrix} \rightarrow B = \begin{pmatrix} 4 & 2 & -3 \\ -5 & 0 & 10 \\ 14 & 0 & 2 \end{pmatrix}$$

- o row2 of B: -2*(row1)+row2
- o row3 of B: 3*(row1)+row3
- o If B is formed from A by type III operation, |B| = |A|
- \circ |A| = |B|

•
$$|B| = (-1)^{1+2}(2)|B_{12}| = -2\begin{vmatrix} -5 & 10\\ 14 & 2 \end{vmatrix} = -2(-1 - 140) = 300$$

$$-2.\begin{vmatrix} -5 & 10 \\ 14 & 2 \end{vmatrix} = -2.(-10-40) = 200$$



Evaluation of Determinants I

$$\mathbf{A} = \begin{pmatrix} -6 & 0 & 1 & 3 & 2 \\ -1 & 5 & 0 & 1 & 7 \\ 8 & 3 & 2 & 1 & 7 \\ 0 & 1 & 5 & -3 & 2 \\ 1 & 15 & -3 & 9 & 4 \end{pmatrix}. \quad \mathbf{B} = \begin{pmatrix} -6 & 0 & 3 & 2 \\ -1 & 5 & 0 & 1 & 7 \\ 20 & 3 & 0 & -5 & 3 \\ 30 & 1 & 0 & -18 & -8 \\ -17 & 15 & 0 & 18 & 10 \end{pmatrix}.$$

$$\mathbf{C} = \begin{pmatrix} -1 & 5 & 1 & 7 \\ 20 & 3 & -5 & 3 \\ 30 & 1 & -18 & -8 \\ -17 & 15 & 18 & 10 \end{pmatrix}. \quad \blacksquare \quad \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 20 & 103 & 15 & 143 \\ 30 & 151 & 12 & 202 \\ -17 & -70 & 1 & -109 \end{pmatrix}.$$

$$\mathbf{E} = \begin{pmatrix} 103 & 15 & 143 \\ 151 & 12 & 202 \\ -70 & 1 & -109 \end{pmatrix}. \quad \mathbf{F} = \begin{pmatrix} 1153 & 0 & 1778 \\ 991 & 0 & 1510 \\ \hline -70 & 1 & -109 \end{pmatrix}.$$



Evaluation of Determinants II

Coloctor expansion

Cofactor expansion

THEOREM 8.2 Cofactor Expansion by a Row

For any k with $1 \le i \le n$.

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}.$$
 (8.4)

THEOREM 8.3 Cofactor Expansion by a Column

For any *j* with $1 \le j \le n$,

$$|\mathbf{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}.$$
 (8.5)



Evaluation of Determinants II

Cofactor expansion

$$|\mathbf{A}| = |[a_{ij}]| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = \frac{afkp - aflo - agjp + agln + ahjo - ahkn - bekp + belo + bgip - bglm - bhio + bhkm \\ + cejp - celn - cfip + cflm + chin - chjm - dejo + dekn + dfio - dfkm - dgin + dgjm$$

$$= a \det \begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \det \begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix} + c \det \begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d \det \begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$

$$= \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} k & l \\ o & p \end{vmatrix} - \begin{vmatrix} a & c \\ e & g \end{vmatrix} \cdot \begin{vmatrix} j & l \\ n & p \end{vmatrix} + \begin{vmatrix} a & d \\ e & h \end{vmatrix} \cdot \begin{vmatrix} j & k \\ n & o \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} \cdot \begin{vmatrix} i & l \\ m & p \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix} \cdot \begin{vmatrix} i & k \\ m & n \end{vmatrix}$$



Evaluation of Determinants II

• Example)
$$A = \begin{pmatrix} -6 & 3 & 7 \\ 12 & -5 & -9 \\ 2 & 4 & -6 \end{pmatrix}$$

• $|A| = -6 \begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} - 3 \begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} + 7 \begin{vmatrix} 12 & -5 \\ 2 & 4 \end{vmatrix} = 172$



A Determinant Formula for A^{-1}

$$(A: I_n) \sim (\widehat{A_R}: A^{-1})$$

Elements of a matrix inverse

THEOREM 8.4 Elements of a Matrix Inverse

Let **A** be a nonsingular $n \times n$ matrix and define an $n \times n$ matrix $\mathbf{B} = [b_{ij}]$ by $\mathbf{C}_{\mathbf{I}} = \begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix}$. $b_{ij} = \frac{1}{|\mathbf{A}|} (-1)^{i+j} M_{ji}.$

ANDE,

Then $\mathbf{B} = \mathbf{A}^{-1}$.

o M_{ji} : determinant of $(n-1)\times(n-1)$ matrix from A removing row j and column i

A3X3 =
$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}$$

A⁻¹ B= $\begin{pmatrix} b_{1j} \\ -1 \end{pmatrix}$ $\begin{pmatrix} b_{1j} \\ -1 \end{pmatrix}$ $\begin{pmatrix} b_{1j} \\ A_{2j} \end{pmatrix}$ $\begin{pmatrix} b_{1j} \\ -1 \end{pmatrix}$ $\begin{pmatrix} b_{1j} \\ A_{2j} \end{pmatrix}$ $\begin{pmatrix} A_{11} & A_{13} \\ A_{21} & A_{22} \end{pmatrix}$

$$A^{-1} = B = [b_{ij}]$$

$$b_{ij} = \frac{1}{|A|} (-1)^{i+j} M_{ji} \rightarrow \text{해결 Aunel}$$

$$j \in \mathcal{A}$$
지한 나와 행정의
$$(deberminant),$$

A Determinant Formula for A^{-1}

• Example)
$$A = \begin{pmatrix} -2 & 4 & 1 \\ 6 & 3 & -3 \\ 2 & -5 \end{pmatrix}$$

$$= \frac{1}{|A|} (-1)^{\frac{1}{2}} \frac{1}{|A|}$$

$$=$$



• A determinant formula for the unique solution of a nonhomogeneous system $\overline{AX} = B$, when \overline{A} is nonsingular

$$x_k = \frac{1}{|A|} |A(k;B)|, \text{ for } k = 1,2,...,n,$$

• A(k; B) is the matrix obtained from A by replacing column k of A with B

Let **A** be a nonsingular $n \times n$ matrix of numbers, and **B** be an $n \times 1$ matrix of numbers. Then the unique solution of $\mathbf{X} = \mathbf{B}$ is determined by

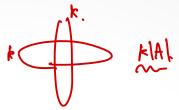
$$X_k = \frac{1}{|\mathbf{A}|} |\mathbf{A}(k; \mathbf{B})| \tag{8.7}$$

for $k = 1, 2, \dots, n$, where $\mathbf{A}(k; \mathbf{B})$ is the matrix obtained from \mathbf{A} by replacing folumn k of \mathbf{A} with \mathbf{B} .

$$AX = B \qquad X = \begin{cases} \frac{2}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{cases} \rightarrow X_{K} = \frac{1}{|A|} A_{1}, A_{2} A_{1} A_{1}$$

$$= \frac{1}{|A|} \left[A_{1}, A_{2}, \dots B_{1}, A_{n} \right].$$

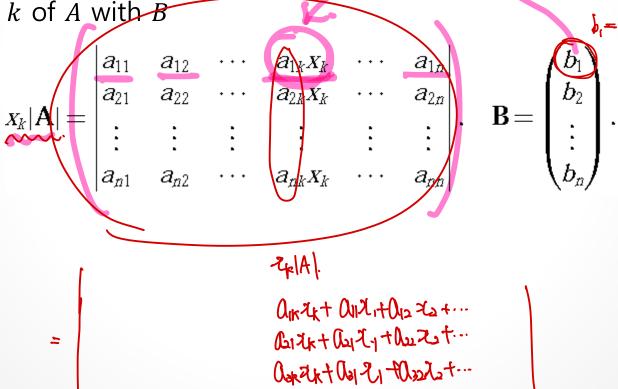




• A determinant formula for the unique solution of a nonhomogeneous system AX = B when A is nonsingular

o
$$x_k = \frac{1}{|A|} |A(k;B)|$$
, for $k = 1,2,...,n$,

• A(k; B) is the matrix obtained from A by replacing column



• A determinant formula for the unique solution of a nonhomogeneous system AX = B, when A is nonsingular

$$x_k = \frac{1}{|A|} |A(k;B)|, \text{ for } k = 1,2,...,n,$$

 A(k; B) is the matrix obtained from A by replacing column k of A with B



$$\begin{pmatrix}
Q^{3} & Q^{32} & Q^{33} \\
Q^{3} & Q^{32} & Q^{33}
\end{pmatrix}
\begin{pmatrix}
\chi_{7} \\
\chi_{1}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\chi_{2} \\
\chi_{1}
\end{pmatrix}$$

• A determinant formula for the unique solution of a nonhomogeneous system AX = B, when A is nonsingular

o
$$x_k = \frac{1}{|A|} |A(k;B)|$$
, for $k = 1,2,...,n$,

A(k; B) is the matrix obtained from A by replacing column
 k of A with B

$$\mathbf{x}_{k}|\mathbf{A}| = \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{11}\mathbf{x}_{1} + \cdots + a_{1n}\mathbf{x}_{2} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{21}\mathbf{x}_{1} + \cdots + a_{2n}\mathbf{x}_{n} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{n1}\mathbf{x}_{1} + \cdots + a_{nn}\mathbf{x}_{n} & \cdots & a_{nn}
\end{vmatrix}$$

$$= \begin{vmatrix}
a_{11} & a_{12} & \cdots & b_{1} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & b_{2} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & b_{n} & \cdots & a_{nn}
\end{vmatrix} = |\mathbf{A}(\mathbf{k}; \mathbf{B})| \mathbf{A}|$$



