Contents

1 Boolean functions and the Fourier expansions

Summary

Definition 1.1. Let n be a positive integer. A **Boolean function** on n variables is a function $f : \{0,1\}^n \to \{0,1\}$ or $f : \{\pm 1\}^n \to \{\pm 1\}$.

List of some Boolean functions

- $\min_n$, $\max_n$: minimum and maximum function
- Maj_n : majority function
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For each point $a = \{a_1, \cdots, a_n\} \in \{\pm 1\}^n$, the indicator polynomial

$$1_{\{a\}}(x) = \prod_i \left(\frac{1 + a_i x_i}{2} \right)$$

takes value 1 when $x = a$ and value 0 otherwise.

Definition 1.2. Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be a Boolean function. The **Fourier expansion** of f is the following expression:

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) x^S$$

where $\hat{f}(S)$ are real numbers called the **Fourier coefficients** of f.

For $S \subseteq [n]$, we define $\chi_S : \mathbb{F}_2^n \to \mathbb{R}$ by

$$\chi_S(x) = (-1)^{\langle x, S \rangle}$$

where $\langle x, S \rangle = \sum_{i \in S} x_i$. We call χ_S the **character** of S. Further, χ_S satisfies $\chi_S(x + y) = \chi_S(x)\chi_S(y)$.

Definition 1.3. Let $f, g : \{\pm 1\}^n \to \{\pm 1\}$ be Boolean functions. The **inner product** of f and g is defined by

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) g(x)$$

Theorem 1.4. *Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be a Boolean function. Then*

$$\hat{f}(S) = \langle f, \chi_S \rangle$$

Proposition 1.5. *Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be a Boolean function. Then*

$$\hat{f}(S) = \langle f, \chi_S \rangle = \mathbb{E}_{\boldsymbol{x}} \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) \chi_S(x)$$

Solutions

1.1 (a) $\min_2(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) = (1,1) \\ -1 & \text{otherwise} \end{cases}$

which means $\min_2(x_1, x_2) = -1 + 2 \times \mathbb{1}_{\{(1,1)\}}(x)$ with $\mathbb{1}_{\{(1,1)\}}(x) = \prod_{i=1}^{2} \frac{1+x_i}{2}$.
Therefore $\min_2(x_1, x_2) = -1 + 2 \times \prod_{i=1}^{2} \frac{1+x_i}{2} = -\frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3$.

(b) Similar to (a),
$\min_3(x_1, x_2, x_3) = -1 + 2 \times \prod_{i=1}^{3} \frac{1+x_i}{2} = -\frac{1}{4} + \frac{1}{4}\sum_i x_i + \frac{1}{4}\sum_{i,j} x_i x_j + x_1 x_2 x_3$

(c) In $\{0,1\}$ setting, $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$. The inner product is

$$\begin{aligned} \langle \mathbb{1}_{\{a\}}, \chi_S \rangle &= \mathbb{E}_{x \sim \{0,1\}^n}\left[\mathbb{1}_{\{a\}}(x)\chi_S(x)\right] \\ &= \frac{1}{2^n}\chi_S(x) \\ &= \frac{1}{2^n}(-1)^{\sum_{i \in S} x_i} \end{aligned}$$

Hence, the Fourier expansion of $\mathbb{1}_{\{a\}}$ is

$$\begin{aligned} \mathbb{1}_{\{a\}}(x) &= \sum_{S \subseteq [n]} \frac{1}{2^n}\chi_S(a)\chi_S(x) \\ &= \frac{1}{2^n}\sum_{S \subseteq [n]} \chi_S(a+x) \\ &= \frac{1}{2^n}\sum_{S \subseteq [n]} (-1)^{\sum_{i \in S}(a_i + x_i)} \end{aligned}$$

(d) $\phi_{\{a\}}(x) = \mathbb{1}_{\{a\}}(x) = \frac{1}{2^n}\sum_{S \subseteq [n]}(-1)^{\sum_{i \in S}(a_i + x_i)}$

(e)

$$\begin{aligned} \phi_{\{a, a+e_i\}} &= \frac{1}{2}\left(\mathbb{1}_{\{a\}} + \mathbb{1}_{\{a+e_i\}}\right) \\ &= \frac{1}{2} \cdot \frac{1}{2^n}\sum_{S \subseteq [n]}\left((-1)^{\sum_{j \in S} a_j} + (-1)^{\sum_{j \in S} a_j + \delta_{ij}}\right)\chi_S(x) \end{aligned}$$

The term $(-1)^{\sum_{j \in S} a_j} + (-1)^{\sum_{j \in S} a_j + \delta_{ij}}$ is 0 if $i \in S$ and $2(-1)^{\sum_{j \in S} a_j}$ otherwise.
Therefore,

$$\begin{aligned} \phi_{\{a, a+e_i\}} &= \frac{1}{2} \cdot \frac{1}{2^n}\sum_{S \subseteq [n] \setminus \{i\}} (-1)^{\sum_{j \in S} a_j}\chi_S(x) \\ &= \frac{1}{2^n}\sum_{S \subseteq [n] \setminus \{i\}} (-1)^{\sum_{j \in S}(a_j + x_j)} \end{aligned}$$

(f) Let ϕ be the probability density function of each x_i, respectively. Then $f =$

$\prod_1^n \phi(x_i)$ and the fourier coefficient is

$$\begin{aligned}
\hat{f}(S) &= \langle f, \chi_S \rangle \\
&= \mathbb{E}_{x\sim\{\pm1\}^n}\left[f(x)\chi_S(x)\right] \\
&= \mathbb{E}_{x\sim\{\pm1\}^n}\left[\prod_{i=1}^{n} \phi(x_i)\chi_S(x)\right]
\end{aligned}$$

1.2 ghi

References