Linear Algebra

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Outline

- Introduction
- Vectors
- Matrices
- Linear Equations
- Solving System of Linear Equations
- Matrix Multiplication
- Inverse
- **Determinant**

Determinant

- The determinant of a **square matrix** is a **scalar** that provides information about the matrix.
  - **Invertibility** of the matrix.
- Learning Target
  - The determinants for 2×2 and 3×3 matrices
  - The properties of Determinants
  - The formula of Determinants
  - Cramer's Rule

Determinant

- 2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad$$
$$-bc$$

- 3×3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

$$det(A) =$$

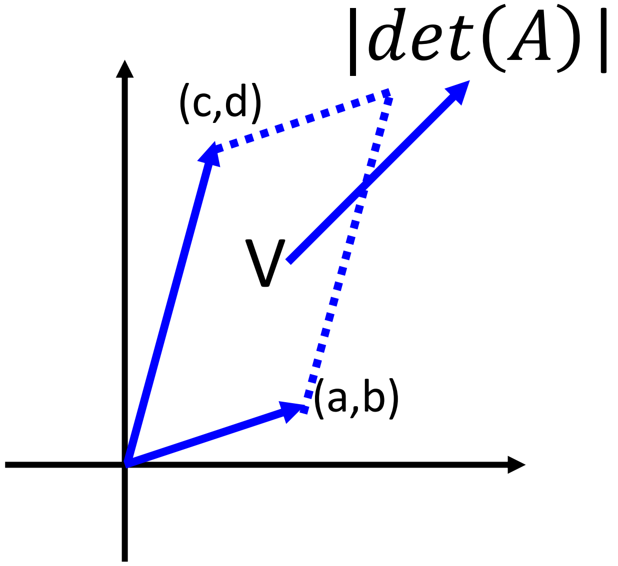
$$a_1 a_5 a_9 + a_2 a_6 a_7 + a_3 a_4 a_8$$

$$-a_3 a_5 a_7 - a_2 a_4 a_9 - a_1 a_6 a_8$$

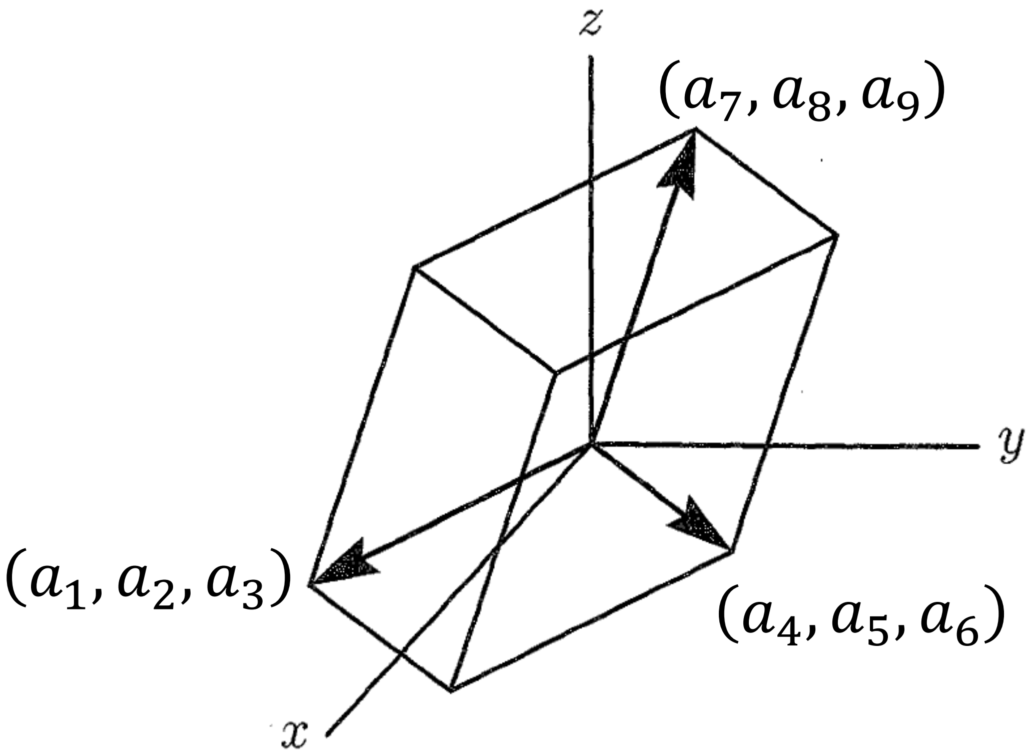
Determinant

- 2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



- 3×3 $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$



Basic Properties of Determinant

- Basic Property 1:
  - $det(I) = 1$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(I_2) = 1 \qquad det(I_3) = 1$$

Basic Properties of Determinant

- Basic Property 2:
  - Exchanging rows reverses the sign of determinant.

$$A \xrightarrow{\text{exchange two rows}} A'$$

$$det(A) = K \quad {\color{red}=} \quad det(A') = -K$$

$$det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) = 1$$

Basic Properties of Determinant

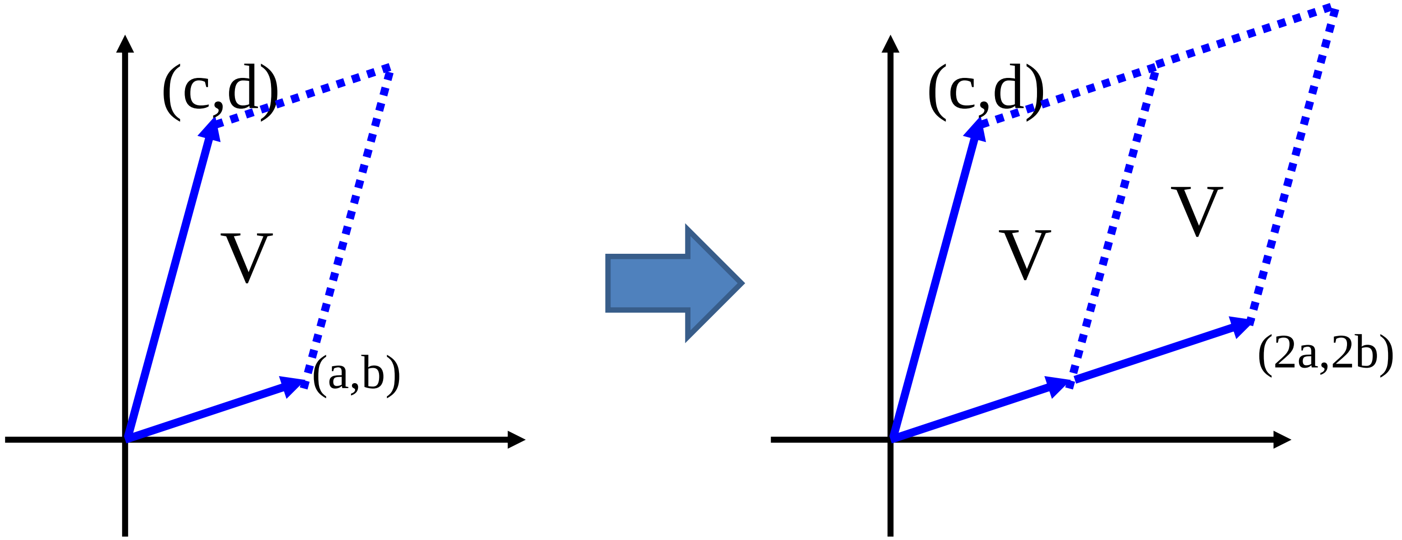
- Basic Property 2:
  - Exchange rows reverse the sign of determinant

If a matrix A has **2 equal rows** ⇨ $det(A) = 0$

Basic Properties of Determinant

- Basic Property 3:
  - Determinant is "linear" for each row

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$



Basic Properties of Determinant

- Basic Property 3:
- If A is $n \times n$ ……, find $det(2A)$?

$$det(2A) = 2^n det(A)$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det(A) = -2$$

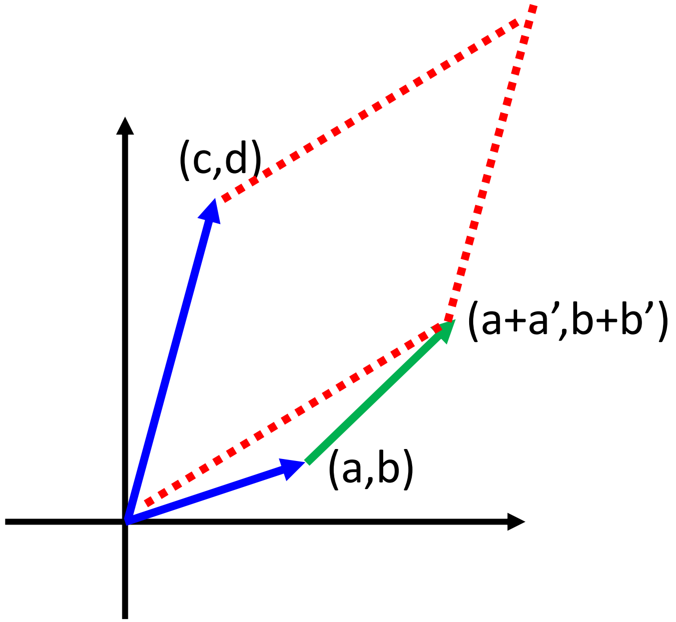
$$t = 4$$

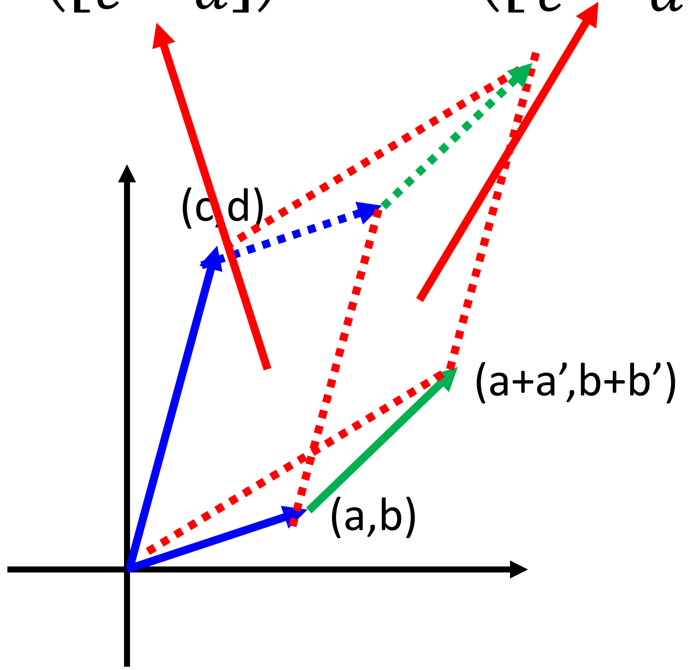
$$\det(tA) = \det\left(\begin{bmatrix} 4*1 & 4*2 \\ 3 & 4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 4 & 8 \\ 3 & 4 \end{bmatrix}\right) = -8$$

Basic Properties of Determinant

- Basic Property 3:
  - Determinant is “linear” for each row

$$det\left(\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$





Basic Properties of Determinant

- Basic Property 3:
  - Determinant is "linear" for each row

$$det\left(\begin{bmatrix} a + a' & b + b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det(A) = -2$$

$$A = \begin{bmatrix} 6 & 7 \\ 3 & 4 \end{bmatrix} \qquad \det(A) = 3$$

$$A = \begin{bmatrix} 5 & 5 \\ 3 & 4 \end{bmatrix} \qquad \det(A) = 5$$

Basic Properties of Determinant

- Basic Property 3:
  - Determinant is "linear" for each row
  - Subtract $k \times row\ i\ from\ row\ j$

Determinant doesn't change

$$det\left(\begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - kdet\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Basic Properties of Determinant

- Basic Property 3:
  - Determinant is "linear" for each row
  - Subtract $k \times row\ i\ from\ row\ j$

Determinant doesn't change

$$det\left(\begin{bmatrix} 1 & 2 \\ 3-(5*1) & 4-(5*2) \end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) + det\left(\begin{bmatrix} 1 & 2 \\ -5*1 & -5*2 \end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) - 5*det\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\right) = det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$$

Basic Properties of Determinant

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows reverse the sign of determinant.
- Basic Property 3: Determinant is “linear” for each row.

Area in 2D and Volume in 3D have the above properties

Can we say determinant is the “Volume” also in high dimension?

Determinants for Upper Triangular Matrix

$$U = \begin{bmatrix} d_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

Does not change the determinant

$$det(U) = det\left(\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}\right)$$

$$= d_1 d_2 \cdots d_n det\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right)$$

$$= 1$$

$$det(U) = d_1 d_2 \cdots d_n$$ (Products of diagonal)

Determinants for Upper Triangular Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\det(A) = 1 \times 4 \times 6 = 24$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}$$

$$\det(B) = 1 \times 6 \times 10 \times 13 \times 15 = 11700$$

Determinants vs. Invertible

A is invertible $\Longleftrightarrow$ $\det(A) \neq 0$

$A \xrightarrow[\text{Elementary row operation}]{} R$

$det(A)$

$det(R)$

$= \pm k_1 k_2 \cdots det(A)$

Exchange: Change sign

Scaling: Multiply k

Add row: nothing

If A is invertible, R is identity

$det(R) = 1 \Rightarrow det(A) \neq 0$

If A is not invertible, R has zero row

$det(R) = 0 \Rightarrow det(A) = 0$

Example

A is invertible $\Longleftrightarrow$ $\det(A) \neq 0$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix}$$

For what scalar c is the matrix not invertible?

$\det(A) = 0$

$$detA = 1 \cdot 0 \cdot 7 + (-1) \cdot c \cdot 2 + 2 \cdot (-1) \cdot 1$$

$$-2 \cdot 0 \cdot 2 - (-1) \cdot (-1) \cdot 7 - 1 \cdot c \cdot 1$$

$$= 0 - 2c - 2 - 7 - c = -3c - 9$$

Not invertible $\Rightarrow$ $-3c - 9 = 0$ $\Rightarrow$ $c = -3$

More Properties of Determinants

$$det(A + B) \neq det(A) + det(B)$$

- $det(AB) = det(A)det(B)$
- $det(A^{-1})$?

$$\because A^{-1}A = I$$
$$\therefore det(A^{-1})det(A) = det(I) = 1$$
$$\therefore \; det(A^{-1}) = 1/det(A)$$

- $det(A^2)$?

$$det(A^2) = det(A)det(A) = det(A)^2$$

- $det(A^T) = det(A)$

Cofactor Expansion

- A: $n \times n$ matrix.
- A_{ij}: the submatrix of A obtained by removing the i^{th} row and the j^{th} column.

$$A_{ij} = \begin{array}{cc} \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{c} \\ \\ i^{th}\text{ row} \\ \\ \\ \end{array} \\ j^{th}\text{row} & \end{array}$$

Cofactor Expansion

c_{ij}: (i, j)-$cofactor$

- Pick row 1

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

- Or pick row i

$$\det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

- Or pick column j

$$\det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$$

$$c_{ij} = (-1)^{i+j} det A_{ij}$$

Determinant for 2×2 matrix

$$c_{ij} = (-1)^{i+j} det(A_{ij})$$

- Define $\det([a]) = a$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$

- Pick the first row

$$det(A) = ac_{11} + bc_{12}$$

$$c_{11} = (-1)^{1+1} det([d]) = d$$

$$c_{12} = (-1)^{1+2} det([c]) = -c$$

Determinant for 3 $\times$ 3 matrix

$$c_{ij} = (-1)^{i+j} det(A_{ij})$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Pick row 2

$$\det(A) = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$$

4 5 6

$$(-1)^{2+1} det(A_{21}) \quad (-1)^{2+2} \det(A_{22}) \quad (-1)^{2+3} det(A_{23})$$

$$A_{21} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Formula for A^{-1}

- $A^{-1} = \frac{1}{det(A)} C^T$

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

  - $det(A)$: scalar
  - C: cofactors of A (C has the same size as A, so does C^T)
  - C^T is adjugate of A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A)$$

$$= ad - bc$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formula for A^{-1}

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $A^{-1} = ?$

$$A^{-1} = \frac{1}{det(A)} C^T$$

$$det(A) = aei + bfg + cdh - ceg - bdi - afh$$

$$C = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} b & c \\ e & f \end{vmatrix} \end{bmatrix}$$

Formula for A^{-1}

- Proof: $AC^T = det(A)I_n$

$$A^{-1} = \frac{1}{det(A)} C^T$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

Transpose

Cramer's Rule

$$A^{-1} = \frac{1}{det(A)} C^T$$

$$Ax = b$$

$$x = A^{-1}b$$

$$= \frac{1}{det(A)} C^T b$$

$$x_1 = \frac{det(B_1)}{det(A)}$$

B_1 =with column 1 replaced by b

$$x_2 = \frac{det(B_2)}{det(A)}$$

$$\vdots$$

$$\begin{bmatrix} b & \boxed{\begin{matrix} n-1 \\ \text{Columns} \\ \text{of } A \end{matrix}} \end{bmatrix}$$

$$x_j = \frac{det(B_j)}{det(A)}$$

B_j =with column j replaced by b

Cramer's Rule

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & -1 \\ 4 & 0 & 1 \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3 \qquad A_{12} = -\begin{vmatrix} -2 & -1 \\ 4 & 1 \end{vmatrix} = -2 \qquad A_{13} = \begin{vmatrix} -2 & 3 \\ 4 & 0 \end{vmatrix} = -12$$

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 0.3 + 1.(-2) + 2.(-12) = -26$$

- $\det(A) \neq 0 \Rightarrow A \ is \ invertible$

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \qquad A_{22} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \qquad A_{23} = -\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = -4$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7 \qquad A_{32} = -\begin{vmatrix} 0 & 2 \\ -2 & -1 \end{vmatrix} = -4 \qquad A_{33} = \begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

$$A^{-1} = -\frac{1}{26}\begin{bmatrix} 3 & -1 & -7 \\ -2 & -8 & -4 \\ -12 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{26} & \frac{1}{26} & \frac{7}{26} \\ \frac{1}{13} & \frac{4}{13} & \frac{2}{13} \\ \frac{6}{13} & -\frac{2}{13} & -\frac{1}{13} \end{bmatrix}$$

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