

해설

고급공학수학1(가)(이동령 교수님) 중간고사

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평균점수 : 16.93/60

오류 제보 : eunsoohong03@soongsil.ac.kr

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문제 1 (6점)

물체의 단위시간당 온도 변화량은 주변 공기의 온도와 물체의 온도 T 의 차이에 정비례한다. 주변 공기의 온도가 300 K 이고, 물체가 15분만에 370 K에서 340 K 까지 식을 때 $T(t)$ 를 구하라.

풀이

$$\frac{dT}{dt} = A(T(t) - 300), \quad T(0) = 370, \quad T(15) = 340$$

$$\frac{1}{T - 300} dT = A dt$$

$$\ln |T - 300| = At + C_1$$

$$Ce^{At} = T - 300$$

$$T(t) = Ce^{At} + 300$$

$$C = 370 - 300 = 70$$

$$70e^{15A} = 340 - 300$$

$$e^{15A} = \frac{4}{7}$$

$$15A = \ln \frac{4}{7}, \quad A = \frac{1}{15} \ln \frac{4}{7}$$

$$T(t) = 70e^{\frac{1}{15} \ln \frac{4}{7} t} + 300$$

$$T(t) = 70 \left(\frac{4}{7} \right)^{\frac{t}{15}} + 300$$

$$T(t) = 70 \left(\frac{4}{7} \right)^{\frac{t}{15}} + 300 \quad \blacktriangleleft$$

문제 2 (10점)

(a) 다음 미분방정식의 양함수 해를 구하라. $y' + \frac{y}{x} - \sqrt{y} = 0$ (7점)

(b) 초기조건 $y(1) = 0$ 을 만족하는 초기값 문제의 해를 구하라. (3점)

풀이

$$\begin{aligned}
 u &= \sqrt{y} \\
 y' &= (u^2)' = 2uu' \\
 2uu' + \frac{u^2}{x} &= u \\
 2xu' + u &= x \\
 (u - x)dx + 2xdu &= 0 \\
 \text{let. } t &= \frac{u}{x}, \quad du = tdx + xdt \\
 x(t - 1)dx + 2x(tdx + xdt) &= 0 \\
 (t - 1)dx + 2(tdx + xdt) &= 0 \\
 (3t - 1)dx + 2xdt &= 0 \\
 -\frac{1}{2x}dx &= \frac{1}{3t - 1}dt \\
 -\frac{1}{2} \ln |x| + C_1 &= \ln |3t - 1| \\
 3t - 1 &= C_2 x^{-\frac{1}{2}} \\
 t &= Cx^{-\frac{1}{2}} + \frac{1}{3} \\
 u = tx &= Cx^{\frac{1}{2}} + \frac{1}{3}x \\
 y = u^2 &= x \left(C + \frac{\sqrt{x}}{3} \right)^2
 \end{aligned}$$

$$(a) \quad y(x) = x \left(C + \frac{\sqrt{x}}{3} \right)^2 \quad \blacktriangleleft$$

$$\begin{aligned}
 y(1) &= \left(C + \frac{1}{3} \right)^2 = 0 \\
 C &= -\frac{1}{3} \\
 y(x) &= \frac{x}{9} (1 - \sqrt{x})^2
 \end{aligned}$$

$$(b) \quad y(x) = \frac{x}{9} (1 - \sqrt{x})^2 \quad \blacktriangleleft$$

문제 3 (7점)

다음 미분방정식의 양함수 해를 구하라. $(2x \sin \frac{y}{x} + 3y \cos \frac{y}{x}) dx - 3x \cos \frac{y}{x} dy = 0$

풀이

$$u = \frac{y}{x}, \quad dy = udx + xdu$$

$$(2x \sin u + 3ux \cos u)dx - 3x \cos u(udx + xdu) = 0$$

$$(2 \sin u + 3u \cos u)dx - 3 \cos u(udx + xdu) = 0$$

$$2 \sin u dx = 3x \cos u du$$

$$\frac{2}{3x} dx = \frac{\cos u}{\sin u} du$$

$$\frac{2}{3} \ln |x| + C_1 = \ln |\sin u|$$

$$\sin u = Cx^{\frac{2}{3}}$$

$$u = \arcsin \left(Cx^{\frac{2}{3}} \right)$$

$$y = ux = x \arcsin \left(Cx^{\frac{2}{3}} \right)$$

$$y(x) = x \arcsin(Cx^{\frac{2}{3}}) \quad \blacktriangleleft$$

문제 4 (10점)

다음 초기값 문제의 해를 구하라.

$$y'' + y = f(t), \quad f(t) = \begin{cases} \sin t, & 0 < t < \frac{\pi}{2} \\ \cos t, & t \geq \frac{\pi}{2} \end{cases}, \quad y(0) = 0, \quad y'(0) = 1$$

풀이

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$y_c = C_1 \cos t + C_2 \sin t$$

$0 < t < \frac{\pi}{2}$ 에서

$$y_{p1} = A_1 t \cos t + B_1 t \sin t$$

$$y'_{p1} = (B_1 t + A_1) \cos t + (-A_1 t + B_1) \sin t$$

$$y''_{p1} = (-A_1 t + 2B_1) \cos t + (-B_1 t - 2A_1) \sin t$$

$$y''_{p1} + y_{p1} = \sin t$$

$$(-A_1 t + 2B_1) \cos t + (-B_1 t - 2A_1) \sin t + A_1 t \cos t + B_1 t \sin t = \sin t$$

$$2B_1 \cos t - 2A_1 \sin t = \sin t$$

$$A_1 = -\frac{1}{2}, \quad B_1 = 0, \quad y_{p1} = -\frac{1}{2}t \cos t$$

$t \geq \frac{\pi}{2}$ 에서

$$y_{p2} = A_2 t \cos t + B_2 t \sin t$$

$$\vdots$$

$$2B_2 \cos t - 2A_2 \sin t = \cos t$$

$$A_2 = 0, \quad B_2 = \frac{1}{2}, \quad y_{p2} = \frac{1}{2}t \sin t$$

따라서

$$y(x) = \begin{cases} C_1 \cos t + C_2 \sin t - \frac{1}{2}t \cos t & \left(0 < t < \frac{\pi}{2}\right) \\ C_3 \cos t + C_4 \sin t + \frac{1}{2}t \sin t & \left(t \geq \frac{\pi}{2}\right) \end{cases}$$

함수 $y(x)$ 가 정의구간에서 미분가능해야 하므로

$$\lim_{k \rightarrow \pi/2} y(k) = \cancel{C_1 \cos \frac{\pi}{2}} + C_2 \sin \frac{\pi}{2} - \cancel{\frac{1}{2} \cdot \frac{\pi}{2} \cos \frac{\pi}{2}} = C_2$$

$$y\left(\frac{\pi}{2}\right) = \cancel{C_3 \cos \frac{\pi}{2}} + C_4 \sin \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} \sin \frac{\pi}{2} = C_4 + \frac{\pi}{4}$$

$$\lim_{k \rightarrow \pi/2} y(k) = y\left(\frac{\pi}{2}\right) \Rightarrow C_2 = C_4 + \frac{\pi}{4} \Rightarrow C_4 = C_2 - \frac{\pi}{4}$$

$$y'(x) = \begin{cases} C_2 \cos t - C_1 \sin t - \frac{1}{2} \cos t + \frac{1}{2} t \sin t & \left(0 < t < \frac{\pi}{2}\right) \\ C_4 \cos t - C_3 \sin t + \frac{1}{2} \sin t + \frac{1}{2} t \cos t & \left(t \geq \frac{\pi}{2}\right) \end{cases}$$

$$\lim_{k \rightarrow \pi/2} y'(k) = \cancel{C_2 \cos \frac{\pi}{2}} - C_1 \sin \frac{\pi}{2} - \cancel{\frac{1}{2} \cos \frac{\pi}{2}} + \frac{1}{2} \cdot \frac{\pi}{2} \sin \frac{\pi}{2} = -C_1 + \frac{\pi}{4}$$

$$y'\left(\frac{\pi}{2}\right) = \cancel{C_4 \cos \frac{\pi}{2}} - C_3 \sin \frac{\pi}{2} + \frac{1}{2} \sin \frac{\pi}{2} + \cancel{\frac{1}{2} \cdot \frac{\pi}{2} \cos \frac{\pi}{2}} = -C_3 + \frac{1}{2}$$

$$\lim_{k \rightarrow \pi/2} y'(k) = y'\left(\frac{\pi}{2}\right) \Rightarrow -C_1 + \frac{\pi}{4} = -C_3 + \frac{1}{2} \Rightarrow C_3 = C_1 + \frac{1}{2} - \frac{\pi}{4}$$

$$y(x) = \begin{cases} C_1 \cos t + C_2 \sin t - \frac{1}{2} t \cos t & \left(0 < t < \frac{\pi}{2}\right) \\ \left(C_1 + \frac{1}{2} - \frac{\pi}{4}\right) \cos t + \left(C_2 - \frac{\pi}{4}\right) \sin t + \frac{1}{2} t \sin t & \left(t \geq \frac{\pi}{2}\right) \end{cases}$$

초기값 문제를 풀면 다음과 같다.

$$y(0) = C_1 \cos 0 + C_2 \sin 0 - \cancel{\frac{1}{2} \cdot 0 \cdot \cos 0} = C_1 = 0 \Rightarrow C_3 = \frac{1}{2} - \frac{\pi}{4}$$

$$y'(0) = C_2 \cos 0 - \cancel{C_1 \sin 0} - \frac{1}{2} \cos 0 + \cancel{\frac{1}{2} \cdot 0 \cdot \sin 0} = C_2 - \frac{1}{2} = 1 \Rightarrow C_2 = \frac{3}{2}, \quad C_4 = \frac{3}{2} - \frac{\pi}{4}$$

$$y(x) = \begin{cases} \frac{3}{2} \sin t - \frac{1}{2} t \cos t & \left(0 < t < \frac{\pi}{2}\right) \\ \left(\frac{1}{2} - \frac{\pi}{4}\right) \cos t + \left(\frac{3}{2} - \frac{\pi}{4}\right) \sin t + \frac{1}{2} t \sin t & \left(t \geq \frac{\pi}{2}\right) \end{cases}$$

$$y(x) = \begin{cases} \frac{3}{2} \sin t - \frac{1}{2} t \cos t & \left(0 < t < \frac{\pi}{2}\right) \\ \left(\frac{1}{2} - \frac{\pi}{4}\right) \cos t + \left(\frac{3}{2} - \frac{\pi}{4}\right) \sin t + \frac{1}{2} t \sin t & \left(t \geq \frac{\pi}{2}\right) \end{cases} \quad \blacktriangleleft$$

문제 5 (7점)

다음 미분방정식의 해를 구하라. $y'' - 9y = \frac{x}{e^{3x}}$

풀이

$$m^2 - 9 = 0$$

$$(m - 3)(m + 3) = 0$$

$$y_c = C_1 e^{3x} + C_2 e^{-3x}$$

$$y_p = (Ax^2 + Bx)e^{-3x}$$

$$y'_p = (-3Ax^2 + 2Ax - 3Bx + B)e^{-3x}$$

$$y''_p = (9Ax^2 - 12Ax + 9Bx + 2A - 6B)e^{-3x}$$

$$(9Ax^2 - 12Ax + 9Bx + 2A - 6B)e^{-3x} - 9(Ax^2 + Bx)e^{-3x} = xe^{-3x}$$

$$-12Ax + 2A - 6B = x$$

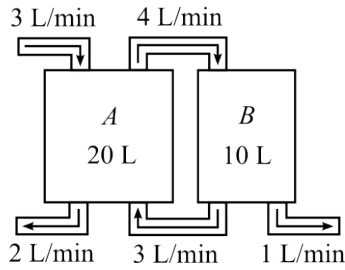
$$A = -\frac{1}{12}, \quad B = -\frac{1}{36}, \quad y_p = -\frac{x}{36}(3x + 1)e^{-3x}$$

$$y(x) = C_1 e^{3x} + C_2 e^{-3x} - \frac{x}{36}(3x + 1)e^{-3x}$$

$$y(x) = C_1 e^{3x} + C_2 e^{-3x} - \frac{x}{36}(3x + 1)e^{-3x} \quad \blacktriangleleft$$

문제 6 (10점)

(pure water)



최초 상태에서 탱크 A의 소금의 양은 150 g, 탱크 B는 50 g 이다. 탱크 A의 소금의 양 $x_1(t)$, 탱크 B의 소금의 양 $x_2(t)$ 를 구하라.

풀이

$$x_1(0) = 150, \quad x_2(0) = 50$$

$$\begin{cases} \frac{dx_1}{dt} = -\frac{x_1}{20} \times (4+2) + \frac{x_2}{10} \times 3 \\ \frac{dx_2}{dt} = \frac{x_1}{20} \times 4 - \frac{x_2}{10} \times (3+1) \end{cases} \Rightarrow \begin{cases} 10x_1' = -3x_1 + 3x_2 \\ 5x_2' = x_1 - 2x_2 \end{cases}$$

$$\begin{cases} (10D+3)x_1 = 3x_2 \\ x_1 = (5D+2)x_2 \end{cases}$$

$$\begin{cases} (10D+3)x_1 = 3x_2 \\ (10D+3)x_1 = (10D+3)(5D+2)x_2 \end{cases}$$

$$0 = (50D^2 + 35D + 3)x_2$$

$$50x_2'' + 35x_2' + 3x_2 = 0$$

$$50m^2 + 35m + 3 = 0$$

$$(10m+1)(5m+3) = 0$$

$$x_2 = C_1 e^{-\frac{1}{10}x} + C_2 e^{-\frac{3}{5}x}$$

$$x_2' = -\frac{C_1}{10} e^{-\frac{1}{10}x} - \frac{3C_2}{5} e^{-\frac{3}{5}x}$$

$$x_1 = 2 \left(C_1 e^{-\frac{1}{10}x} + C_2 e^{-\frac{3}{5}x} \right) + 5 \left(-\frac{C_1}{10} e^{-\frac{1}{10}x} - \frac{3C_2}{5} e^{-\frac{3}{5}x} \right)$$

$$x_1 = \frac{3}{2} C_1 e^{-\frac{1}{10}x} - C_2 e^{-\frac{3}{5}x}$$

$$x_1(0) = \frac{3}{2} C_1 - C_2 = 150$$

$$x_2(0) = C_1 + C_2 = 50$$

$$\frac{5}{2} C_1 = 200, \quad C_1 = 80, \quad C_2 = -30$$

$$\begin{cases} x_1(t) = 120e^{-\frac{1}{10}t} + 30e^{-\frac{3}{5}t} \\ x_2(t) = 80e^{-\frac{1}{10}t} - 30e^{-\frac{3}{5}t} \end{cases}$$

$$x_1(t) = 120e^{-\frac{1}{10}t} + 30e^{-\frac{3}{5}t} \quad \blacktriangleleft$$

$$x_2(t) = 80e^{-\frac{1}{10}t} - 30e^{-\frac{3}{5}t} \quad \blacktriangleleft$$

문제 7 (10점)

다음 연계 미분방정식의 해를 구하라. $\begin{cases} x' = y + t^2 \\ y' = z + t \\ z' = x \end{cases}$

풀이

$$\begin{cases} Dx = y + t^2 \\ Dy = z + t \\ Dz = x \end{cases} \Rightarrow \begin{cases} D^3x = D^2y + D^2t^2 \\ D^2y = Dz + Dt \\ Dz = x \end{cases} \Rightarrow D^3x = x + Dt + D^2t^2$$

$$x''' - x = \frac{d}{dt}(t) + \frac{d^2}{dt^2}(t^2)$$

$$x''' - x = 3$$

$$m^3 - 1 = 0$$

$$(m-1)(m^2 + m + 1) = 0$$

$$(m-1) \left\{ \left(m - \frac{1}{2} \right)^2 + \frac{3}{4} \right\} = 0$$

$$m_1 = 1, \quad m_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad m_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$x(t) = C_1 e^t + e^{-\frac{1}{2}t} \left(C_2 \cos \frac{\sqrt{3}}{2}t + C_3 \sin \frac{\sqrt{3}}{2}t \right)$$

$$y = x' - t^2$$

$$y(t) = C_1 e^t + e^{-\frac{1}{2}t} \left\{ \left(-\frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 \right) \cos \frac{\sqrt{3}}{2}t + \left(-\frac{\sqrt{3}}{2}C_2 - \frac{1}{2}C_3 \right) \sin \frac{\sqrt{3}}{2}t \right\} - t^2$$

$$z = y' - t$$

$$z(t) = C_1 e^t + e^{-\frac{1}{2}t} \left\{ \left(-\frac{1}{2}C_2 - \frac{\sqrt{3}}{2}C_3 \right) \cos \frac{\sqrt{3}}{2}t + \left(-\frac{3+\sqrt{3}}{4}C_3 + \frac{1+\sqrt{3}}{4}C_2 \right) \sin \frac{\sqrt{3}}{2}t \right\} - 3t$$

$$x(t) = C_1 e^t + e^{-\frac{1}{2}t} \left(C_2 \cos \frac{\sqrt{3}}{2}t + C_3 \sin \frac{\sqrt{3}}{2}t \right) - 3 \quad \blacktriangleleft$$

$$y(t) = C_1 e^t + e^{-\frac{1}{2}t} \left\{ \left(-\frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 \right) \cos \frac{\sqrt{3}}{2}t + \left(-\frac{\sqrt{3}}{2}C_2 - \frac{1}{2}C_3 \right) \sin \frac{\sqrt{3}}{2}t \right\} - t^2 \quad \blacktriangleleft$$

$$z(t) = C_1 e^t + e^{-\frac{1}{2}t} \left\{ \left(-\frac{1}{2}C_2 - \frac{\sqrt{3}}{2}C_3 \right) \cos \frac{\sqrt{3}}{2}t + \left(-\frac{3+\sqrt{3}}{4}C_3 + \frac{1+\sqrt{3}}{4}C_2 \right) \sin \frac{\sqrt{3}}{2}t \right\} - 3t \quad \blacktriangleleft$$