해설

고급공학수학1(가)(이동령 교수님) 중간고사

시험실시 : 2025-04-30 12:00-13:15 (75분)

평균점수 : 16.93/60

2025-12-27

문제 1 (6점)

물체의 단위시간당 온도 변화량은 주변 공기의 온도와 물체의 온도 T의 차이에 정비례한다. 주변 공기의 온도가 300 K 이고, 물체가 15분만에 370 K에서 340 K 까지 식을 때 $T(t)$를 구하라.

풀이

$$\frac{dT}{dt} = A(T(t) - 300), \quad T(0) = 370,\ T(15) = 340$$
$$\frac{1}{T-300}dT = Adt$$
$$\ln|T-300| = At + C_1$$
$$Ce^{At} = T - 300$$
$$T(t) = Ce^{At} + 300$$
$$C = 370 - 300 = 70$$
$$70e^{15A} = 340 - 300$$
$$e^{15A} = \frac{4}{7}$$
$$15A = \ln\frac{4}{7}, \quad A = \frac{1}{15}\ln\frac{4}{7}$$
$$T(t) = 70e^{\frac{1}{15}\ln\frac{4}{7}t} + 300$$
$$T(t) = 70\left(\frac{4}{7}\right)^{\frac{t}{15}} + 300$$

$T(t) = 70\left(\frac{4}{7}\right)^{\frac{t}{15}} + 300$ ◀

문제 2 (10점)

(a) 다음 미분방정식의 양함수 해를 구하라. $y' + \frac{y}{x} - \sqrt{y} = 0$ (7점)

(b) 초기조건 $y(1) = 0$을 만족하는 초기값 문제의 해를 구하라. (3점)

풀이

$$
\begin{aligned}
&u = \sqrt{y} \\
&y' = (u^2)' = 2uu' \\
&2uu' + \frac{u^2}{x} = u \\
&2xu' + u = x \\
&(u - x)dx + 2xdu = 0 \\
&\text{let.} \quad t = \frac{u}{x}, \quad du = tdx + xdt \\
&x(t-1)dx + 2x(tdx + xdt) = 0 \\
&(t-1)dx + 2(tdx + xdt) = 0 \\
&(3t-1)dx + 2xdt = 0 \\
&-\frac{1}{2x}dx = \frac{1}{3t-1}dt \\
&-\frac{1}{2}\ln|x| + C_1 = \ln|3t-1| \\
&3t - 1 = C_2 x^{-\frac{1}{2}} \\
&t = Cx^{-\frac{1}{2}} + \frac{1}{3} \\
&u = tx = Cx^{\frac{1}{2}} + \frac{1}{3}x \\
&y = u^2 = x\left(C + \frac{\sqrt{x}}{3}\right)^2
\end{aligned}
$$

$(a) \quad y(x) = x\left(C + \frac{\sqrt{x}}{3}\right)^2$ ◀

$$
\begin{aligned}
&y(1) = \left(C + \frac{1}{3}\right)^2 = 0 \\
&C = -\frac{1}{3} \\
&y(x) = \frac{x}{9}\left(1 - \sqrt{x}\right)^2
\end{aligned}
$$

$(b) \quad y(x) = \frac{x}{9}\left(1 - \sqrt{x}\right)^2$ ◀

문제 3 (7점)

다음 미분방정식의 양함수 해를 구하라. $\left(2x\sin\frac{y}{x} + 3y\cos\frac{y}{x}\right)dx - 3x\cos\frac{y}{x}dy = 0$

풀이

$$u = \frac{y}{x}, \quad dy = udx + xdu$$
$$(2x\sin u + 3ux\cos u)dx - 3x\cos u(udx + xdu) = 0$$
$$(2\sin u + 3u\cos u)dx - 3\cos u(udx + xdu) = 0$$
$$2\sin udx = 3x\cos udu$$
$$\frac{2}{3x}dx = \frac{\cos u}{\sin u}du$$
$$\frac{2}{3}\ln|x| + C_1 = \ln|\sin u|$$
$$\sin u = Cx^{\frac{2}{3}}$$
$$u = \arcsin\left(Cx^{\frac{2}{3}}\right)$$
$$y = ux = x\arcsin\left(Cx^{\frac{2}{3}}\right)$$

$$y(x) = x\arcsin(Cx^{\frac{2}{3}}) \quad ◀$$

문제 4 (10점)

다음 초기값 문제의 해를 구하라.

$y'' + y = f(t),\ f(t) = \begin{cases} \sin t, & 0 < t < \frac{\pi}{2} \\ \cos t, & t \geq \frac{\pi}{2} \end{cases},\ y(0) = 0,\ y'(0) = 1$

풀이

$$\begin{aligned} &m^2 + 1 = 0 \\ &m = \pm i \\ &y_c = C_1 \cos t + C_2 \sin t \end{aligned}$$

$0 < t < \frac{\pi}{2}$ 에서

$$\begin{aligned} &y_{p1} = A_1 t \cos t + B_1 t \sin t \\ &y'_{p1} = (B_1 t + A_1) \cos t + (-A_1 t + B_1) \sin t \\ &y''_{p1} = (-A_1 t + 2B_1) \cos t + (-B_1 t - 2A_1) \sin t \\ &y''_{p1} + y_{p1} = \sin t \\ &(-A_1 t + 2B_1) \cos t + (-B_1 t - 2A_1) \sin t + A_1 t \cos t + B_1 t \sin t = \sin t \\ &2B_1 \cos t - 2A_1 \sin t = \sin t \\ &A_1 = -\frac{1}{2}, \quad B_1 = 0, \quad y_{p1} = -\frac{1}{2} t \cos t \end{aligned}$$

$t \geq \frac{\pi}{2}$ 에서

$$\begin{aligned} &y_{p2} = A_2 t \cos t + B_2 t \sin t \\ &\qquad \vdots \\ &2B_2 \cos t - 2A_2 \sin t = \cos t \\ &A_2 = 0, \quad B_2 = \frac{1}{2}, \quad y_{p2} = \frac{1}{2} t \sin t \end{aligned}$$

따라서

$$y(x) = \begin{cases} C_1 \cos t + C_2 \sin t - \dfrac{1}{2} t \cos t & \left(0 < t < \dfrac{\pi}{2}\right) \\ C_3 \cos t + C_4 \sin t + \dfrac{1}{2} t \sin t & \left(t \geq \dfrac{\pi}{2}\right) \end{cases}$$

함수 $y(x)$가 정의구간에서 미분가능해야 하므로

$$\lim_{k\to\pi/2} y(k) = \cancelto{0}{C_1\cos\frac{\pi}{2}} + C_2\sin\frac{\pi}{2} - \cancelto{0}{\frac{1}{2}\cdot\frac{\pi}{2}\cos\frac{\pi}{2}} = C_2$$

$$y\left(\frac{\pi}{2}\right) = \cancelto{0}{C_3\cos\frac{\pi}{2}} + C_4\sin\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2}\sin\frac{\pi}{2} = C_4 + \frac{\pi}{4}$$

$$\lim_{k\to\pi/2} y(k) = y\left(\frac{\pi}{2}\right) \quad\Rightarrow\quad C_2 = C_4 + \frac{\pi}{4} \quad\Rightarrow\quad C_4 = C_2 - \frac{\pi}{4}$$

$$y'(x) = \begin{cases} C_2\cos t - C_1\sin t - \dfrac{1}{2}\cos t + \dfrac{1}{2}t\sin t & \left(0 < t < \dfrac{\pi}{2}\right) \\ C_4\cos t - C_3\sin t + \dfrac{1}{2}\sin t + \dfrac{1}{2}t\cos t & \left(t \geq \dfrac{\pi}{2}\right) \end{cases}$$

$$\lim_{k\to\pi/2} y'(k) = \cancelto{0}{C_2\cos\frac{\pi}{2}} - C_1\sin\frac{\pi}{2} - \cancelto{0}{\frac{1}{2}\cos\frac{\pi}{2}} + \frac{1}{2}\cdot\frac{\pi}{2}\sin\frac{\pi}{2} = -C_1 + \frac{\pi}{4}$$

$$y'\left(\frac{\pi}{2}\right) = \cancelto{0}{C_4\cos\frac{\pi}{2}} - C_3\sin\frac{\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2} + \cancelto{0}{\frac{1}{2}\cdot\frac{\pi}{2}\cos\frac{\pi}{2}} = -C_3 + \frac{1}{2}$$

$$\lim_{k\to\pi/2} y'(k) = y'\left(\frac{\pi}{2}\right) \quad\Rightarrow\quad -C_1 + \frac{\pi}{4} = -C_3 + \frac{1}{2} \quad\Rightarrow\quad C_3 = C_1 + \frac{1}{2} - \frac{\pi}{4}$$

$$y(x) = \begin{cases} C_1\cos t + C_2\sin t - \dfrac{1}{2}t\cos t & \left(0 < t < \dfrac{\pi}{2}\right) \\ \left(C_1 + \dfrac{1}{2} - \dfrac{\pi}{4}\right)\cos t + \left(C_2 - \dfrac{\pi}{4}\right)\sin t + \dfrac{1}{2}t\sin t & \left(t \geq \dfrac{\pi}{2}\right) \end{cases}$$

초기값 문제를 풀면 다음과 같다.

$$y(0) = C_1\cos 0 + \cancelto{0}{C_2\sin 0 - \frac{1}{2}\cdot 0\cdot\cos 0} = C_1 = 0 \quad\Rightarrow\quad C_3 = \frac{1}{2} - \frac{\pi}{4}$$

$$y'(0) = C_2\cos 0 - \cancelto{0}{C_1\sin 0} - \frac{1}{2}\cos 0 + \cancelto{0}{\frac{1}{2}\cdot 0\cdot\sin 0} = C_2 - \frac{1}{2} = 1 \quad\Rightarrow\quad C_2 = \frac{3}{2}, \quad C_4 = \frac{3}{2} - \frac{\pi}{4}$$

$$y(x) = \begin{cases} \dfrac{3}{2}\sin t - \dfrac{1}{2}t\cos t & \left(0 < t < \dfrac{\pi}{2}\right) \\ \left(\dfrac{1}{2} - \dfrac{\pi}{4}\right)\cos t + \left(\dfrac{3}{2} - \dfrac{\pi}{4}\right)\sin t + \dfrac{1}{2}t\sin t & \left(t \geq \dfrac{\pi}{2}\right) \end{cases}$$

$$y(x) = \begin{cases} \dfrac{3}{2}\sin t - \dfrac{1}{2}t\cos t & \left(0 < t < \dfrac{\pi}{2}\right) \\ \left(\dfrac{1}{2} - \dfrac{\pi}{4}\right)\cos t + \left(\dfrac{3}{2} - \dfrac{\pi}{4}\right)\sin t + \dfrac{1}{2}t\sin t & \left(t \geq \dfrac{\pi}{2}\right) \end{cases} \quad ◀$$

문제 5 (7점)

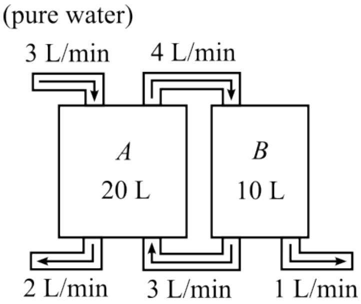
다음 미분방정식의 해를 구하라. $y'' - 9y = \dfrac{x}{e^{3x}}$

풀이

$$
\begin{aligned}
&m^2 - 9 = 0\\
&(m-3)(m+3) = 0\\
&y_c = C_1 e^{3x} + C_2 e^{-3x}\\
&y_p = (Ax^2 + Bx)e^{-3x}\\
&y_p' = (-3Ax^2 + 2Ax - 3Bx + B)e^{-3x}\\
&y_p'' = (9Ax^2 - 12Ax + 9Bx + 2A - 6B)e^{-3x}\\
&(9Ax^2 - 12Ax + 9Bx + 2A - 6B)e^{-3x} - 9(Ax^2 + Bx)e^{-3x} = xe^{-3x}\\
&-12Ax + 2A - 6B = x\\
&A = -\frac{1}{12}, \quad B = -\frac{1}{36}, \quad y_p = -\frac{x}{36}(3x+1)e^{-3x}\\
&y(x) = C_1 e^{3x} + C_2 e^{-3x} - \frac{x}{36}(3x+1)e^{-3x}
\end{aligned}
$$

$y(x) = C_1 e^{3x} + C_2 e^{-3x} - \frac{x}{36}(3x+1)e^{-3x}$ ◀

문제 6 (10점)



최초 상태에서 탱크 A의 소금의 양은 150 g, 탱크 B는 50 g 이다. 탱크 A의 소금의 양 $x_1(t)$, 탱크 B의 소금의 양 $x_2(t)$를 구하라.

풀이

$$x_1(0) = 150, \quad x_2(0) = 50$$

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{x_1}{20} \times (4+2) + \dfrac{x_2}{10} \times 3 \\ \dfrac{dx_2}{dt} = \dfrac{x_1}{20} \times 4 - \dfrac{x_2}{10} \times (3+1) \end{cases} \Rightarrow \begin{cases} 10x_1' = -3x_1 + 3x_2 \\ 5x_2' = x_1 - 2x_2 \end{cases}$$

$$\begin{cases} (10D+3)x_1 = 3x_2 \\ x_1 = (5D+2)x_2 \end{cases}$$

$$\begin{cases} (10D+3)x_1 = 3x_2 \\ (10D+3)x_1 = (10D+3)(5D+2)x_2 \end{cases}$$

$$0 = (50D^2 + 35D + 3)x_2$$
$$50x_2'' + 35x_2' + 3x_2 = 0$$
$$50m^2 + 35m + 3 = 0$$
$$(10m+1)(5m+3) = 0$$
$$x_2 = C_1 e^{-\frac{1}{10}x} + C_2 e^{-\frac{3}{5}x}$$
$$x_2' = -\frac{C_1}{10} e^{-\frac{1}{10}x} - \frac{3C_2}{5} e^{-\frac{3}{5}x}$$
$$x_1 = 2\left(C_1 e^{-\frac{1}{10}x} + C_2 e^{-\frac{3}{5}x}\right) + 5\left(-\frac{C_1}{10} e^{-\frac{1}{10}x} - \frac{3C_2}{5} e^{-\frac{3}{5}x}\right)$$
$$x_1 = \frac{3}{2} C_1 e^{-\frac{1}{10}x} - C_2 e^{-\frac{3}{5}x}$$
$$x_1(0) = \frac{3}{2} C_1 - C_2 = 150$$
$$x_2(0) = C_1 + C_2 = 50$$
$$\frac{5}{2} C_1 = 200, \quad C_1 = 80, \quad C_2 = -30$$
$$\begin{cases} x_1(t) = 120e^{-\frac{1}{10}x} + 30e^{-\frac{3}{5}x} \\ x_2(t) = 80e^{-\frac{1}{10}x} - 30e^{-\frac{3}{5}x} \end{cases}$$

$$x_1(t) = 120e^{-\frac{1}{10}x} + 30e^{-\frac{3}{5}x} \quad ◀$$

$$x_2(t) = 80e^{-\frac{1}{10}x} - 30e^{-\frac{3}{5}x} \quad ◀$$

문제 7 (10점)

다음 연계 미분방정식의 해를 구하라. $\begin{cases} x' = y + t^2 \\ y' = z + t \\ z' = x \end{cases}$

풀이

$$\begin{cases} Dx = y + t^2 \\ Dy = z + t \\ Dz = x \end{cases} \quad \Rightarrow \quad \begin{cases} D^3x = D^2y + D^2t^2 \\ D^2y = Dz + Dt \\ Dz = x \end{cases} \quad \Rightarrow \quad D^3x = x + Dt + D^2t^2$$

$$x''' - x = \frac{d}{dt}(t) + \frac{d^2}{dt^2}(t^2)$$

$$x''' - x = 3$$

$$m^3 - 1 = 0$$

$$(m-1)(m^2 + m + 1) = 0$$

$$(m-1)\left\{\left(m - \frac{1}{2}\right)^2 + \frac{3}{4}\right\} = 0$$

$$m_1 = 1, \quad m_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad m_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$x(t) = C_1e^x + e^{-\frac{1}{2}}\left(C_2 \cos\frac{\sqrt{3}}{2}t + C_3 \sin\frac{\sqrt{3}}{2}t\right)$$

$$y = x' - t^2$$

$$y(t) = C_1e^t + e^{-\frac{1}{2}t}\left\{\left(-\frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3\right)\cos\frac{\sqrt{3}}{2}t + \left(-\frac{\sqrt{3}}{2}C_2 - \frac{1}{2}C_3\right)\sin\frac{\sqrt{3}}{2}t\right\} - t^2$$

$$z = y' - t$$

$$z(t) = C_1e^t + e^{-\frac{1}{2}t}\left\{\left(-\frac{1}{2}C_2 - \frac{\sqrt{3}}{2}C_3\right)\cos\frac{\sqrt{3}}{2}t + \left(-\frac{3+\sqrt{3}}{4}C_3 + \frac{1+\sqrt{3}}{4}C_2\right)\sin\frac{\sqrt{3}}{2}t\right\} - 3t$$

$$x(t) = C_1e^t + e^{-\frac{1}{2}t}\left(C_2 \cos\tfrac{\sqrt{3}}{2}t + C_3 \sin\tfrac{\sqrt{3}}{2}t\right) - 3 \quad ◀$$

$$y(t) = C_1e^t + e^{-\frac{1}{2}t}\left\{\left(-\tfrac{1}{2}C_2 + \tfrac{\sqrt{3}}{2}C_3\right)\cos\tfrac{\sqrt{3}}{2}t + \left(-\tfrac{\sqrt{3}}{2}C_2 - \tfrac{1}{2}C_3\right)\sin\tfrac{\sqrt{3}}{2}t\right\} - t^2 \quad ◀$$

$$z(t) = C_1e^t + e^{-\frac{1}{2}t}\left\{\left(-\tfrac{1}{2}C_2 - \tfrac{\sqrt{3}}{2}C_3\right)\cos\tfrac{\sqrt{3}}{2}t + \left(-\tfrac{3+\sqrt{3}}{4}C_3 + \tfrac{1+\sqrt{3}}{4}C_2\right)\sin\tfrac{\sqrt{3}}{2}t\right\} - 3t \quad ◀$$