

《量子信息基础》2021.5.20 随堂作业:

1. (Text book* Problem 3.27)

Let \hat{Q} be an operator with a complete set of orthonormal eigenvectors:

$$\hat{Q}|e_n\rangle = q_n|e_n\rangle$$

已知题目给的这个式子，实际上就是说，Q 是一个算符

(a) Show that \hat{Q} can be written in terms of its spectral decomposition:

$$\hat{Q} = \sum_n q_n |e_n\rangle\langle e_n|$$

要证明这个式子

Hint: An operator is characterized by its action on all possible vectors, so what you must show is that

$$\hat{Q}|\alpha\rangle = \left\{ \sum_n q_n |e_n\rangle\langle e_n| \right\} |\alpha\rangle$$

for any $|\alpha\rangle$.

Define an arbitrary vector $|\alpha\rangle$

$$|\alpha\rangle = \sum_n c_n |e_n\rangle$$

$$\hat{Q}|\alpha\rangle = \sum_n c_n \hat{Q}|e_n\rangle = \sum_n \langle e_n|\alpha\rangle \cdot q_n |e_n\rangle = \left\{ \sum_n q_n |e_n\rangle\langle e_n| \right\} |\alpha\rangle$$

$$\hat{Q} = \sum_n q_n |e_n\rangle\langle e_n|$$

推导和答案正确给 30 分

(b) Another way to define a function of \hat{Q} is via the spectral decomposition:

$$f(\hat{Q}) = \sum_n f(q_n) |e_n\rangle\langle e_n|$$

Show that this is equivalent to

$$e^{\hat{Q}} \equiv 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \dots$$

when $f(\hat{Q}) = e^{\hat{Q}}$.

$$\begin{aligned}
f(\hat{Q}) = e^{\hat{Q}} &= \sum_n e^{q_n} |e_n\rangle\langle e_n| = \sum_n \left(1 + q_n + \frac{1}{2}q_n^2 + \frac{1}{3!}q_n^3 + \dots\right) |e_n\rangle\langle e_n| \\
&= \sum_n |e_n\rangle\langle e_n| + \sum_n q_n |e_n\rangle\langle e_n| + \frac{1}{2} \sum_n q_n^2 |e_n\rangle\langle e_n| + \frac{1}{3!} \sum_n q_n^3 |e_n\rangle\langle e_n| \\
&= 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \dots
\end{aligned}$$

推导和答案正确给 30 分

2. An operator \hat{Q} has the complete sets of Eigen wave functions $\{|a_n\rangle\}$ and $\{|b_n\rangle\}$ in the A and B representations respectively. Assuming $\{|a_n\rangle\}$ and $\{|b_n\rangle\}$ are connected by unitary transformation

$$|b_n\rangle = \hat{U}|a_n\rangle$$

prove that

$$\hat{Q}_{(B)} = \hat{U}\hat{Q}_{(A)}\hat{U}^\dagger$$

$$\langle b_n|\hat{Q}_{(B)}|b_n\rangle = \langle a_n|\hat{U}^\dagger\hat{Q}_{(B)}\hat{U}|a_n\rangle = \langle a_n|\hat{Q}_{(A)}|a_n\rangle$$

$$\hat{U}^\dagger\hat{Q}_{(B)}\hat{U} = \hat{Q}_{(A)}$$

$$\therefore \hat{Q}_{(B)} = \hat{U}\hat{Q}_{(A)}\hat{U}^\dagger$$

推导和答案正确给 40 分

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).