Laplace Transform.

$\boxed{f(t) = C}$

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} c \, dt = c\int_0^\infty e^{-st}\, dt$$

$$= c\left(-\frac{1}{s}\right)\cdot\left[e^{-st}\right]_0^\infty \quad (s>0)$$

$$(0-1)$$

$$= \frac{c}{s}$$

Laplace transform.

$\boxed{f(t) = e^{at}}$

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t}\, dt$$

$$= \frac{1}{(a-s)}\left[e^{(a-s)t}\right]_0^\infty$$

$a-s<0$

$s>a$

$$= \frac{1}{s-a}$$

$\boxed{f(t) = t}$

$$\mathcal{L}[f](s) = \int_0^\infty t e^{-st}\, dt = \left[-\frac{1}{s} t e^{-st}\right]_0^\infty - \int_0^\infty\left(-\frac{1}{s}e^{-st}\right) dt$$

$$= +\frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$-\frac{1}{s^2}\left[e^{-st}\right]_0^\infty$$

$$= \frac{1}{s^2}$$

$f(t)=t^2$. $\quad L[f](s) = \int_0^\infty e^{-st}\cdot t^2\,dt = \left[-\frac{1}{s}t^2e^{-st}\right]_0^\infty - \int_0^\infty \left(-\frac{2}{s}te^{-st}\right)dt$

$$= \lim_{t\to\infty}\left(-\frac{t^2}{se^{st}}\right) + \frac{2}{s}\int_0^\infty e^{-st}\,dt$$

$$\int_0^\infty e^{-st} = \frac{1}{s}$$

$$= \frac{2}{s^3}$$

$$\boxed{f(t)=t^n \longrightarrow F(s) = \frac{n!}{s^{n+1}}}$$

$$\therefore f(t)=t^3 \longrightarrow F(s) = \frac{6}{s^4}$$

$\boxed{f(t) = \sin(at)}$ $\quad L[f](s) = \int_0^\infty \sin(at)e^{-st}\,dt$

$$= \cancel{\left[-\frac{1}{s}\sin(at)e^{-st}\right]_0^\infty} \quad 0$$

$$- \int_0^\infty -\frac{a}{s}\cos(at)e^{-st}\,dt$$

$\sin(at) \quad e^{-st}$

$a\cos(at) \quad -\frac{1}{s}e^{-st}$

$\int$

$$\frac{a}{s}\int_0^\infty \cos(at)e^{-st}\,dt$$

$\cos(at) \quad e^{-st}$

$-a\sin(at) \quad -\frac{1}{s}e^{-st}$

$\int$

$$\frac{a}{s}\left[\left[-\frac{1}{s}e^{-st}\cos(at)\right]_0^\infty - \int_0^\infty \frac{a}{s}\sin(at)e^{-st}\,dt\right.$$

$$\frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}\int_0^\infty \sin(at)e^{-st}\,dt\right] = \int_0^\infty \sin(at)e^{-st}\,dt$$

$$\frac{a}{s^2} - \frac{a^2}{s^2}\int_0^\infty \sin(at)e^{-st}\,dt = \int_0^\infty \sin(at)e^{-st}\,dt$$

$$\left(1+\frac{a^2}{s^2}\right)\int_0^\infty \sin(at)e^{-st}\,dt = \frac{a}{s^2}$$

$$\int_0^\infty \sin(at)e^{-st}\,dt = \frac{\frac{a}{s^2}}{1+\frac{a^2}{s^2}} = \frac{a}{\cancel{s^2}} \times \frac{\cancel{s^2}}{a^2+s^2} = \boxed{\frac{a}{a^2+s^2}}$$

$$f(t) = \cos(at) \qquad F(s) = \boxed{\frac{s}{a^2+s^2}}$$

---

e^{iat} 오일러 법칙 정리

$$\mathcal{L}[f](s) = \int_0^\infty e^{iat}e^{-st}\,dt = \int_0^\infty e^{-(s-ai)t}\,dt$$

$$\frac{1}{s-ai} = \frac{s}{s^2+a^2} + \frac{a}{s^2+a^2}i$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2} + \frac{a}{s^2+a^2}i\right]$$

$$= \mathcal{L}[\cos(at)] + i\,\mathcal{L}[\sin(at)]$$

$f(s) = e^{at}\sin(bt)$

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} e^{at} \sin(bt)\,dt = \int_0^\infty e^{(a-s)t} \sin(bt)\,dt$$

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2+b^2}$$ → (a) 만큼 평행이동

$$\mathcal{L}[e^{at}\sin(bt)](s) = \boxed{\frac{1}{(s-a)^2+b^2}} \quad (s>a)$$

Laplace Transform properties.

- Linearity

$$\mathcal{L}[f+g](s) = \mathcal{L}[f](s) + \mathcal{L}[g](s) = F + G$$

$$\mathcal{L}[cf](s) = c\,\mathcal{L}[f](s) = cF$$

- Inverse of Laplace transform

$\mathcal{L}^{-1}[F] = f$ when $\mathcal{L}[f] = F$.

- $\mathcal{L}^{-1}[F+G] = f+g$

- $\mathcal{L}^{-1}[c\,F] = cf$

$$1 \longrightarrow \frac{1}{s}$$

$$t^n \longrightarrow \frac{n!}{s^{n+1}}$$

$$e^{at} \longrightarrow \frac{1}{s-a}$$

$$t^n e^{at} \longrightarrow \frac{n!}{(s-a)^{n+1}}$$

$$e^{at} - e^{bt} \longrightarrow \frac{1}{s-a} - \frac{1}{s-b} = \frac{a-b}{(s-a)(s-b)}$$

$$\sin(at) \longrightarrow \frac{a}{s^2+a^2}$$

$$\cos(at) \longrightarrow \frac{s}{s^2+a^2}$$

$$\sinh(at) \longrightarrow \frac{a}{s^2-a^2}$$

$$\cosh(at) \longrightarrow \frac{s}{s^2-a^2}$$

$$\delta(t-a) \longrightarrow e^{-as}$$

Solution of initial Value problem.

$f(t)$ $t \geq 0$ continuous.

$$\mathcal{L}[f'(t)](s) = \int_0^\infty f'(t) e^{-st}\,dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\boxed{\lim_{t\to\infty} \cancel{\frac{f(t)}{e^{st}}} - f(0)}$$

$\to 0$

$$s\int_0^\infty e^{-st} f(t)\,dt = s\mathcal{L}[f](s) = sF(s)$$

initial value given.

$$\mathcal{L}[f'(t)](s) = sF(s) - f(0)$$

☆ 공식 정리

$$\mathcal{L}[f^{n}(t)](s) = s^n F(s) - s^{n-1} f(0) - \cdots - f^{n-1}(0)$$

$$\mathcal{L}[f''(t)](s) = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}[f'''(t)](s) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

Solution of Initial Value problem.

ex:) 1

$y' - 4y = 1$ $\quad y(0) = 1$.

$L[y'](s) = sY(s) - y(0)$

$L[-4y](s) = -4Y(s)$

$sY(s) - \cancel{y(0)}^{1} - 4Y(s) = \frac{1}{s}$

$Y(s)(s-4) = \frac{s+1}{s}$

$Y(s) = \frac{s+1}{s(s-4)} = \left(1+\frac{1}{s}\right)\frac{1}{s-4}$

$L^{-1}[Y](s) = L^{-1}\left[\frac{1}{s-4}\right] + L^{-1}\left[\frac{1}{s(s-4)}\right]$

$\frac{a}{s} + \frac{b}{s-4}$

$as - 4a + bs = 1$

$s(a+b) - 4a = 1$

$a + b = 0$

$a = -\frac{1}{4}$ $\quad b = \frac{1}{4}$

e^{4t}

$L^{-1}\left[-\frac{1}{4}\cdot\frac{1}{s}\right] + L^{-1}\left[\frac{1}{4}\,\frac{1}{(s-4)}\right]$

$-\frac{1}{4}L^{-1}\left[\frac{1}{s}\right]$ $\quad \frac{1}{4}L^{-1}\left[\frac{1}{(s-4)}\right]$

$+$

$-\frac{1}{4}$ $+$ $\frac{1}{4}e^{4t}$

$\therefore -\frac{1}{4} + \frac{5}{4}e^{4t}$

ex:) 2.

$$y'' + 4y' + 3y = e^t$$

solve 1) associated homogeneous equation

$y'' + 4y' + 3y = 0$ 의 해

$r^2 + 4r + 3 = 0$

"$r = -1$" or "$r = -3$"

e^{-t} , e^{-3t}

$$\begin{vmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{vmatrix}$$

$-3e^{-4t} + e^{-4t} \neq$ non-zero

linearly independent.

particular solution of non-homogeneous equation

$y_p = ae^t$

$y_p' = ae^t$

$y_p'' = ae^t$

$8ae^t = e^t$

$a = \frac{1}{8}$

$y_p = \frac{1}{8}e^t$

$$y(t) = C_1e^{-t} + C_2e^{-3t} + \frac{1}{8}e^t$$

$y(0) = C_1 + C_2 + \frac{1}{8} = 0$

$y'(0) = -C_1 - 3C_2 + \frac{1}{8} = 2$

$C_1 = \frac{7}{8}$, $C_2 = -\frac{7}{8}$

$$y(t) = \boxed{\frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t} + \frac{1}{8}e^{t}}$$

particular solution.

Solve 2 :)

$$y'' + 4y' + 3y = e^{t} \qquad y(0)=0 \qquad y'(0)=2$$

$$L[y''] = s^2 Y(s) - s\cancel{y(0)} - y'(0)$$

$$L[y'] = sY(s) - y(0)$$

$$\therefore\ s^2 Y(s) - s\cancel{y(0)} - y'(0) + 4sY(s) - 4\cancel{y(0)} + 3Y(s) = \frac{1}{s-1}$$

$$s^2 Y(s) + 4sY(s) + 3Y(s) - 2 = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{8}\cdot\frac{1}{(s-1)} - \frac{3}{4}\cdot\frac{1}{(s+1)} - \frac{7}{8}\,\frac{1}{(s+3)}$$

$$L^{-1}[Y(s)]$$

$$y(t) = \boxed{\frac{1}{8}e^{t} - \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}}$$

같다.

Shifting Theorms.

$f(t) \longrightarrow f(t-a)$

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st}\left(e^{at} f(t)\right)dt$$

$$= \mathcal{L}[e^{at} f(t)](s)$$

$$\therefore \mathcal{L}^{-1}[F(s-a)] = e^{at} f(t).$$

The pulse Function

어차피 $t \geq a$ 에서 유효

$$\mathcal{L}[H(t-a)](s) = \int_0^\infty e^{-st} H(t-a)\,dt = \int_a^\infty e^{-st} \underbrace{H(t-a)}_{1}\,dt$$

$$= \int_a^\infty e^{-st}\,dt = -\frac{1}{s}\left[e^{-st}\right]_a^\infty = \frac{e^{-as}}{s}$$

$$\frac{1}{s} = \mathcal{L}[1]$$

$$= e^{-as} \cdot \mathcal{L}[1](s)$$

$$\mathcal{L}[H(t-a)f(t-a)](s) = \int_0^\infty e^{-st} H(t-a)f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

$$T = t-a \qquad t = T+a.$$

$$= \int_0^\infty e^{-(T+a)t} f(T)\,dT,$$

$$= e^{-at}\int_0^\infty e^{-Tt} f(T)\,dT,$$

$$e^{-at}\,\mathcal{L}[f](s)$$

using pulse function

Heaviside function.

The Laplace transform of pulse function

(shifting theorm)

ex) Compute $L[g(t)](s)$ for $g(t) = \begin{cases} 0 & t<2 \\ t^2+1 & t\geq 2 \end{cases}$

$$L[g(t)](s) = L[H(t-2)(t^2+1)](s) = L[H(t-2)((t-2)^2+4(t-2)+5)]$$

$$L[H(t-2)(t-2)^2] = L[t^2]\cdot e^{-2s} = \frac{2}{s^3}e^{-2s}$$

$$+\ L[H(t-2)\cdot 4(t-2)] = 4L[t]\cdot e^{-2s} = \frac{4}{s^2}e^{-2s}$$

$$+\ L[H(t-2)\cdot 5] = \frac{5}{s}e^{-2s}$$

$$L[g(t)](s) = \boxed{e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{5}{s}\right)}$$

Compute $L^{-1}\left[\frac{se^{-3s}}{s^2+4}\right] = L^{-1}\left[\frac{s}{s^2+4}e^{-3s}\right]$

$e^{3s}\quad \frac{1}{s-3}$

$$L[\ ] \longrightarrow \frac{s}{s^2+4}e^{-3s}$$

$$e^{-3s}L[\ ] = \frac{s}{s^2+4}e^{-3s}$$

$$= \cos(2t)$$

$$\therefore \boxed{H(t-3)\cos(2(t-3))}$$

$y'' + 4y = f(t)$; $y(0) = y'(0) = 0$ $\quad f(t) = \begin{cases} 0 & t < 3 \\ t & t \geq 3 \end{cases}$

$t \to H(t-3)$

$\mathcal{L}[y'' + 4y] = s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)} + 4Y(s)$

$\boxed{= Y(s)(s^2+4)}$

$\mathcal{L}[f(t)] = \mathcal{L}[H(t-3)\,t] = \mathcal{L}[H(t-3)(t-3) + 3H(t-3)]$

$= \dfrac{e^{-3s}}{s^2} + \dfrac{3e^{-3s}}{s} = \boxed{\dfrac{e^{-3s}(1+3s)}{s^2}}$

$Y(s) = \left(\dfrac{3s+1}{(s^2+4)\cdot s^2}\right) e^{-3s}$

$\dfrac{a}{s^2} \quad \dfrac{b}{s^2+4}$

$a(s^2+4) + bs^2 = 1$

$(a+b)s^2 + 4a = 1$

$\left(a = \frac{1}{4},\ b = -\frac{1}{4}\right)$

$\dfrac{3s}{(s^2+4)s^2} + \dfrac{1}{(s^2+4)s^2}$

$Y(s) = e^{-3s}\left[\dfrac{1}{4}\left(\dfrac{1}{s^2} - \dfrac{1}{s^2+4}\right) + \dfrac{3}{4}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right)\right]$

$(a+b)s^2 + 4a = 3s$

$\left(a = \frac{3}{4}s,\ b = -\frac{3}{4}s\right)$

$\Downarrow$

$$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{4s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-3s}}{4(s^2+4)}\right]$$

$$e^{-3s}\mathcal{L}[\,\square\,] = \frac{e^{-3s}}{4s^2} \qquad e^{-3s}\mathcal{L}[\,\square\,] = \frac{e^{-3s}}{8(s^2+4)}$$

$$\frac{t}{4}$$

$$\boxed{\frac{1}{4}H(t-3)(t-3)} \qquad -\boxed{\frac{1}{8}\sin(2(t-3))(H(t-3))}$$

$$y(t) = \frac{1}{4}H(t-3)(t-3) - \frac{1}{8}H(t-3)\sin(2(t-3)) + \cdots$$

Heaviside's Formula.

$$\mathcal{L}^{-1}[F(s)] = \frac{p(a_1)}{Q_1(a_1)}e^{a_1 t} + \frac{p(a_2)}{Q_2(a_2)}e^{a_2 t} + \cdots$$

$$F(s) = \frac{p(s)}{Q(s)} = \frac{p(s)}{(s-a_1)(s-a_2)\cdots(s-a_n)} \quad \cdots \text{①}$$

$\cdots$③

$p(s)$의 차수 $<$ $Q(s)$의 차수

$Q_i(a_i)$: a_i :

$$\underbrace{(s-a_1)\boxed{(s-a_2)\cdots(s-a_n)}}_{}$$

$$= Q_1(a_1)$$

$$F(s)=\frac{s}{(s^2+4)(s-1)}=\frac{s}{(s-2i)(s+2i)(s-1)}$$

$a_1=2i \quad a_2=-2i \quad a_3=1$

$Q_1(2i)=(2i+2i)(2i-1)=-8-4i$

$Q_2(-2i)=(-2i-2i)(-2i-1)=-8+4i$

$Q_3(1)=(1-2i)(1+2i)=5$

$\mathcal{L}^{-1}[F(s)](t)=$

$$\frac{2i}{-8-4i}e^{2it}+\frac{-2i}{-8+4i}e^{-2it}+\frac{1}{5}e^{t}$$

$$=\frac{-1-2i}{10}e^{2it}+\frac{-1+2i}{10}e^{-2it}+\frac{1}{5}e^{t}$$

$$e^{iat}=\cos(at)+i\sin(at)$$

$$-\frac{1}{10}\left(\underbrace{e^{2it}+e^{-2it}}_{=2\cos(2t)}\right)+\frac{2i}{10}\left(\underbrace{e^{2it}-e^{-2it}}_{+2i\sin(2t)}\right)+\frac{1}{5}e^{t}$$

$\cos(2t)+\cancel{i\sin(2t)}$

$+\cos(-2t)+\cancel{i\sin(-2t)}$

$$-\frac{1}{5}\cos(2t)+\frac{2}{5}\sin(2t)+\frac{1}{5}e^{t}$$

$$e^{iat}=\int_0^\infty e^{-st}e^{iat}\,dt$$

$$=\int_0^\infty e^{-(s-ai)t}\,dt$$

$$=\frac{1}{s-ia}=\frac{s+ia}{s^2+a^2}$$

Convolution

$$f*g = g*f$$

$$t*1 \neq t$$

$$\int_0^t \tau\, d\tau = \frac{t^2}{2}$$

---

proof of $L[f*g] = L[f]\,L[g]$

$$L[f*g] = \int_0^\infty \left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)e^{-st}\,dt$$

$$= \int_0^\infty \left(\int_0^t f(\tau)\,g(t-\tau)\,e^{-st}\,d\tau\right)dt$$

$-s(t-\tau+\tau)$

$$= \int_0^\infty \int_0^\infty f(\tau)\,g(t-\tau)\,e^{-st}\,dt\,d\tau$$

$t-\tau=\alpha$

$$= \int_0^\infty f(\tau)e^{-st}\left(\int_0^\infty g(\alpha)\,e^{-s(\alpha)}\,dt\right)d\tau$$

$\times e^{-s\tau}$ $\quad 0 \sim \infty \quad 0 \sim \infty$

$$\int_0^\infty f(\tau)e^{-s\tau}\left(\int_0^\infty g(\alpha)e^{-s\alpha}\,d\alpha\right)d\tau$$

$$\int_0^\infty f(\tau)e^{-s\tau}\,L[g](s)\,d\tau$$

$$L[f](s)\,L[g](s)$$

Compute. $\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right] = \frac{1}{16}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s-4} + \frac{4}{(s-4)^2}\right]$

$= \frac{1}{16}(1 - e^{4t} + 4te^{4t})$

by convolution theory

$$\frac{1}{s(s-4)^2} = \frac{1}{s}\frac{1}{(s-4)^2}$$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 \qquad \mathcal{L}^{-1}\left[\frac{1}{(s-4)^2}\right] = te^{4t}$

$\mathcal{L}[\ ]\,\mathcal{L}[\ ]$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right] = 1 * te^{4t} = \int_0^t \tau e^{4\tau}\,d\tau$$

$$= \frac{1}{4}\left[\tau e^{4\tau}\right]_0^t - \frac{1}{16}\left[e^{4\tau}\right]_0^t$$

$$= \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t} + \frac{1}{16}$$

$\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right]$

$\mathcal{L}[\cancel{\bigcirc}]\,\mathcal{L}[\bigcirc] = \frac{1}{s(s-4)^2}$ (te^{4t})

$1 * te^{4t}$

$$y'' - 2y' - 8y = f(t) \qquad y(0) = 1 \quad y'(0) = 0.$$

$$s^2 Y(s) - s y(0) - \cancel{y'(0)} - 2(sY(s) - y(0)) - 8Y(s)$$

$$= \cancel{s^2 Y(s)} - s \cancel{- 2sY(s)} + 2 \cancel{- 8Y(s)}$$

$$= (s^2 - 2s - 8) Y(s) - s + 2 = F(s)$$

$$(s-4)(s+2) \cdot Y(s)$$

$$Y(s) = \frac{F(s) + s - 2}{(s-4)(s+2)} = \frac{F(s)}{(s-4)(s+2)} + \frac{s-2}{(s-4)(s+2)}$$

$$\frac{1}{6}\left(\frac{1}{s-4} - \frac{1}{s+2}\right) F(s) + \frac{1}{3}\left(\frac{1}{(s-4)} + \frac{2}{s+2}\right).$$

$$\mathcal{L}^{-1}[Y](s) = y(t) = \frac{1}{3} e^{4t} + \frac{2}{3} e^{-2t}$$

$$\frac{1}{6} \frac{1}{(s-4)} F(s) - \frac{1}{6} \cdot \frac{1}{(s+2)} F(s)$$

$$\frac{1}{6} \mathcal{L}[e^{4t}] \cdot \mathcal{L}[f(t)]$$

$$\frac{1}{6}\left(e^{4t} * f(t)\right) - \frac{1}{6}\left(e^{-2t} * f(t)\right).$$

$$f(t) = 2t^2 + \int_0^t f(t-\tau) e^{-\tau} d\tau$$

$$f(t) = 2t^2 + (f * e^{-\tau})$$

$$F(s) = \frac{4}{s^3} + \frac{F(s)}{s+1}$$

$$F(s)\left(1 - \frac{1}{s+1}\right) = \frac{4}{s^3}$$

$$F = \frac{4}{s^4} + \frac{4}{s^3} \qquad \frac{4}{6} t^3 \rightarrow \frac{6}{s^4}$$

$$2t^2 \rightarrow 2\frac{2}{t^3}$$

$$f(t) = \frac{2}{3} t^3 + 2t^2$$

Impulses and the Dirac Delta Function.

$$\frac{1}{\varepsilon}[H(t)-H(t-\varepsilon)]$$

$$\delta_\varepsilon(t-a) = \frac{1}{\varepsilon}[H(t-a) - H(t-a-\varepsilon)]$$

$$\mathcal{L}[\delta_\varepsilon(t-a)] = \mathcal{L}\left[\frac{1}{\varepsilon}[H(t-a) - H(t-a-\varepsilon)]\right]$$

$$= \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$= \frac{e^{-as}(1-e^{-\varepsilon s})}{\varepsilon s}$$

$$\lim_{\varepsilon\to 0} \frac{\cancel{d}e^{-\varepsilon s}}{\cancel{d}} \quad 1 \quad 1$$

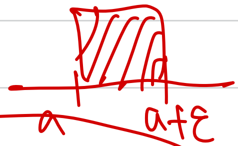
$$\lim_{\varepsilon\to 0} \mathcal{L}[\delta_\varepsilon(t-a)]$$

$$= e^{-as}$$

$$\boxed{\text{In particular } \mathcal{L}[\delta(t)] = 1}$$

let $a \geq 0$ $[0, \infty]$ integrable and continuous

$$\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$$



$$\int_0^\infty f(t)\delta(t-a)\,dt = \int_0^\infty \frac{1}{\varepsilon}\left[H(t-a) - H(t-a-\varepsilon)\right] f(t)\,dt$$

$$= \frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(t)\,dt = \frac{1}{\varepsilon}\left(\varepsilon f(t_\varepsilon)\right) \quad (a < t_\varepsilon < a+\varepsilon)$$

$\varepsilon \to 0^+$

$t_\varepsilon = a$

$$y''+2y'+2y=\boxed{\delta(t-3)} \qquad y(0)=y'(0)=0$$

$$\mathcal{L}[\delta(t-3)]=e^{-3s}$$

$$s^2Y(s)+2sY(s)+2Y(s)=(s^2+2s+2)Y(s)=e^{-3s}$$

$$Y(s)=\frac{e^{-3s}}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}[Y(s)] \to \bigcirc$$

$$e^{-3s}\mathcal{L}[\bigcirc]=\frac{e^{-3s}\cdot 1}{(s+1)^2+1}$$

$$e^{-t}\sin(t)$$

$$y(t)=H(t-3)e^{-(t-3)}\sin(t-3)$$