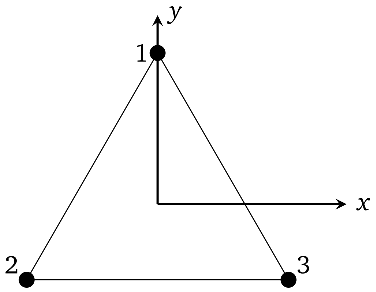
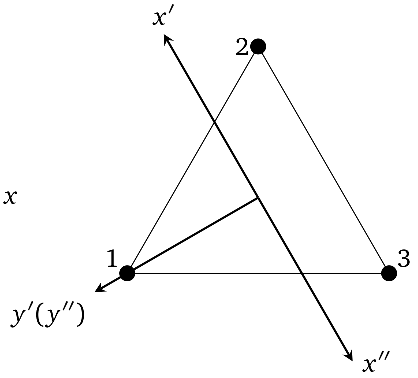
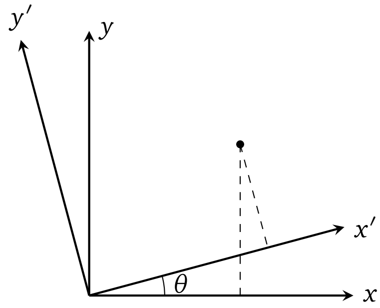






It is obvious that S_3 should be an invariance group of the system, but the "natural" representation is not so trivial to figure out. This is because the natural representation involves coordinates of a particular frame. First we will figure out the natural representation of $g = (12)$.

Given the original pose of the system, when we exchange 1 and 2, the coordinate frame has to be transformed to maintain invariance. First, $x - y$ needs to rotate by $\theta = 2\pi/3$ to $x' - y'$, and then x' needs to reflect to get to $x'' - y''$.

The two transforms (rotation and reflection) bring

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix} \to \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} -x\cos\theta - y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

In other words, the following matrix operation is the natural representation of $g = (12)$:

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} \to \begin{bmatrix} x_2 \\ y_2 \\ x_1 \\ y_1 \\ x_3 \\ y_3 \end{bmatrix} \to \begin{bmatrix} -x_2\cos\theta - y_2\sin\theta \\ -x_2\sin\theta + y_2\cos\theta \\ -x_1\cos\theta - y_1\sin\theta \\ -x_1\sin\theta + y_1\cos\theta \\ -x_3\cos\theta - y_3\sin\theta \\ -x_3\sin\theta + y_3\cos\theta \end{bmatrix} \Longrightarrow D((12)) = \begin{bmatrix} 0 & 0 & -\cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 & 0 \\ -\cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

With $\theta = 2\pi/3$, we have

$$D((12)) = \begin{bmatrix} 0 & 0 & 1/2 & -\sqrt{3}/2 & 0 & 0 \\ 0 & 0 & -\sqrt{3}/2 & -1/2 & 0 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 & 0 & 0 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & 0 & -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

For $g = (123)$, a simple rotation of $\theta = 2\pi/3$ without reflection will do, it's easy to see that

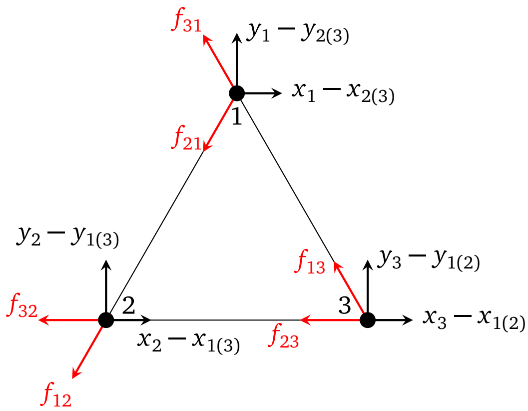
$$D((123)) = \begin{bmatrix} 0 & 0 & -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & -\sqrt{3}/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & \sqrt{3}/2 \\ 0 & 0 & 0 & 0 & -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 & 0 & 0 & 0 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can verify that $D((12))^2 = D((123))^3 = I$ (see octave scripts).

It is also clear that for this natural representation, the characters for the 3 equivalent classes of S_3 are $(6, 0, 0)$ (agreeing with the errata), hence the natural representation falls apart as $6 = 1 + \bar{1} + 2 + 2$. That is, without looking at the "dynamics" H of the system, as long as the system is invariant under S_3, the system should have 4 eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ with degeneracy of 1, 1, 2, 2 respectively.

When H is invariant under S_3, it means $[H, D(g)] = 0$ for all $g \in S_3$, which means for any given g, by choosing suitable basis, we can simultaneously diagonalize H and $D(g)$.

Denote such set of basis as $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_{3,1}, \mathbf{u}_{3,2}, \mathbf{u}_{4,1}, \mathbf{u}_{4,2}$, which are the corresponding eigenvectors of λ_{1-4}. Note within subspace spanned by $\mathbf{u}_{3,1}, \mathbf{u}_{3,2}$, we are free to choose another pair of basis without violating the diagonalization of H, but doing so will destroy the diagonal form of $D(g)$, similarly for $\mathbf{u}_{4,1}, \mathbf{u}_{4,2}$. In other words, if we do eigenvalue decomposition of H and come up with *an* orthonormal set $\{\mathbf{u}\}$, it may not always diagonalize $D(g)$, but it is guaranteed to transform $D(g)$ into block diagonal form. Running further orthogonalization within the two 2-dimensional subspaces will make $D(g)$ diagonal, and such orthogonalization may be g-dependent.



Now write the dynamics of the system. Let x_i, y_i be particle i's offset from its equilibrium position. For small offsets, the positive directions of the spring force are defined by the diagram. For example, f_{31} is the push force from 3 on 1, due to the relative offset $(x_1 - x_3, y_1 - y_3)$.

Projecting the relative offsets along the springs, we have the following

$$
\begin{aligned}
f_{21} &= k\left[\frac{x_1 - x_2}{2} + \frac{\sqrt{3}}{2}(y_1 - y_2)\right] \\
f_{31} &= k\left[\frac{x_1 - x_3}{2} - \frac{\sqrt{3}}{2}(y_1 - y_3)\right] \\
f_{12} &= k\left[\frac{x_2 - x_1}{2} + \frac{\sqrt{3}}{2}(y_2 - y_1)\right] \\
f_{32} &= k(x_2 - x_3) \\
f_{13} &= k\left[\frac{x_3 - x_1}{2} - \frac{\sqrt{3}}{2}(y_3 - y_1)\right] \\
f_{23} &= k(x_3 - x_2)
\end{aligned}
$$

Projecting back to x, y directions,

$$
\begin{aligned}
\frac{d^2 x_1}{dt^2} &= \frac{1}{m}\left(-\frac{1}{2}f_{21} - \frac{1}{2}f_{31}\right) = \frac{k}{m}\left(-\frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{\sqrt{3}}{4}y_2 - \frac{\sqrt{3}}{4}y_3\right) \\
\frac{d^2 y_1}{dt^2} &= \frac{1}{m}\left(-\frac{\sqrt{3}}{2}f_{21} + \frac{\sqrt{3}}{2}f_{31}\right) = \frac{k}{m}\left(\frac{\sqrt{3}}{4}x_2 - \frac{\sqrt{3}}{4}x_3 - \frac{3}{2}y_1 + \frac{3}{4}y_2 + \frac{3}{4}y_3\right) \\
\frac{d^2 x_2}{dt^2} &= \frac{1}{m}\left(-\frac{1}{2}f_{12} - f_{32}\right) = \frac{k}{m}\left(\frac{1}{4}x_1 - \frac{5}{4}x_2 + x_3 + \frac{\sqrt{3}}{4}y_1 - \frac{\sqrt{3}}{4}y_2\right) \\
\frac{d^2 y_2}{dt^2} &= \frac{1}{m}\left(-\frac{\sqrt{3}}{2}f_{12}\right) = \frac{k}{m}\left(\frac{\sqrt{3}}{4}x_1 - \frac{\sqrt{3}}{4}x_2 + \frac{3}{4}y_1 - \frac{3}{4}y_2\right) \\
\frac{d^2 x_3}{dt^2} &= \frac{1}{m}\left(-\frac{1}{2}f_{13} - f_{23}\right) = \frac{k}{m}\left(\frac{1}{x}x_1 + x_2 - \frac{5}{4}x_3 - \frac{\sqrt{3}}{4}y_1 + \frac{\sqrt{3}}{4}y_3\right) \\
\frac{d^2 y_3}{dt^2} &= \frac{1}{m}\left(\frac{\sqrt{3}}{2}f_{13}\right) = \frac{k}{m}\left(-\frac{\sqrt{3}}{4}x_1 + \frac{\sqrt{3}}{4}x_3 + \frac{3}{4}y_1 - \frac{3}{4}y_3\right)
\end{aligned}
$$

Letting $k/m = 1$ and use the definition of H

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{x}_3 \\ \ddot{y}_3 \end{bmatrix} = -H \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

we have

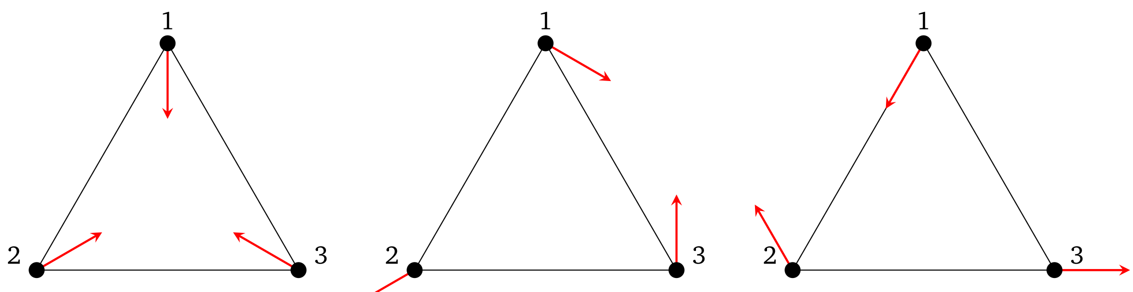
$$H = \begin{bmatrix} 1/2 & 0 & -1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ 0 & 3/2 & -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 5/4 & \sqrt{3}/4 & -1 & 0 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 & 0 & 0 \\ -1/4 & \sqrt{3}/4 & -1 & 0 & 5/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & 0 & 0 & -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

Resorting to octave, we can readily verify that $[H, D((12))] = [H, D((123))] = 0$. Also running eigenvalue decomposition on H, we identify H's eigenvalues as 3, 3/2, 3/2, 0, 0, 0. From the discussion above on the degeneracy, we identify $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 3/2$ with λ_3, λ_4 having degeneracy of 2. In fact, for this particular H, λ_2 and λ_3 happen to coincide which makes its eigenspace 3-dimensional.

Now for $\lambda_1 = 3, \lambda_4 = 3/2$, the numerical eigenvectors are recognized as

$$\mathbf{u}_{\lambda=3} = \begin{bmatrix} 0 \\ -1/\sqrt{3} \\ 1/2 \\ \sqrt{3}/6 \\ -1/2 \\ \sqrt{3}/6 \end{bmatrix} \propto \begin{bmatrix} 0 \\ -1 \\ \sqrt{3}/2 \\ 1/2 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \qquad \mathbf{u}_{\lambda=3/2,1} = \begin{bmatrix} 1/2 \\ -\sqrt{3}/6 \\ -1/2 \\ -\sqrt{3}/6 \\ 0 \\ 1/\sqrt{3} \end{bmatrix} \propto \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{u}_{\lambda=3/2,2} = \begin{bmatrix} -\sqrt{3}/6 \\ -1/2 \\ -\sqrt{3}/6 \\ 1/2 \\ 1/\sqrt{3} \\ 0 \end{bmatrix} \propto \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ 1 \\ 0 \end{bmatrix}$$

which describe the following modes.



$\mathbf{u}_{\lambda=3}$ $\mathbf{u}_{\lambda=3/2,1}$ $\mathbf{u}_{\lambda=3/2,2}$

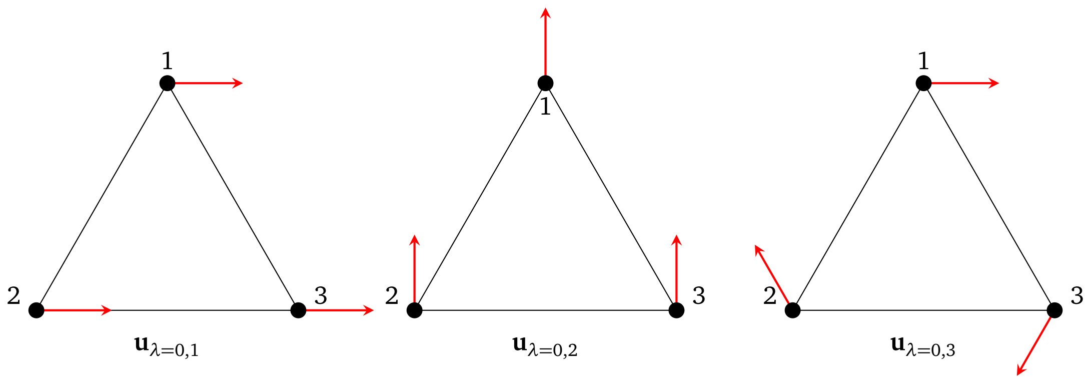
The numerical eigenvectors for the 0 eigenvalue are not recognizable as closed form. But we know they can be rotated as long as they form the orthonormal basis of the 3-dimensional subspace of eigenvalue 0. For the zero-frequency mode, two of them can be constructed easily (corresponding to translation in x and y direction)

$$\mathbf{u}_{\lambda=0,1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \propto \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 0 \end{bmatrix} \qquad \mathbf{u}_{\lambda=0,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \propto \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}$$

Now with 5 out of 6 orthonormal vectors figured out, the 6th is determined up to an overall sign:

$$\mathbf{u}_{\lambda=0,3} = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -\sqrt{3}/6 \\ 1/2 \\ -\sqrt{3}/6 \\ -1/2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ \sqrt{3}/2 \\ -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

These correspond to the following modes



Now the complete U matrix is

$$\begin{bmatrix} 0 & 1/2 & -\sqrt{3}/6 & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ -1/\sqrt{3} & -\sqrt{3}/6 & -1/2 & 0 & 1/\sqrt{3} & 0 \\ 1/2 & -1/2 & -\sqrt{3}/6 & 1/\sqrt{3} & 0 & -\sqrt{3}/6 \\ \sqrt{3}/6 & -\sqrt{3}/6 & 1/2 & 0 & 1/\sqrt{3} & 1/2 \\ -1/2 & 0 & 1/\sqrt{3} & 1/\sqrt{3} & 0 & -\sqrt{3}/6 \\ \sqrt{3}/6 & 1/\sqrt{3} & 0 & 0 & 1/\sqrt{3} & -1/2 \end{bmatrix}$$

Now it's easy to verify that $U^\dagger U = I$ and $U^\dagger H U = \mathrm{diag}(3, 3/2, 3/2, 0, 0, 0)$. But $U^\dagger D((12))U$ is only block diagonal $\mathrm{diag}(1, M_{2\times 2}, N_{2\times 2}, -1)$, where

$$M = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \qquad N = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

Since both M and N are similar to the 2×2 representation of (12) in S_3 (as can be told from the same trace and determinant):

$$D^{(2)}((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

it is clear that $D((12))$ is indeed similar to the direct sum $D^{(1)} \oplus D^{(2)} \oplus D^{(2)} \oplus D^{(\bar{1})}$

In fact, let's be more explicit, i.e., let's find another set of two basis for the subspace spanned by $\mathbf{u}_{\lambda=3/2,1}, \mathbf{u}_{\lambda=3/2,2}$, and the subspace spanned by $\mathbf{u}_{\lambda=0,1}, \mathbf{u}_{\lambda=0,2}$ so $D((12))$ is diagonalized (in other words, to diagonalize the two 2×2 blocks in the block diagonal $U^\dagger D((12))U$).

To do this, let U_M be the eigen decomposition of M so $U_M^\dagger M U_M$ is diagonalized. Recall that

$$M = \begin{bmatrix} \mathbf{u}_{\lambda=3/2,1}^\dagger \\ \mathbf{u}_{\lambda=3/2,2}^\dagger \end{bmatrix} D((12)) \begin{bmatrix} \mathbf{u}_{\lambda=3/2,1} & \mathbf{u}_{\lambda=3/2,2} \end{bmatrix}$$

so if we replace column 2 and 3 in U with the two columns

$$\begin{bmatrix} \mathbf{u}_{\lambda=3/2,1} & \mathbf{u}_{\lambda=3/2,2} \end{bmatrix} U_M$$

the new U should diagonalize the M block when conjugating $D((12))$. Similarly column 4 and 5 are to be replaced with

$$\begin{bmatrix} \mathbf{u}_{\lambda=0,1} & \mathbf{u}_{\lambda=0,2} \end{bmatrix} U_N$$

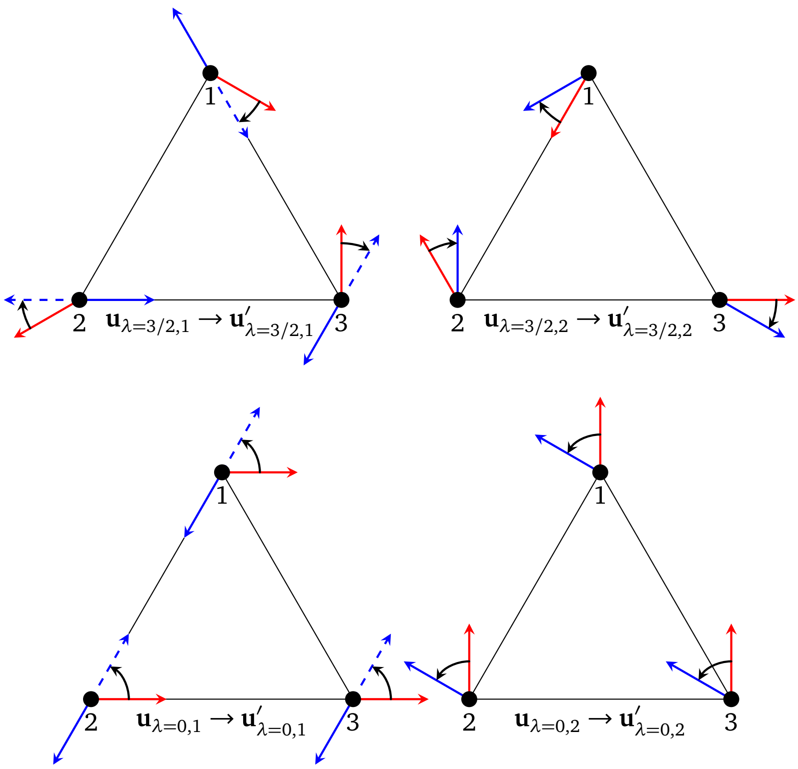
The new U matrix should simultaneously diagonlize H and $D((12))$.

Resorting to the octave script, we obtain the replacement columns as the following:

$$\mathbf{u}'_{\lambda=3/2,1} = \begin{bmatrix} -\sqrt{3}/6 \\ 1/2 \\ 1/\sqrt{3} \\ 0 \\ -\sqrt{3}/6 \\ -1/2 \end{bmatrix} \propto \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \\ 1 \\ 0 \\ -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \qquad \mathbf{u}'_{\lambda=3/2,2} = \begin{bmatrix} -1/2 \\ -\sqrt{3}/6 \\ 0 \\ 1/\sqrt{3} \\ 1/2 \\ -\sqrt{3}/6 \end{bmatrix} \propto \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \\ 0 \\ 1 \\ \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$

$$\mathbf{u}'_{\lambda=0,1} = \begin{bmatrix} -\sqrt{3}/6 \\ -1/2 \\ -\sqrt{3}/6 \\ -1/2 \\ -\sqrt{3}/6 \\ -1/2 \end{bmatrix} \propto \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \qquad \mathbf{u}'_{\lambda=0,2} = \begin{bmatrix} -1/2 \\ \sqrt{3}/6 \\ -1/2 \\ \sqrt{3}/6 \\ -1/2 \\ \sqrt{3}/6 \end{bmatrix} \propto \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \\ -\sqrt{3}/2 \\ 1/2 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

They can be visualized as the blue vectors below, where the U_M, U_N matrices are to be visualized as a rotation followed by reflection (where the reflection is not essential, it is a result of an arbitrary sign freedom chosen by octave).



This rotated U matrix can be verified to produce $U'^{\dagger}D((12))U' = \text{diag}(1, -1, 1, -1, 1, -1)$.

We can do the same exercise for $D((123))$, but we lose the nice visualization since U_M, U_N now have complex numbers as they must since they are supposed to transform M, N into a similar matrix of the representation of (123) in S_3, which is complex.