《量子信息基础》2022.4.28 随堂作业：

1. Prove that the binary entropy $S_{bin}(p)$ attains its maximum value of one at $p = 1/2$.

$$S_{bin}(p) \equiv -p\log p - (1-p)\log(1-p)$$

$$\frac{d\,S_{bin}(p)}{dp} = -\log p - \frac{1}{\ln 2} + \log(1-p) + \frac{1}{\ln 2} = -\log p + \log(1-p)$$

$$= \log\left(\frac{1-p}{p}\right) = 0$$

$$\therefore\ p = \frac{1}{2}$$

推导和答案正确给 30 分

2. Calculate the Von Neumann entropy $S(\rho)$ for the following density matrix:

(1) $\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(2) $\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(3) $\rho = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

关键是把 sp 这个公式记住

（1） The eigenvalues of matrix ρ are

$\lambda_1 = 1; \lambda_2 = 0$

$$S(\rho) = -\sum_x \lambda_x \log \lambda_x = -1\log 1 - 0\log 0 = 0$$

推导和答案正确给 15 分

（2） The eigenvalues of matrix ρ satisfy that

$$\left(\frac{1}{2} - \lambda\right)\left(\frac{1}{2} - \lambda\right) - \frac{1}{4} = 0$$

$\lambda_1 = 1; \lambda_1 = 0$

$$S(\rho) = -\sum_x \lambda_x \log \lambda_x = -1\log 1 - 0\log 0 = 0$$

推导和答案正确给 15 分

（3） The eigenvalues of matrix ρ satisfy that

$$\left(\frac{2}{3} - \lambda\right)\left(\frac{1}{3} - \lambda\right) - \frac{1}{9} = 0$$

$\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{6}; \lambda_1 = \frac{1}{2} - \frac{\sqrt{5}}{6}$

$$S(\rho) = -\sum_x \lambda_x \log \lambda_x = -\left(\frac{1}{2} + \frac{\sqrt{5}}{6}\right) \log\left(\frac{1}{2} + \frac{\sqrt{5}}{6}\right) - \left(\frac{1}{2} - \frac{\sqrt{5}}{6}\right) \log\left(\frac{1}{2} - \frac{\sqrt{5}}{6}\right)$$
$$\cong 0.55$$

换底公式，算 log2，logab=logcb/logca

推导正确给 15 分（最后答案没有给出具体数字也算正确）