Let $A^{i_{n+1}i_{n+2}\cdots i_{2n}}$ be an antisymmetric tensor with rank-n, and let

$$B^{i_1 i_2\cdots i_n} = \frac{1}{n!}\epsilon^{i_1 i_2\cdots i_n i_{n+1} i_{n+2}\cdots i_{2n}} A^{i_{n+1}i_{n+2}\cdots i_{2n}} \tag{1}$$

It's easy to see that B is also antistymmetric.

Now we shall show that

$$A^{i_{n+1}i_{n+2}\cdots i_{2n}} = \frac{1}{n!}\epsilon^{i_1 i_2\cdots i_n i_{n+1} i_{n+2}\cdots i_{2n}} B^{i_1 i_2\cdots i_n} \tag{2}$$

Plugging (1) into (2), we want to show

$$\begin{aligned} A^{i_{n+1}i_{n+2}\cdots i_{2n}} &= \frac{1}{n!}\epsilon^{i_1 i_2\cdots i_n i_{n+1} i_{n+2}\cdots i_{2n}} B^{i_1 i_2\cdots i_n} \\ &= \frac{1}{n!}\frac{1}{n!}\epsilon^{i_1 i_2\cdots i_n i_{n+1} i_{n+2}\cdots i_{2n}} \left(\epsilon^{i_1 i_2\cdots i_n j_1 j_2\cdots j_n} A^{j_1 j_2\cdots j_n}\right) \end{aligned} \tag{3}$$

On the RHS, the sum is over the indices $i_1, i_2, \cdots, i_n, j_1, j_2, \cdots, j_n$. Let the "outer" sum be over i_{1-n}'s, and let the "inner" sum be over j_{1-n}'s.

Given any combination of distinct $i_{n+1}, i_{n+2}, \cdots, i_{2n}$ chosen from the $2n$ values, there are $n!$ ways to choose $i_1, i_2, \cdots, i_n$ so the first ϵ on RHS is non-zero. For each such choice of i_{1-n}, there are $n!$ choices for j_{1-n} in the inner sum to make the second ϵ non-zero. We know that all these j_{1-n} values must be a permutation of $i_{n+1}, \cdots, i_{2n}$. Since A is antisymmetric, $A^{j_1 j_2\cdots j_n}$ is just $A^{i_{n+1}i_{n+2}\cdots i_{2n}}$ with a sign determined by whether the aforementioned permutation is even or odd. But multiplying $A^{i_{n+1}i_{n+2}\cdots i_{2n}}$ with $\epsilon^{i_1 i_2\cdots i_n j_1 j_2\cdots j_n}$ will exactly cancel this sign for all permutations. This means the inner sum can be determined to be

$$\epsilon^{i_1 i_2\cdots i_n j_1 j_2\cdots j_n} A^{j_1 j_2\cdots j_n} = n! \cdot \epsilon^{i_1 i_2\cdots i_n i_{n+1} i_{n+2}\cdots i_{2n}} A^{i_{n+1}i_{n+2}\cdots i_{2n}}$$

Now plugging this into the outer sum and use $\epsilon\epsilon = 1$ (for non-zero ϵ's), we obtain (3).

Now notice that in (1), A's indices are contracting with the second half of ϵ's indices, while in (2), B's indices are contracting with the first half of ϵ's indices. If we define "dual" as either (1) or (2), it's not going to be well defined since the two definitions differed by a sign when n is odd.

But we do want to maintain the invariance that "the dual of the dual is identity". This can be fixed by defining "dual" as (1) with an extra constant c (to be determined).

Plugging c into the "dual-of-dual", we have

$$\begin{aligned} D^{i_1 i_2\cdots i_n} &= \text{dual-of}(A^{j_1 j_2\cdots j_n}) = c\cdot\frac{1}{n!}\epsilon^{i_1 i_2\cdots i_n j_1 j_2\cdots j_n} A^{j_1 j_2\cdots j_n} \\ E^{i_{n+1}i_{n+2}\cdots i_{2n}} &= \text{dual-of}(D^{i_1 i_2\cdots i_n}) = c\cdot\frac{1}{n!}\epsilon^{i_{n+1}i_{n+2}\cdots i_{2n} i_1 i_2\cdots i_n} D^{i_1 i_2\cdots i_n} \\ &= \left(c\cdot\frac{1}{n!}\right)^2 \epsilon^{i_{n+1}i_{n+2}\cdots i_{2n} i_1 i_2\cdots i_n}\left(\epsilon^{i_1 i_2\cdots i_n j_1 j_2\cdots j_n} A^{j_1 j_2\cdots j_n}\right) \\ &= \left(c\cdot\frac{1}{n!}\right)^2 (-1)^n \epsilon^{i_1 i_2\cdots i_n i_{n+1} i_{n+2}\cdots i_{2n}}\left(\epsilon^{i_1 i_2\cdots i_n j_1 j_2\cdots j_n} A^{j_1 j_2\cdots j_n}\right) \qquad \text{(by (3))} \\ &= c^2(-1)^n A^{i_{n+1}i_{n+2}\cdots i_{2n}} \end{aligned}$$

So if we want "dual-of-dual" to be identity, we must have $c^2 = (-1)^n$, which can be satisfied by setting $c = i^n$.

Then it's straightforward to see that

$$\begin{aligned} T_+^{i_1 i_2\cdots i_n} &= \left(A^{i_1 i_2\cdots i_n} + \text{dual-of}(A^{i_1 i_2\cdots i_n})\right) \\ T_-^{i_1 i_2\cdots i_n} &= \left(A^{i_1 i_2\cdots i_n} - \text{dual-of}(A^{i_1 i_2\cdots i_n})\right) \end{aligned}$$

are self-dual and anti-self-dual respectively.