Homework #1
-121

현재 32.2°C
주변의 환경 15.5°C) Newton's coling raw

10분 후 31.1°C가 됨

$\boxed{\text{그럼 20분 후는 몇도, 16°C 때 몇분 후}}$

T(t)를 t초 후에 사물의 온도라고 하자.
그럼 주어진 조건과 뉴턴의 냉각법칙에 의해

$$\frac{dT}{dt} = k(T-15.5)$$

$$\frac{dT}{T-15.5} = k\,dt$$

$$\int \frac{dT}{T-15.5} = \int k\,dT$$

$$\ln|T(t)-15.5| = kt + C$$

$$T(t) - 15.5 = e^{kt+C}$$

$$T(t) = e^{kt+C} + 15.5$$

$$T(t) = Ae^{kt} + 15.5$$

$$T(0) = 32.2$$

$$T(0) = A + 15.5 = 32.2$$

$$A = 16.7$$

또한 10분후 31.1 이니까

$$T(10) = 31.1$$

$$31.1 = (16.7)e^{10k} + 15.5$$

$$0.9341 = e^{10k}$$

$$10k = \ln(0.9341)$$

$$10k \approx -0.068$$

$$k \approx -0.0068$$

즉 $\underline{T(t) = 16.7e^{-0.0068t} + 15.5}$

$$T(20) = (16.7)e^{-0.0068\times 20} + 15.5$$

$$= 16.7\times e^{-1.136} + 15.5$$

$$\approx 16.7\times(0.8728) + 15.5$$

$$\approx 30.01$$

$T(20) \approx \boxed{30^\circ C}$

즉 20분 후면 약 30°C 일 것이다.

또한 18°C가 되려면

$$18 = (16.7) \times e^{(-0.0068)t} + 15.5$$

$$18 - 15.5 = (16.7) \times e^{-0.0068t}$$

$$\frac{2.5}{16.7} = e^{-0.0068 \times t}$$

$$\ln(0.1497) = \ln\left(e^{-0.0068 \times t}\right)$$

$$\approx -1.8991 = -0.0068t$$

$$t = \frac{1.8991}{0.0068}$$

$$t \approx 279.28$$

$$\approx 280\text{min}$$

즉 대략 280분 후면 18°C가 될 것이다.

Homework #1: Q2

Evaluate $\int_0^\infty e^{-t^2-\left(\frac{a}{t}\right)^2}\,dt$

$I(x) = \int_0^\infty e^{-t^2-\left(\frac{x}{t}\right)^2}\,dt.$ $\quad I(a) = ?$

$I(0) = \int_0^\infty e^{-t^2}\,dt$

$= \frac{1}{2}\sqrt{\pi}$

$I'(x) = \frac{d}{dx}\int_0^\infty e^{-t^2-\left(\frac{x}{t}\right)^2}\,dt$ $\qquad \frac{x^2}{t^2}$

$= \int_0^\infty \frac{\partial}{\partial x}\left(e^{-t^2-\left(\frac{x}{t}\right)^2}\right)dt$

$= \int_0^\infty e^{-t^2}\frac{\partial}{\partial x}\left(e^{-\left(\frac{x}{t}\right)^2}\right)dt.$

$\int_0^\infty e^{-t^2}\cdot\left(e^{-\left(\frac{x}{t}\right)^2}\right)\cdot\frac{\partial}{\partial x}\cdot\left(-\frac{x}{t}\right)^2 dt.$

$= \int_0^\infty e^{-t^2-\left(\frac{x}{t}\right)^2}\cdot\left(-\frac{1}{t^2}\right)\cdot 2x\cdot dt.$

$I'(x) = \int_0^\infty e^{-t^2-\left(\frac{x}{t}\right)^2}\cdot\left(\frac{-2x}{t^2}\right)dt \quad \cdots (1)$

$\Downarrow$

$$I'(x)=\int_0^\infty e^{-t^2-\left(\frac{x}{t}\right)^2}\cdot\left(\frac{-2x}{t^2}\right)dt \quad \cdots ①$$

y에 대한 식으로.

$\frac{x}{t}=k$.

$\frac{dk}{dt}=-\frac{x}{t^2}$

$\left(\lim_{t\to\infty}\frac{x}{t}=0 \quad \text{"} \quad k=0, \quad \lim_{t\to 0}k=\infty\right)$

$\frac{-2x}{t^2}=\frac{2dk}{dt}$

① $\rightarrow$ $I'(x)=\int_\infty^0 e^{-\left(\frac{x}{k}\right)^2-k^2}\cdot\frac{2dk}{\cancel{dt}}\cdot\cancel{dt}$

$=2\int_\infty^0 e^{-k^2-\left(\frac{x}{k}\right)^2}dk.$

$=-2\int_0^\infty e^{-k-\left(\frac{x}{k}\right)^2}dk.$

$I(x)=\int_0^\infty e^{-t^2-\left(\frac{x}{t}\right)^2}dt$ in the equation

$\Rightarrow I'(x)=-2I(x)$ / $I'=-2I \quad \cdots ②$

(well-known-condition)

$\int_0^\infty e^{-t^2}dt=\frac{1}{2}\sqrt{\pi}=I(0)$

$\cdots$ ② $\frac{dI}{dx}=-2I.$

$\frac{dI}{I}=-2dx$

$\int\frac{dI}{I}=\int-2\,dx.$

$\ln|I|=-2x+C \quad \cdots ③$

$\ln[I(0)]=C$

$C=\ln\left(\frac{1}{2}\sqrt{\pi}\right)$

$\ln|I(x)|=-2x+\ln\left(\frac{1}{2}\sqrt{\pi}\right)$

$e^{-2x+\ln\left(\frac{1}{2}\sqrt{\pi}\right)}=I(x)$

$I(x)=e^{-2x}\cdot\frac{1}{2}\sqrt{\pi}$

$=\frac{1}{2}\sqrt{\pi}e^{-2x}.$

$I(a)$ $\Leftarrow$ we want this value

$I(a)=\frac{1}{2}\sqrt{\pi}e^{-16}$

$\therefore \frac{1}{2}\sqrt{\pi}e^{-16}$

Homework #1 : Q3

$p'(t) = ap(t) - bp(t)^2$ with initial condition $p(0) = p_0$.

to obtain $p(t) = \dfrac{ap_0}{a - bp_0 + bp_0 e^{at}}\, e^{at}$ and to show that $p(t)$ determined by this equation is bounded above and asymptotically approaches the limit a/b as $t \to \infty$

$$p'(t) = ap(t) - bp(t)^2$$

$$\frac{dp(t)}{dt} = ap(t) - bp(t)^2$$

$$\frac{dp(t)}{ap(t) - bp(t)^2} = dt$$

$$p(t) = k$$

$$\frac{dk}{ak - bk^2} = dt$$

$$-b\left(k^2 - \frac{a}{b}k\right) = -b\left[\left(k - \frac{a}{2b}\right)^2 - \left(\frac{a}{2b}\right)^2\right]$$

$$\frac{dk}{-b\left[\left(k - \frac{a}{2b}\right)^2 - \left(\frac{a}{2b}\right)^2\right]} = dt$$

$$\frac{dk}{\left(k - \frac{a}{2b}\right)^2 - \left(\frac{a}{2b}\right)^2} = -b\,dt$$

$$k - \frac{a}{2b} = u \qquad \frac{a}{2b} = v$$

$$dk = du$$

$$\frac{dk}{u^2 - v^2} = -b\,dt$$

$$\frac{1}{2v}\int \left(\frac{-1}{(u+v)} + \frac{1}{(u-v)}\right) \overset{du}{dk} = \int -b\,dt$$

$$\frac{1}{2v}\left(\int \frac{du}{u-v} - \int \frac{du}{u+v}\right) = \int -b\,dt$$

$$\frac{1}{2v}\left(-\ln|u+v| + \ln|u-v|\right) = -bt + C$$

$$\frac{1}{2v} \ln\left(\frac{u-v}{u+v}\right) = -bt + C$$

$$\frac{1}{\cancel{2}\cdot\frac{a}{\cancel{2}b}}\ln\left(\frac{k-\frac{a}{2b}-\frac{a}{2b}}{k-\frac{a}{2b}+\frac{a}{2b}}\right)=-bt+C$$

$$\frac{b}{a}\ln\left(\frac{k-\frac{a}{b}}{k}\right)=-bt+C$$

$$\frac{b}{a}\ln\left(\frac{bk-a}{bk}\right)=-bt+C$$

$P(0)=P_0$

$t=0 \longrightarrow k=P_0$

$$\frac{b}{a}\ln\left(\frac{bk-a}{bk}\right)=C$$

$$\boxed{\frac{b}{a}\ln\left(\frac{bP_0-a}{bP_0}\right)=C} \quad \therefore ①$$

$$\frac{a}{b}\left(\frac{b}{a}\ln\left(\frac{bk-a}{bk}\right)\right)=\left(-bt+\frac{b}{a}\ln\left(\frac{bP_0-a}{bP_0}\right)\right)\frac{a}{b}$$

$$\ln\left(\frac{bk-a}{bk}\right)=-at+\ln\left(\frac{bP_0-a}{bP_0}\right)$$

$$\ln\left(\frac{bk-a}{bk}\right)=\ln e^{-at}+\ln\left(\frac{bP_0-a}{bP_0}\right)$$

$$\frac{bk-a}{bk}=\left(e^{-at}\right)\cdot\left(\frac{bP_0-a}{bP_0}\right)$$

$$\cancel{b}P_0(bk-a)=e^{-at}\cdot(bP_0-a)\cdot\cancel{b}k$$

$$P_0e^{at}(bk-a)=k(bP_0-a)$$

$$bP_0e^{at}k-aP_0e^{at}=kbP_0-ak$$

$\Downarrow$

(k에 대해 정리 해야 함)

$$k(a-bP_0+bP_0e^{at})=aP_0e^{at}$$

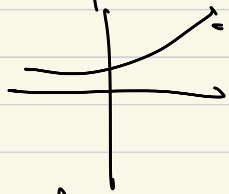
$$k=\frac{aP_0e^{at}}{a-bP_0+bP_0e^{at}}$$

$$\therefore P(t)=\frac{aP_0}{a-bP_0+bP_0e^{-at}}$$

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\left(\frac{a p_0}{a - b p_0 + b p_0 e^{at}}\, e^{at}\right)$$

$a \to$ positive

$b \to$ positive



$$= \lim_{t\to\infty} \frac{a p_0}{a\cdot e^{-at} - b p_0 \cdot e^{-at} + b p_0}$$

$$\lim_{t\to\infty} e^{-at} = 0$$

$$= \lim_{t\to\infty} \frac{a p_0}{\cancel{a\cdot 0} - \cancel{b\cdot p_0 \cdot 0} + b p_0} = \frac{a p_0}{b p_0} \quad \frac{a}{b}$$

we solved $\lim_{t\to\infty} p(t) = \frac{a}{b}$

Homework #1 : Q4

$Q'(t) = kQ(t)$

$$\frac{dQ(t)}{dt} = kQ(t)$$

$$\frac{dQ(t)}{Q(t)} = k\,dt$$

$$\int \frac{dQ(t)}{Q(t)} = \int k\,dt$$

$$\ln(Q(t)) = kt + C \quad (C = \ln c)$$

Let: Let the year 1890, $t = 0$

$$\ln\left(\frac{Q(t)}{c}\right) = kt$$

$$e^{kt} = \frac{Q(t)}{c}$$

$$Q(t) = ce^{kt}$$

$$Q(0) = c \cdot 1 = 62979766$$

$$\therefore Q(t) = 62979766e^{kt}$$

1790 $t = -100$ $\rightarrow$ 3929213

$$62979766e^{-100k} = 3929213$$

$$e^{100k} = \frac{62979766}{3929213}$$

$$k = \frac{1}{100}\ln\left(\frac{62979766}{3929213}\right)$$

$100k \approx 2.77437$

$$\boxed{k \approx 0.02774}$$

$$\therefore Q(t) = 62979766e^{0.02774t}$$

new model

또한 $t \to \infty$ 이면 $\frac{a}{b}$ 로 $P(t)$는 수렴하니까.

따라서 limit of population 은

$$\frac{0.03134}{1.5887 \times E^{-10}}$$

$\boxed{\approx 197268207}$ 이다.

$\boxed{\text{Assignment \# 1.}}$

$$y' + p(x)y = \underline{Q(x)}$$

y-intercept $\underset{\sim}{2x^2}$

$$\underline{\text{"} y = y'x + 2x^2 \text{"}}$$

(양변을 $\frac{1}{x}$로 나누자)

$$\frac{y}{x} = y' + 2x.$$

$$y' - \frac{y}{x} = -2x.$$

$$p(x) = -\frac{1}{x}$$

$$Q(x) = -2x.$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$e^{\int -\frac{1}{x}dx}\left(y' - \frac{y}{x}\right) = e^{\int -\frac{1}{x}dx}(-2x)$$

$$\frac{1}{x}\left(y' - \frac{y}{x}\right) = \frac{1}{x}\cdot(-2x)$$

$$= \frac{1}{x}y' - \frac{y}{x^2} = -2.$$

$$\left(\frac{1}{x}y\right)' = -2$$

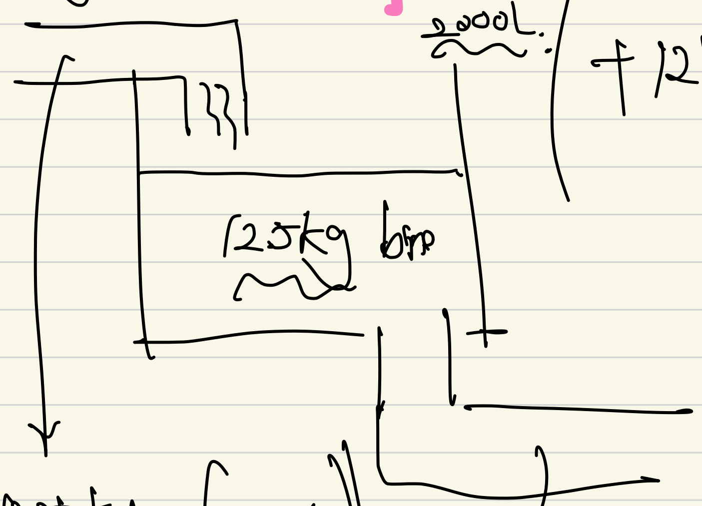
$$\frac{d\left(\frac{y}{x}\right)}{dx} = -2$$

$$\int \frac{d}{dx}\left(\frac{y}{x}\right)\cdot dx = \int -2\,dx$$

$$\frac{y}{x} = -2x + C$$

$$\boxed{y = -2x^2 + Cx}$$

Assignment #2.



$$\left(+12\text{kg} - 8\text{kg} \,/\, \text{min.} \right)$$

$$\left(\frac{0.25\text{kg}}{L} \times \left(\frac{12L}{\text{min.}} \right) \right)$$

$$(0.25 \times 12) \quad \left(\frac{8L}{\text{min.}} \right)$$

부피 일정
4L 씩 증가.

$Q(t)$: the amount of salt in the tank at time t

$\frac{dQ(t)}{dt}$: the rate of change with respect to time

$$\left(\frac{0.25\text{kg}}{1L} \times \frac{12L}{\text{min}} \right) - \left(\frac{Q(t)\text{kg}}{200+4t} \times \frac{\cancel{8}L}{\text{min}} \right)$$

$$Q'(t) = 3 - \frac{4Q(t)}{200+4t}.$$

$t=0 \rightarrow 12.5\text{kg}$

$$Q(0) = 12.5 \quad \cdots ①$$

$$\frac{dQ(t)}{dt} = 3 - \frac{2Q(t)}{100+t}.$$

$$Q'(t) + \frac{2}{100+t} Q(t) = 3.$$

$p(x) = \frac{2}{50+t}$

$e^{\int \frac{2}{50+t}\,dt} = e^{\cancel{2}\ln(50+t)^2} = (50+t)^2$

↓

Integrating factor

$(50+t)^2\left(Q'(t) + \frac{2}{50+t}Q(t)\right) = 3$

$(50+t)^2 Q'(t) + 2(50+t)Q(t) = 3(50+t)^2$

$(50+t)^2 Q(t) = 3\int (50+t)^2\,dt$

$Q(t)(50+t)^2 = 3\left[\frac{1}{3}(50+t)^3\right] + C$

$Q(t)(50+t)^2 = (50+t)^3 + C$

$Q(t) = (50+t) + \frac{C}{(50+t)^2}$

by --- ①

$Q(0) = 12.5$

$Q(0) = 50 + \frac{C}{(50)^2} = 12.5$

$\frac{C}{2500} = -37.5$

$C = -37.5(2500)$

$$Q(t) = (50+t) + \frac{-31.5 \times 2500}{(50+t)^2}$$

200L →
/min → 4L
50min
∴ $t = 50$ 일때
brine → kg

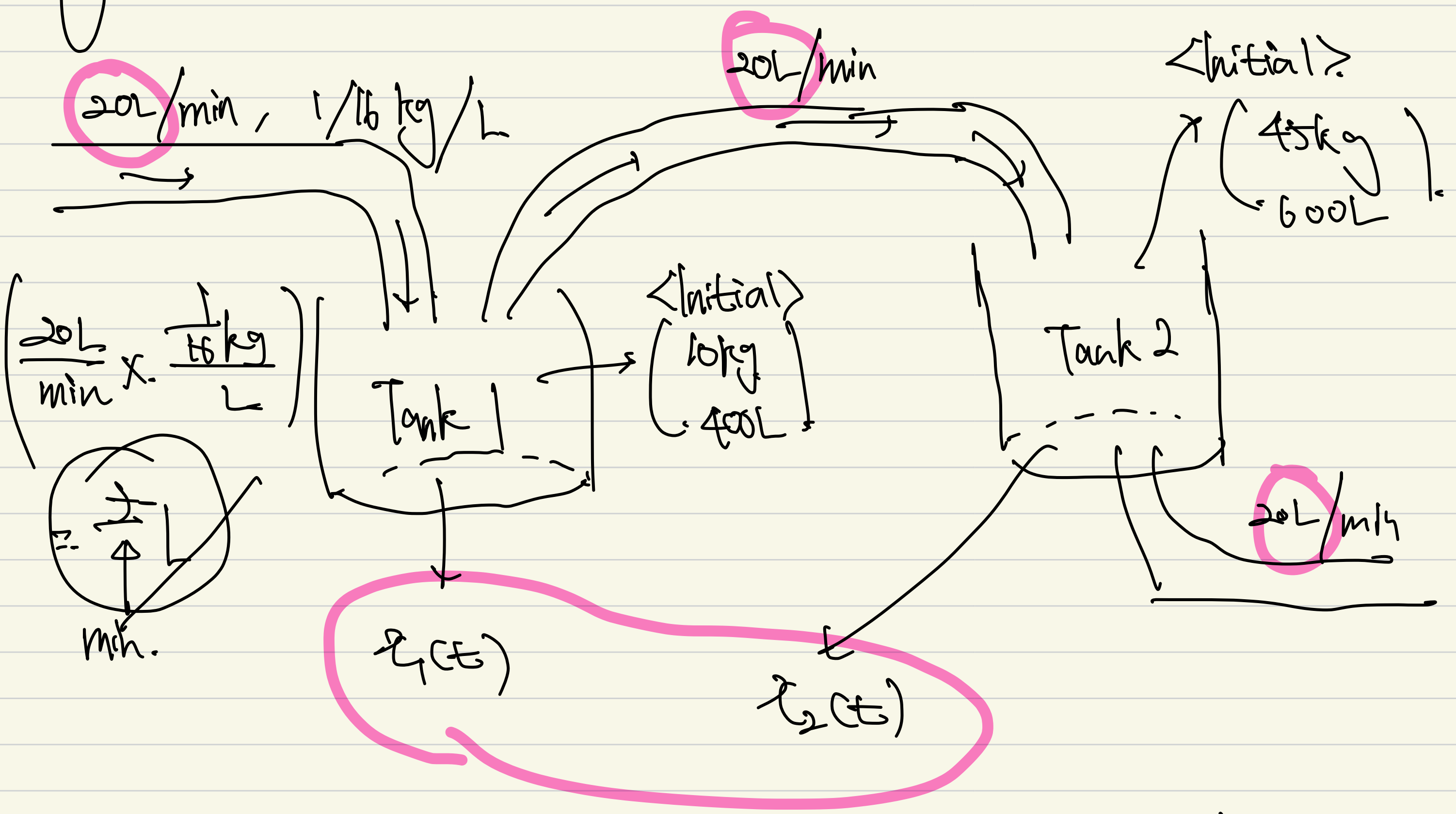
$$Q(50) = 100 + \frac{-31.5 \times 2500}{10000}$$

$$= 100 - \frac{93750}{10000}$$

$$= 100 - 9.375$$

$$\boxed{= 90.625\,\text{kg}}$$

Assignment # 3.



= t에서 각각 Tank1, Tank2의 소금 양

소금의 양 t가 변화하고 풀면.

<tank 1>

$$\frac{dx_1(t)}{dt} = \frac{5}{4} - \left(\frac{x_1(t)}{\cancel{400}_{20}} \times \cancel{20}\right)$$

$$= \frac{5}{4} - \frac{x_1(t)}{20}$$

$$\frac{dx_1(t)}{dt} + \frac{x_1(t)}{20} = \frac{5}{4}$$

$$(x_1(t))' + \frac{x_1(t)}{20} = \frac{5}{4}$$

Integral factor.

$$= e^{\int \frac{1}{20} dx} = e^{\frac{t}{20}}$$

<tank 2>

$$\frac{dx_2(t)}{dt} = \left(20 \cdot \times \frac{x_1(t)}{400}\right) - \left(20 \times \frac{x_2(t)}{600}\right)$$

$$= \frac{x_1(t)}{20} - \frac{x_2(t)}{30}$$

$$\frac{dx_2(t)}{dt} + \frac{x_2(t)}{30} = \frac{x_1(t)}{20}$$

$$(x_2(t))' + \frac{x_2(t)}{30} = \frac{x_1(t)}{20}$$

$$e^{\frac{t}{20}}\left((x_1(t))' + \frac{x_1(t)}{20}\right) = \frac{5}{4}\cdot e^{\frac{t}{20}}$$

$$e^{\frac{t}{20}}(x_1(t))' + e^{\frac{t}{20}}\cdot\frac{x_1(t)}{20} = \frac{5}{4}e^{\frac{t}{20}}$$

$$\left(x_1(t)\cdot e^{\frac{t}{20}}\right)' = \frac{5}{4}e^{\frac{t}{20}}$$

$$\int \frac{d\left(x_1(t)\cdot e^{\frac{t}{20}}\right)}{dt}\cdot dt = \frac{5}{4}\int e^{\frac{t}{20}}\,dt$$

$$x_1(t)\times e^{\frac{t}{20}} = \frac{5}{4}\cdot 20\cdot e^{\frac{t}{20}} + C$$

$$x_1(t)\times e^{\frac{t}{20}} = 25e^{\frac{t}{20}} + C$$

$$x_1(t) = 25 + C\cdot e^{-\frac{t}{20}}$$

Init value

$$x_1(0) = 25 + C\cdot e^{0} = 10$$

$$25 + C = 10$$

$$C = -15$$

$$x_1(t) = 25 - 15e^{-\frac{t}{20}}$$

$$(x_2(t))' + \frac{x_2(t)}{30} = \frac{x_1(t)}{20}$$

$$= \frac{1}{20}\left(25 - 15e^{-\frac{t}{20}}\right)$$

$$\frac{dx_2(t)}{dt}+\frac{x_2(t)}{30}=\frac{25-15e^{-\frac{t}{20}}}{20}$$

$$\frac{dx_2(t)}{dt}+\frac{x_2(t)}{30}=\frac{5}{4}-\frac{3}{4}e^{-\frac{t}{20}}$$

Integral factor.

$$e^{\int \frac{1}{30}dt}=e^{\frac{1}{30}t}$$

$$\left((x_2(t))'+\frac{1}{30}x_2(t)\right)e^{\frac{1}{30}t}=e^{\frac{1}{30}t}\left(\frac{5}{4}-\frac{3}{4}e^{-\frac{t}{20}}\right)$$

$$\int\left(x_2(t)\cdot e^{\frac{1}{30}t}\right)'=\int e^{\frac{1}{30}t}\left(\frac{5}{4}-\frac{3}{4}e^{-\frac{t}{20}}\right)$$

$$x_2(t)\cdot e^{\frac{1}{30}t}=\frac{5}{4}\int e^{\frac{t}{30}}dt-\frac{3}{4}\int e^{\left(\frac{t}{30}-\frac{t}{20}\right)}dt$$

$$=\frac{5\cdot 30}{4}\cdot e^{\frac{t}{30}}-\frac{-3\cdot 60}{4}\cdot e^{-\frac{t}{60}}+C$$

$$x_2(t)\cdot e^{\frac{t}{30}}=\frac{75}{2}e^{\frac{t}{30}}+45e^{-\frac{t}{60}}+C$$

$$x_2(t)=\frac{75}{2}+45e^{-\frac{t}{20}}+C\cdot e^{-\frac{t}{30}}$$

Initial value.

$x_2(0)=45$

$$\frac{75}{2}+45e^{0}+C\cdot e^{0}=45$$

$$32.5+\cancel{45}+C=\cancel{45}$$

$$C=-32.5$$

$$x_2=\frac{75}{2}+45e^{-\frac{t}{20}}-32.5e^{-\frac{t}{30}}$$

$$\frac{dx_2(t)}{dt}=0-\frac{45}{20}e^{-\frac{t}{20}}-\frac{65}{2}\cdot\left(-\frac{1}{30}\right)e^{-\frac{t}{30}}$$

(이중 미분해서 >0 이여야 최소의 정의)

$$\frac{dx_2(t)}{dt}=-\frac{9}{4}e^{-\frac{t}{20}}+\frac{5}{4}e^{-\frac{t}{30}}=0$$

우리는 언제 Tank 2의 brine 최소.

$$\frac{9}{4}e^{-\frac{t}{20}}=\frac{5}{4}e^{-\frac{t}{30}}$$

$$9\cdot e^{-\frac{t}{20}}=5\cdot e^{-\frac{t}{30}}$$

$$\frac{e^{-\frac{t}{20}}}{e^{-\frac{t}{30}}}=\frac{5}{9}$$

$$e^{-\frac{t}{20}+\frac{t}{30}}=\frac{5}{9}$$

$$e^{-\frac{t}{60}}=\frac{5}{9}$$

$$\ln\frac{5}{9}=-\frac{t}{60}$$

$$-60\ln\frac{5}{9}=t$$

$$t=\boxed{60\ln\frac{9}{5}}$$

$$\frac{d^2 x_2(t)}{dt} = -\frac{9}{4}\left(-\frac{1}{20}\right)e^{-\frac{t}{20}} + \frac{5}{4}\left(-\frac{1}{30}\right)e^{-\frac{t}{30}}$$

$$= \frac{9}{80}e^{-\frac{t}{20}} - \frac{1}{24}e^{-\frac{t}{30}}$$

at $t = 60\ln\frac{9}{5}$

$$\frac{d^2 x_2\left(60\ln\frac{9}{5}\right)}{dt} = \frac{9}{80}\cdot e^{-\frac{1}{20}\cdot 60\ln\frac{9}{5}} - \frac{1}{24}\cdot e^{-\frac{1}{30}\cdot 60\ln\frac{9}{5}}$$

$$= \frac{9}{80}\cdot\left(\frac{5}{9}\right)^3 - \frac{1}{24}\left(\frac{5}{9}\right)^2$$

$$= \frac{1}{16}\left(\frac{5}{9}\right)^2 - \frac{1}{24}\left(\frac{5}{9}\right)^2$$

$$= \left(\frac{5}{9}\right)^2\left(\frac{1}{16} - \frac{1}{24}\right) > 0$$

$\therefore\ t = 60\ln\frac{9}{5}$ 에서 Min

concentration of salt in tank2 is minimum at $60\ln\left(\frac{9}{5}\right)$ minutes, and it has $\frac{2725}{81}$ kg of salt at that time

이때 tank 안의 brine 의 양

$$x_2 = \frac{75}{2} + 45\cdot e^{-\frac{60\ln\frac{9}{5}}{20}} - \frac{75}{2}\cdot e^{-\frac{60\ln\frac{9}{5}}{30}}$$

$$= \frac{75}{2} + 45\cdot\left(\frac{5}{9}\right)^3 - \frac{75}{2}\left(\frac{5}{9}\right)^2$$

$$= \frac{2725}{81}$$