Let's explicitly find the representation of the subgroup $SU(n)$ in $SO(2n)$.

Starting with simplest case of $n=2$. Let $U \in SU(2)$ be written as

$$U = \begin{bmatrix} \alpha_R + i\alpha_I & \beta_R + i\beta_I \\ \gamma_R + i\gamma_I & \delta_R + i\delta_I \end{bmatrix}$$

Now apply U to $\begin{bmatrix} z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{bmatrix}$, we have

$$U\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \alpha_R + i\alpha_I & \beta_R + i\beta_I \\ \gamma_R + i\gamma_I & \delta_R + i\delta_I \end{bmatrix}\begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{bmatrix} = \begin{bmatrix} \alpha_R x_1 - \alpha_I y_1 + i(\alpha_I x_1 + \alpha_R y_1) + \beta_R x_2 - \beta_I y_2 + i(\beta_I x_2 + \beta_R y_2) \\ \gamma_R x_1 - \gamma_I y_1 + i(\gamma_I x_1 + \gamma_R y_1) + \delta_R x_2 - \delta_I y_2 + i(\delta_I x_2 + \delta_R y_2) \end{bmatrix} \equiv \begin{bmatrix} x_1' + iy_1' \\ x_2' + iy_2' \end{bmatrix}$$

We could make a mapping $C^2 \to R^4$ via

$$\begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}$$

Then when

$$\begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{bmatrix} \xrightarrow{U} \begin{bmatrix} x_1' + iy_1' \\ x_2' + iy_2' \end{bmatrix}$$

we would expect

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix}$$

where $R \in SO(4)$ be U's injection image into $SO(4)$.

In other words, we expect

$$R\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \alpha_R x_1 - \alpha_I y_1 + \beta_R x_2 - \beta_I y_2 \\ \gamma_R x_1 - \gamma_I y_1 + \delta_R x_2 - \delta_I y_2 \\ \alpha_I x_1 + \alpha_R y_1 + \beta_I x_2 + \beta_R y_2 \\ \gamma_I x_1 + \gamma_R y_1 + \delta_I x_2 + \delta_R y_2 \end{bmatrix}$$

or

$$R = \begin{bmatrix} \alpha_R & \beta_R & -\alpha_I & -\beta_I \\ \gamma_R & \delta_R & -\gamma_I & -\delta_I \\ \alpha_I & \beta_I & \alpha_R & \beta_R \\ \gamma_I & \delta_I & \gamma_R & \delta_R \end{bmatrix} = \begin{bmatrix} \mathrm{Re}U & -\mathrm{Im}U \\ \mathrm{Im}U & \mathrm{Re}U \end{bmatrix}$$

Since U^* is an equally valid representation of $SU(2)$, then we could also make

$$R = \begin{bmatrix} \mathrm{Re}U & \mathrm{Im}U \\ -\mathrm{Im}U & \mathrm{Re}U \end{bmatrix}$$

Which is also a valid injection into $SO(4)$.