We show that the construction

$$T_{klm} = \psi^T C \gamma_k \gamma_l \gamma_m \psi$$

transforms like a tensor in $SO(2n)$.

In fact, under the unitary transform $U = e^{\frac{i}{4}\omega_{ij}\sigma_{ij}} P_+$,

$$\psi \to e^{\frac{i}{4}\omega_{ij}\sigma_{ij}} P_+ \psi$$

which implies

$$\begin{aligned} T_{klm} &\to \psi^T P_+ e^{\frac{i}{4}\omega_{ij}\sigma_{ij}^T} C \gamma_k \gamma_l \gamma_m e^{\frac{i}{4}\omega_{ij}\sigma_{ij}} P_+ \psi \qquad \text{(by eq 34)} \\ &= \psi^T P_+ C \left[e^{-\frac{i}{4}\omega_{ij}\sigma_{ij}} \gamma_k \gamma_l \gamma_m e^{\frac{i}{4}\omega_{ij}\sigma_{ij}} \right] P_+ \psi \end{aligned}$$

Expanding the term inside the bracket (keeping up to 1st order of ω), we have

$$\begin{aligned} e^{-\frac{i}{4}\omega_{ij}\sigma_{ij}} \gamma_k \gamma_l \gamma_m e^{\frac{i}{4}\omega_{ij}\sigma_{ij}} &= \left(1 - \frac{i}{4}\omega_{ij}\sigma_{ij}\right) \gamma_k \gamma_l \gamma_m \left(1 + \frac{i}{4}\omega_{ij}\sigma_{ij}\right) \\ &= \gamma_k \gamma_l \gamma_m - \frac{i}{4}\omega_{ij}\left[\sigma_{ij}, \gamma_k \gamma_l \gamma_m\right] \end{aligned}$$

where by eq (8),

$$\begin{aligned} -\frac{i}{4}\omega_{ij}\left[\sigma_{ij}, \gamma_k\gamma_l\gamma_m\right] &= -\frac{i}{4}\omega_{ij}\left(\left[\sigma_{ij}, \gamma_k\right]\gamma_l\gamma_m + \gamma_k\left[\sigma_{ij}, \gamma_l\right]\gamma_m + \gamma_k\gamma_l\left[\sigma_{ij}, \gamma_m\right]\right) \\ &= -\frac{i}{4}\omega_{ij} \cdot (2i) \cdot \left[\left(\delta_{ik}\gamma_j - \delta_{jk}\gamma_i\right)\gamma_l\gamma_m + \gamma_k\left(\delta_{il}\gamma_j - \delta_{jl}\gamma_i\right)\gamma_m + \gamma_k\gamma_l\left(\delta_{im}\gamma_j - \delta_{jm}\gamma_i\right)\right] \\ &= \frac{1}{2}\left(\omega_{kj}\gamma_j - \omega_{ik}\gamma_i\right)\gamma_l\gamma_m + \frac{1}{2}\gamma_k\left(\omega_{lj}\gamma_j - \omega_{il}\gamma_i\right)\gamma_m + \frac{1}{2}\gamma_k\gamma_l\left(\omega_{mj}\gamma_j - \omega_{im}\gamma_i\right) \\ (\omega \text{ antisymmetric}) \qquad &= \omega_{kj}\gamma_j\gamma_l\gamma_m + \omega_{lj}\gamma_k\gamma_j\gamma_m + \omega_{mj}\gamma_k\gamma_l\gamma_j \end{aligned}$$

Combining the above, we eventually have

$$T_{klm} \to T_{klm} + \omega_{kj} T_{jlm} + \omega_{lj} T_{kjm} + \omega_{mj} T_{klj}$$

which indicates the 3-index object T_{klm} transforms like a tensor in $SO(2n)$

To be more explicit, we want to show that the above transformation is equivalent to $T_{klm} \to R_{kr}R_{ls}R_{mt}T_{rst}$, for some $R \in SO(2n)$.

Indeed, if we construct R from U as

$$R = e^{\omega_{ij}J_{(ij)}}$$

where $J_{(ij)}$'s are defined as $2n \times 2n$ matrix with all zeros except at $(i,j) = 1$. (This is similar to eq (14) on pp76 but we make ω_{ij} carry the negative signs). We have a mapping between $Spin(2n) \to SO(2n)$ (with former being the double covering of the latter).

Expanding near identity,

$$R = e^{\omega_{ij}J_{(ij)}} = I + \omega_{ij}J_{(ij)}$$

which gives

$$R_{kr}R_{ls}R_{mt} = (\delta_{kr} + \omega_{kr})(\delta_{ls} + \omega_{ls})(\delta_{mt} + \omega_{mt}) = \delta_{kr}\delta_{ls}\delta_{mt} + \omega_{kr}\delta_{ls}\delta_{mt} + \omega_{ls}\delta_{kr}\delta_{mt} + \omega_{mt}\delta_{kr}\delta_{ls}$$

Then we have

$$R_{kr}R_{ls}R_{mt}T_{rst} = T_{klm} + \omega_{kr}T_{rlm} + \omega_{ls}T_{ksm} + \omega_{mt}T_{klt}$$

as desired.

The important part of this exercise is that U must not be recognized as the rotation matrix that transforms T_{klm}. U is a unitary group generated from the σ_{ij}'s, while R is an orthogonal matrix generated from the $J_{(ij)}$'s. The connection between U and R is that they have the same set of ω_{ij}'s.