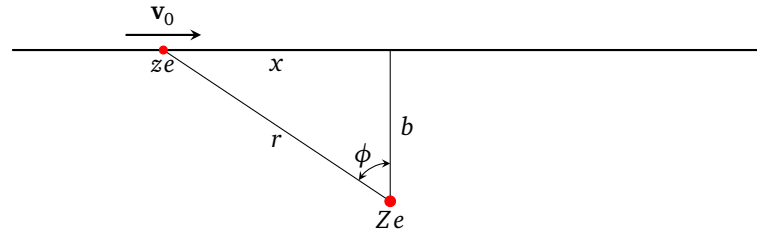


1. Prob 14.7



(a) At the location indicated by the diagram, the acceleration of the incoming charge is

$$|\mathbf{a}| = \frac{zZe^2}{mr^2} \quad (1)$$

Larmor formula gives the power radiated at this location

$$P = \frac{2}{3} \frac{z^2 e^2}{c^3} |\mathbf{a}|^2 = \frac{2}{3} \frac{z^4 Z^2 e^6}{m^2 c^3 r^4} \quad (2)$$

With the (somewhat inconsistent) assumption that the charge is moving with uniform speed v_0 on a straight line, we have

$$r^2 = b^2 + x^2 \quad \Rightarrow \quad x dx = r dr \quad \Rightarrow \quad dt = \frac{dx}{v_0} = \frac{r dr}{x v_0} = \frac{dr}{v_0 \sin \phi} \quad (3)$$

The total radiated energy can be found by

$$\Delta W = \int_{-\pi/2}^{\pi/2} \frac{2}{3} \frac{z^4 Z^2 e^6}{m^2 c^3 v_0} \frac{1}{(b/\cos \phi)^4} \frac{d\left(\frac{b}{\cos \phi}\right)}{\sin \phi} d\phi = \frac{2}{3} \frac{z^4 Z^2 e^6}{m^2 c^3 v_0 b^3} \int_{-\pi/2}^{\pi/2} \cos^2 \phi d\phi = \frac{\pi z^4 Z^2 e^6}{3 m^2 c^3 v_0 b^3} \quad (4)$$

(b) Replacing b by

$$r_c = \frac{2zZe^2}{mv_0^2} \quad (5)$$

in (4) gives

$$\Delta W = \frac{\pi z^4 Z^2 e^6}{3 m^2 c^3 v_0} \left(\frac{mv_0^2}{2zZe^2} \right)^3 = \frac{\pi}{24} \frac{zm v_0^5}{Z c^3} \quad (6)$$

Comparing this with the calculation of the exact non-relativistic head-on collision in problem 14.5, we see the coefficient changes from $8/45 \approx 0.18$ to $\pi/24 \approx 0.13$. The straight-line approximation error is about 30%.

(c) The radiation cross section can be calculated via

$$\chi = \int_{b_{\min}}^{\infty} \Delta W \cdot 2\pi b db = \frac{2\pi^2 z^4 Z^2 e^6}{3 m^2 c^3 v_0} \int_{b_{\min}}^{\infty} \frac{db}{b^2} = \frac{2\pi^2 z^4 Z^2 e^6}{3 m^2 c^3 v_0} \cdot \frac{1}{b_{\min}} \quad (7)$$

Using uncertainty principle as in chapter 13, we set $b_{\min} = \hbar/mv_0$, then

$$\chi = \frac{2\pi^2 z^4 Z^2 e^6}{3 \hbar c^3 m} \quad (8)$$

where the result given by (15.30) is

$$\chi = \frac{16 z^4 Z^2 e^6}{3 \hbar c^3 m} \quad (9)$$

The numerical coefficient has moderate difference: $2\pi^2/3 \approx 6.58$ from (10) versus $16/3 \approx 5.33$ from (11).

2. Prob 14.8

First recall the invariant form of Larmor formula (14.24), (14.25) for total radiated power

$$P = \frac{2}{3} \frac{z^2 e^2}{m^2 c^3} \left[\left(\frac{d\mathbf{p}}{d\tau} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{d\tau} \right)^2 \right] \quad (10)$$

Let \mathbf{F} be the force acting on the particle, we have

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \quad \Rightarrow \quad \frac{d\mathbf{p}}{d\tau} = \gamma \mathbf{F} \quad (11)$$

$$E = \sqrt{p^2 c^2 + m^2 c^4} \quad \Rightarrow \quad \frac{dE}{d\tau} = (\nabla_{\mathbf{p}} E) \cdot \frac{d\mathbf{p}}{d\tau} = \left(\frac{\mathbf{p} c^2}{E} \right) \cdot \gamma \mathbf{F} = \gamma \mathbf{v} \cdot \mathbf{F} \quad (12)$$

turning (10) into

$$P = \frac{2}{3} \frac{z^2 e^2}{m^2 c^3} \gamma^2 [F^2 - (\boldsymbol{\beta} \cdot \mathbf{F})^2] \quad (13)$$

Invoking the straight-line, constant-speed assumption and referring to the diagram in problem 14.7, we know

$$F^2 - (\boldsymbol{\beta} \cdot \mathbf{F})^2 = F^2 - \beta_0^2 F^2 \sin^2 \phi = \left(\frac{z Z e^2}{r^2} \right)^2 (1 - \beta_0^2 \sin^2 \phi) \quad (14)$$

With (3), the total radiated energy is the time integral

$$\begin{aligned} \Delta W &= \int_{-\infty}^{\infty} P dt = \frac{2}{3} \frac{z^2 e^2}{m^2 c^3} \gamma_0^2 (z Z e^2)^2 \int_{-\pi/2}^{\pi/2} \left[\frac{1}{(b/\cos \phi)^4} \right] (1 - \beta_0^2 \sin^2 \phi) \frac{d \left(\frac{b}{\cos \phi} \right)}{v_0 \sin \phi} d\phi \\ &= \frac{2}{3} \frac{z^4 Z^2 e^6}{m^2 c^3} \frac{1}{b^3 v_0} \gamma_0^2 \underbrace{\int_{-\pi/2}^{\pi/2} (1 - \beta_0^2 \sin^2 \phi) \cos^2 \phi d\phi}_{\sin \left(\frac{1}{\gamma_0^2} + 3 \right)} \\ &= \frac{\pi z^4 Z^2 e^6}{4 m^2 c^4 \beta_0} \frac{1}{b^3} \left(\gamma_0^2 + \frac{1}{3} \right) \end{aligned} \quad (15)$$

3. Prob 14.9

(a) Using (13) and the fact that the magnetic force is perpendicular to the velocity, we have the radiation power

$$P = \frac{2}{3} \frac{q^2}{m^2 c^3} \gamma^2 (q \beta B)^2 = \frac{2}{3} \frac{q^4 B^2}{m^2 c^3} \gamma^2 \beta^2 = \frac{2}{3} \frac{q^4 B^2}{m^2 c^3} (\gamma^2 - 1) \quad (16)$$

(b) With $\gamma \gg 1$, (16) becomes

$$-\frac{dE}{dt} = -mc^2 \frac{d\gamma}{dt} \approx \frac{2}{3} \frac{q^4 B^2}{m^2 c^3} \gamma^2 \quad \Rightarrow \quad -\frac{d\gamma}{\gamma^2} \approx \frac{2}{3} \frac{q^4 B^2}{m^3 c^5} dt \quad \Rightarrow \quad t \approx \frac{3}{2} \frac{m^3 c^5}{q^4 B^2} \left(\frac{1}{\gamma} - \frac{1}{\gamma_0} \right) \quad (17)$$

(c) Without the approximation $\gamma \gg 1$, we would have

$$-\left(\frac{d\gamma}{\gamma^2 - 1} \right) = \frac{2}{3} \frac{q^4 B^2}{m^3 c^5} dt \quad \Rightarrow \quad t = \frac{3}{2} \frac{m^3 c^5}{q^4 B^2} \cdot \frac{1}{2} \ln \left(\frac{\gamma + 1}{\gamma - 1} \right) \Bigg|_{\gamma_0}^{\gamma_t} \quad (18)$$

Define

$$\lambda \equiv \frac{4}{3} \frac{q^4 B^2}{m^3 c^5} \quad (19)$$

then from (18), we have

$$\ln \left(\frac{\gamma_t + 1}{\gamma_t - 1} \right) = \lambda t + \ln \left(\frac{\gamma_0 + 1}{\gamma_0 - 1} \right) \quad \Rightarrow \quad \gamma_t = \frac{e^{\lambda t} \left(\frac{\gamma_0 + 1}{\gamma_0 - 1} \right) + 1}{e^{\lambda t} \left(\frac{\gamma_0 + 1}{\gamma_0 - 1} \right) - 1} \quad (20)$$

The kinetic energy at time t is

$$T_t = T_0 \cdot \left(\frac{\gamma_t - 1}{\gamma_0 - 1} \right) = \frac{2T_0}{e^{\lambda t} (\gamma_0 + 1) - (\gamma_0 - 1)} \quad \longrightarrow \quad T_0 e^{-\lambda t} \quad \text{as } \gamma_0 \rightarrow 1 \quad (21)$$

(d) For magnetic dipole \mathbf{m} , the magnetic induction is

$$\mathbf{B} = \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \quad (22)$$

Let \mathbf{m} be along the z -axis, and with $\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}$, the above can be written as

$$\mathbf{B} = \frac{m}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) \implies B^2 = \frac{m^2}{r^6} (1 + 3 \cos^2 \theta) \quad (23)$$

By (16), the radiated power is proportional to B^2 , which is greater at the turning point $\theta \neq \pi/2$ than at the equator $\theta = \pi/2$.

4. Prob 14.10

(a) For linear motion, the radiation power's angular distribution is given by (14.39)

$$\frac{dP}{d\Omega} = \frac{e^2 \dot{v}^2}{4\pi c^3} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5} \quad (24)$$

where

$$\dot{v} = -\frac{\beta_0 c}{\Delta t} \quad \beta = \beta_0 \left(1 - \frac{t}{\Delta t}\right) \quad \text{for } 0 \leq t \leq \Delta t \quad (25)$$

The radiated energy per unit solid angle is the time integral

$$\begin{aligned} \frac{dE}{d\Omega} &= \int_0^{\Delta t} \frac{dP}{d\Omega} dt = \frac{e^2}{4\pi c} \left(\frac{\beta_0}{\Delta t}\right)^2 \sin^2 \theta \int_0^{\Delta t} \frac{dt}{\left[1 - \beta_0 \left(1 - \frac{t}{\Delta t}\right) \cos \theta\right]^5} \\ &= \frac{e^2}{4\pi c} \left(\frac{\beta_0}{\Delta t}\right)^2 \sin^2 \theta \cdot \frac{\Delta t}{\beta_0 \cos \theta} \int_{1-\beta_0 \cos \theta}^1 \frac{du}{u^5} \\ &= \frac{e^2 \beta_0}{4\pi c \Delta t} \frac{\sin^2 \theta}{\cos \theta} \cdot \frac{1}{4} \left[\frac{1}{(1 - \beta_0 \cos \theta)^4} - 1 \right] \\ &= \frac{e^2 \beta_0}{16\pi c \Delta t} \frac{\sin^2 \theta}{\cos \theta} \left\{ \frac{[1 + (1 - \beta_0 \cos \theta)^2][1 - (1 - \beta_0 \cos \theta)^2]}{(1 - \beta_0 \cos \theta)^4} \right\} \\ &= \frac{e^2 \beta_0^2}{16\pi c \Delta t} \frac{(2 - \beta_0 \cos \theta)[1 + (1 - \beta_0 \cos \theta)^2] \sin^2 \theta}{(1 - \beta_0 \cos \theta)^4} \end{aligned} \quad (26)$$

(b) Up to order $O(\gamma^{-2}, \theta^2)$, we have

$$\beta = \left(1 - \frac{1}{\gamma^2}\right)^{1/2} \approx 1 - \frac{1}{2\gamma^2} \quad \cos \theta \approx 1 - \frac{\theta^2}{2} \quad \sin \theta \approx \theta \quad (27)$$

then (26) can be approximated as

$$\frac{dE}{d\Omega} = \frac{e^2 \beta^2}{16\pi c \Delta t} \frac{\overbrace{\left(1 + \frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right)}^{\approx 1} \left[1 + \left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right)^2\right]^{\approx 1}}{\left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right)^4} \theta^2 \approx \frac{e^2 \beta^2}{\pi c \Delta t} \frac{\gamma^6 (\gamma \theta)^2}{(1 + \gamma^2 \theta^2)^4} = \frac{e^2 \beta^2 \gamma^6}{\pi c \Delta t} \frac{\xi}{(1 + \xi)^4} \quad (28)$$

From $\xi = \gamma^2 \theta^2$, we have $d\xi = 2\gamma^2 \theta d\theta$, therefore

$$\frac{dE}{d\xi} = \frac{dE}{2\gamma^2 \theta d\theta} = \frac{1}{2\gamma^2} \cdot 2\pi \frac{dE}{d\Omega} = \frac{e^2 \beta^2 \gamma^4}{c \Delta t} \frac{\xi}{(1 + \xi)^4} \quad (29)$$

By definition (see 13.60),

$$\langle \theta^2 \rangle = \frac{\int_{-\infty}^{\infty} \frac{\xi \theta^2 \theta d\theta}{(1 + \xi)^4}}{\int_{-\infty}^{\infty} \frac{\xi \theta d\theta}{(1 + \xi)^4}} = \frac{\frac{1}{2\gamma^4} \cdot 2 \int_0^{\infty} \frac{\xi^2 d\xi}{(1 + \xi)^4}}{\frac{1}{2\gamma^2} \cdot 2 \int_0^{\infty} \frac{\xi d\xi}{(1 + \xi)^4}} = \frac{2}{\gamma^2} \implies \sqrt{\langle \theta^2 \rangle} = \frac{\sqrt{2}}{\gamma} \quad (30)$$