1. In the rest frame of the magnetic moment, the electric and magnetic fields are

$$\mathbf{E}'=0 \qquad \mathbf{B}'=\frac{3\left(\boldsymbol{\mu}\cdot\hat{\mathbf{n}}'\right)\hat{\mathbf{n}}'-\boldsymbol{\mu}}{r'^3}=\frac{\mu}{r'^5}\left[3x'z'\hat{\mathbf{x}}+3y'z'\hat{\mathbf{y}}+\left(3z'^2-r'^2\right)\hat{\mathbf{z}}\right] \tag{1}$$

Transforming the fields into the lab frame using Lorentz transformation

$$\mathbf{E}_{\parallel}=\mathbf{E}'_{\parallel} \qquad \mathbf{E}_{\perp}=\gamma\left(\mathbf{E}'_{\perp}-\boldsymbol{\beta}\times\mathbf{B}'\right) \tag{2}$$

gives the electric field

$$\mathbf{E}=\frac{3\gamma\beta\mu z'}{r'^5}\left(y'\hat{\mathbf{x}}-x'\hat{\mathbf{y}}\right)=-\frac{3\gamma\beta\mu\rho z'}{r'^5}\hat{\boldsymbol{\phi}}=-\frac{3\gamma\beta\mu\rho\cdot\gamma\left(z-ut\right)}{\left[\rho^2+\gamma^2\left(z-ut\right)^2\right]^{5/2}}\hat{\boldsymbol{\phi}}=-\frac{3\gamma^2\beta\mu\rho\left(z-ut\right)}{\left[\rho^2+\gamma^2\left(z-ut\right)^2\right]^{5/2}}\hat{\boldsymbol{\phi}} \tag{3}$$

of which the Fourier component is

$$\begin{aligned}
E_{\phi}(\omega)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}E_{\phi}e^{i\omega t}dt\\
&=\frac{1}{\sqrt{2\pi}}\left(-3\gamma^2\beta\mu\rho\right)\int_{-\infty}^{\infty}\frac{\left(z-ut\right)e^{i\omega t}dt}{\left[\rho^2+\gamma^2\left(z-ut\right)^2\right]^{5/2}} && \text{let } \tau=\frac{z}{u}-t\\
&=\frac{-3\gamma^2\beta\mu\rho}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{u\tau e^{i\omega(z/u-\tau)}d\tau}{(\gamma u)^5\left[\left(\frac{\rho}{\gamma u}\right)^2+\tau^2\right]^{5/2}} && \text{let } s=\frac{\rho}{\gamma u}\\
&=\frac{-3\beta\mu\rho}{\sqrt{2\pi}\gamma^3u^4}e^{i\omega z/u}\left[\overbrace{\int_{-\infty}^{\infty}\frac{\tau\cos(\omega\tau)d\tau}{(s^2+\tau^2)^{5/2}}}^{0}-i\int_{-\infty}^{\infty}\frac{\tau\sin(\omega\tau)d\tau}{(s^2+\tau^2)^{5/2}}\right]\\
&=\frac{3i\beta\mu\rho}{\sqrt{2\pi}\gamma^3u^4}e^{i\omega z/u}\int_{-\infty}^{\infty}\left(-\frac{1}{3}\right)\left[\frac{d\left(s^2+\tau^2\right)^{-3/2}}{d\tau}\right]\sin(\omega\tau)d\tau\\
&=\frac{i\beta\mu\rho\omega}{\sqrt{2\pi}\gamma^3u^4}e^{i\omega z/u}\int_{-\infty}^{\infty}\frac{\cos(\omega\tau)d\tau}{(s^2+\tau^2)^{3/2}}
\end{aligned} \tag{4}$$

From the integral representation of the modified Bessel function of the second kind (see DLMF 10.32.E11)

$$K_{\nu}(\omega s)=\frac{\Gamma\left(\nu+\frac{1}{2}\right)(2s)^{\nu}}{\sqrt{\pi}\omega^{\nu}}\int_0^{\infty}\frac{\cos\omega t\,dt}{(s^2+t^2)^{\nu+\frac{1}{2}}} \tag{5}$$

the integral in (4) can be evaluated as

$$\int_{-\infty}^{\infty}\frac{\cos(\omega\tau)d\tau}{(s^2+\tau^2)^{3/2}}=\frac{2\sqrt{\pi}\omega K_1(\omega s)}{\Gamma(3/2)\cdot 2s}=\frac{2\omega\gamma u}{\rho}K_1\left(\frac{\omega\rho}{\gamma u}\right) \tag{6}$$

giving

$$E_{\phi}(\omega)=i\sqrt{\frac{2}{\pi}}\frac{\omega^2\beta\mu}{\gamma^2u^3}K_1\left(\frac{\omega\rho}{\gamma u}\right)e^{i\omega z/u} \tag{7}$$

which is exactly $\beta\mu/\gamma ze$ times the partial derivative in the $z$ direction of $E_{\rho}$ in (13.80).

2. The intensity distribution follows (13.79)

$$\frac{d^2I}{d\omega d\Omega}=\frac{c}{32\pi^2}\left(\frac{\omega_p}{c}\right)^4|\mathbf{F}|^2 \tag{8}$$

where

$$\mathbf{F}=\frac{i}{\left(\frac{\omega}{u}-k\cos\theta\right)}\iint dxdy\left(\hat{\mathbf{k}}\times\mathbf{E}_i\right)|_{z=0}\times\hat{\mathbf{k}}e^{-ikx\sin\theta} \tag{9}$$

Given

$$\mathbf{E}_i=E_\phi(\omega)\,\hat{\boldsymbol{\phi}} \tag{10}$$

and that $\hat{\mathbf{k}}\perp\hat{\boldsymbol{\phi}}$, the triple cross product in (9) is just $E_\phi(\omega)|_{z=0}\hat{\boldsymbol{\phi}}$, therefore

$$\begin{aligned}
\mathbf{F}&=\frac{i}{\left(\frac{\omega}{u}-k\cos\theta\right)}\iint dxdyE_\phi(\omega)|_{z=0}\left(-\sin\phi\hat{\mathbf{x}}+\cos\phi\hat{\mathbf{y}}\right)e^{-ikx\sin\theta}\\
&=\frac{-\sqrt{\frac{2}{\pi}}\frac{\omega^2\beta\mu}{\gamma^2u^3}}{\left(\frac{\omega}{u}-k\cos\theta\right)}\iint dxdyK_1\left(\frac{\omega\sqrt{x^2+y^2}}{\gamma u}\right)\left(\overbrace{-\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}}}^{\text{no contrib.}}+\frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}}\right)e^{-ikx\sin\theta}\\
&=\frac{-\hat{\mathbf{y}}\sqrt{\frac{2}{\pi}}\frac{\omega^2\beta\mu}{\gamma^2u^3}}{\left(\frac{\omega}{u}-k\cos\theta\right)}\iint dxdye^{-ikx\sin\theta}\left(-\frac{\gamma u}{\omega}\right)\frac{\partial}{\partial x}K_0\left(\frac{\omega\sqrt{x^2+y^2}}{\gamma u}\right)\\
&=\frac{\hat{\mathbf{y}}\sqrt{\frac{2}{\pi}}\frac{\omega^2\beta\mu}{\gamma^2u^3}}{\left(\frac{\omega}{u}-k\cos\theta\right)}\left(\frac{\gamma u}{\omega}\right)\cdot ik\sin\theta\iint dxdyK_0\left(\frac{\omega\sqrt{x^2+y^2}}{\gamma u}\right)e^{-ikx\sin\theta} &&\text{see (13.82)}\\
&=i\hat{\mathbf{y}}\cdot 2\sqrt{2\pi}\frac{\omega\beta\mu}{\gamma u}\cdot\frac{k\sin\theta}{u\left(\frac{\omega}{u}-k\cos\theta\right)\left(\frac{\omega^2}{\gamma^2u^2}+k^2\sin^2\theta\right)} &&\text{up to } O\left(\frac{1}{\gamma^2},\theta^2\right)\\
&\approx i\hat{\mathbf{y}}\cdot 4\sqrt{2\pi}\frac{\gamma^2\mu}{\omega}\frac{\sqrt{\eta}}{\left(1+\frac{1}{\nu^2}+\eta\right)(1+\eta)}\\
&=i\hat{\mathbf{y}}\cdot 4\sqrt{2\pi}\frac{\omega\mu}{\omega_p^2}\frac{1}{\nu^2}\frac{\sqrt{\eta}}{\left(1+\frac{1}{\nu^2}+\eta\right)(1+\eta)}
\end{aligned} \tag{11}$$

Its modulus is the modulus of (13.83) times a factor of $\omega\mu/zec\gamma$, so (13.84), (13.85) gain a factor of $(\omega\mu/zec\gamma)^2$.

3. Indeed, with $\alpha=e^2/\hbar c$, $a_0=\hbar^2/me^2$,

$$\frac{\alpha^4}{4}\left(\frac{\mu}{\mu_B}\right)^2\left(\frac{\hbar\omega_p}{\hbar\omega_0}\right)^2\nu^2=\frac{\left(\frac{e^2}{\hbar c}\right)^4}{4}\left(\frac{\mu}{\frac{e\hbar}{2mc}}\right)^2\left[\frac{\omega_p}{\frac{e^2}{\hbar\left(\frac{\hbar^2}{me^2}\right)}}\right]^2\left(\frac{\omega}{\gamma\omega_p}\right)^2=\left(\frac{\omega\mu}{ec\gamma}\right)^2=\frac{dI_\mu(\nu)}{dI_e(\nu)} \tag{12}$$

4. Explicitly, for electron and magnetic moment,

$$\frac{d^2I_e}{d\nu d\eta}=\frac{e^2\gamma\omega_p}{\pi c}\left[\frac{\eta}{\nu^4\left(1+\frac{1}{\nu^2}+\eta\right)^2(1+\eta)^2}\right] \tag{13}$$

$$\frac{d^2I_\mu}{d\nu d\eta}=\left(\frac{\omega\mu}{ec\gamma}\right)^2\frac{d^2I_e}{d\nu d\eta}=\left(\frac{\omega_p^3\mu^2\gamma}{\pi c^3}\right)\cdot\nu^2\left[\frac{\eta}{\nu^4\left(1+\frac{1}{\nu^2}+\eta\right)^2(1+\eta)^2}\right] \tag{14}$$

Then from (13.85), we have

$$I_{\mathrm{e}} = \int_0^{\nu_{\max}} \frac{dI_{\mathrm{e}}}{d\nu} d\nu = \frac{e^2\gamma\omega_p}{\pi c} \overbrace{\int_0^{\nu_{\max}} \left[\left(1+2\nu^2\right)\ln\left(1+\frac{1}{\nu^2}\right)-2\right] d\nu}^{J_{\mathrm{e}}} \tag{15}$$

$$I_\mu = \int_0^{\nu_{\max}} \frac{dI_\mu}{d\nu} d\nu = \left(\frac{\omega_p^3\mu^2\gamma}{\pi c^3}\right) \underbrace{\int_0^{\nu_{\max}} \nu^2\left[\left(1+2\nu^2\right)\ln\left(1+\frac{1}{\nu^2}\right)-2\right] d\nu}_{J_\mu} \tag{16}$$

After tedious calculations, both $J_{\mathrm{e}}$ and $J_\mu$ can be expressed in closed form (see appendix below)

$$\begin{aligned} J_{\mathrm{e}} &= \left(\nu_{\max} + \frac{2\nu_{\max}^3}{3}\right)\ln\left(1+\frac{1}{\nu_{\max}^2}\right) - \frac{2\nu_{\max}}{3} + \frac{2}{3}\tan^{-1}\nu_{\max} \\ &= \frac{1}{3}\left[\left(3\nu_{\max}+2\nu_{\max}^3\right)\ln\left(1+\frac{1}{\nu_{\max}^2}\right) - 2\nu_{\max} + 2\tan^{-1}\nu_{\max}\right] \end{aligned} \tag{17}$$

$$\begin{aligned} J_\mu &= \left(\frac{\nu_{\max}^3}{3} + \frac{2\nu_{\max}^5}{5}\right)\ln\left(1+\frac{1}{\nu_{\max}^2}\right) - \frac{2\nu_{\max}^3}{5} - \frac{2\nu_{\max}}{15} + \frac{2}{15}\tan^{-1}\nu_{\max} \\ &= \frac{1}{15}\left[\left(5\nu_{\max}^3+6\nu_{\max}^5\right)\ln\left(1+\frac{1}{\nu_{\max}^2}\right) - 6\nu_{\max}^3 - 2\nu_{\max} + 2\tan^{-1}\nu_{\max}\right] \end{aligned} \tag{18}$$

The ratio of the total intensities is

$$\begin{aligned} \frac{I_\mu}{I_{\mathrm{e}}} &= \frac{1}{5}\left(\frac{\dfrac{\omega_p^3\mu^2\gamma}{\pi c^3}}{\dfrac{e^2\gamma\omega_p}{\pi c}}\right) \overbrace{\left[\frac{\left(5\nu_{\max}^3+6\nu_{\max}^5\right)\ln\left(1+\frac{1}{\nu_{\max}^2}\right) - 6\nu_{\max}^3 - 2\nu_{\max} + 2\tan^{-1}\nu_{\max}}{\left(3\nu_{\max}+2\nu_{\max}^3\right)\ln\left(1+\frac{1}{\nu_{\max}^2}\right) - 2\nu_{\max} + 2\tan^{-1}\nu_{\max}}\right]}^{G(\nu_{\max})} \\ &= \frac{\alpha^4}{20}\left(\frac{\mu}{\mu_B}\right)^2\left(\frac{\hbar\omega_p}{\hbar\omega_0}\right)^2 G(\nu_{\max}) \end{aligned} \tag{19}$$

- **Appendix: Direct evaluation of $J_{\mathrm{e}}$ and $J_\mu$**

Note that

$$\ln\left(1+\frac{1}{\nu^2}\right) = \ln\left(\nu^2+1\right) - 2\ln\nu \tag{20}$$

then

$$J_{\mathrm{e}} = \overbrace{\int_0^{\nu_{\max}} (1+2\nu^2)\ln\left(\nu^2+1\right) d\nu}^{J_{\mathrm{e}1}} - \overbrace{\int_0^{\nu_{\max}} 2\ln\nu\, d\nu}^{J_{\mathrm{e}2}} - \overbrace{\int_0^{\nu_{\max}} 4\nu^2\ln\nu\, d\nu}^{J_{\mathrm{e}3}} - \overbrace{\int_0^{\nu_{\max}} 2 d\nu}^{J_{\mathrm{e}4}} \tag{21}$$

Using integration by parts, we have

$$\begin{aligned} J_{\mathrm{e}1} &= \left(\nu_{\max}+\frac{2\nu_{\max}^3}{3}\right)\ln\left(\nu_{\max}^2+1\right) - \int_0^{\nu_{\max}} \left(\nu+\frac{2\nu^3}{3}\right)\left(\frac{2\nu}{\nu^2+1}\right) d\nu \\ &= \left(\nu_{\max}+\frac{2\nu_{\max}^3}{3}\right)\ln\left(\nu_{\max}^2+1\right) - \int_0^{\nu_{\max}} \left[\frac{4\nu^2}{3}+\frac{2}{3}-\frac{2}{3(\nu^2+1)}\right] d\nu \\ &= \left(\nu_{\max}+\frac{2\nu_{\max}^3}{3}\right)\ln\left(\nu_{\max}^2+1\right) - \frac{4\nu_{\max}^3}{9} - \frac{2\nu_{\max}}{3} + \frac{2\tan^{-1}\nu_{\max}}{3} \\ &= \left(\nu_{\max}+\frac{2\nu_{\max}^3}{3}\right)\left[\ln\left(1+\frac{1}{\nu_{\max}^2}\right) + 2\ln\nu_{\max}\right] - \frac{4\nu_{\max}^3}{9} - \frac{2\nu_{\max}}{3} + \frac{2\tan^{-1}\nu_{\max}}{3} \end{aligned} \tag{22}$$

and similarly,

$$J_{\mathrm{e}2} = 2\nu_{\max}\ln\nu_{\max} - 2\nu_{\max} \qquad J_{\mathrm{e}3} = \frac{4\nu_{\max}^3}{3}\ln\nu_{\max} - \frac{4\nu_{\max}^3}{9} \qquad J_{\mathrm{e}4} = 2\nu_{\max} \tag{23}$$

Putting these back to (21) gives (17).

The evaluation of $J_\mu$ is similar.