해설

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1-(a). 다음 행렬의 행렬식을 계산하시오. $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -4 & 6 \\ 3 & -2 & 0 \end{pmatrix}$

$$|\mathbf{A}| = 3 \begin{vmatrix} -1 & 2 \\ -4 & 6 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 3 \cdot 2 + 2 \cdot 2 = 10$$

1-(b). 딸림 행렬 $\operatorname{adj} \mathbf{A}$를 구하시오.

$$\operatorname{adj} \mathbf{A} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \begin{pmatrix} 12 & 18 & 8 \\ -4 & -6 & -1 \\ 2 & -2 & -2 \end{pmatrix}^T = \begin{pmatrix} 12 & -4 & 2 \\ 18 & -6 & -2 \\ 8 & -1 & -2 \end{pmatrix}$$

1-(c). 다음 연립방정식을 푸시오. $\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ 2x_1 - 4x_2 + 6x_3 = -3 \\ 3x_1 - 2x_2 = 7 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & -4 & 6 & -3 \\ 3 & -2 & 0 & 7 \end{array}\right) \begin{array}{c} -2R_1 + R_2 \\ -3R_1 + R_3 \\ \Longrightarrow \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -2 & 2 & -5 \\ 0 & 1 & -6 & 4 \end{array}\right) \begin{array}{c} -\frac{1}{2}R_2 \\ -R_2 + R_3 \\ \Longrightarrow \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & -5 & \frac{3}{2} \end{array}\right)$$

$$\Rightarrow \quad \begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ x_2 - x_3 &= \tfrac{5}{2} \\ -5x_3 &= \tfrac{3}{2} \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3.8 \\ 2.2 \\ -0.3 \end{pmatrix}$$

2-(a). 다음 행렬의 고유값을 모두 구하시오. $\mathbf{A} = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & -2 & -2 \\ -2 & 1-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)\{(1-\lambda)^2 - 4\} + 2(2\lambda - 2 - 4) - 2(4 - 2\lambda + 2)$$

$$= -(\lambda-1)(\lambda-3)(\lambda+1) + 8(\lambda-3) = -(\lambda-3)(\lambda^2 - 1 - 8) = -(\lambda-3)^2(\lambda+3) = 0$$

$\lambda_1 = -3, \quad \lambda_2 = 3$

2-(b). **A**의 고유벡터들을 이용해서 직교행렬을 만드시오.

for $\lambda_1 = -3$,

$$(\mathbf{A} + 3\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array}\right) \overset{\substack{\frac{1}{2}R_1 + R_2 \\ \frac{1}{2}R_1 + R_3}}{\Longrightarrow} \left(\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right) \overset{\substack{R_2 + R_3 \\ \frac{1}{3}R_2,\ \frac{1}{2}R_1}}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} 2k_1 - k_2 - k_3 &= 0 \\ k_2 - k_3 &= 0 \end{aligned} \Rightarrow k_1 = k_2 = k_3 \Rightarrow \mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for $\lambda_2 = 3$,

$$(\mathbf{A} - 3\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array}\right) \overset{\substack{-R_1 + R_2 \\ -R_1 + R_3 \\ -\frac{1}{2}R_1}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow k_1 + k_2 + k_3 = 0$$

$$\Rightarrow \begin{aligned} k_1 &= t \\ k_2 &= u \\ k_3 &= -t - u \end{aligned} \Rightarrow \mathbf{K} = \begin{pmatrix} t \\ u \\ -t-u \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\Rightarrow \mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_* = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\mathbf{K}_*$와 $\mathbf{K}_2$가 직교하지 않음.

$$\mathbf{K}_3' = \mathbf{K}_* - \frac{\mathbf{K}_*^T\mathbf{K}_2}{|\mathbf{K}_*||\mathbf{K}_2|}\mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}\cdot\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}, \quad \mathbf{K}_3 = -2\mathbf{K}_3' = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{K}_1}{|\mathbf{K}_1|} & \frac{\mathbf{K}_2}{|\mathbf{K}_2|} & \frac{\mathbf{K}_3}{|\mathbf{K}_3|} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \mathbf{P}^{-1} = \mathbf{P}^T$$

2-(c). 대각행렬 **D**에 대해 $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$를 만족시키는 **P**, **D**, $\mathbf{P}^{-1}$을 쓰시오.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

3. $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$에 대해 $\mathbf{A}^{20}$을 구하시오.

풀이 1 — 대각화를 이용하여

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda-2)(\lambda-1) = 0 \quad \Rightarrow \quad \begin{array}{c} \lambda_1 = 1 \\ \lambda_2 = 2 \end{array} \quad \Rightarrow \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(\mathbf{A} - \mathbf{I}|\mathbf{0}) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad k_1 + k_2 = 0 \quad \Rightarrow \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(\mathbf{A} - 2\mathbf{I}|\mathbf{0}) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right) \quad \Rightarrow \quad k_2 = 0 \quad \Rightarrow \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

$$\mathbf{A}^{20} = \mathbf{PD}^{20}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{20} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2^{20} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{20} & 2^{20}-1 \\ 0 & 1 \end{pmatrix}$$

풀이 2 — Cayley-Hamilton 정리를 이용하여

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0 \quad \Rightarrow \quad \begin{array}{c} \lambda_1 = 1 \\ \lambda_2 = 2 \end{array}$$

$$\lambda^m = c_1\lambda + c_0\lambda$$

$$1^{20} = c_1 + c_0$$

$$2^{20} = 2c_1 + c_0$$

$$c_1 = 2^{20} - 1, \quad c_0 = -2^{20} + 2$$

$$\mathbf{A}^{20} = c_1\mathbf{A} + c_0\mathbf{I} = (2^{20}-1)\mathbf{A} - (2^{20}-2)\mathbf{I}$$

$$= \begin{pmatrix} 2^{21}-2 & 2^{20}-1 \\ 0 & 2^{20}-1 \end{pmatrix} - \begin{pmatrix} 2^{20}-2 & 0 \\ 0 & 2^{20}-2 \end{pmatrix} = \begin{pmatrix} 2^{20} & 2^{20}-1 \\ 0 & 1 \end{pmatrix}$$

4. 다음 행렬을 단위위삼각행렬을 포함하도록 LU 인수분해하라. $\mathbf{A} = \begin{pmatrix} -2 & 0 & 3 \\ 6 & 1 & -4 \\ -4 & 4 & 30 \end{pmatrix}$

$$\begin{pmatrix} -2 & 0 & 3 \\ 6 & 1 & -4 \\ -4 & 4 & 30 \end{pmatrix} \overset{\substack{3R_1 + R_2 \\ -2R_1 + R_3}}{\Longrightarrow} \begin{pmatrix} -2 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 4 & 24 \end{pmatrix} \overset{-4R_2 + R_3}{\Longrightarrow} \begin{pmatrix} -2 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{pmatrix} = \mathbf{U}'$$

$$\mathbf{L}' = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{L}'\mathbf{U}' = \mathbf{L}'\mathbf{D}\mathbf{D}^{-1}\mathbf{U}' = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -2 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\mathbf{L}'\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 \\ 6 & 1 & 0 \\ -4 & 4 & 4 \end{pmatrix} = \mathbf{L}$$

$$\mathbf{D}^{-1}\mathbf{U}' = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{U}$$

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 0 \\ 6 & 1 & 0 \\ -4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

5. $t=\pi/2$에서 $\mathbf{r}(t)=2\cos t\mathbf{i}+2\sin t\mathbf{j}+3t\mathbf{k}$의 단위접선벡터, 단위법선벡터, 단위종법선벡터, 접촉평면, 법평면, 전직평면의 방정식을 구하시오.

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{-2\sin t\mathbf{i}+2\cos t\mathbf{j}+3\mathbf{k}}{\sqrt{13}},\quad \mathbf{T}\left(\frac{\pi}{2}\right)=\frac{-2\mathbf{i}+3\mathbf{k}}{\sqrt{13}}\quad \blacktriangleleft\ \text{단위접선벡터}$$

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\frac{\frac{1}{\sqrt{13}}(-2\cos t\mathbf{i}-2\sin t\mathbf{j})}{\frac{1}{\sqrt{13}}\cdot 2}=-\cos t\mathbf{i}-\sin t\mathbf{j},\quad \mathbf{N}\left(\frac{\pi}{2}\right)=-\mathbf{j}\quad \blacktriangleleft\ \text{단위법선벡터}$$

$$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\frac{1}{\sqrt{13}}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&0&3\\0&-1&0\end{vmatrix}=\frac{3\mathbf{i}+2\mathbf{k}}{\sqrt{13}}\quad \blacktriangleleft\ \text{단위종법선벡터}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right)=2\mathbf{j}+\frac{3}{2}\pi\mathbf{k}=<0,2,\frac{3}{2}\pi>$$

$$\left(<x,y,z>-<0,2,\frac{3}{2}\pi>\right)\cdot\mathbf{B}=0\quad\Rightarrow\quad\frac{1}{\sqrt{13}}\left\{3x+2\left(z-\frac{3}{2}\pi\right)\right\}=0\quad\Rightarrow\quad 3x+2z=3\pi$$

▲ 접촉평면

$$\left(<x,y,z>-<0,2,\frac{3}{2}\pi>\right)\cdot\mathbf{T}=0\quad\Rightarrow\quad\frac{1}{\sqrt{13}}\left\{-2x+3\left(z-\frac{3}{2}\pi\right)\right\}=0\quad\Rightarrow\quad -4x+6z=9\pi$$

▲ 법평면

$$\left(<x,y,z>-<0,2,\frac{3}{2}\pi>\right)\cdot\mathbf{N}=0\quad\Rightarrow\quad -(y-2)=0\quad\Rightarrow\quad y=2\quad \blacktriangleleft\ \text{전직평면}$$

6. 점 (0,2,1)에서 함수 $f(x,y,z) = 2xy + z^2e^y$의 $\mathbf{v} = 1\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$방향으로의 방향도함수를 구하시오.

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$
$$\nabla f(x,y,z) = 2y\mathbf{i} + (2x + z^2e^y)\mathbf{j} + 2ze^y\mathbf{k}$$
$$\nabla f(0,2,1) = 4\mathbf{i} + e^2\mathbf{j} + 2e^2\mathbf{k}$$
$$D_{\mathbf{u}}f(0,2,1) = \mathbf{u} \cdot \nabla f(0,2,1) = \frac{1}{3}(4 + 2e^2 - 4e^2) = \frac{2}{3}(2 - e^2)$$