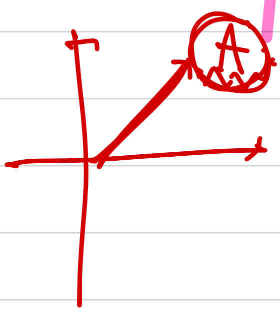
Eigenvalues and Diagonalization and special Matrices

$A\vec{E} = \lambda \vec{E}$



$A(c\vec{E}) = \lambda(c\vec{E})$

E의 multiple을 넣어야 하든 λ는 변하지 않는다.

$A\vec{E} = \lambda\vec{E}$

$A\vec{E} - \lambda\vec{E} = 0$

$(A - I_n\lambda)\vec{E} = 0$

→ 0이 되어야 한다

BX = 0 의

B가 non-singular면 X

singular면 행렬식 0

$\cancel{B^{-1}B}X = 0$

singular이라면

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(A - \lambda I)\xi = 0$$

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{pmatrix}.$$

$$(-1)^2 (1-\lambda)(1-\lambda)(-1-\lambda)$$

$$= -(1-\lambda)^2(1+\lambda)$$

$\lambda = 1$ or $\lambda = -1$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2e_1 - e_2 = 0$$
$$2e_2 + e_3 = 0$$
$$e_3 = e_3$$

$$\begin{pmatrix} -\frac{1}{4}e_3 \\ -\frac{1}{2}e_3 \\ e_3 \end{pmatrix}$$

$$-e_1 = 0$$
$$e_3 = 0$$
$$e_3 = 0$$

$$= \begin{pmatrix} e_1 \\ 0 \\ 0 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$e_3 = -2e_2 = -4e_1$$

$$e_2 = 2e_1$$

$$\boxed{\lambda = 1, \ \xi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$-4e_3\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

$$\lambda = -1, \ \xi = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

Let A be a real number matrix

$$\boxed{\lambda = \alpha + i\beta}$$

the $\bar{\lambda}$ 도 eigen value라 하자.

$\bar{E} \to \bar{\lambda}$ 과 대응하는 eigenvector라 하자.

$$\overline{AE} = \bar{\lambda}\bar{E}$$

eigen vector E를 알면 → eigen value 알 수 있게 된다

$$\Big(\big(1 \quad 2 \big) \quad \big(3 \quad 4 \big) \Big)$$

$$AE = \lambda E$$

$$\underbrace{\bar{E}^t (AE)}_{\text{vector}} = \bar{E}^t \lambda E = \underbrace{\lambda \bar{E}^t E}_{\text{scalar}}$$

scalar

$$= \underbrace{(\bar{E} \cdot E)}_{\text{scalar}} \lambda$$

scalar

$$\left(\lambda = \frac{\bar{E}^t A E}{\bar{E}^t E} \right)$$

$$A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$$

$(A - \lambda I)E =$

-3

1

$$\begin{pmatrix} 4 & -4 & 4 \\ 12 & -12 & 12 \\ 4 & -4 & 4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$e_1 - e_2 + e_3 = 0$

$e_1 = e_2 - e_3$

$$E = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_2 - e_3 \\ e_2 \\ e_3 \end{pmatrix} = \boxed{e_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + e_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

2개나 나올 수 있다

$\lambda = 1 \rightarrow$ $E_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = -3 \rightarrow \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

A be real symmetric matrix.

$$\begin{cases} \lambda_1 \to E_1 \\ \lambda_2 \to E_2 \end{cases}$$

E_1와 E_2는 orthogonal

$E \cdot G = E^t G$

$$\lambda E^t G = (AE)^t G$$

$$= \underline{E^t A G} = \lambda_2 E^t G$$

$$(\lambda - \lambda_2) E^t G = 0$$

$$\underline{E \cdot G}$$

E와 G는 orthogonal 함.

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$\lambda_1 = 2 \quad \lambda_2 = -1 \quad \lambda_3 = 4$

$$\begin{matrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{matrix} \to \text{mutually orthogonal}$$

Diagonalization.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

$\lambda_1 = a_{11}$ $\quad \lambda_2 = a_{22}$ $\quad \lambda_3 = a_{33}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22}-a_{11} & 0 \\ 0 & 0 & a_{33}-a_{11} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$e_2 = 0 \quad e_3 = 0$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ 얻다.

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & & d_n \end{pmatrix}.$$

$P \rightarrow$ formed from eigenvectors.

$P^{-1}AP \rightarrow P$ diagonalizes A

↳ P를 통해 A를 대각화 한다.

Ex.

$A = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}$ $(A - I_n\lambda)$

$$\begin{pmatrix} -1-\lambda & 4 \\ 0 & 3-\lambda \end{pmatrix}$$

$(-1-\lambda)(3-\lambda) = 0$

$\lambda_1 = -1$ $\lambda_2 = 3$

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$e_1 = e_2$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$e_1 = e_1$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -1 \rightarrow E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = 3 \rightarrow E_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =$$

$$\rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 0 & 3 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}}$$

eigen value

Orthogonal matrices

$A^{-1} = A^t$ $|A| = \pm 1$

$|I_n| = |AA^{-1}| = |AA^t| = |A||A^t| = |A|^2$

n×n real symmetric matrix with distinct eigenvalues can be diagonalized by an orthogonal matrix.

$$S = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$2 \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$-1 \to \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$4 \to \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$AE = \lambda E$

$\lambda E \to \lambda E$ 만족하는 x

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

↓ orthogonal matrix로

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$A = Q^{-1}AQ$
$= Q^tAQ$
$= \boxed{QAQ^t}$

이렇게 decomposition 할 수 있다.

정규화 할 수 있다.

unitary matrices.

$U^{-1} = \bar{U}^{t}$

λ가 unitary matrix의 eigen.value라면.

$|\lambda| = 1$

$UE = \lambda E$

$\overline{UE} = \bar{\lambda}\bar{E}$

$(\overline{UE})^{t} = \bar{E}^{t}\bar{U}^{t} = \bar{E}^{t}U^{-1}$

$\quad\| \qquad\qquad\qquad = (\bar{\lambda}\bar{E})^{t}$

$(\bar{\lambda}\bar{E})^{t} \qquad\qquad = \bar{\lambda}\bar{E}^{t}$

$\bar{E}^{t}\cancel{U^{-1}U}E = \bar{\lambda}\bar{E}^{t}UE$

$\bar{E}^{t}E = \bar{\lambda}\bar{E}^{t}(UE)$

$= \bar{\lambda}\bar{E}^{t}\lambda E$

$= \bar{\lambda}\lambda\bar{E}^{t}E$

$\bar{\lambda}\lambda = 1$

$\lambda = a+bi$

$\bar{\lambda} = a-bi$

$\bar{\lambda}\lambda = a^{2}+b^{2} = 1$

Hermitian

$\overline{H} = H^t$ → Hermitian

$\overline{S} = -S^t$ → skew-Hermitian.

$Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

$\overline{H} = H^t$

$\overline{\overline{Z}^t H Z} = \underbrace{Z^t \overline{H} \overline{Z}}$ → B

B^t

$(Z^t \overline{H} \overline{Z})^t = \overline{Z}^t \overline{H}^t Z$

$= \overline{Z}^t H Z$

conjugate is equal to itself.

$\overline{A} = B \quad B^t = A$

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→ The eigenvalues of Hermitian matrix are real

skew-Hermitian matrix → pure imaginary

$= \begin{pmatrix} a+bi \\ a-bi \end{pmatrix}$

$Q(x) = \cancel{5x_1^2} + \cancel{3x_2^2} + \cancel{2x_3^2} - \cancel{x_1x_2} + 8x_2x_3$

$$\rightarrow \begin{bmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

Quadratic form as $x^T A x$

$$(x_1 \quad x_2 \quad x_3) \begin{pmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$

$$(x_1 x_2) \begin{pmatrix} 1 & -4 \\ -4 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$(-3 \quad 1) \begin{pmatrix} 1 & -4 \\ -4 & -5 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$(-7 \quad 7) \begin{pmatrix} -3 \\ 1 \end{pmatrix} = 21 + 7 = 28$$

$9 + 24 - 5$

$= 28$

~~28~~

principal axis theorem.

$A \rightarrow$ real symmetric matrix.

$\rightarrow \quad X = QY$

QY → orthogonal matrix Q.

$Q^{-1} = Q^{t}$

우리는 이미 알고 있음

$$Y = Q^{-1}X = Q^{t}X.$$

$$\alpha x_1^2 + \cancel{\boxed{\beta x_1 x_2}} + \gamma x_2^2.$$

$$\hookrightarrow \alpha' y_1^2 + \beta' y_2^2$$ → 이렇게 간단하게 표현이 가능해 짐!

$$X^tAX = (QY)^tAQY = (Y^tQ^t)AQY$$

$$= Y^t(Q^tAQ)Y$$

└ 대각화 되어진 matrix가 됨

Ex $x_1^2 - 9x_1x_2 + x_2^2 \rightarrow X^T \begin{pmatrix} 1 & -\frac{9}{2} \\ -\frac{9}{2} & 1 \end{pmatrix} X$

Eigen Values of A

$$\begin{vmatrix} 1-\lambda & -\frac{9}{2} \\ -\frac{9}{2} & (1-\lambda) \end{vmatrix}$$

$$(1-\lambda)^2 - \frac{49}{4}$$

$$= 1 - 2\lambda + \lambda^2 - \frac{49}{4}$$

$$= \lambda^2 - 2\lambda - \frac{45}{4}$$

$$4\lambda^2 - 8\lambda - 45$$

$$\begin{matrix} 2 & -9 \\ 2 & 5 \end{matrix}$$

$$(2\lambda - 9)(2\lambda + 5)$$

$\lambda_1 = \frac{9}{2}$ $\quad$ $\lambda_2 = -\frac{5}{2}$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\rightarrow X = QY$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2 \\ \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} y_1 - y_2 \\ y_1 + y_2 \end{pmatrix}$$

$$(y_1 \; y_2) \begin{pmatrix} -\frac{9}{2} & 0 \\ 0 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$-\frac{5}{2}y_1^2 + \frac{9}{2}y_2^2 \; //$$

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$\rightarrow \underline{X^t A X} = (QY)^t A (QY)$

$X = QY \qquad = Y^t \boxed{Q^t A Q} Y.$

$Q \rightarrow$ orthogonal matrix.

$Q^{-1} A Q$ ☆ $\rightarrow$ 대각화 과정 ☆

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8장

$X' = AX + G$, $X(t_0) = X^0$ $\rightarrow$ "initial value"

"$y'' + p(x)y' + Q(x)y = 0$"

$X' = AX$

$x_1'(t) = a_{11}(t)x_1 + a_{12}(t)x_2(t)$

$x_2'(t) = a_{21}(t)x_1 + a_2$

Ex

$X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X$

$X_1' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X_1$

$\begin{pmatrix} -6e^{3t} \\ 3e^{3t} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix}$

$= \begin{pmatrix} -6e^{3t} \\ 3e^{3t} \end{pmatrix}$

$X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X.$

$\phi_1(t) = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix}, \quad \phi_2(t) = \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}.$

$\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 \neq 0$

2개의 solution vector는

linearly independent 하다.

$X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X.$ $\quad \boxed{c_1 \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}}.$

$\hookrightarrow$ general solution

$\boxed{\begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\downarrow$

fundamental matrix.

$$X(t_0) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \rightarrow$$

0

$$\underset{\sim}{\Omega}(0)C = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= -\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$X(t) = \underset{\sim}{\Omega}(t)\, C$$

$$= \begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t}(-2-8t) \\ e^{3t}(3+4t) \end{pmatrix}$$

↳ particular solution

$y'' + p(x)y' + Q(x)y = f(x)$.

$\underline{Y_1 - Y_2}$ → associated homogeneous equation of solution

↓

linear Comb

$Y_1 = Y_2 + \square$

$$\boxed{X' = AX + G}$$

$X = \underbrace{\Phi C}_{\text{homogeneous}} + \underbrace{\Psi_p}_{\text{particular solution (nonhomogeneous)}}$

$X' = AX$

$y'' + P(x)y' + Q(x)y = 0$

$y = e^{rx}$ → 놓고 풀었었다

$X = E e^{\lambda t}$

vector

scalar

→ 이러한 형태라 가정하자,,

$X' = \lambda E e^{\lambda t} = A E e^{\lambda t} = AX$

행렬 A의 (eigen value, eigen vector)를 구하면 된다.

$Ee^{\lambda t}$ → $X' = AX$ 의 solution 이다.

suppose A has $\lambda_1, \dots, \lambda_n$ eigenvalues →

Then $\left(E_1 e^{\lambda_1 t}, \dots, E_n e^{\lambda_n t} \right)$ are linearly independent solutions,,

〃 characteristic equation의 solution의 종류에 따라 달라짐 〃

① $\lambda =$ distinct real

〃 repeated

〃 complex.

ex

$X' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} X$

$\begin{pmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{pmatrix}$

$(12 - 7\lambda + \lambda^2 - 6)$

$= \lambda^2 - 7\lambda + 6$

$(\lambda - 1)(\lambda - 6)$

① $\lambda = 1$

$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3x_1 + 2x_2 = 0$

$\begin{pmatrix} x_1 \\ -\frac{3}{2} x_1 \end{pmatrix}$

$\begin{pmatrix} -2 \\ +3 \end{pmatrix}$

$\underline{K e^{\lambda t}}$

$X_1 = \underline{\begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{t}}$

② $\lambda = 6$

$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 = x_2$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$X_2 = \underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}}$

$X = \begin{pmatrix} -2e^{t} & e^{6t} \\ 3e^{t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Complex eigen value

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$$

$$y_2 = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$$

$$\vec{t} e^{(\alpha+\beta i)t} \rightarrow e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$\begin{pmatrix} 2+4i \\ 5 \\ 1-2i \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}}_{V} + \underbrace{\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} i}_{V}$$

$$\phi_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(V + iV)$$

$$= e^{\alpha t}(\cos(\beta t)V - \sin(\beta t)V) + ie^{\alpha t}(\sin(\beta t)V + \cos(\beta t)V)$$

$$\tfrac{1}{2}(\phi_1 + \phi_2) = e^{\alpha t}(\cos(\beta t)V - \sin(\beta t)V)$$

$$\tfrac{1}{2}(\phi_1 - \phi_2) = ie^{\alpha t}(\sin(\beta t)V + \cos(\beta t)V)$$

$X' = AX$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

$2, \quad -1+\sqrt{3}i \quad -1-\sqrt{3}i \to$ eigen value.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -2\sqrt{3}i \\ -3+\sqrt{3}i \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2\sqrt{3}i \\ -3-\sqrt{3}i \end{pmatrix}$$

$\xi e^{\lambda t}$ $\qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$

$$\overset{U}{\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}} + \overset{V}{\begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix}} i = U + Vi$$

$$e^{\alpha t}(\cos(\beta t)U - \sin(\beta t)V)$$

$$\phi_3 = e^{-t}\left[\cos(\sqrt{3}t)\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} - \sin(\sqrt{3}t)\begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix}\right]$$

$$\phi_4 = e^{-t}\left[\sin(\sqrt{3}t)\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \cos(\sqrt{3}t)\begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix}\right]$$

$$Q(t) = \begin{pmatrix} e^{t} & & '' \\ 0 & & \sim \\ & & \\ 0 & & \end{pmatrix}$$

$X' = AX$ → repeated

$$A = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix} \rightarrow \begin{pmatrix} 1-\lambda & 3 \\ -3 & 7-\lambda \end{pmatrix}$$

$$7 - 8\lambda + \lambda^2 + 9$$

$$\lambda^2 - 8\lambda + 16$$

$$(\lambda - 4)^2 = 0$$

$$\lambda = 4 \qquad \xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\phi_2 = \underline{E_1 t e^{\lambda t}} + \underline{E_2 e^{\lambda t}}.$

$\phi_2' = E_1 e^{\lambda t} + E_1 t \lambda e^{\lambda t} + \lambda E_2 e^{\lambda t}.$

$\underline{X'} = \underline{AX}$

$E_1 e^{\lambda t} + E_1 t \lambda e^{\lambda t} + \lambda E_2 e^{\lambda t}$

$= A E_1 t e^{\lambda t} + A E_2 e^{\lambda t}.$

$E_1 + E_1 \lambda t + \lambda E_2$

$= A E_1 t + A E_2.$

$n!$ $120n$ E

$\lambda = 4 \quad E = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{(A - \lambda) E_2 = E_1}$

$(A - 4I) E_2 = E_1$

$\begin{pmatrix} 1-4 & 3 \\ -3 & 7-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}!$

$\underline{-3x_1 + 3x_2 = 1}$

$3x_1 = +1 - 3x_2.$

$\underline{x_1 = \frac{1}{3} - x_2}$

$\begin{pmatrix} \frac{1}{3} - x_2 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

$\alpha - \frac{1}{3}$

$\alpha.$

$E_2 = \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix}$

$$\phi_2 = E_1 t e^{\lambda t} + E_2 e^{\lambda t}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{4t} + \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix} e^{4t}$$

$$= \begin{pmatrix} t+1 \\ t+\frac{4}{3} \end{pmatrix} e^{4t}$$