해설

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1-(a). 다음 행렬의 행렬식을 계산하시오. $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -4 & 6 \\ 3 & -2 & 0 \end{pmatrix}$

$$|\mathbf{A}| = 3 \begin{vmatrix} -1 & 2 \\ -4 & 6 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 3 \cdot 2 + 2 \cdot 2 = 10$$

1-(b). 딸림 행렬 $\mathrm{adj}\,\mathbf{A}$를 구하시오.

$$\mathrm{adj}\,\mathbf{A} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^{T} = \begin{pmatrix} 12 & 18 & 8 \\ -4 & -6 & -1 \\ 2 & -2 & -2 \end{pmatrix}^{T} = \begin{pmatrix} 12 & -4 & 2 \\ 18 & -6 & -2 \\ 8 & -1 & -2 \end{pmatrix}$$

1-(c). 다음 연립방정식을 푸시오. $\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ 2x_1 - 4x_2 + 6x_3 = -3 \\ 3x_1 - 2x_2 = 7 \end{cases}$

$$\begin{pmatrix} 1 & -1 & 2 & | & 1 \\ 2 & -4 & 6 & | & -3 \\ 3 & -2 & 0 & | & 7 \end{pmatrix} \begin{array}{c} -2R_1 + R_2 \\ -3R_1 + R_3 \\ \Longrightarrow \end{array} \begin{pmatrix} 1 & -1 & 2 & | & 1 \\ 0 & -2 & 2 & | & -5 \\ 0 & 1 & -6 & | & 4 \end{pmatrix} \begin{array}{c} -\frac{1}{2}R_2 \\ -R_2 + R_3 \\ \Longrightarrow \end{array} \begin{pmatrix} 1 & -1 & 2 & | & 1 \\ 0 & 1 & -1 & | & \frac{5}{2} \\ 0 & 0 & -5 & | & \frac{3}{2} \end{pmatrix}$$

$$\Rightarrow \quad \begin{array}{r} x_1 - x_2 + 2x_3 = 1 \\ x_2 - x_3 = \frac{5}{2} \\ -5x_3 = \frac{3}{2} \end{array} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3.8 \\ 2.2 \\ -0.3 \end{pmatrix}$$

2-(a). 다음 행렬의 고유값을 모두 구하시오. $\mathbf{A} = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & -2 & -2 \\ -2 & 1-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)\{(1-\lambda)^2 - 4\} + 2(2\lambda - 2 - 4) - 2(4 - 2\lambda + 2)$$

$$= -(\lambda-1)(\lambda-3)(\lambda+1) + 8(\lambda-3) = -(\lambda-3)(\lambda^2 - 1 - 8) = -(\lambda-3)^2(\lambda+3) = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = 3$$

2-(b). $\mathbf{A}$의 고유벡터들을 이용해서 직교행렬을 만드시오.

for $\lambda_1 = -3$,

$$(\mathbf{A} + 3\mathbf{I}|\mathbf{0}) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \begin{matrix} \frac{1}{2}R_1 + R_2 \\ \frac{1}{2}R_1 + R_3 \\ \Longrightarrow \end{matrix} \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ 0 & 3 & -3 & | & 0 \\ 0 & -3 & 3 & | & 0 \end{pmatrix} \begin{matrix} R_2 + R_3 \\ \frac{1}{3}R_2, \frac{1}{2}R_1 \\ \Longrightarrow \end{matrix}$$

$$\begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow \begin{matrix} 2k_1 - k_2 - k_3 = 0 \\ k_2 - k_3 = 0 \end{matrix} \Rightarrow k_1 = k_2 = k_3 \Rightarrow \mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for $\lambda_2 = 3$,

$$(\mathbf{A} - 3\mathbf{I}|\mathbf{0}) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \begin{matrix} -R_1 + R_2 \\ -R_1 + R_3 \\ -\frac{1}{2}R_1 \\ \Longrightarrow \end{matrix} \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow k_1 + k_2 + k_3 = 0$$

$$\Rightarrow \begin{matrix} k_1 = t \\ k_2 = u \\ k_3 = -t-u \end{matrix} \Rightarrow \mathbf{K} = \begin{pmatrix} t \\ u \\ -t-u \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_* = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \mathbf{K}_*\text{와 } \mathbf{K}_2\text{가 직교하지 않음.}$$

$$\mathbf{K}_3' = \mathbf{K}_* - \frac{\mathbf{K}_*^T\mathbf{K}_2}{|\mathbf{K}_*||\mathbf{K}_2|}\mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}\cdot\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}, \quad \mathbf{K}_3 = -2\mathbf{K}_3' = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} \dfrac{\mathbf{K}_1}{|\mathbf{K}_1|} & \dfrac{\mathbf{K}_2}{|\mathbf{K}_2|} & \dfrac{\mathbf{K}_3}{|\mathbf{K}_3|} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \mathbf{P}^{-1} = \mathbf{P}^T$$

2-(c). 대각행렬 $\mathbf{D}$에 대해 $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$를 만족시키는 $\mathbf{P}, \mathbf{D}, \mathbf{P}^{-1}$을 쓰시오.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

3. $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$에 대해 $\mathbf{A}^{20}$을 구하시오.

풀이 1 — 대각화를 이용하여

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda-2)(\lambda-1) = 0 \quad \Rightarrow \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \end{matrix} \quad \Rightarrow \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(\mathbf{A} - \mathbf{I}|\mathbf{0}) = \left(\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right) \quad \Rightarrow \quad k_1 + k_2 = 0 \quad \Rightarrow \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(\mathbf{A} - 2\mathbf{I}|\mathbf{0}) = \left(\begin{matrix} 0 & 1 \\ 0 & -1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right) \quad \Rightarrow \quad k_2 = 0 \quad \Rightarrow \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

$$\mathbf{A}^{20} = \mathbf{P}\mathbf{D}^{20}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{20} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2^{20} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{20} & 2^{20} - 1 \\ 0 & 1 \end{pmatrix}$$

풀이 2 — Cayley-Hamilton 정리를 이용하여

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0 \quad \Rightarrow \quad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \end{matrix}$$

$$\lambda^m = c_1\lambda + c_0\lambda$$

$$1^{20} = c_1 + c_0$$

$$2^{20} = 2c_1 + c_0$$

$$c_1 = 2^{20} - 1, \quad c_0 = -2^{20} + 2$$

$$\mathbf{A}^{20} = c_1\mathbf{A} + c_0\mathbf{I} = (2^{20} - 1)\mathbf{A} - (2^{20} - 2)\mathbf{I}$$

$$= \begin{pmatrix} 2^{21} - 2 & 2^{20} - 1 \\ 0 & 2^{20} - 1 \end{pmatrix} - \begin{pmatrix} 2^{20} - 2 & 0 \\ 0 & 2^{20} - 2 \end{pmatrix} = \begin{pmatrix} 2^{20} & 2^{20} - 1 \\ 0 & 1 \end{pmatrix}$$

4. 다음 행렬을 단위위삼각행렬을 포함하도록 LU 인수분해하라. $\mathbf{A} = \begin{pmatrix} -2 & 0 & 3 \\ 6 & 1 & -4 \\ -4 & 4 & 30 \end{pmatrix}$

$$\begin{pmatrix} -2 & 0 & 3 \\ 6 & 1 & -4 \\ -4 & 4 & 30 \end{pmatrix} \begin{array}{c} 3R_1 + R_2 \\ -2R_1 + R_3 \\ \Longrightarrow \end{array} \begin{pmatrix} -2 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 4 & 24 \end{pmatrix} \begin{array}{c} -4R_2 + R_3 \\ \Longrightarrow \end{array} \begin{pmatrix} -2 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{pmatrix} = \mathbf{U}'$$

$$\mathbf{L}' = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{L}'\mathbf{U}' = \mathbf{L}'\mathbf{D}\mathbf{D}^{-1}\mathbf{U}' = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -2 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\mathbf{L}'\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 6 & 1 & 0 \\ -4 & 4 & 4 \end{pmatrix} = \mathbf{L}$$

$$\mathbf{D}^{-1}\mathbf{U}' = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{U}$$

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 0 \\ 6 & 1 & 0 \\ -4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

5. $t = \pi/2$에서 $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t\mathbf{k}$의 단위접선벡터, 단위법선벡터, 단위종법선벡터, 접촉평면, 법평면, 전직평면의 방정식을 구하시오.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 3\mathbf{k}}{\sqrt{13}}, \quad \mathbf{T}\left(\frac{\pi}{2}\right) = \frac{-2\mathbf{i} + 3\mathbf{k}}{\sqrt{13}} \quad \blacktriangleleft \quad \text{단위접선벡터}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\frac{1}{\sqrt{13}}(-2\cos t\mathbf{i} - 2\sin t\mathbf{j})}{\frac{1}{\sqrt{13}}\cdot 2} = -\cos t\mathbf{i} - \sin t\mathbf{j}, \quad \mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j} \quad \blacktriangleleft \quad \text{단위법선벡터}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{13}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 3 \\ 0 & -1 & 0 \end{vmatrix} = \frac{3\mathbf{i} + 2\mathbf{k}}{\sqrt{13}} \quad \blacktriangleleft \quad \text{단위종법선벡터}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = 2\mathbf{j} + \frac{3}{2}\pi\mathbf{k} = \ <0, 2, \frac{3}{2}\pi>$$

$$\left(<x, y, z> - <0, 2, \frac{3}{2}\pi>\right)\cdot\mathbf{B} = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{13}}\left\{3x + 2\left(z - \frac{3}{2}\pi\right)\right\} = 0 \quad \Rightarrow \quad 3x + 2z = 3\pi$$

$$\blacktriangle \quad \text{접촉평면}$$

$$\left(<x, y, z> - <0, 2, \frac{3}{2}\pi>\right)\cdot\mathbf{T} = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{13}}\left\{-2x + 3\left(z - \frac{3}{2}\pi\right)\right\} = 0 \quad \Rightarrow \quad -4x + 6z = 9\pi$$

$$\blacktriangle \quad \text{법평면}$$

$$\left(<x, y, z> - <0, 2, \frac{3}{2}\pi>\right)\cdot\mathbf{N} = 0 \quad \Rightarrow \quad -(y - 2) = 0 \quad \Rightarrow \quad y = 2 \quad \blacktriangleleft \quad \text{전직평면}$$

6. 점 $(0,2,1)$에서 함수 $f(x,y,z) = 2xy + z^2 e^y$의 $\mathbf{v} = 1\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$방향으로의 방향도함수를 구하시오.

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

$$\nabla f(x,y,z) = 2y\mathbf{i} + (2x + z^2 e^y)\mathbf{j} + 2ze^y\mathbf{k}$$

$$\nabla f(0,2,1) = 4\mathbf{i} + e^2\mathbf{j} + 2e^2\mathbf{k}$$

$$D_{\mathbf{u}}f(0,2,1) = \mathbf{u} \cdot \nabla f(0,2,1) = \frac{1}{3}(4 + 2e^2 - 4e^2) = \frac{2}{3}(2 - e^2)$$