We shall derive the conformal Killing condition

$$g_{\mu\sigma}\partial_\rho \xi^\mu + g_{\rho\nu}\partial_\sigma \xi^\nu + \xi^\lambda \partial_\lambda g_{\rho\sigma} + \kappa g_{\rho\sigma} = 0$$

from the following relations

$$g'_{\rho\sigma}(x') = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \tag{1}$$
$$g'_{\rho\sigma}(x') = \Omega^2(x')g_{\rho\sigma}(x') = [1+\epsilon\kappa(x)]g_{\rho\sigma}(x') \tag{2}$$
$$x'^\mu = x^\mu + \epsilon\xi^\mu(x) \tag{3}$$

First for (2), by expanding $g_{\rho\sigma}(x')$ around x, we have

$$\begin{aligned} g'_{\rho\sigma}(x') &= [1+\epsilon\kappa(x)]g_{\rho\sigma}(x') = [1+\epsilon\kappa(x)]\left[g_{\rho\sigma}(x) + \frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\cdot(x'^\lambda - x^\lambda)\right] \\ &= [1+\epsilon\kappa]\left[g_{\rho\sigma} + \epsilon\xi^\lambda\partial_\lambda g_{\rho\sigma}\right] \\ &\approx g_{\rho\sigma} + \epsilon\kappa g_{\rho\sigma} + \epsilon\xi^\lambda\partial_\lambda g_{\rho\sigma} \end{aligned} \tag{4}$$

To see what (1) gives, we need to evaluate $\partial x^\mu/\partial x'^\rho$.
From (3), we have

$$\frac{\partial x^\mu}{\partial x'^\rho} = \frac{\partial x'^\mu}{\partial x'^\rho} - \epsilon\frac{\partial \xi^\mu(x)}{\partial x'^\rho} = \delta^\mu_\rho - \epsilon\frac{\partial \xi^\mu}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x'^\rho}$$

Now if we view $\partial x^\mu/\partial x'^\rho$ as element $X^\mu{}_\rho$ of unknown matrix X, and view $\partial\xi^\mu/\partial x^\sigma$ as element $\Xi^\mu{}_\sigma$ of a known matrix Ξ, the above is expressing a linear relation

$$\begin{aligned} X^\mu{}_\rho &= I^\mu{}_\rho - \epsilon\Xi^\mu{}_\sigma X^\sigma{}_\rho \qquad \text{or equivalently} \\ X &= I - \epsilon\Xi X \end{aligned}$$

which can be solved as

$$\begin{aligned} X &= (I+\epsilon\Xi)^{-1} \approx I - \epsilon\Xi \qquad \text{or equivalently} \\ X^\mu{}_\rho &= I^\mu{}_\rho - \epsilon\Xi^\mu{}_\rho \qquad \text{or} \\ \frac{\partial x^\mu}{\partial x'^\rho} &= \delta^\mu_\rho - \epsilon\partial_\rho\xi^\mu \end{aligned}$$

Now with this plugged into (1), we have

$$\begin{aligned} g'_{\rho\sigma}(x') &= g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \\ &= g_{\mu\nu}\left(\delta^\mu_\rho - \epsilon\partial_\rho\xi^\mu\right)\left(\delta^\nu_\sigma - \epsilon\partial_\sigma\xi^\nu\right) \\ &\approx g_{\rho\sigma} - \epsilon g_{\rho\nu}\partial_\sigma\xi^\nu - \epsilon g_{\mu\sigma}\partial_\rho\xi^\mu \end{aligned} \tag{5}$$

The desired Killing condition is obtained by equalizing (4) and (5).

Next we show that the inversion

$$\xi^\mu = a_\lambda(\eta^{\mu\lambda}x^2 - 2x^\mu x^\lambda)$$

satisfies the Killing condition for conformally flat spacetime

$$\partial_\rho\xi_\sigma + \partial_\sigma\xi_\rho = \frac{2}{d}\eta_{\rho\sigma}\partial\cdot\xi \tag{6}$$

To see this, note that

$$\partial_\rho\xi^\mu = \overbrace{a_\lambda\partial_\rho(\eta^{\mu\lambda}x^2)}^{A} - \overbrace{\partial_\rho(2a_\lambda x^\mu x^\lambda)}^{B}$$

where in A, $\partial_\rho x^2 = 2x_\rho$ (this can be seen by observing $x^2 = (x^0)^2 - (x^i)^2$, so $\partial_0 x^2 = 2x^0 = 2x_0$, $\partial_i x^2 = -2x^i = 2x_i$), so

$$A = 2a_\lambda\eta^{\mu\lambda}x_\rho$$

Differentiating by part in B,

$$B = 2a_\rho x^\mu + 2\delta^\mu_\rho a_\lambda x^\lambda$$

Therefore

$$\partial_\rho \xi^\mu = 2a_\lambda \eta^{\mu\lambda} x_\rho - 2(a_\rho x^\mu + \delta^\mu_\rho a_\lambda x^\lambda) \tag{7}$$

where λ is to be summed over.

Now the RHS of (6) becomes (making explicit sum signs to emphasize which index are to be summed over)

$$\begin{aligned}
\frac{2}{d}\eta_{\rho\sigma}\partial_\mu \xi^\mu &= \frac{2}{d}\eta_{\rho\sigma}\left(\sum_{\mu,\lambda} 2a_\lambda \eta^{\mu\lambda} x_\mu - \sum_\mu 2a_\mu x^\mu - \sum_{\mu,\lambda} 2a_\lambda x^\lambda\right) \\
&= \frac{2}{d}\eta_{\rho\sigma}\left(\sum_{\lambda} 2a_\lambda x^\lambda - \sum_\mu 2a_\mu x^\mu - \sum_{\mu,\lambda} 2a_\lambda x^\lambda\right) \\
&= \frac{2}{d}\eta_{\rho\sigma}(-2d)a_\lambda x^\lambda \\
&= -4\eta_{\rho\sigma} a_\lambda x^\lambda
\end{aligned}$$

In the mean time, on the LHS of (6),

$$\begin{aligned}
\partial_\rho \xi_\sigma &= \partial_\rho(\eta_{\mu\sigma}\xi^\mu) \\
&= \eta_{\mu\sigma}\left[2a_\lambda \eta^{\mu\lambda} x_\rho - 2(a_\rho x^\mu + \delta^\mu_\rho a_\lambda x^\lambda)\right] \\
&= 2a_\lambda x_\rho \delta^\lambda_\sigma - 2\eta_{\mu\sigma} a_\rho x^\mu - 2\eta_{\rho\sigma} a_\lambda x^\lambda \\
&= 2a_\sigma x_\rho - 2a_\rho x_\sigma - 2\eta_{\rho\sigma} a_\lambda x^\lambda
\end{aligned}$$

With the first two terms cancel those from $\partial_\sigma \xi_\rho$, it's clear the LHS of (6) is also equal to $-4\eta_{\rho\sigma} a_\lambda x^\lambda$.

Now we proceed to the (very tedious) proof of the following non-trivial commutator relations of the conformal algebra

$$\begin{aligned}
[J^{\mu\nu}, K^\lambda] &= -\eta^{\mu\lambda} K^\nu + \eta^{\nu\lambda} K^\mu &(8)\\
[K^\mu, P^\lambda] &= -2(\eta^{\mu\nu} x^\lambda - \eta^{\lambda\mu} x^\nu - x^\mu \eta^{\lambda\nu})\partial_\nu \\
&= 2(J^{\mu\lambda} + \eta^{\lambda\mu} D) &(9)\\
[K^\mu, K^\nu] &= 0 &(10)
\end{aligned}$$

where

$$K^\lambda = (\eta^{\lambda\tau} x^2 - 2x^\lambda x^\tau)\partial_\tau$$

First, we take note that since $(\partial_\mu x^\nu)(f) = \delta^\nu_\mu \cdot f + x^\nu \partial_\mu f$, as differential operator,

$$\begin{aligned}
\partial_\mu x^\nu &= \delta^\nu_\mu + x^\nu \partial_\mu \qquad \Longrightarrow &(11)\\
\partial_\mu x_\nu &= \eta_{\nu\rho}\partial_\mu x^\rho = \eta_{\nu\rho}(\delta^\rho_\mu + x^\rho \partial_\mu) = \eta_{\mu\nu} + x_\nu \partial_\mu &(12)
\end{aligned}$$

or in commutator form

$$\begin{aligned}
[\partial_\mu, x^\nu] &= \delta^\nu_\mu &(13)\\
[\partial_\mu, x_\nu] &= \eta_{\mu\nu} &(14)
\end{aligned}$$

Also note the important commutators

$$\begin{aligned}
[\partial_\mu, x^2] &= [\partial_\mu, x_\lambda]x^\lambda + x_\lambda[\partial_\mu, x^\lambda] \\
&= \eta_{\mu\lambda} x^\lambda + x_\lambda \delta^\lambda_\mu = 2x_\mu &(15)\\
[\partial_\rho, x^\mu x^\nu] &= [\partial_\rho, x^\mu]x^\nu + x^\mu[\partial_\rho, x^\nu] \\
&= \delta^\mu_\rho x^\nu + \delta^\nu_\rho x^\mu &(16)
\end{aligned}$$

To see (8), we first establish the following

$$
\begin{aligned}
[J_{\mu\nu}, x^\lambda] &= [x_\mu \partial_\nu - x_\nu \partial_\mu, x^\lambda] = x_\mu[\partial_\nu, x^\lambda] - x_\nu[\partial_\mu, x^\lambda] \\
&= \delta^\lambda_\nu x_\mu - \delta^\lambda_\mu x_\nu & (17) \\
[J_{\mu\nu}, \partial_\tau] &= [x_\mu \partial_\nu - x_\nu \partial_\mu, \partial_\tau] = [x_\mu, \partial_\tau]\partial_\nu - [x_\nu, \partial_\tau]\partial_\mu \\
&= -\eta_{\mu\tau}\partial_\nu + \eta_{\nu\tau}\partial_\mu & (18)
\end{aligned}
$$

Then raising J's indices, we get

$$
\begin{aligned}
[J^{\mu\nu}, x^\lambda] &= \eta^{\mu\rho}\eta^{\nu\sigma}[J_{\rho\sigma}, x^\lambda] = \eta^{\mu\rho}\eta^{\nu\sigma}(\delta^\lambda_\sigma x_\rho - \delta^\lambda_\rho x_\sigma) \\
&= x^\mu \eta^{\nu\lambda} - x^\nu \eta^{\mu\lambda} & (19) \\
[J^{\mu\nu}, \partial_\tau] &= \eta^{\mu\rho}\eta^{\nu\sigma}[J_{\rho\sigma}, \partial_\tau] = \eta^{\mu\rho}\eta^{\nu\sigma}[-\eta_{\rho\tau}\partial_\sigma + \eta_{\sigma\tau}\partial_\rho] \\
&= -\delta^\mu_\tau \partial^\nu + \delta^\nu_\tau \partial^\mu & (20)
\end{aligned}
$$

Subsequently

$$
\begin{aligned}
[J^{\mu\nu}, x^2] &= [J^{\mu\nu}, x_\lambda x^\lambda] = [J^{\mu\nu}, x_\lambda]x^\lambda + x_\lambda[J^{\mu\nu}, x^\lambda] \\
&= \eta_{\lambda\tau}[J^{\mu\nu}, x^\tau]x^\lambda + x_\lambda[J^{\mu\nu}, x^\lambda] \\
&= \eta_{\lambda\tau}(x^\mu\eta^{\nu\tau} - x^\nu\eta^{\mu\tau})x^\lambda + x_\lambda(x^\mu\eta^{\nu\lambda} - x^\nu\eta^{\mu\lambda}) \\
&= \delta^\nu_\lambda x^\mu x^\lambda - \delta^\mu_\lambda x^\nu x^\lambda + x^\nu x^\mu - x^\mu x^\nu \\
&= 0 & (21)
\end{aligned}
$$

Finally, towards (8)

$$
[J^{\mu\nu}, K^\lambda] = [J^{\mu\nu}, (\eta^{\lambda\tau}x^2 - 2x^\lambda x^\tau)\partial_\tau] = \overbrace{\eta^{\lambda\tau}[J^{\mu\nu}, x^2\partial_\tau]}^{A} - 2\overbrace{[J^{\mu\nu}, x^\lambda x^\tau \partial_\tau]}^{B}
$$

where

$$
\begin{aligned}
A = \eta^{\lambda\tau}[J^{\mu\nu}, x^2\partial_\tau] &= \eta^{\lambda\tau}\left([J^{\mu\nu}, x^2]\partial_\tau + x^2[J^{\mu\nu}, \partial_\tau]\right) = \eta^{\lambda\tau}x^2(-\delta^\mu_\tau\partial^\nu + \delta^\nu_\tau\partial^\mu) \\
&= x^2(-\eta^{\lambda\mu}\partial^\nu + \eta^{\lambda\nu}\partial^\mu) & (22) \\
B = [J^{\mu\nu}, x^\lambda x^\tau\partial_\tau] &= [J^{\mu\nu}, x^\lambda]x^\tau\partial_\tau + x^\lambda[J^{\mu\nu}, x^\tau]\partial_\tau + x^\lambda x^\tau[J^{\mu\nu}, \partial_\tau] \\
&= (x^\mu\eta^{\nu\lambda} - x^\nu\eta^{\mu\lambda})x^\tau\partial_\tau + x^\lambda(x^\mu\eta^{\nu\tau} - x^\nu\eta^{\mu\tau})\partial_\tau + x^\lambda x^\tau(-\delta^\mu_\tau\partial^\nu + \delta^\nu_\tau\partial^\mu) \\
&= (x^\mu\eta^{\nu\lambda} - x^\nu\eta^{\mu\lambda})x^\tau\partial_\tau + 0 & (23)
\end{aligned}
$$

Finally

$$
\begin{aligned}
[J^{\mu\nu}, K^\lambda] = A - 2B &= x^2(-\eta^{\lambda\mu}\partial^\nu + \eta^{\lambda\nu}\partial^\mu) - 2(x^\mu\eta^{\nu\lambda} - x^\nu\eta^{\mu\lambda})x^\tau\partial_\tau \\
&= -\eta^{\mu\lambda}(x^2\eta^{\nu\tau} - 2x^\nu x^\tau)\partial_\tau + \eta^{\nu\lambda}(x^2\eta^{\mu\tau} - 2x^\mu x^\tau)\partial_\tau \\
&= -\eta^{\mu\lambda}K^\nu + \eta^{\nu\lambda}K^\mu
\end{aligned}
$$

Coming to (9),

$$
\begin{aligned}
[K^\mu, P^\lambda] &= \eta^{\lambda\tau}[(\eta^{\mu\nu}x^2 - 2x^\mu x^\nu)\partial_\nu, \partial_\tau] \\
&= \eta^{\lambda\tau}\left(\overbrace{\eta^{\mu\nu}[x^2\partial_\nu, \partial_\tau]}^{A} - 2\overbrace{[x^\mu x^\nu\partial_\nu, \partial_\tau]}^{B}\right) & (24)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \eta^{\mu\nu}[x^2, \partial_\tau]\partial_\nu + \eta^{\mu\nu}x^2[\partial_\nu, \partial_\tau] \\
&= -2\eta^{\mu\nu}x_\tau\partial_\nu & (25) \\
B &= [x^\mu x^\nu, \partial_\tau]\partial_\nu \\
&= -\delta^\mu_\tau x^\nu\partial_\nu - x^\mu\delta^\nu_\tau\partial_\nu \\
&= -\delta^\mu_\tau x^\nu\partial_\nu - x^\mu\partial_\tau & (26)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
[K^\mu, P^\lambda] &= \eta^{\lambda\tau}(A - 2B) \\
&= \eta^{\lambda\tau}(-2\eta^{\mu\nu}x_\tau\partial_\nu + 2\delta^\mu_\tau x^\nu\partial_\nu + 2x^\mu\partial_\tau) \\
&= -2(x^\lambda\eta^{\mu\nu}\partial_\nu - \eta^{\mu\lambda}x^\nu\partial_\nu - x^\mu\eta^{\lambda\tau}\partial_\tau)
\end{aligned}
$$

which is exactly (9) with dummy index exchange $\tau \to \nu$.

Now coming to (10), we have

$$
\begin{aligned}
[K^\mu, K^\nu] &= \left[(\eta^{\mu\rho}x^2 - 2x^\mu x^\rho)\partial_\rho, (\eta^{\nu\sigma}x^2 - 2x^\nu x^\sigma)\partial_\sigma\right] \\
&= \overbrace{\eta^{\mu\rho}\eta^{\nu\sigma}[x^2\partial_\rho, x^2\partial_\sigma]}^{A} - \overbrace{2\eta^{\nu\sigma}[x^\mu x^\rho\partial_\rho, x^2\partial_\sigma]}^{B} \\
&\quad - \overbrace{2\eta^{\mu\rho}[x^2\partial_\rho, x^\nu x^\sigma\partial_\sigma]}^{C} + \overbrace{4[x^\mu x^\rho\partial_\rho, x^\nu x^\sigma\partial_\sigma]}^{D}
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
A &= \eta^{\mu\rho}\eta^{\nu\sigma}[x^2\partial_\rho, x^2\partial_\sigma] \\
&= \eta^{\mu\rho}\eta^{\nu\sigma}\left([x^2\partial_\rho, x^2]\partial_\sigma + x^2[x^2\partial_\rho, \partial_\sigma]\right) \\
&= \eta^{\mu\rho}\eta^{\nu\sigma}\left(x^2[\partial_\rho, x^2]\partial_\sigma + x^2[x^2, \partial_\sigma]\partial_\rho\right) \\
&= 2\eta^{\mu\rho}\eta^{\nu\sigma}x^2(x_\rho\partial_\sigma - x_\sigma\partial_\rho) \\
&= 2x^2(x^\mu\partial^\nu - x^\nu\partial^\mu)
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
B &= 2\eta^{\nu\sigma}[x^\mu x^\rho\partial_\rho, x^2\partial_\sigma] \\
&= 2\eta^{\nu\sigma}\left([x^\mu x^\rho\partial_\rho, x^2]\partial_\sigma + x^2[x^\mu x^\rho\partial_\rho, \partial_\sigma]\right) \\
&= 2\eta^{\nu\sigma}\left(x^\mu x^\rho[\partial_\rho, x^2]\partial_\sigma + x^2[x^\mu x^\rho, \partial_\sigma]\partial_\rho\right) \\
&= 2\eta^{\nu\sigma}\left(x^\mu x^\rho 2x_\rho\partial_\sigma - x^2(\delta^\mu_\sigma x^\rho + \delta^\rho_\sigma x^\mu)\partial_\rho\right) \\
&= 2\eta^{\nu\sigma}\left(2x^2x^\mu\partial_\sigma - x^2\delta^\mu_\sigma x^\rho\partial_\rho - x^2x^\mu\partial_\rho\right) \\
&= 2x^2(x^\mu\partial^\nu - \eta^{\mu\nu}x^\rho\partial_\rho)
\end{aligned} \tag{29}
$$

$$
C = -2x^2(x^\nu\partial^\mu - \eta^{\mu\nu}x^\sigma\partial_\sigma) \tag{30}
$$

where C is obtained from B by $\rho \leftrightarrow \sigma, \mu \leftrightarrow \nu$ and flipping the sign.

We can already see that $A - B - C = 0$, it remains to show $D = 0$.

Indeed

$$
\begin{aligned}
\frac{D}{4} &= [x^\mu x^\rho\partial_\rho, x^\nu x^\sigma\partial_\sigma] \\
&= [x^\mu x^\rho\partial_\rho, x^\nu x^\sigma]\partial_\sigma + x^\nu x^\sigma[x^\mu x^\rho\partial_\rho, \partial_\sigma] \\
&= x^\mu x^\rho[\partial_\rho, x^\nu x^\sigma]\partial_\sigma + x^\nu x^\sigma[x^\mu x^\rho, \partial_\sigma]\partial_\rho \\
&= x^\mu x^\rho(\delta^\nu_\rho x^\sigma + \delta^\sigma_\rho x^\nu)\partial_\sigma - x^\nu x^\sigma(\delta^\mu_\sigma x^\rho + \delta^\rho_\sigma x^\mu)\partial_\rho \\
&= x^\mu x^\nu x^\sigma\partial_\sigma + x^\mu x^\sigma x^\nu\partial_\sigma - x^\nu x^\mu x^\rho\partial_\rho - x^\nu x^\rho x^\mu\partial_\rho \\
&= 0
\end{aligned} \tag{31}
$$