Here we derive an explicit traceless form for the tensor $T^{ik}_{jl} = \psi^i_j \chi^k_l$.

Note here since we assume ψ^i_j and χ^k_l are standard-form traceless $(1,1)$ tensors, we already have zero "parallel" traces

$$T^{hk}_{hl} = T^{ih}_{jh} = 0$$

First we try to get rid of the "diagonal" traces for T. It was quite easy to guess its form

$$D^{ik}_{jl} = T^{ik}_{jl} - \frac{1}{N}\left(\delta^i_l T^{hk}_{jh} + \delta^k_j T^{ih}_{hl}\right) + \frac{1}{N^2}\delta^i_l \delta^k_j T^{hm}_{mh} \tag{1}$$

However, by adding these "correction" terms onto T, the parallel traces are no longer zero. We can do the same trick to get rid of D's parallel traces:

$$P^{ik}_{jl} = D^{ik}_{jl} - \frac{1}{N}\left(\delta^i_j D^{hk}_{hl} + \delta^k_l D^{ih}_{jh}\right) + \frac{1}{N^2}\delta^i_j \delta^k_l D^{hm}_{hm} \tag{2}$$

Now, the diagonal trace of P is no longer zero. We can keep iterating this process. Intuition indicates the alternate application of eliminating diagonal trace and parallel trace should eventually converge to the traceless tensor we are seeking.

The octave script shows this iterative process converges quite fast.

In principle, if we keep recursing the D formula and P formula, we may be able to see the pattern of a few infinite series, and then we can get the limit of those series which gives us the closed-form formula. But in reality, there are just too many terms to track. Here, we try to express P in T.

First, let's express D^{qk}_{ql}. Using (1), we have

$$\begin{aligned} D^{qk}_{ql} &= 0 - \sum_q \frac{1}{N}\left(\delta^q_l T^{hk}_{qh} + \delta^k_q T^{qh}_{hl}\right) + \sum_q \frac{1}{N^2}\delta^q_l \delta^k_q T^{hm}_{mh} \\ &= -\frac{1}{N}\left(T^{hk}_{lh} + T^{kh}_{hl}\right) + \frac{1}{N^2}\delta^k_l T^{hm}_{mh} \end{aligned} \tag{3}$$

By symmetry between ij and kl, we have

$$D^{iq}_{jq} = -\frac{1}{N}\left(T^{ih}_{hj} + T^{hi}_{jh}\right) + \frac{1}{N^2}\delta^i_j T^{hm}_{mh} \tag{4}$$

Next, using (1), we see that

$$\begin{aligned} D^{qr}_{qr} &= -\sum_{q,r} \frac{1}{N}\left(\delta^q_r T^{hr}_{qh} + \delta^r_q T^{qh}_{hr}\right) + \sum_{q,r} \frac{1}{N^2}\delta^q_r \delta^r_q T^{hm}_{mh} \\ &= -\frac{2}{N} T^{hm}_{mh} + \frac{N}{N^2} T^{hm}_{mh} = -\frac{1}{N} T^{hm}_{mh} \end{aligned} \tag{5}$$

Now look at the 3 terms in (2), the 1st term is given by (1), the 2nd term is given by (3) and (4), the 3rd term is given by (5). We can express P in terms of T after all. This is only the second step of an infinite series of iterations, it's already too complicated to track.

But this exercise gives us insights to guess the final format of the converged series.

Plugging (3) and (4) into the 2nd term of (2), we expect the following term to show up in the converged traceless tensor:

$$\delta^i_j\left(T^{hk}_{lh} + T^{kh}_{hl}\right) + \delta^k_l\left(T^{ih}_{hj} + T^{hi}_{jh}\right) \tag{6}$$

By looking at the 2nd term of (1), we expect something similar to (6) to show up in the final result, i.e.

$$\delta^i_l\left(T^{hk}_{jh} + T^{kh}_{hj}\right) + \delta^k_j\left(T^{ih}_{hl} + T^{hi}_{lh}\right) \tag{7}$$

By the 3rd term of (1), we would also expect this term

$$\delta^i_l \delta^k_j T^{hm}_{mh} \tag{8}$$

Plugging (5) into 3rd term of (2), we would also expect

$$\delta^i_j \delta^k_l T^{hm}_{mh} \tag{9}$$

Finally we have the 0th order term T^{ik}_{jl} from (1), but since all the rest of the terms in (6)-(9) are symmetric in the upstairs and symmetric in the downstairs, we speculate this will apply to the 0th order term as well

$$T^{ik}_{jl} + T^{ki}_{lj} \tag{10}$$

Therefore, we speculate the iterative process will eventually converge to the traceless tensor, which is a linear combination of (6) - (10):

$$\begin{aligned} S^{ik}_{jl} &= T^{ik}_{jl} + T^{ki}_{lj} \\ &+ A \cdot \left[\delta^i_l \left(T^{hk}_{jh} + T^{kh}_{hj} \right) + \delta^k_j \left(T^{ih}_{hl} + T^{hi}_{lh} \right) \right] \\ &+ B \cdot \left[\delta^i_j \left(T^{hk}_{lh} + T^{kh}_{hl} \right) + \delta^k_l \left(T^{ih}_{hj} + T^{hi}_{jh} \right) \right] \\ &+ C \cdot \delta^i_l \delta^k_j T^{hm}_{mh} \\ &+ D \cdot \delta^i_j \delta^k_l T^{hm}_{mh} \end{aligned}$$

To determine the coefficients A, B, C, D, we enforce that both the parallel and diagonal trace of S must be zero.

First, when we contract i, j to get parallel trace for S, the 0th order terms vanish.

The A term becomes

$$A \cdot \left[T^{hk}_{lh} + T^{kh}_{hl} + T^{hk}_{lh} + T^{kh}_{hl} \right]$$

The B term becomes

$$B \cdot \left[N \left(T^{hk}_{lh} + T^{kh}_{hl} \right) + 2\delta^k_l T^{hm}_{mh} \right]$$

The C term becomes

$$C \delta^k_l T^{hm}_{mh}$$

The D term becomes

$$DN \delta^k_l T^{hm}_{mh}$$

Enforcing the zero parallel trace yields the following constraints

$$2A + BN = 0 \tag{11}$$

$$2B + C + DN = 0 \tag{12}$$

Now contract i, l to get the diagonal trace.

The 0th order term becomes

$$T^{hk}_{jh} + T^{kh}_{hj}$$

The A term becomes

$$A \cdot \left[N \left(T^{hk}_{jh} + T^{kh}_{hj} \right) + 2\delta^k_j T^{hm}_{mh} \right]$$

The B term becomes

$$B \cdot \left[T^{hk}_{jh} + T^{kh}_{hj} + T^{hk}_{jh} + T^{kh}_{hj} \right]$$

The C term becomes

$$CN \delta^k_j T^{hm}_{mh}$$

The D term becomes

$$D \delta^k_j T^{hm}_{mh}$$

Enforcing the zero diagonal trace yields the following constraints

$$1 + AN + 2B = 0 \tag{13}$$

$$2A + CN + D = 0 \tag{14}$$

(11) - (14) are easily solved as

$$\begin{aligned} A &= \frac{-N}{N^2 - 4} = -\frac{3}{5} \\ B &= \frac{2}{N^2 - 4} = \frac{2}{5} \\ C &= \frac{2N^2 + 4}{(N^2 - 1)(N^2 - 4)} = \frac{11}{20} \\ D &= \frac{-6N}{(N^2 - 1)(N^2 - 4)} = -\frac{9}{20} \end{aligned}$$

Note that the S thus constructed is symmetric in upstairs and symmetric in downstairs, it must correspond to the 27 of the $8 \otimes 8$, which is confirmed by the octave script.

With this, we can try to adjust the signs in the 0th order term, as well as the A, B terms, and see if we can obtain other solutions. In fact, the following combination

$$
\begin{aligned}
A^{ik}_{jl} = & \, T^{ik}_{jl} - T^{ki}_{lj} \\
& + A \cdot \left[\delta^i_l \left(T^{hk}_{jh} - T^{kh}_{hj} \right) + \delta^k_j \left(T^{ih}_{hl} - T^{hi}_{lh} \right) \right] \\
& + B \cdot \left[\delta^i_j \left(T^{hk}_{lh} + T^{kh}_{hl} \right) - \delta^k_l \left(T^{ih}_{hj} + T^{hi}_{jh} \right) \right] \\
& + C \cdot \delta^i_l \delta^k_j T^{hm}_{mh} \\
& + D \cdot \delta^i_j \delta^k_l T^{hm}_{mh}
\end{aligned}
$$

yields a solution with

$$
A = -\frac{1}{3} \qquad B = C = D = 0
$$

which corresponds to the antisymmetric tensors (the $10 \oplus 10^*$ of $8 \otimes 8$, as can be verified by the octave script).

A^{ik}_{jl} flips sign only when both ik and jl flip signs. Thus we can divide it into two groups $A^{[ik]}_{jl}$ and $A^{ik}_{[jl]}$, which can be verified to each have rank 10 and represents the 10 and 10^* of $8 \otimes 8$.

Also note what we said earlier that the iterative process will converge to S was not true. It turns out iterative process converges to a tensor with 47 independent columns (which is $27 + 10 + 10^*$), but the iterative process gave us insights into the potential form of the traceless tensor which gave rise to S earlier.