First we will prove eq (41).
Recall that

$$J_{\mu\nu} = i\big(x_\mu \partial_\nu - x_\nu \partial_\mu\big) \tag{1}$$
$$\partial_\mu x_\nu = \eta_{\mu\nu} \tag{2}$$

Now

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= \big[i\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big), i\big(x_\rho\partial_\sigma - x_\sigma\partial_\rho\big)\big] = -\big[\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big), \big(x_\rho\partial_\sigma - x_\sigma\partial_\rho\big)\big] \\
&= -\big[\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big)\big(x_\rho\partial_\sigma - x_\sigma\partial_\rho\big) - \big(x_\rho\partial_\sigma - x_\sigma\partial_\rho\big)\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big)\big] \\
&= -\Bigg[\Bigg(\underbrace{x_\mu\partial_\nu x_\rho\partial_\sigma}_{A} - \underbrace{x_\mu\partial_\nu x_\sigma\partial_\rho}_{B} - \underbrace{x_\nu\partial_\mu x_\rho\partial_\sigma}_{C} + \underbrace{x_\nu\partial_\mu x_\sigma\partial_\rho}_{D}\Bigg) - \\
&\qquad \Bigg(\underbrace{x_\rho\partial_\sigma x_\mu\partial_\nu}_{E} - \underbrace{x_\rho\partial_\sigma x_\nu\partial_\mu}_{F} - \underbrace{x_\sigma\partial_\rho x_\mu\partial_\nu}_{G} + \underbrace{x_\sigma\partial_\rho x_\nu\partial_\mu}_{H}\Bigg)\Bigg]
\end{aligned}$$

where

$$\begin{aligned}
A &= x_\mu\partial_\nu x_\rho\partial_\sigma = x_\mu\partial_\nu\left(x_\rho\partial_\sigma\right) = x_\mu\left[\left(\partial_\nu x_\rho\right)\partial_\sigma + x_\rho\partial_\nu\partial_\sigma\right] \\
&= x_\mu\eta_{\nu\rho}\partial_\sigma + x_\mu x_\rho\partial_\nu\partial_\sigma \\
B &= x_\mu\eta_{\nu\sigma}\partial_\rho + x_\mu x_\sigma\partial_\nu\partial_\rho \\
C &= x_\nu\eta_{\mu\rho}\partial_\sigma + x_\nu x_\rho\partial_\mu\partial_\sigma \\
D &= x_\nu\eta_{\mu\sigma}\partial_\rho + x_\nu x_\sigma\partial_\mu\partial_\rho \\
E &= x_\rho\eta_{\sigma\mu}\partial_\nu + x_\rho x_\mu\partial_\sigma\partial_\nu \\
F &= x_\rho\eta_{\sigma\nu}\partial_\mu + x_\rho x_\nu\partial_\sigma\partial_\mu \\
G &= x_\sigma\eta_{\rho\mu}\partial_\nu + x_\sigma x_\mu\partial_\rho\partial_\nu \\
H &= x_\sigma\eta_{\rho\nu}\partial_\mu + x_\sigma x_\nu\partial_\rho\partial_\mu
\end{aligned}$$

The second derivatives cancel between A/E, B/G, C/F and D/H, which leaves us with

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= -\Bigg[\underbrace{\eta_{\nu\rho}\left(x_\mu\partial_\sigma - x_\sigma\partial_\mu\right)}_{A-H} - \underbrace{\eta_{\nu\sigma}\left(x_\mu\partial_\rho - x_\rho\partial_\mu\right)}_{B-F} - \underbrace{\eta_{\mu\rho}\left(x_\nu\partial_\sigma - x_\sigma\partial_\nu\right)}_{C-G} + \underbrace{\eta_{\mu\sigma}\left(x_\nu\partial_\rho - x_\rho\partial_\nu\right)}_{D-E}\Bigg] \\
&= -i\left[-\eta_{\nu\rho}\cdot i\left(x_\mu\partial_\sigma - x_\sigma\partial_\mu\right) + \eta_{\nu\sigma}\cdot i\left(x_\mu\partial_\rho - x_\rho\partial_\mu\right) + \eta_{\mu\rho}\cdot i\left(x_\nu\partial_\sigma - x_\sigma\partial_\nu\right) - \eta_{\mu\sigma}\cdot i\left(x_\nu\partial_\rho - x_\rho\partial_\nu\right)\right] \\
&= -i\left(\eta_{\nu\sigma}J_{\mu\rho} + \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}\right)
\end{aligned}$$

Next we show eq (46):

$$\begin{aligned}
[J_{\mu\nu}, P_\rho] &= i\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big)i\partial_\rho - i\partial_\rho i\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big) \\
&= i^2\left[x_\mu\partial_\nu\partial_\rho - x_\nu\partial_\mu\partial_\rho - \partial_\rho\left(x_\mu\partial_\nu\right) + \partial_\rho\left(x_\nu\partial_\mu\right)\right] \\
&= i^2\left[x_\mu\partial_\nu\partial_\rho - x_\nu\partial_\mu\partial_\rho - \left(\eta_{\rho\mu}\partial_\nu + x_\mu\partial_\rho\partial_\nu\right) + \left(\eta_{\rho\nu}\partial_\mu + x_\nu\partial_\rho\partial_\mu\right)\right] \\
&= -i\left(\eta_{\rho\mu}i\partial_\nu - \eta_{\rho\nu}i\partial_\mu\right) \\
&= -i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right)
\end{aligned}$$

Next we show that $P_\rho P^\rho$ is a Casimir invariant of the Poincaré algebra, i.e., it commutes with every generator $J_{\mu\nu}$ (it commutes with the P's obviously).

Indeed

$$\begin{aligned}
[J_{\mu\nu}, P_\rho P^\rho] &= [J_{\mu\nu}, P_\rho]P^\rho + P_\rho[J_{\mu\nu}, P^\rho] \\
&= [J_{\mu\nu}, P_\rho]\eta^{\sigma\rho}P_\sigma + P_\rho[J_{\mu\nu}, \eta^{\sigma\rho}P_\sigma] \\
&= \left[-i\big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\big)\eta^{\sigma\rho}P_\sigma\right] + P_\rho\left[-i\big(\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu\big)\eta^{\sigma\rho}\right] \\
&= -i\left(\eta_{\mu\rho}\eta^{\sigma\rho}P_\nu P_\sigma - \eta_{\nu\rho}\eta^{\sigma\rho}P_\mu P_\sigma + \eta_{\mu\sigma}\eta^{\sigma\rho}P_\rho P_\nu - \eta_{\nu\sigma}\eta^{\sigma\rho}P_\rho P_\mu\right) \\
&= -i\left(\delta^\sigma_\mu P_\nu P_\sigma - \delta^\sigma_\nu P_\mu P_\sigma + \delta^\rho_\mu P_\rho P_\nu - \delta^\rho_\nu P_\rho P_\mu\right) \\
&= 0
\end{aligned}$$

Next we show eq (48):

$$\begin{aligned}
W_\sigma P^\sigma &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}P^\rho P^\sigma \\
&= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\cdot i\,(x^\mu\partial^\nu - x^\nu\partial^\mu)\cdot i\partial^\rho\cdot i\partial^\sigma \\
&= \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\cdot(x^\mu\partial^\nu\partial^\rho\partial^\sigma - x^\nu\partial^\mu\partial^\rho\partial^\sigma)
\end{aligned}$$

It's easy to see all six terms corresponding to the same x^μ sum to zero because the ∂'s commute, but the ϵ changes sign upon exchange of two indices.

Next we show that eq (48) implies W_σ transforms like a vector (with some caveats). In the following all the digits actually mean an index subscript so $1 \longleftrightarrow \mu_1$.

By definition

$$W_4 = -\frac{1}{2}\epsilon_{1234}J^{12}P^3$$

We would like to raise the lower index to upper one by contracting it with η^{45}, so

$$\begin{aligned}
W^5 = \eta^{45}W_4 &= -\frac{1}{2}\epsilon_{1234}\eta^{45}J^{12}P^3 \\
&= -\frac{1}{2}\epsilon_{1234}\eta^{45}\eta^{16}\eta^{27}J_{67}\eta^{38}P_8 \\
&= -\frac{1}{2}\left(\epsilon_{1234}\eta^{45}\eta^{16}\eta^{27}\eta^{38}\right)J_{67}P_8 \\
&= -\frac{1}{2}\epsilon_{6785}\mathrm{det}\eta J_{67}P_8 \qquad (\text{define } E^{6785} \equiv \epsilon_{6785}\mathrm{det}\eta) \\
&= -\frac{1}{2}E^{6785}J_{67}P_8
\end{aligned}$$

The caveat is, the "upper index" flavor of W will have to use E^{6785} which differs by the regular ϵ_{6785} by a factor of $\mathrm{det}\eta = -1$.

Now we show the commutator relationship between W and Poincaré generators.

First,

$$[P_1, W_2] = 0$$

In fact

$$
\begin{aligned}
[P_1, W_2] &= \left[P_1, -\frac{1}{2}\epsilon_{3452} J^{34} P^5\right] \\
&= -\frac{1}{2}\epsilon_{3452}[P_1, J^{34}P^5] \\
&= -\frac{1}{2}\epsilon_{3452}\left\{[P_1, J^{34}]P^5 + J^{34}[P_1, P^5]\right\} \\
&= -\frac{1}{2}\epsilon_{3452}\left\{[P_1, \eta^{36}\eta^{47}J_{67}]P^5 + J^{34}[P_1, \eta^{58}P_8]\right\} \qquad ([P_1, P_8] = 0) \\
&= -\frac{1}{2}\epsilon_{3452}\eta^{36}\eta^{47}[P_1, J_{67}]P^5 \\
&= -\frac{1}{2}\epsilon_{3452}\eta^{36}\eta^{47}\left[i\left(\eta_{61}P_7 - \eta_{71}P_6\right)\right]P^5 \\
&= -\frac{i}{2}\epsilon_{3452}\left[(\eta^{36}\eta_{61})(\eta^{47}P_7) - (\eta^{47}\eta_{71})(\eta^{36}P_6)\right]P^5 \\
&= -\frac{i}{2}\epsilon_{3452}\left[\delta^3_1 P^4 - \delta^4_1 P^3\right]P^5 \\
&= -\frac{i}{2}\epsilon_{1452}P^4P^5 + \frac{i}{2}\epsilon_{3152}P^3P^5
\end{aligned}
$$

which is easily seen to vanish since P^4P^5 commute but ϵ_{1452} changes sign upon exchanging index 4 and 5.

Second,

$$[J_{\mu\nu}, W_\rho] = i\left(\eta_{\nu\rho}W_\mu - \eta_{\mu\rho}W_\nu\right)$$

To see this, note that since $W_\rho P^\rho = 0$, we have

$$
\begin{aligned}
0 &= [J_{\mu\nu}, W_\rho P^\rho] \\
&= [J_{\mu\nu}, W_\rho]P^\rho + W_\rho[J_{\mu\nu}, P^\rho] \\
&= [J_{\mu\nu}, W_\rho]P^\rho + W_\rho\eta^{\rho\sigma}[J_{\mu\nu}, P_\sigma] \\
&= [J_{\mu\nu}, W_\rho]P^\rho + W_\rho\eta^{\rho\sigma}(-i)\left(\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu\right) \\
&= [J_{\mu\nu}, W_\rho]P^\rho + W_\rho(-i)\left(\delta^\rho_\mu P_\nu - \delta^\rho_\nu P_\mu\right) \qquad \Longrightarrow \\
[J_{\mu\nu}, W_\rho]P^\rho &= i\left(W_\mu P_\nu - W_\nu P_\mu\right) \\
&= i\left(W_\mu\eta_{\nu\rho} - W_\nu\eta_{\mu\rho}\right)P^\rho
\end{aligned}
$$

If they equal as differential operators for arbitrary states, we must have $[J_{\mu\nu}, W_\rho] = i\left(\eta_{\nu\rho}W_\mu - \eta_{\mu\rho}W_\nu\right)$.

Now we show that $W_\rho W^\rho$ is the second Casimir invariant of Poincaré algebra.

First

$$[P_\mu, W_\rho W^\rho] = [P_\mu, W_\rho]W^\rho + W_\rho[P_\mu, \eta^{\rho\sigma}W_\sigma] = 0$$

Next, we see that

$$
\begin{aligned}
[J_{\mu\nu}, W_\rho W^\rho] &= [J_{\mu\nu}, W_\rho]W^\rho + W_\rho[J_{\mu\nu}, W^\rho] \\
&= [J_{\mu\nu}, W_\rho]W^\rho + W_\rho[J_{\mu\nu}, \eta^{\rho\sigma}W_\sigma] \\
&= i\left(\eta_{\nu\rho}W_\mu - \eta_{\mu\rho}W_\nu\right)W^\rho + W_\rho\eta^{\rho\sigma}(i)\left(\eta_{\nu\sigma}W_\mu - \eta_{\mu\sigma}W_\nu\right) \\
&= i\left(W_\mu W_\nu - W_\nu W_\mu\right) + iW_\rho\left(\delta^\rho_\nu W_\mu - \delta^\rho_\mu W_\nu\right) \\
&= i\left(W_\mu W_\nu - W_\nu W_\mu\right) + i\left(W_\nu W_\mu - W_\mu W_\nu\right) = 0
\end{aligned}
$$

Lastly we show eq (50).

$$
\begin{aligned}
[W_1, W_2] &= -\frac{1}{2}\epsilon_{3451}\left[J^{34}P^5, W_2\right] \\
&= -\frac{1}{2}\epsilon_{3451}\left\{J^{34}\left[P^5, W_2\right] + \left[J^{34}, W_2\right]P^5\right\} \\
&= -\frac{1}{2}\epsilon_{3451}\left\{J^{34}\left[\eta^{56}P_6, W_2\right] + \left[J^{34}, W_2\right]P^5\right\} \\
&= -\frac{1}{2}\epsilon_{3451}\left[J^{34}, W_2\right]P^5 \\
&= -\frac{1}{2}\epsilon_{3451}\eta^{36}\eta^{47}\left[J_{67}, W_2\right]P^5 \\
&= -\frac{1}{2}\epsilon_{3451}\eta^{36}\eta^{47}(i)\left(\eta_{72}W_6 - \eta_{62}W_7\right)P^5 \\
&= -\frac{i}{2}\epsilon_{3451}\left(\eta^{36}\delta_2^4 W_6 - \eta^{47}\delta_2^3 W_7\right)P^5 \\
&= -\frac{i}{2}\left(\epsilon_{3251}W^3P^5 - \epsilon_{2451}W^4P^5\right) \\
&= -\frac{i}{2}\left(\epsilon_{3251}W^3P^5 + \epsilon_{4251}W^4P^5\right)
\end{aligned}
$$

Now remember the sum ranges over indices 3,4 and 5, when indices 1,2,5 have taken unique value, 3 and 4 must be equal to the only remaining value to make ϵ non-vanishing. We rename this as index 6, so the above becomes

$$
\begin{aligned}
[W_1, W_2] &= -\frac{i}{2}\left(\epsilon_{6251}W^6P^5 + \epsilon_{6251}W^6P^5\right) \\
&= -i\epsilon_{6251}W^6P^5 \qquad \text{(applying two transpositions (16)(56))} \\
&= -i\epsilon_{1265}W^6P^5
\end{aligned}
$$

renaming $1 \longleftrightarrow \mu, 2 \longleftrightarrow \nu, 6 \longleftrightarrow \rho, 5 \longleftrightarrow \sigma$, we get

$$
[W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma
$$