| $S_4$ | $n_c$ | | 1 | $\bar{1}$ | 2 | 3 | $\bar{3}$ |
|---|---|---|---|---|---|---|---|
| | 1 | I | 1 | 1 | 2 | 3 | 3 |
| $Z_2$ | 3 | (12)(34) | 1 | 1 $_a$ | 2 $_a$ | -1 $_a$ | -1 $_a$ |
| $Z_3$ | 8 | (123) | 1 | 1 $_b$ | -1 $_b$ | 0 $_b$ | 0 $_b$ |
| $Z_2$ | 6 | (12) | 1 | -1 $_c$ | 0 $_c$ | 1 $_c$ | -1 $_c$ |
| $Z_4$ | 6 | (1234) | 1 | -1 $_d$ | 0 $_d$ | -1 $_d$ | 1 $_d$ |

First we determine the irreducible representations:

$$1 + r^2 + s^2 + t^2 + u^2 = 24$$

It's not hard to come up with integer solutions $r = 1, s = 2, t = u = 3$.

Then we determine column $\bar{1}$, by column orthogonality,

$$1 + 3a + 8b + 6c + 6d = 0 \tag{1}$$
$$1 + 3a^2 + 8b^2 + 6c^2 + 6d^2 = 24 \tag{2}$$

where $a^2 = b^3 = c^2 = d^4 = 1$ (by group structure). Then (2) becomes

$$8b^2 + 6d^2 = 14 \tag{3}$$

But by looking at 1 dimensional representation product for (12)(34) we have $c^2 = a = 1$, so (1) becomes

$$8b + 6c + 6d = -4 \tag{4}$$

Combining (3), (4) and $b^3 = c^2 = d^4 = 1$, the only admissible solution is $b = 1, c = d = -1$.

Now column 2. Column orthogonality implies

$$2 + 3a + 8b + 6c + 6d = 0 \tag{5}$$
$$2 + 3a + 8b - 6c - 6d = 0 \tag{6}$$
$$4 + 3a^2 + 8b^2 + 6c^2 + 6d^2 = 24 \tag{7}$$

Which implies

$$2 + 3a + 8b = 0 \tag{8}$$
$$c + d = 0 \tag{9}$$
$$3a^2 + 8b^2 + 6c^2 + 6d^2 = 20 \tag{10}$$

Look at class of (12), since (12) and (34) commute, their 2-dimensional representations can be diagonalized simultaneously, let

$$(12) \sim \text{diag}(\lambda_1, \lambda_2)$$
$$(34) \sim \text{diag}(\lambda_3, \lambda_4)$$

by group structure, we know $\lambda_i^2 = 1$, as well as $c = \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$. Also $(12)(34) \sim \text{diag}(\lambda_1\lambda_3, \lambda_2\lambda_4)$ and $a = \lambda_1\lambda_3 + \lambda_2\lambda_4$.

At this point, $c$ can only be $\pm 2, 0$.

When $c = 2$, we have $\lambda_i = 1$, so $d = -2$ (by 9), $a = 2$, there is no $b$ that satisfies (8) and (10) simultaneously.

When $c = -2$, all $\lambda_i = -1$, so $d = 2$, $a = 2$. There is still no satisfiable $b$.

When $c = 0$, we can either have $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = 1$, in which case $a = -2, d = 0$, and no $b$ satisfies the constraints. The only case left is $\lambda_1 = \lambda_3 = 1, \lambda_2 = \lambda_4 = -1$, in which case $a = 2, d = 0$, and $b = -1$ is a satisfiable solution.

Now column 3. By column orthogonality

$$3 + 3a + 8b + 6c + 6d = 0 \tag{11}$$
$$3 + 3a + 8b - 6c - 6d = 0 \tag{12}$$
$$6 + 6a - 8b = 0 \tag{13}$$
$$9 + 3a^2 + 8b^2 + 6c^2 + 6d^2 = 24 \tag{14}$$

(11) and (12) entail

$$3+3a+8b=0 \tag{15}$$
$$6+6a-8b=0 \tag{16}$$
$$c=-d \tag{17}$$

So we nailed $a=-1, b=0$, and $c^2=d^2=1, c=-d$. It's clear that the sign of $c$ determines column 3 or $\bar{3}$.