

The charge's position  $\mathbf{r}(t)$  can be expanded at  $t = 0$  as

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}(0)t + \frac{\mathbf{a}(0)}{2}t^2 + \frac{\mathbf{j}(0)}{6}t^3 + \dots \quad (1)$$

where  $\mathbf{v}, \mathbf{a}, \mathbf{j}$  are the velocity, acceleration, and jerk of the charge respectively.

When the charge moves ultrarelativistically, we can approximately treat its speed as constant, therefore

$$0 = \frac{d(\mathbf{v} \cdot \mathbf{v})}{dt} = 2\mathbf{v} \cdot \mathbf{a} \quad \Rightarrow \quad 0 = \frac{d^2(\mathbf{v} \cdot \mathbf{v})}{dt^2} = 2a^2 + 2\mathbf{v} \cdot \mathbf{j} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{j} = -a^2 = -\left(\frac{v^2}{\rho}\right)^2 = -\frac{v^4}{\rho^2} \quad (2)$$

In ultrarelativistic beaming, the photon's direction  $\mathbf{k}$  is approximately along  $\mathbf{v}$ , so we have

$$\begin{aligned} \mathbf{k} \cdot \mathbf{r}(t) &\approx \mathbf{k} \cdot \mathbf{r}(0) + \mathbf{k} \cdot \mathbf{v}(0)t + \frac{\mathbf{k} \cdot \mathbf{a}(0)}{2}t^2 + \frac{\mathbf{k} \cdot \mathbf{j}(0)}{6}t^3 && \text{up to global phase factor} \\ &\approx \mathbf{k} \cdot \mathbf{v}(0)t + \frac{k\mathbf{v}(0) \cdot \mathbf{j}(0)}{6\mathbf{v}(0)}t^3 && \text{by (2)} \\ &= \mathbf{k} \cdot \mathbf{v}(0)t - \frac{k[v(0)]^3}{6\rho^2}t^3 && \text{take } v(0) \approx c \text{ in higher order term} \\ &\approx \mathbf{k} \cdot \mathbf{v}(0)t - \frac{\omega c^2}{6\rho^2}t^3 \end{aligned} \quad (3)$$

The phase (14.72) becomes

$$\omega \left[ t - \frac{\mathbf{n} \cdot \mathbf{r}(t)}{c} \right] = \omega \left[ t - \frac{\mathbf{k} \cdot \mathbf{r}(t)}{kc} \right] = \omega t - \mathbf{k} \cdot \mathbf{r}(t) \approx [\omega - \mathbf{k} \cdot \mathbf{v}(0)]t + \frac{\omega c^2}{6\rho^2}t^3 = \frac{p^\mu k_\mu}{\gamma m}t + \frac{\omega c^2}{6\rho^2}t^3 \quad (4)$$

where we can see the identical coefficient of the  $t^3$  term as (14.73).

For this problem, the more useful form is

$$\omega \left[ t - \frac{\mathbf{n} \cdot \mathbf{r}(t)}{c} \right] \approx \frac{p^\mu k_\mu}{\gamma m}t - \frac{\mathbf{k} \cdot \mathbf{j}(0)}{6}t^3 \quad (5)$$

because we can express  $\mathbf{k} \cdot \mathbf{j}$  in terms of  $p^\mu k_\mu$  as follows:

$$\frac{d^2(p^\mu k_\mu)}{d\tau^2} = \frac{d^2}{d\tau^2} [\gamma m (\omega - \mathbf{k} \cdot \mathbf{v})] \quad \Rightarrow \quad \frac{1}{\gamma^3 m} \frac{d^2(p^\mu k_\mu)}{d\tau^2} = -\mathbf{k} \cdot \mathbf{j} = \frac{\omega c^2}{\rho^2} \quad (6)$$

where we have used the approximation that  $\gamma$  is constant and that  $d\tau = dt/\gamma$ .

From (6), we get

$$\rho^2 = \frac{\omega \gamma^3 m c^2}{d^2(p^\mu k_\mu)/d\tau^2} = \frac{\omega E^3}{m^2 c^4 [d^2(p^\mu k_\mu)/d\tau^2]} \quad (7)$$

and the phase can be written explicitly as

$$\omega \left[ t - \frac{\mathbf{n} \cdot \mathbf{r}(t)}{c} \right] = \overbrace{\left( \frac{p^\mu k_\mu}{\gamma m} \right) t}^a + \frac{1}{6} \overbrace{\left[ \frac{1}{\gamma^3 m} \frac{d^2(p^\mu k_\mu)}{d\tau^2} \right] t^3}^b \quad (8)$$

Referencing figure 14.9, we see that the polarization vectors of this problem are  $\epsilon_{\parallel} = \epsilon_1$ ,  $\epsilon_{\perp} = \mathbf{n} \times \epsilon_1 = -\epsilon_2$ , and

$$\mathbf{p} \cdot \epsilon_2 = \mathbf{p} \cdot (\epsilon_1 \times \mathbf{n}) = \mathbf{n} \cdot (\mathbf{p} \times \epsilon_1) = \gamma m v \cos\left(\frac{vt}{\rho}\right) \sin \theta \quad (9)$$

$$\frac{d(\epsilon_1 \cdot \mathbf{p})}{d\tau} \approx \gamma^2 m \epsilon_1 \cdot \mathbf{a} = \gamma^2 m \frac{v^2}{\rho} \quad (10)$$

Thus from (14.71)

$$\begin{aligned} \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) &= \boldsymbol{\beta} \left[ -\epsilon_{\parallel} \sin\left(\frac{vt}{\rho}\right) + \epsilon_{\perp} \cos\left(\frac{vt}{\rho}\right) \sin \theta \right] \\ &\approx \frac{1}{c} \left[ -\epsilon_1 \left( \frac{v^2}{\rho} \right) t - \epsilon_2 v \cos\left(\frac{vt}{\rho}\right) \sin \theta \right] \\ &= -\epsilon_1 \left[ \frac{d(\epsilon_1 \cdot \mathbf{p})/d\tau}{\gamma^2 m c} \right] t - \epsilon_2 \left( \frac{\mathbf{p} \cdot \epsilon_2}{\gamma m c} \right) \end{aligned} \quad (11)$$

Putting (11) and (8) into (14.67), we get

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2 c} |-\epsilon_1 A_1(\omega) - \epsilon_2 A_2(\omega)|^2 \quad (12)$$

where

$$A_1(\omega) \approx \left[ \frac{d(\epsilon_1 \cdot \mathbf{p})/d\tau}{\gamma^2 mc} \right] \overbrace{\int_{-\infty}^{\infty} t \exp \left[ i \left( at + \frac{bt^3}{6} \right) \right] dt}^{I_1} \quad (13)$$

$$A_2(\omega) \approx \left( \frac{\mathbf{p} \cdot \epsilon_2}{\gamma mc} \right) \underbrace{\int_{-\infty}^{\infty} \exp \left[ i \left( at + \frac{bt^3}{6} \right) \right] dt}_{I_2} \quad (14)$$

Taking parity into consideration, the two integrals are

$$I_1 = 2 \int_0^{\infty} t \sin \left( at + \frac{bt^3}{6} \right) dt \quad (15)$$

$$I_2 = 2 \int_0^{\infty} \cos \left( at + \frac{bt^3}{6} \right) dt \quad (16)$$

With variable change

$$t = \left( \frac{3\xi}{2a} \right) x \quad \text{where} \quad \xi = \frac{2\sqrt{2}a^{3/2}}{3b^{1/2}} = \frac{2\sqrt{2}}{3m} \frac{(p^\mu k_\mu)^{3/2}}{[d^2(p^\mu k_\mu)/d\tau^2]^{1/2}} \quad (17)$$

the two integrals become

$$I_1 = 2 \left( \frac{3\xi}{2a} \right)^2 \int_0^{\infty} x \sin \left[ \frac{3}{2} \xi \left( x + \frac{x^3}{3} \right) \right] dx = 2 \left( \frac{3\xi}{2a} \right)^2 \cdot \frac{1}{\sqrt{3}} K_{2/3}(\xi) \quad (18)$$

$$I_2 = 2 \left( \frac{3\xi}{2a} \right) \int_0^{\infty} \cos \left[ \frac{3}{2} \xi \left( x + \frac{x^3}{3} \right) \right] dx = 2 \left( \frac{3\xi}{2a} \right) \cdot \frac{1}{\sqrt{3}} K_{1/3}(\xi) \quad (19)$$

Putting these into (12) yields

$$\begin{aligned} \frac{d^2 I}{d\omega d\Omega} &= \frac{e^2 \omega^2}{4\pi^2 c} [A_1^2(\omega) + A_2^2(\omega)] \\ &= \frac{e^2 \omega^2}{4\pi^2 c} \left\{ \left[ \frac{d(\epsilon_1 \cdot \mathbf{p})/d\tau}{\gamma^2 mc} \right]^2 \left( \frac{3\sqrt{3}\xi^2}{2a^2} \right)^2 K_{2/3}^2(\xi) + \left( \frac{\mathbf{p} \cdot \epsilon_2}{\gamma mc} \right)^2 \left( \frac{\sqrt{3}\xi}{a} \right)^2 K_{1/3}^2(\xi) \right\} \end{aligned} \quad (20)$$

By (17) and (8), we have

$$\frac{\xi}{a} = \frac{2\sqrt{2}}{3} \left( \frac{a}{b} \right)^{1/2} = \frac{2\sqrt{2}\gamma}{3} \left[ \frac{p^\mu k_\mu}{d^2(p^\mu k_\mu)/d\tau^2} \right]^{1/2} \quad (21)$$

which turns (20) into

$$\frac{d^2 I}{d\omega d\Omega} = \frac{4e^2 \omega^2}{3\pi^2 m^2 c^3} \left\{ \left[ \frac{d(\epsilon_1 \cdot \mathbf{p})}{d\tau} \right]^2 \left[ \frac{p^\mu k_\mu}{d^2(p^\mu k_\mu)/d\tau^2} \right]^2 K_{2/3}^2(\xi) + \frac{(\mathbf{p} \cdot \epsilon_2)^2}{2} \left[ \frac{p^\mu k_\mu}{d^2(p^\mu k_\mu)/d\tau^2} \right] K_{1/3}^2(\xi) \right\} \quad (22)$$

Substituting the phase space volume element transformation

$$d^3 k = k^2 dk d\Omega = \frac{\omega^2}{c^3} d\omega d\Omega \quad (23)$$

into (22) gives the desired result

$$\begin{aligned} \hbar \omega \frac{d^3 N}{d^3 k} &= \frac{c^3}{\omega^2} \cdot \frac{d^2 I}{d\omega d\Omega} \\ &= \frac{4e^2}{3\pi^2 m^2} \left\{ \left[ \frac{d(\epsilon_1 \cdot \mathbf{p})}{d\tau} \right]^2 \left[ \frac{p^\mu k_\mu}{d^2(p^\mu k_\mu)/d\tau^2} \right]^2 K_{2/3}^2(\xi) + \frac{(\mathbf{p} \cdot \epsilon_2)^2}{2} \left[ \frac{p^\mu k_\mu}{d^2(p^\mu k_\mu)/d\tau^2} \right] K_{1/3}^2(\xi) \right\} \end{aligned} \quad (24)$$