First we show how the isomorphism between $SO(3,1) \cong SU(2) \otimes SU(2)$ works.

Any group element $g \in SO(3,1)$ can be assumed to take the form $g = e^{i(\theta_i J_i + \phi_i K_i)}$, let this be expressed in $J_{\pm i} = J_i \pm iK_i$ generators as

$$e^{i(\theta_i J_i + \phi_i K_i)} = e^{i(\alpha_i J_{+i} + \beta_i J_{-i})}$$

Then we know

$$\alpha_i(J_i + iK_i) + \beta_i(J_i - iK_i) = \theta_i J_i + \phi_i K_i \implies \begin{cases} \alpha_i + \beta_i = \theta_i \\ \alpha_i - \beta_i = -i\phi_i \end{cases} \implies \begin{cases} \alpha_i = \frac{\theta_i - i\phi_i}{2} \\ \beta_i = \frac{\theta_i + i\phi_i}{2} \end{cases}$$

Now given g, we can use its α_i, β_i to construct two elements $(u, v) \in SU(2) \otimes SU(2)$ where

$$\begin{aligned} u &= e^{i\alpha_i \sigma_i} \\ v &= e^{i\beta_i \sigma_i} \end{aligned}$$

This is how an element of the Lorentz group $SO(3,1)$ can be represented by a pair of Weyl spinors (u, v). On the other hand, given two spinors u, v, we can reverse the procedure to determine θ_i, ϕ_i hence $g \in SO(3,1)$. The fact that the construction is homomorphic is not shown here, but the octave scripts has a demonstration.

Now we elaborate on the point that $(u^\dagger u, u^\dagger \vec{\sigma} u)$ transforms like 4-vector.

Consider rotation $e^{i\theta_3 J_3}$. Recall that

$$J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad iJ_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_3^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then expanding the exponential and focusing on the x, y components, we have (as 2×2 matrix)

$$\begin{aligned} e^{i\theta_3 J_3} &= \left[1 + \frac{(i\theta_3)^2}{2!} + \frac{(i\theta_3)^4}{4!} + \cdots\right] I + \left[\frac{\theta_3}{1!} - \frac{\theta_3^3}{3!} + \cdots\right] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{bmatrix} = R(-\theta_3) \end{aligned}$$

Now consider the boost $e^{i\phi_3 K_3}$, recall that

$$K_3 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad iK_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (iK_3)^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Expanding,

$$\begin{aligned} e^{i\phi_3 K_3} &= \left[1 + \frac{\phi_3^2}{2!} + \frac{\phi_3^4}{4!} + \cdots\right] I + \left[\frac{\phi_3}{1!} + \frac{\phi_3^3}{3!} + \cdots\right] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cosh\phi_3 & \sinh\phi_3 \\ \sinh\phi_3 & \cosh\phi_3 \end{bmatrix} = B(\phi_3) \end{aligned}$$

A spacetime 4-vector (t, x, y, z) under $R(-\theta_3)$ transforms like

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 & \theta_3 \\ -\theta_3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \theta_3 y \\ y - \theta_3 x \end{bmatrix}$$

We now use the isomorphic construction above to see the action of the same rotation on u, and how it effectively transforms $(u^\dagger u, u^\dagger \vec{\sigma} u)$. Since

$$\begin{aligned} u &\to e^{i\theta_3 \sigma_3 / 2} u \\ u^\dagger &\to u^\dagger e^{-i\theta_3 \sigma_3 / 2} \end{aligned}$$

we know that the "time component" $u^\dagger u$ is invariant, so is the σ_3 component of $\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix}$. But the x and y components transform like the following

$$\begin{bmatrix}u^\dagger\sigma_1 u\\ u^\dagger\sigma_2 u\end{bmatrix} \to \begin{bmatrix}u^\dagger e^{-i\theta_3\sigma_3/2}\sigma_1 e^{i\theta_3\sigma_3/2}u\\ u^\dagger e^{-i\theta_3\sigma_3/2}\sigma_2 e^{i\theta_3\sigma_3/2}u\end{bmatrix}$$

Now

$$\begin{aligned}
e^{-i\theta_3\sigma_3/2}\sigma_1 e^{i\theta_3\sigma_3/2} &\approx (I - i\theta_3\sigma_3/2)\sigma_1(I + i\theta_3\sigma_3/2)\\
&\approx \sigma_1 + i\frac{\theta_3}{2}[\sigma_1,\sigma_3]\\
&= \sigma_1 + i\frac{\theta_3}{2}(-2i\sigma_2) = \sigma_1 + \theta_3\sigma_2
\end{aligned}$$

similarly,

$$\begin{aligned}
e^{-i\theta_3\sigma_3/2}\sigma_2 e^{i\theta_3\sigma_3/2} &\approx (I - i\theta_3\sigma_3/2)\sigma_2(I + i\theta_3\sigma_3/2)\\
&\approx \sigma_2 + i\frac{\theta_3}{2}[\sigma_2,\sigma_3]\\
&= \sigma_2 + i\frac{\theta_3}{2}(2i\sigma_1) = \sigma_2 - \theta_3\sigma_1
\end{aligned}$$

Which is precisely how (x, y) transforms under $R(-\theta_3)$.
Under $B(\phi_3)$, the spacetime 4-vector transforms as

$$\begin{bmatrix}t\\ z\end{bmatrix} \to \begin{bmatrix}\cosh\phi_3 & \sinh\phi_3\\ \sinh\phi_3 & \cosh\phi_3\end{bmatrix}\begin{bmatrix}t\\ z\end{bmatrix} \approx \begin{bmatrix}1 & \phi_3\\ \phi_3 & 1\end{bmatrix}\begin{bmatrix}t\\ z\end{bmatrix} = \begin{bmatrix}t+\phi_3 z\\ z+\phi_3 t\end{bmatrix}$$

And similarly, under boost

$$\begin{aligned}
u &\to e^{\phi_3\sigma_3/2}u\\
u^\dagger &\to u^\dagger e^{\phi_3\sigma_3/2}
\end{aligned}$$

$$\begin{bmatrix}u^\dagger\sigma_0 u\\ u^\dagger\sigma_1 u\\ u^\dagger\sigma_2 u\\ u^\dagger\sigma_3 u\end{bmatrix} \to \begin{bmatrix}u^\dagger e^{\phi_3\sigma_3/2}\sigma_0 e^{\phi_3\sigma_3/2}u\\ u^\dagger e^{\phi_3\sigma_3/2}\sigma_1 e^{\phi_3\sigma_3/2}u\\ u^\dagger e^{\phi_3\sigma_3/2}\sigma_2 e^{\phi_3\sigma_3/2}u\\ u^\dagger e^{\phi_3\sigma_3/2}\sigma_3 e^{\phi_3\sigma_3/2}u\end{bmatrix}$$

Picking i,

$$\begin{aligned}
e^{\phi_3\sigma_3/2}\sigma_i e^{\phi_3\sigma_3/2} &\approx (I + \phi_3\sigma_3/2)\sigma_i(I + \phi_3\sigma_3/2)\\
&\approx \sigma_i + \frac{\phi_3}{2}\{\sigma_i,\sigma_3\}
\end{aligned}$$

which leaves σ_1, σ_2 unchanged, and transforms

$$I = \sigma_0 \to \sigma_0 + \phi_3\sigma_3 \qquad \sigma_3 \to \sigma_3 + \phi_3\sigma_0$$

which is precisely how (t, z) transforms under $B(\phi_3)$.

Next let's show that the Weyl Lagrangian denstiy

$$\mathscr{L} = iu^\dagger\sigma^\mu\partial_\mu u$$

is Lorentz invariant.

We do this by looking at how it transforms under infinitesimal rotation $e^{i\theta_3 J_3} \sim R(-\theta_3)$ and infinitesimal boost $e^{i\phi_3 K_3} \sim B(\phi_3)$ respectively.

Under rotation $R(-\theta_3)$, focusing on the x, y components,

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix} = R(-\theta_3)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta_3 + y\sin\theta_3 \\ -x\sin\theta_3 + y\cos\theta_3 \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = R^{-1}(-\theta_3)\begin{bmatrix} x' \\ y' \end{bmatrix}\begin{bmatrix} x'\cos\theta_3 - y'\sin\theta_3 \\ x'\sin\theta_3 + y'\cos\theta_3 \end{bmatrix} \quad \Longrightarrow$$

$$\partial'_\mu = \begin{bmatrix} \partial_{x'} \\ \partial_{y'} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} \\ \frac{\partial}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial y'} \end{bmatrix} = \begin{bmatrix} \partial_x\cos\theta_3 + \partial_y\sin\theta_3 \\ -\partial_x\sin\theta_3 + \partial_y\cos\theta_3 \end{bmatrix} = R^{-1T}(-\theta_3)\partial_\mu$$

Restoring to the full 4-vector, under rotation $e^{i\theta_3 J_3} \sim R(-\theta_3)$

$$\partial_\mu \to \partial_\mu + \theta_3 \begin{bmatrix} 0 \\ \partial_y \\ -\partial_x \\ 0 \end{bmatrix}$$

Combining with $u \to e^{i\theta_3\sigma_3/2}u, u^\dagger \to u^\dagger e^{-i\theta_3\sigma_3/2}$, the Lagrangian $\mathscr{L}$ is transformed by

$$\mathscr{L} \to \mathscr{L}' = iu^\dagger e^{-i\theta_3\sigma_3/2}\sigma^\mu \left(\partial_\mu + \theta_3 \begin{bmatrix} 0 \\ \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \right) e^{i\theta_3\sigma_3/2}u$$

$$\approx iu^\dagger \left(I - \frac{i\theta_3\sigma_3}{2} \right)\sigma^\mu \left(\partial_\mu + \theta_3 \begin{bmatrix} 0 \\ \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \right)\left(I + \frac{i\theta_3\sigma_3}{2} \right)u$$

$$\approx \mathscr{L} + iu^\dagger \left[\frac{i\theta_3}{2}\left(\sigma^\mu\partial_\mu\sigma_3 - \sigma_3\sigma^\mu\partial_\mu \right) + \theta_3(\sigma^1\partial_y - \sigma^2\partial_x) \right]u$$

If $\mathscr{L}$ were to be Lorentz invariant, the operator inside the brackets will have to be zero. Indeed

$$\begin{aligned}
\sigma^\mu\partial_\mu\sigma_3 u &= \sigma^\mu\partial_\mu\sigma_3 \begin{bmatrix} u_1(t,\vec{x}) \\ u_2(t,\vec{x}) \end{bmatrix} = \sigma^\mu\partial_\mu \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} \\
&= I\begin{bmatrix} \partial_t u_1 \\ -\partial_t u_2 \end{bmatrix} + \sigma^1\begin{bmatrix} \partial_x u_1 \\ -\partial_x u_2 \end{bmatrix} + \sigma^2\begin{bmatrix} \partial_y u_1 \\ -\partial_y u_2 \end{bmatrix} + \sigma^3\begin{bmatrix} \partial_z u_1 \\ -\partial_z u_2 \end{bmatrix} \\
&= \begin{bmatrix} \partial_t u_1 \\ -\partial_t u_2 \end{bmatrix} + \begin{bmatrix} -\partial_x u_2 \\ \partial_x u_1 \end{bmatrix} + \begin{bmatrix} i\partial_y u_2 \\ i\partial_y u_1 \end{bmatrix} + \begin{bmatrix} \partial_z u_1 \\ \partial_z u_2 \end{bmatrix} \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\sigma_3\sigma^\mu\partial_\mu u &= \sigma_3\left(I\begin{bmatrix} \partial_t u_1 \\ \partial_t u_2 \end{bmatrix} + \sigma^1\begin{bmatrix} \partial_x u_1 \\ \partial_x u_2 \end{bmatrix} + \sigma^2\begin{bmatrix} \partial_y u_1 \\ \partial_y u_2 \end{bmatrix} + \sigma^3\begin{bmatrix} \partial_z u_1 \\ \partial_z u_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} \partial_t u_1 \\ -\partial_t u_2 \end{bmatrix} + \begin{bmatrix} \partial_x u_2 \\ -\partial_x u_1 \end{bmatrix} + \begin{bmatrix} -i\partial_y u_2 \\ -i\partial_y u_1 \end{bmatrix} + \begin{bmatrix} \partial_z u_1 \\ \partial_z u_2 \end{bmatrix} \qquad (2)
\end{aligned}$$

Therefore

$$\frac{i\theta_3}{2}\left(\sigma^\mu\partial_\mu\sigma_3 - \sigma_3\sigma^\mu\partial_\mu \right)u = \theta_3\begin{bmatrix} (-i\partial_x - \partial_y)u_2 \\ (i\partial_x - \partial_y)u_1 \end{bmatrix}$$

which exactly cancels

$$\theta_3(\sigma^1\partial_y - \sigma^2\partial_x)u = \theta_3\left(\sigma^1\begin{bmatrix} \partial_y u_1 \\ \partial_y u_2 \end{bmatrix} - \sigma^2\begin{bmatrix} \partial_x u_1 \\ \partial_x u_2 \end{bmatrix} \right) = \theta_3\left(\begin{bmatrix} \partial_y u_2 \\ \partial_y u_1 \end{bmatrix} + \begin{bmatrix} i\partial_x u_2 \\ -i\partial_x u_1 \end{bmatrix} \right)$$

Similarly under boost $e^{i\phi_3 K_3} \sim B(\phi_3)$

$$\begin{bmatrix} t \\ z \end{bmatrix} \to \begin{bmatrix} t' \\ z' \end{bmatrix} = B(\phi_3)\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} t\cosh\phi_3 + z\sinh\phi_3 \\ t\sinh\phi_3 + z\cosh\phi_3 \end{bmatrix} \quad \Longrightarrow$$

$$\begin{bmatrix} t \\ z \end{bmatrix} = B^{-1}(\phi_3)\begin{bmatrix} t' \\ z' \end{bmatrix} = \begin{bmatrix} t'\cosh\phi_3 - z'\sinh\phi_3 \\ -t'\sinh\phi_3 + z'\cosh\phi_3 \end{bmatrix} \quad \Longrightarrow$$

$$\partial'_\mu = \begin{bmatrix} \partial_{t'} \\ \partial_{z'} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t'} \\ \frac{\partial}{\partial t}\frac{\partial t}{\partial z'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z'} \end{bmatrix} = \begin{bmatrix} \partial_t\cosh\phi_3 - \partial_z\sinh\phi_3 \\ -\partial_t\sinh\phi_3 + \partial_z\cosh\phi_3 \end{bmatrix} = B^{-1T}(\phi_3)\partial_\mu$$

$$\partial_\mu \to \partial_\mu - \phi_3 \begin{bmatrix} \partial_z \\ 0 \\ 0 \\ \partial_t \end{bmatrix}$$

Combining with $u \to e^{\phi_3\sigma_3/2}u, u^\dagger \to u^\dagger e^{\phi_3\sigma_3/2}$, the Lagrangian $\mathscr{L}$ is transformed by

$$\begin{aligned}
\mathscr{L} \to \mathscr{L}' &= iu^\dagger e^{\phi_3\sigma_3/2}\sigma^\mu \left(\partial_\mu - \phi_3 \begin{bmatrix} \partial_z \\ 0 \\ 0 \\ \partial_t \end{bmatrix} \right) e^{\phi_3\sigma_3/2}u \\
&\approx iu^\dagger \left(I + \frac{\phi_3\sigma_3}{2} \right) \sigma^\mu \left(\partial_\mu - \phi_3 \begin{bmatrix} \partial_z \\ 0 \\ 0 \\ \partial_t \end{bmatrix} \right) \left(I + \frac{\phi_3\sigma_3}{2} \right) u \\
&\approx \mathscr{L} + u^\dagger \left[\frac{\phi_3}{2} \left(\sigma^\mu \partial_\mu \sigma_3 + \sigma_3 \sigma^\mu \partial_\mu \right) - \phi_3 (\partial_z + \sigma^3 \partial_t) \right] u
\end{aligned}$$

From (1) and (2), we know that

$$\frac{\phi_3}{2} \left(\sigma^\mu \partial_\mu \sigma_3 + \sigma_3 \sigma^\mu \partial_\mu \right) u = \phi_3 \begin{bmatrix} (\partial_t + \partial_z)u_1 \\ (-\partial_t + \partial_z)u_2 \end{bmatrix}$$

which exactly cancels

$$-\phi_3(\partial_z + \sigma^3\partial_t)u = -\phi_3 \begin{bmatrix} (\partial_z + \partial_t)u_1 \\ (\partial_z - \partial_t)u_2 \end{bmatrix}$$

Now we prove the useful "Weyl map". Let $\sigma^\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$, and $\bar{\sigma}^\mu = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)$. The Weyl map claims

$$\bar{\sigma}^\mu \Lambda_\mu{}^\nu = (S_R^{-1})^\dagger \bar{\sigma}^\nu S_R^{-1} \tag{3}$$
$$\sigma^\mu \Lambda_\mu{}^\nu = (S_L^{-1})^\dagger \sigma^\nu S_L^{-1} \tag{4}$$
$$\Lambda^\nu{}_\mu \bar{\sigma}^\mu = S_R^{-1} \bar{\sigma}^\nu (S_R^{-1})^\dagger \tag{5}$$
$$\Lambda^\nu{}_\mu \sigma^\mu = S_L^{-1} \sigma^\nu (S_L^{-1})^\dagger \tag{6}$$

where S_R, S_L are the representations of Λ in the right/left-handed Weyl spinor respectively, i.e.,

$$\begin{aligned}
S_R &= e^{i\alpha_i\sigma_i} = e^{i(\theta_i - i\phi_i)\sigma_i/2} \\
S_L &= e^{i\beta_i\sigma_i} = e^{i(\theta_i + i\phi_i)\sigma_i/2}
\end{aligned}$$

note that

$$\begin{aligned}
(S_R^{-1})^\dagger &= \left(e^{-i(\theta_i - i\phi_i)\sigma_i/2} \right)^\dagger = e^{i(\theta_i + i\phi_i)\sigma_i/2} = S_L \qquad \text{and vice versa} \\
(S_L^{-1})^\dagger &= S_R
\end{aligned}$$

We prove (3)-(6) with infinitesimal rotation around z-axis and infinitesimal boost along z-axis.
First, with rotation $\Lambda = e^{i\theta_3 J_3}$, we have

$$\Lambda \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta_3 & 0 \\ 0 & -\theta_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad S_R = e^{i\theta_3\sigma_3/2} \approx I + \frac{i\theta_3\sigma_3}{2}, \qquad S_L = e^{i\theta_3\sigma_3/2} \approx I + \frac{i\theta_3\sigma_3}{2}$$

For (3), $\nu = 1$,

$$\begin{aligned}
\text{LHS} &= \bar{\sigma}^1 \Lambda_1{}^1 + \bar{\sigma}^2 \Lambda_2{}^1 = -\sigma_1 + \theta_3\sigma_2 \\
\text{RHS} &= \left(I + \frac{i\theta_3\sigma_3}{2} \right) (-\sigma_1) \left(I - \frac{i\theta_3\sigma_3}{2} \right) = -\sigma_1 - \frac{i\theta_3}{2}[\sigma_3, \sigma_1] = -\sigma_1 + \theta_3\sigma_2
\end{aligned}$$

$\nu = 2$,

$$\begin{aligned}
\text{LHS} &= \bar{\sigma}^1 \Lambda_1{}^2 + \bar{\sigma}^2 \Lambda_2{}^2 = -\theta_3\sigma_1 - \sigma_2 \\
\text{RHS} &= \left(I + \frac{i\theta_3\sigma_3}{2} \right) (-\sigma_2) \left(I - \frac{i\theta_3\sigma_3}{2} \right) = -\sigma_2 - \frac{i\theta_3}{2}[\sigma_3, \sigma_2] = -\sigma_2 - \theta_3\sigma_1
\end{aligned}$$

For (4), $\nu = 1$,

$$\begin{aligned}\text{LHS} &= \sigma^1 {\Lambda_1}^1 + \sigma^2 {\Lambda_2}^1 = \sigma_1 - \theta_3\sigma_2 \\ \text{RHS} &= \left(I + \frac{i\theta_3\sigma_3}{2}\right)(\sigma_1)\left(I - \frac{i\theta_3\sigma_3}{2}\right) = \sigma_1 + \frac{i\theta_3}{2}[\sigma_3, \sigma_1] = \sigma_1 - \theta_3\sigma_2\end{aligned}$$

$\nu = 2$,

$$\begin{aligned}\text{LHS} &= \sigma^1 {\Lambda_1}^2 + \sigma^2 {\Lambda_2}^2 = \theta_3\sigma_1 + \sigma_2 \\ \text{RHS} &= \left(I + \frac{i\theta_3\sigma_3}{2}\right)(\sigma_2)\left(I - \frac{i\theta_3\sigma_3}{2}\right) = \sigma_2 + \frac{i\theta_3}{2}[\sigma_3, \sigma_2] = \sigma_2 + \theta_3\sigma_1\end{aligned}$$

For (5), $\nu = 1$,

$$\begin{aligned}\text{LHS} &= {\Lambda^1}_1 \bar{\sigma}^1 + {\Lambda^1}_2 \bar{\sigma}^2 = -\sigma_1 - \theta_3\sigma_2 \\ \text{RHS} &= \left(I - \frac{i\theta_3\sigma_3}{2}\right)(-\sigma_1)\left(I + \frac{i\theta_3\sigma_3}{2}\right) = -\sigma_1 + \frac{i\theta_3}{2}[\sigma_3, \sigma_1] = -\sigma_1 - \theta_3\sigma_2\end{aligned}$$

$\nu = 2$,

$$\begin{aligned}\text{LHS} &= {\Lambda^2}_1 \bar{\sigma}^1 + {\Lambda^2}_2 \bar{\sigma}^2 = \theta_3\sigma_1 - \sigma_2 \\ \text{RHS} &= \left(I - \frac{i\theta_3\sigma_3}{2}\right)(-\sigma_2)\left(I + \frac{i\theta_3\sigma_3}{2}\right) = -\sigma_2 + \frac{i\theta_3}{2}[\sigma_3, \sigma_2] = -\sigma_2 + \theta_3\sigma_1\end{aligned}$$

For (6), $\nu = 1$,

$$\begin{aligned}\text{LHS} &= {\Lambda^1}_1 \sigma^1 + {\Lambda^1}_2 \sigma^2 = \sigma_1 + \theta_3\sigma_2 \\ \text{RHS} &= \left(I - \frac{i\theta_3\sigma_3}{2}\right)(\sigma_1)\left(I + \frac{i\theta_3\sigma_3}{2}\right) = \sigma_1 - \frac{i\theta_3}{2}[\sigma_3, \sigma_1] = \sigma_1 + \theta_3\sigma_2\end{aligned}$$

$\nu = 2$,

$$\begin{aligned}\text{LHS} &= {\Lambda^2}_1 \sigma^1 + {\Lambda^2}_2 \sigma^2 = -\theta_3\sigma_1 + \sigma_2 \\ \text{RHS} &= \left(I - \frac{i\theta_3\sigma_3}{2}\right)(\sigma_2)\left(I + \frac{i\theta_3\sigma_3}{2}\right) = \sigma_2 - \frac{i\theta_3}{2}[\sigma_3, \sigma_2] = \sigma_2 - \theta_3\sigma_1\end{aligned}$$

Similarly, for boost $\Lambda = e^{i\phi_3 K_3}$, we have

$$\Lambda \approx \begin{bmatrix} 1 & 0 & 0 & \phi_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \phi_3 & 0 & 0 & 1 \end{bmatrix}, \qquad S_R = e^{\phi_3\sigma_3/2} \approx I + \frac{\phi_3\sigma_3}{2}, \qquad S_L = e^{-\phi_3\sigma_3/2} \approx I - \frac{\phi_3\sigma_3}{2}$$

For (3), $\nu = 0$,

$$\begin{aligned}\text{LHS} &= \bar{\sigma}^0 {\Lambda_0}^0 + \bar{\sigma}^3 {\Lambda_3}^0 = I - \phi_3\sigma_3 \\ \text{RHS} &= \left(I - \frac{\phi_3\sigma_3}{2}\right)(\sigma_0)\left(I - \frac{\phi_3\sigma_3}{2}\right) = I - \phi_3\sigma_3\end{aligned}$$

$\nu = 3$,

$$\begin{aligned}\text{LHS} &= \bar{\sigma}^0 {\Lambda_0}^3 + \bar{\sigma}^3 {\Lambda_3}^3 = \phi_3 I - \sigma_3 \\ \text{RHS} &= \left(I - \frac{\phi_3\sigma_3}{2}\right)(-\sigma_3)\left(I - \frac{\phi_3\sigma_3}{2}\right) = -\sigma_3 + \phi_3 I\end{aligned}$$

For (4), $\nu = 0$,

$$\begin{aligned}\text{LHS} &= \sigma^0 {\Lambda_0}^0 + \sigma^3 {\Lambda_3}^0 = I + \phi_3\sigma_3 \\ \text{RHS} &= \left(I + \frac{\phi_3\sigma_3}{2}\right)(\sigma_0)\left(I + \frac{\phi_3\sigma_3}{2}\right) = I + \phi_3\sigma_3\end{aligned}$$

$\nu = 3$,

$$\begin{aligned}\text{LHS} &= \sigma^0 {\Lambda_0}^3 + \sigma^3 {\Lambda_3}^3 = \phi_3 I + \sigma_3 \\ \text{RHS} &= \left(I + \frac{\phi_3\sigma_3}{2}\right)(\sigma_3)\left(I + \frac{\phi_3\sigma_3}{2}\right) = \sigma_3 + \phi_3 I\end{aligned}$$

For (5), $\nu = 0$,

$$\text{LHS} = \Lambda^0{}_0\bar{\sigma}^0 + \Lambda^0{}_3\bar{\sigma}^3 = I - \phi_3\sigma_3$$
$$\text{RHS} = \left(I - \frac{\phi_3\sigma_3}{2}\right)(\sigma_0)\left(I - \frac{\phi_3\sigma_3}{2}\right) = I - \phi_3\sigma_3$$

$\nu = 3$,

$$\text{LHS} = \Lambda^3{}_0\bar{\sigma}^0 + \Lambda^3{}_3\bar{\sigma}^3 = \phi_3 I - \sigma_3$$
$$\text{RHS} = \left(I - \frac{\phi_3\sigma_3}{2}\right)(-\sigma_3)\left(I - \frac{\phi_3\sigma_3}{2}\right) = -\sigma_3 + \phi_3 I$$

For (6), $\nu = 0$,

$$\text{LHS} = \Lambda^0{}_0\sigma^0 + \Lambda^0{}_3\sigma^3 = I + \phi_3\sigma_3$$
$$\text{RHS} = \left(I + \frac{\phi_3\sigma_3}{2}\right)(\sigma_0)\left(I + \frac{\phi_3\sigma_3}{2}\right) = I + \phi_3\sigma_3$$

$\nu = 3$,

$$\text{LHS} = \Lambda^3{}_0\sigma^0 + \Lambda^3{}_3\sigma^3 = \phi_3 I + \sigma_3$$
$$\text{RHS} = \left(I + \frac{\phi_3\sigma_3}{2}\right)(\sigma_3)\left(I + \frac{\phi_3\sigma_3}{2}\right) = \sigma_3 + \phi_3 I$$

At last we show the Lorentz invariance of $\mathscr{L}$ in index notation. Given $\Lambda = e^{i(\theta_i J_i + \phi_i K_i)} \in SO(3,1)$, let $S = e^{i(\theta_i - i\phi_i)\sigma_i/2}$ be the corresponding transformation to be applied to u.

First observe that

$$\sigma_i \partial_\nu \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sigma_i \begin{bmatrix} \partial_\nu u_1 \\ \partial_\nu u_2 \end{bmatrix} = \partial_\nu \sigma_i \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \Longrightarrow \qquad [\sigma_i, \partial_\nu] = 0 \qquad \Longrightarrow \qquad S\partial_\nu = \partial_\nu S$$

Now the differential operator ∂_μ transforms like

$$\partial_\mu \to \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu}$$

By definition of Λ, we have

$$x^\nu = (\Lambda^{-1})^\nu{}_\mu x'^\mu$$

so

$$\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu \partial_\nu = (\Lambda^{-1T})_\mu{}^\nu \partial_\nu$$

In fact

$$\Lambda^T \eta \Lambda = \eta \qquad \Longrightarrow \qquad \Lambda^{-1T} = \eta \Lambda^T \eta \qquad \Longrightarrow \qquad (\Lambda^{-1T})_\mu{}^\nu = \eta_{\mu\rho}\Lambda^\rho{}_\tau \eta^{\tau\nu}$$

therefore the Lagrangian transforms as

$$\begin{aligned}
\mathscr{L} = iu^\dagger \sigma^\mu \partial_\mu u \to \mathscr{L}' &= iu^\dagger S^\dagger \left[\sigma^\mu (\Lambda^{-1T})_\mu{}^\nu \partial_\nu\right] Su \\
&= iu^\dagger S^\dagger \left[\sigma^\mu \eta_{\mu\rho}\Lambda^\rho{}_\tau \eta^{\tau\nu}\partial_\nu\right] Su \\
&= iu^\dagger S^\dagger \left[\bar{\sigma}_\rho \Lambda^\rho{}_\tau \partial^\tau\right] Su \qquad \text{(applying Weyl map)} \\
&= iu^\dagger S^\dagger \left[(S^{-1})^\dagger \bar{\sigma}_\tau S^{-1} \partial^\tau\right] Su \qquad ([S, \partial_\tau] = 0) \\
&= iu^\dagger \bar{\sigma}_\tau \partial^\tau u = iu^\dagger \sigma^\tau \partial_\tau u
\end{aligned}$$

Another way of saying $\mathscr{L}$ is Lorentz invariant is that the Weyl equation

$$\sigma^\mu \partial_\mu u = 0$$

transforms under S by

$$\sigma^\mu \partial'_\mu S u = (S^{-1})^\dagger \sigma^\mu \partial_\mu u = 0$$