1. The energy spectrum is given in (13.85)

$$\frac{dI}{d\nu} = \frac{z^2 e^2 \gamma \omega_p}{\pi c}\left[\left(1+2\nu^2\right)\ln\left(1+\frac{1}{\nu^2}\right)-2\right] \tag{1}$$

With

$$\nu = \frac{\omega}{\gamma \omega_p} \tag{2}$$

the total number of transition radiation quanta with $\omega > \omega_p$ is thus the integral

$$N_\gamma = \int_{1/\gamma}^{\infty} \frac{dI/\hbar\omega}{d\nu} d\nu = \frac{z^2 e^2}{\pi \hbar c} \overbrace{\int_{1/\gamma}^{\infty}\left[\left(1+2\nu^2\right)\ln\left(1+\frac{1}{\nu^2}\right)-2\right]\frac{d\nu}{\nu}}^{A} \tag{3}$$

Denote $u = 1/\nu^2$, the integral A becomes

$$\begin{aligned} A &= \int_{\gamma^2}^{0}\left[\left(1+\frac{2}{u}\right)\ln(1+u)-2\right]u^{1/2}\left(-\frac{1}{2}u^{-3/2}\right)du \\ &= \frac{1}{2}\int_0^{\gamma^2}\frac{1}{u}\left[\left(1+\frac{2}{u}\right)\ln(1+u)-2\right]du \\ &= \frac{1}{2}\left[\overbrace{\int_0^{\gamma^2}\frac{\ln(1+u)}{u}du}^{A_1}+\overbrace{\int_0^{\gamma^2}\frac{2\ln(1+u)}{u^2}du}^{A_2}-\overbrace{\int_0^{\gamma^2}\frac{2}{u}du}^{A_3}\right] \end{aligned} \tag{4}$$

From the integral representation of dilogarithm function (see DLMF 25.12.2), we have

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}dt \tag{5}$$

we have

$$A_1 = -\mathrm{Li}_2\left(-\gamma^2\right) \tag{6}$$

Using DLMF 25.12.4

$$\mathrm{Li}_2(-z)+\mathrm{Li}_2\left(-\frac{1}{z}\right) = -\frac{\pi^2}{6}-\frac{1}{2}(\ln z)^2 \tag{7}$$

we can rewrite A_1 as

$$A_1 = \frac{\pi^2}{6}+\frac{1}{2}\left(\ln\gamma^2\right)^2-\mathrm{Li}_2\left(-\frac{1}{\gamma^2}\right) \tag{8}$$

With the series expansion of dilogarithm function (see DLMF 25.12.1)

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^2} \tag{9}$$

we have

$$A_1 = \frac{\pi^2}{6}+2(\ln\gamma)^2+O\left(\frac{1}{\gamma^2}\right) \approx \frac{\pi^2}{6}+2(\ln\gamma)^2 \tag{10}$$

$A_2 - A_3$ can be computed by treating A_2 by parts,

$$\begin{aligned}
A_2 - A_3 &= \int_0^{\gamma^2} \frac{2\ln(1+u)}{u^2}du - \int_0^{\gamma^2} \frac{2}{u}du \\
&= 2\left\{\left[-\frac{\ln(1+u)}{u}\right]_0^{\gamma^2} + \int_0^{\gamma^2}\left[\frac{1}{u(1+u)} - \frac{1}{u}\right]du\right\} \\
&= 2\left\{\overbrace{\lim_{u\to 0}\left[\frac{\ln(1+u)}{u}\right]}^{1} - \frac{\ln\left(1+\gamma^2\right)}{\gamma^2} - \int_0^{\gamma^2}\frac{du}{1+u}\right\} \\
&= 2\left[1 - \frac{\ln\left(1+\gamma^2\right)}{\gamma^2} - \ln\left(1+\gamma^2\right)\right] \\
&= 2\left[1 - \ln\left(1+\gamma^2\right)\overbrace{\left(1+\frac{1}{\gamma^2}\right)}^{\approx 1}\right] \\
&\approx 2\left[1 - \ln\gamma^2 - \underbrace{\ln\left(1+\frac{1}{\gamma^2}\right)}_{\approx 0}\right]
\end{aligned} \tag{11}$$

Putting (8) and (10) back to (4), and then to (3), we have

$$N_\gamma = \frac{z^2e^2}{\pi\hbar c}\left[(\ln\gamma - 1)^2 + \frac{\pi^2}{12}\right] \tag{12}$$

2. By (13.87) the total energy emitted in transition radiation is

$$I = \frac{z^2e^2\gamma\omega_p}{3c} \tag{13}$$

This gives the mean energy of the radiated photons as

$$\langle\hbar\omega\rangle = \frac{I}{N_\gamma} = \frac{\pi\gamma\hbar\omega_p}{3\left[(\ln\gamma - 1)^2 + \frac{\pi^2}{12}\right]} \tag{14}$$

With $\hbar\omega_p = 20\text{eV}$, we have

$$\langle\hbar\omega\rangle = \begin{cases} 0.586\text{keV} & \text{for } \gamma = 10^3 \\ 3.07\text{keV} & \text{for } \gamma = 10^4 \\ 18.8\text{keV} & \text{for } \gamma = 10^5 \end{cases} \tag{15}$$