1. **Steps from (13.82) to (13.83)**

We have already established the exact amplitude **F** in (13.82) (here we are using u for velocity since v and ν are easily confused)

$$\mathbf{F} = \epsilon_a \frac{2\sqrt{2\pi} ze \overbrace{\sin\theta\left(k\cos\theta - \frac{\omega}{u\gamma^2}\right)}^{A}}{\underbrace{u\left(\frac{\omega}{u} - k\cos\theta\right)}_{B}\underbrace{\left(\frac{\omega^2}{\gamma^2 u^2} + k^2\sin^2\theta\right)}_{C}} \tag{1}$$

the goal is to approximate, under the limit $\gamma \gg 1$ and $\theta \ll 1$, to get (13.83)

$$\mathbf{F} = \epsilon_a 4\sqrt{2\pi}\frac{ze}{c}\left(\frac{c}{\omega_p}\right)^2 \frac{\gamma}{\nu^2}\frac{\sqrt{\eta}}{\left(1+\frac{1}{\nu^2}+\eta\right)(1+\eta)} \tag{2}$$

where

$$\eta = \gamma^2\theta^2 \qquad \nu = \frac{\omega}{\gamma\omega_p} \tag{3}$$

The strategy is to approximate (1) to the order of $O\left(1/\gamma^2, \theta^2\right)$.

First note that

$$k = \frac{\omega n(\omega)}{c} = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{\omega}{c}\sqrt{1-\frac{1}{\gamma^2\nu^2}} \approx \frac{\omega}{c}\left(1-\frac{1}{2\gamma^2\nu^2}\right) \tag{4}$$

$$u = \beta c = c\sqrt{1-\frac{1}{\gamma^2}} \approx c\left(1-\frac{1}{2\gamma^2}\right) \tag{5}$$

With this, we can approximate A, B and C separately. For A, since $\sin\theta \approx \theta$, we can keep $k\cos\theta - \omega/u\gamma^2$ at the 0th order, i.e.,

$$A \approx \theta\frac{\omega}{c} \tag{6}$$

For B, C, we have

$$\begin{aligned} B = u\left(\frac{\omega}{u} - k\cos\theta\right) &= \omega - ku\cos\theta \\ &\approx \omega\left[1 - \frac{u}{c}\left(1-\frac{1}{2\gamma^2\nu^2}\right)\cos\theta\right] \\ &\approx \omega\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{1}{2\gamma^2\nu^2}\right)\left(1-\frac{\theta^2}{2}\right)\right] \\ &\approx \frac{\omega}{2\gamma^2}\left(1+\frac{1}{\nu^2}+\gamma^2\theta^2\right) \end{aligned} \tag{7}$$

$$C = \frac{\omega^2}{\gamma^2 u^2} + k^2\sin^2\theta \approx \omega^2\left[\frac{1}{\gamma^2 u^2} + \frac{\theta^2}{c^2}\left(1-\frac{1}{2\gamma^2\nu^2}\right)\right] \approx \frac{\omega^2}{c^2\gamma^2}\left(1+\gamma^2\theta^2\right) \tag{8}$$

This gives the approximated **F** as desired

$$\begin{aligned} \mathbf{F} &\approx \epsilon_a \frac{2\sqrt{2\pi} ze \frac{\theta\omega}{c}}{\frac{\omega}{2\gamma^2}\left(1+\frac{1}{\nu^2}+\eta\right)\frac{\omega^2}{\gamma^2 c^2}(1+\eta)} \\ &= \epsilon_a 4\sqrt{2\pi}\frac{ze}{c}\frac{c^2\gamma^2\theta}{\frac{\omega^2}{\gamma^2}\left(1+\frac{1}{\nu^2}+\eta\right)(1+\eta)} \\ &= \epsilon_a 4\sqrt{2\pi}\frac{ze}{c}\left(\frac{c}{\omega_p}\right)^2\frac{\gamma}{\nu^2}\frac{\sqrt{\eta}}{\left(1+\frac{1}{\nu^2}+\eta\right)(1+\eta)} \end{aligned} \tag{9}$$

2. **Energy spectrum distribution (13.85)**

From the energy distribution in ν and η (13.84)

$$\frac{d^2 I}{d\nu d\eta} = \frac{z^2 e^2 \gamma \omega_p}{\pi c} \left[\frac{\eta}{\nu^4 \left(1 + \frac{1}{\nu^2} + \eta\right)^2 (1+\eta)^2} \right] \tag{10}$$

let's integrate over the angular variable η and obtain (13.85)

$$\frac{dI}{d\nu} = \frac{z^2 e^2 \gamma \omega_p}{\pi c} \left[\left(1 + 2\nu^2\right) \ln\left(1 + \frac{1}{\nu^2}\right) - 2 \right] \tag{11}$$

Let $a \equiv 1/\nu^2$, and define the integral

$$X = \int_0^\infty \frac{\eta d\eta}{(1+a+\eta)^2 (1+\eta)^2} \tag{12}$$

Write the integrand as

$$\frac{\eta}{(1+a+\eta)^2 (1+\eta)^2} = \frac{A}{1+\eta} + \frac{B}{(1+\eta)^2} + \frac{C}{1+a+\eta} + \frac{D}{(1+a+\eta)^2} \tag{13}$$

or

$$\eta = A(1+\eta)(1+a+\eta)^2 + B(1+a+\eta)^2 + C(1+\eta)^2(1+a+\eta) + D(1+\eta)^2 \tag{14}$$

The coefficients can be solved by assigning special values to η:

$$\begin{aligned} \eta &= -1 & &\Longrightarrow & B &= -\frac{1}{a^2} \\ \eta &= -1-a & &\Longrightarrow & D &= -\left(\frac{1+a}{a^2}\right) \end{aligned} \tag{15}$$

Comparing the η^3 coefficients of (14) gives

$$A + C = 0 \tag{16}$$

Setting $\eta = 0$ in (14) gives

$$0 = A(1+a)^2 - \frac{(1+a)^2}{a^2} + C(1+a) - \left(\frac{1+a}{a^2}\right) \tag{17}$$

from which we can solve for A and C

$$A = \frac{a+2}{a^3} \qquad C = -\frac{a+2}{a^3} \tag{18}$$

Now the integral X can be computed as

$$\begin{aligned} X &= A[\ln(1+\eta) - \ln(1+a+\eta)]\Big|_0^\infty - B\left(\frac{1}{1+\eta}\right)\Big|_0^\infty - D\left(\frac{1}{1+a+\eta}\right)\Big|_0^\infty \\ &= \left(\frac{a+2}{a^3}\right)\ln(1+a) - \frac{1}{a^2} - \left(\frac{1+a}{a^2}\right)\left(\frac{1}{1+a}\right) \\ &= \left(\frac{a+2}{a^3}\right)\ln(1+a) - \frac{2}{a^2} \end{aligned} \tag{19}$$

Putting $a = 1/\nu^2$ back, we have

$$\frac{dI}{d\nu} = \frac{z^2 e^2 \gamma \omega_p}{\pi c} \frac{X}{\nu^4} = \frac{z^2 e^2 \gamma \omega_p}{\pi c} \left[\left(1 + 2\nu^2\right) \ln\left(1 + \frac{1}{\nu^2}\right) - 2 \right] \tag{20}$$

3. **Total radiation (13.87)**

To calculate total radiation, we will subject (10) to double integral. Define

$$Y = \int_0^\infty d\nu \int_0^\infty \frac{\eta d\eta}{\nu^4 \left(1 + \frac{1}{\nu^2} + \eta\right)^2 (1+\eta)^2}$$

$$= \int_0^\infty \frac{\eta d\eta}{(1+\eta)^2} \underbrace{\int_0^\infty \frac{d\nu}{[(1+\eta)\nu^2 + 1]^2}}_{Y'} \tag{21}$$

With $t = \sqrt{1+\eta}\,\nu$, the inner integral Y' can be obtained directly

$$Y' = \frac{1}{\sqrt{1+\eta}} \int_0^\infty \frac{dt}{(1+t^2)^2} \qquad \text{let } t = \tan\theta$$

$$= \frac{1}{\sqrt{1+\eta}} \int_0^{\pi/2} \cos^2\theta d\theta = \frac{\pi}{4\sqrt{1+\eta}} \tag{22}$$

Thus

$$Y = \frac{\pi}{4} \int_0^\infty \frac{\eta d\eta}{(1+\eta)^{5/2}} \qquad \text{let } u = \eta + 1$$

$$= \frac{\pi}{4} \int_1^\infty \left(\frac{du}{u^{3/2}} - \frac{du}{u^{5/2}}\right)$$

$$= \frac{\pi}{4} \left(-2u^{-1/2} + \frac{2}{3}u^{-3/2}\right)\Bigg|_1^\infty = \frac{\pi}{3} \tag{23}$$

giving the total radiation (13.87)

$$I = \frac{z^2 e^2 \gamma \omega_p}{\pi c} \cdot Y = \frac{z^2 e^2 \gamma \omega_p}{3c} \tag{24}$$