Given a Lorentz transformation $\Lambda = e^{i\omega_{\mu\nu}J^{\mu\nu}/2}$, we will see how this transforms the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Keep in mind that the Dirac spinor ψ is composed of the two Weyl spinors, i.e., $\psi = \begin{bmatrix} u \\ v \end{bmatrix}$, and that

$$\gamma^\mu = \begin{bmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{bmatrix}$$

The generators $J^{\mu\nu}$ of Λ are the following aliases

$$J^{0i} = K_i \qquad J^{ij} = \epsilon_{ijk} J_k$$

then considering antisymmetry in the exponent of Λ,

$$\Lambda = e^{i(\omega_{23}J_1 + \omega_{31}J_2 + \omega_{12}J_3 + \omega_{01}K_1 + \omega_{02}K_2 + \omega_{03}K_3)}$$

Now following the standard procedure to map this to Weyl spinor representations, i.e., write $\Lambda = e^{i[\alpha_i(J_i+iK_i)+\beta_i(J_i-iK_i)]}$, we have

$$\begin{cases} \alpha_1 + \beta_1 = \omega_{23} \\ \alpha_1 - \beta_1 = -i\omega_{01} \end{cases} \implies \begin{cases} \alpha_1 = \frac{\omega_{23} - i\omega_{01}}{2} \\ \beta_1 = \frac{\omega_{23} + i\omega_{01}}{2} \end{cases} \qquad \text{etc.}$$

So under Λ, the right-handed Weyl spinor u and the left-handed Weyl spinor v transform as

$$u \to S_R u = e^{i\alpha_i \sigma_i} u$$
$$v \to S_L v = e^{i\beta_i \sigma_i} v$$

Note that

$$(S_R^{-1})^\dagger = (e^{-i\alpha_i\sigma_i})^\dagger = e^{i\alpha_i^*\sigma_i} = S_L \qquad \text{and vice versa}$$
$$(S_L^{-1})^\dagger = S_R$$

Now for the Dirac spinor $\psi = \begin{bmatrix} u \\ v \end{bmatrix}$, under Λ, it must transform as

$$\psi = \begin{bmatrix} u \\ v \end{bmatrix} \to \begin{bmatrix} S_R u \\ S_L v \end{bmatrix} = \begin{bmatrix} S_R & 0 \\ 0 & S_L \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \equiv S\psi$$

We want to show that when $\psi \to S\psi$ under Λ, the Dirac equation still holds, i.e.

$$(i\gamma^\mu \partial'_\mu - m)S\psi = 0$$

This is equivalent to claiming that

$$\left(i \begin{bmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{bmatrix} \partial'_\mu - m \right) \begin{bmatrix} S_R & 0 \\ 0 & S_L \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

which is equivalent to two claims

$$i\sigma^\mu \partial'_\mu S_R u - m S_L v = 0 \tag{1}$$
$$i\bar{\sigma}^\mu \partial'_\mu S_L v - m S_R u = 0 \tag{2}$$

From the discussion of Weyl spinor's Lorentz invariance, we know

$$i\sigma^\mu \partial'_\mu S_R u = (S_R^{-1})^\dagger \cdot i\sigma^\mu \partial_\mu u = S_L \cdot i\sigma^\mu \partial_\mu u$$

therefore, (1) is indeed true since

$$i\sigma^\mu \partial'_\mu S_R u - m S_L v = S_L(i\sigma^\mu \partial_\mu u - mv) = 0$$

and of course, similarly

$$i\bar{\sigma}^{\mu}\partial'_{\mu}S_L v - mS_R u = S_R(i\bar{\sigma}^{\mu}\partial_{\mu}v - mu) = 0$$

More compactly, we write the transformation of Dirac equation under Λ (hence $S(\Lambda)$) as

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \rightarrow (i\gamma^{\mu}\partial'_{\mu} - m)S\psi = S(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

Note the relation in the book (pp 476) should be corrected in our context as

$$S^{-1}\gamma^{\lambda}S = \Lambda^{\lambda}{}_{\mu}\gamma^{\mu}$$

which can be seen by

$$\begin{aligned}
\begin{bmatrix} S_R^{-1} & 0 \\ 0 & S_L^{-1} \end{bmatrix}\begin{bmatrix} 0 & \bar{\sigma}^{\lambda} \\ \sigma^{\lambda} & 0 \end{bmatrix}\begin{bmatrix} S_R & 0 \\ 0 & S_L \end{bmatrix} = \Lambda^{\lambda}{}_{\mu}\begin{bmatrix} 0 & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & 0 \end{bmatrix} \quad &\Longleftrightarrow \\
\begin{bmatrix} 0 & S_R^{-1}\bar{\sigma}^{\lambda} \\ S_L^{-1}\sigma^{\lambda} & 0 \end{bmatrix}\begin{bmatrix} S_R & 0 \\ 0 & S_L \end{bmatrix} = \Lambda^{\lambda}{}_{\mu}\begin{bmatrix} 0 & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & 0 \end{bmatrix} \quad &\Longleftrightarrow \\
\begin{bmatrix} 0 & S_R^{-1}\bar{\sigma}^{\lambda}S_L \\ S_L^{-1}\sigma^{\lambda}S_R & 0 \end{bmatrix} = \begin{bmatrix} \Lambda^{\lambda}{}_{\mu}\bar{\sigma}^{\mu} \\ \Lambda^{\lambda}{}_{\mu}\sigma^{\mu} \end{bmatrix}
\end{aligned}$$

or

$$S_R^{-1}\bar{\sigma}^{\lambda}S_L = \Lambda^{\lambda}{}_{\mu}\bar{\sigma}^{\mu}, \qquad S_L^{-1}\sigma^{\lambda}S_R = \Lambda^{\lambda}{}_{\mu}\sigma^{\mu}$$

which are the "Weyl map" equation (5-6) proved in the last document.