《量子信息基础》2021.4.22 随堂作业：

1. (Text book* Problem 5.4)
   (a) If ψ_a and ψ_b are orthogonal, and both are normalized, what is the constant A in Equation 5.17 ?
   (b) If $\psi_a = \psi_b$ (and it is normalized), what is A? (This case, of course, occurs only for bosons.)

(a) $\psi_{\pm}(\boldsymbol{r_1}, \boldsymbol{r_2}) = A[\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2}) \pm \psi_b(\boldsymbol{r_1})\psi_a(\boldsymbol{r_2})]$(5.17)

$$\int \left|\psi_{\pm}(\boldsymbol{r_1}, \boldsymbol{r_2})\right|^2 d\boldsymbol{r_1} d\boldsymbol{r_2}$$
$$= |\boldsymbol{A}|^2 \int [\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2}) \pm \psi_b(\boldsymbol{r_1})\psi_a(\boldsymbol{r_2})]^*[\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2}) \pm \psi_b(\boldsymbol{r_1})\psi_a(\boldsymbol{r_2})] d\boldsymbol{r_1} d\boldsymbol{r_2}$$
$$= |\boldsymbol{A}|^2 \left[\int |\psi_a(\boldsymbol{r_1})|^2 d\boldsymbol{r_1} \int |\psi_b(\boldsymbol{r_2})|^2 d\boldsymbol{r_2} \pm \int \psi_b^*(\boldsymbol{r_1})\psi_a(\boldsymbol{r_1}) d\boldsymbol{r_1} \int \psi_a^*(\boldsymbol{r_2})\psi_b(\boldsymbol{r_2}) d\boldsymbol{r_2}\right.$$
$$\left.\pm \int \psi_a^*(\boldsymbol{r_1})\psi_b(\boldsymbol{r_1}) d\boldsymbol{r_1} \int \psi_b^*(\boldsymbol{r_2})\psi_a(\boldsymbol{r_2}) d\boldsymbol{r_2} + \int |\psi_b(\boldsymbol{r_1})|^2 d\boldsymbol{r_1} \int |\psi_a(\boldsymbol{r_2})|^2 d\boldsymbol{r_2}\right]$$
$$= |\boldsymbol{A}|^2(1 \pm 0 \pm 0 + 1) = 2|A|^2 = 1$$

$$\mathrm{A} = \frac{1}{\sqrt{2}}$$

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(b) If $\psi_a = \psi_b$

$$\psi_+(\boldsymbol{r_1}, \boldsymbol{r_2}) = 2A\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2})$$
$$\int |\psi_+(\boldsymbol{r_1}, \boldsymbol{r_2})|^2 d\boldsymbol{r_1} d\boldsymbol{r_2} = |\boldsymbol{A}|^2 \int [2\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2})]^*[2\psi_a(\boldsymbol{r_1})\psi_b(\boldsymbol{r_2})] d\boldsymbol{r_1} d\boldsymbol{r_2}$$
$$= 4|\boldsymbol{A}|^2 \int |\psi_a(\boldsymbol{r_1})|^2 d\boldsymbol{r_1} \int |\psi_b(\boldsymbol{r_2})|^2 d\boldsymbol{r_2} = 4|\boldsymbol{A}|^2 = \mathbf{1}$$
$$\mathrm{A} = \frac{1}{2}$$

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2. (Text book* Problem 5.6)
   Imagine two non-interacting particles, each of mass *m*, in the infinite square well. If one is in the state ψ_n ($\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$), and the other in state $\psi_l (l \neq n)$, calculate $\langle(x_1 - x_2)^2\rangle$, assuming (a) they are distinguishable particles, (b) they are identical bosons, and (c) they are identical fermions.

The wave functions are

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

两个不相互作用的粒子，质量为 m，告知了状态，分三种情况求 x1-x2 平方的均值

$$\psi_l(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{l\pi}{a}x\right)$$

(a)

If the two particles are distinguishable

$$\langle (x_1 - x_2)^2\rangle = \langle x^2\rangle_n + \langle x^2\rangle_l - 2\langle x\rangle_n\langle x\rangle_l$$

$$\langle x^2\rangle_n = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{2}{a}\int_0^a x^2\frac{1-\cos(2n\pi x/a)}{2}dx$$

$$= \frac{1}{a}\int_0^a x^2 dx - \frac{1}{a}\int_0^a x^2\cos\left(\frac{2n\pi}{a}x\right)dx = \frac{a^2}{3} - \frac{a^2}{2(n\pi)^2}$$

$$\langle x^2\rangle_l = \frac{a^2}{3} - \frac{a^2}{2(l\pi)^2}$$

$$\langle x\rangle_n = \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{1}{a}\int_0^a x\left(1-\cos\left(\frac{2n\pi}{a}x\right)\right)dx = \frac{a}{2}$$

$$\langle x\rangle_l = \frac{a}{2}$$

$$\therefore\ \langle (x_1 - x_2)^2\rangle = \langle x^2\rangle_n + \langle x^2\rangle_l - 2\langle x\rangle_n\langle x\rangle_l = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{l^2}\right)\right]$$

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(b)

If the two particles are indistinguishable bosons

$$\langle (x_1 - x_2)^2\rangle = \langle x^2\rangle_n + \langle x^2\rangle_l - 2\langle x\rangle_n\langle x\rangle_l - 2|\langle x\rangle_{nl}|^2$$

$$\langle x\rangle_{nl} = \frac{2}{a}\int_0^a x\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{l\pi}{a}x\right)dx$$

$$= \frac{1}{a}\int_0^a x\left[\cos\left(\frac{(n-l)\pi}{a}x\right) - \cos\left(\frac{(n+l)\pi}{a}x\right)\right]dx$$

$$= \frac{1}{a}\left[\frac{a}{(n-l)\pi}\right]^2\cos\left(\frac{(n-l)\pi}{a}x\right) + \frac{ax}{(n-l)\pi}\sin\left(\frac{(n-l)\pi}{a}x\right)$$

$$- \left[\frac{a}{(n+l)\pi}\right]^2\cos\left(\frac{(n+l)\pi}{a}x\right) - \frac{ax}{(n+l)\pi}\sin\left(\frac{(n+l)\pi}{a}x\right)\Bigg|_0^a$$

$$= \frac{1}{a}\left\{\left[\frac{a}{(n-l)\pi}\right]^2(\cos[(n-l)\pi]-1)\right.$$

$$\left. - \left[\frac{a}{(n+l)\pi}\right]^2(\cos[(n+l)\pi]-1)\right\}$$

$$= \frac{a}{\pi^2}[(-1)^{n+l}-1]\left[\frac{1}{(n-l)^2} - \frac{1}{(n+l)^2}\right]$$

$$= \begin{cases} \dfrac{-8nla}{\pi^2(n^2-l^2)^2} & when\ n+l = 2m+1,\ \ m\ is\ an\ integer \\ 0 & when\ n+l = 2m \end{cases}$$

$when\ n+l = 2m$

$$\langle (x_1 - x_2)^2\rangle = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{l^2}\right)\right]$$

$when\, n + l = 2m + 1$

$$\langle (x_1 - x_2)^2 \rangle = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right] - \frac{128 n^2 l^2 a^2}{\pi^4 (n^2 - l^2)^4}$$

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(c)

If the two particles are indistinguishable fermions

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_n + \langle x^2 \rangle_l - 2\langle x \rangle_n \langle x \rangle_l + 2|\langle x \rangle_{nl}|^2$$

$when\, n + l = 2m$

$$\langle (x_1 - x_2)^2 \rangle = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right]$$

$when\, n + l = 2m + 1$

$$\langle (x_1 - x_2)^2 \rangle = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right] + \frac{128 n^2 l^2 a^2}{\pi^4 (n^2 - l^2)^4}$$

理解，并且会算这四个积分就算成功

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* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).