Contents

1 Boolean functions and the Fourier expansions

Summary

Definition 1.1. Let n be a positive integer. A **Boolean function** on n variables is a function $f : \{0,1\}^n \to \{0,1\}$ or $f : \{\pm 1\}^n \to \{\pm 1\}$.

List of some Boolean functions

- $\min_n$, $\max_n$: minimum and maximum function
- Maj_n : majority function
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For each point $a = \{a_1, \cdots, a_n\} \in \{\pm 1\}^n$, the indicator polynomial

$$1_{\{a\}}(x) = \prod_i \left(\frac{1 + a_i x_i}{2} \right)$$

takes value 1 when $x = a$ and value 0 otherwise.

Definition 1.2. Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be a Boolean function. The **Fourier expansion** of f is the following expression:

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) x^S$$

where $\hat{f}(S)$ are real numbers called the **Fourier coefficients** of f.

For $S \subseteq [n]$, we define $\chi_S : \mathbb{F}_2^n \to \mathbb{R}$ by

$$\chi_S(x) = (-1)^{\langle x, S \rangle}$$

where $\langle x, S \rangle = \sum_{i \in S} x_i$. We call χ_S the **character** of S. Further, χ_S satisfies $\chi_S(x + y) = \chi_S(x)\chi_S(y)$.

Definition 1.3. Let $f, g : \{\pm 1\}^n \to \{\pm 1\}$ be Boolean functions. The **inner product** of f and g is defined by

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) g(x)$$

Theorem 1.4. *Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be a Boolean function. Then*

$$\hat{f}(S) = \langle f, \chi_S \rangle$$

Proposition 1.5. *Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be a Boolean function. Then*

$$\hat{f}(S) = \langle f, \chi_S \rangle = \mathbb{E}_{\boldsymbol{x}} \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) \chi_S(x)$$

Solutions

1.1(a). $min_2(x) = 1 \quad \Longleftrightarrow \quad x = (1, 1).$

A (title)**.** This is a theorem.