1. From Prob 14.13, the time-averaged power radiated per unit solid angle in the m-th harmonic by a single charge e in periodic circular motion is

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = \frac{e^2 m^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^T e^{im\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]} \mathbf{n} \times \mathbf{v}(t)\, dt \right|^2 \tag{1}$$

For N charges q_j at fixed angular positions θ_j on the same circle of radius R, the j-th charge has

$$\mathbf{x}_j(t) = R\left[\cos\left(\omega_0 t + \theta_j\right)\hat{\mathbf{x}} + \sin\left(\omega_0 t + \theta_j\right)\hat{\mathbf{y}}\right] \tag{2}$$
$$\mathbf{v}_j(t) = \beta c\left[-\sin\left(\omega_0 t + \theta_j\right)\hat{\mathbf{x}} + \cos\left(\omega_0 t + \theta_j\right)\hat{\mathbf{y}}\right] \tag{3}$$

Since the radiation fields add coherently, the total amplitude for the m-th harmonic is the charge-weighted sum over all particles. Replacing $e \to q_j$ in (1), the N-particle power in the m-th harmonic is

$$\left\langle \frac{dP_m(N)}{d\Omega} \right\rangle = \frac{m^2 \omega_0^4}{(2\pi c)^3} \left| \sum_{j=1}^N q_j \overbrace{\int_0^T e^{im\omega_0\left[t-\mathbf{n}\cdot\mathbf{x}_j(t)/c\right]} \mathbf{n} \times \mathbf{v}_j(t)\, dt}^{\mathbf{I}_j} \right|^2 \tag{4}$$

Since we are averaging over a period, where all particles complete one circle, we can choose $\mathbf{n}$ to have no azimuthal component without loss of generality,

$$\mathbf{n} = \sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}} \tag{5}$$

Then

$$\mathbf{n}\cdot\mathbf{x}_j(t) = R\sin\theta\cos\left(\omega_0 t + \theta_j\right) \tag{6}$$
$$\mathbf{n}\times\mathbf{v}_j(t) = \beta c\left[-\cos\theta\cos\left(\omega_0 t + \theta_j\right)\hat{\mathbf{x}} - \cos\theta\sin\left(\omega_0 t + \theta_j\right)\hat{\mathbf{y}} + \sin\theta\cos\left(\omega_0 t + \theta_j\right)\hat{\mathbf{z}}\right] \tag{7}$$

The integral $\mathbf{I}_j$ is therefore $\mathbf{I}_j = I_{jx}\hat{\mathbf{x}} + I_{jy}\hat{\mathbf{y}} + I_{jz}\hat{\mathbf{z}}$, where

$$I_{jx} = -\beta c\cos\theta \int_0^T e^{im\omega_0 t} e^{-im\beta\sin\theta\cos\left(\omega_0 t+\theta_j\right)} \cos\left(\omega_0 t + \theta_j\right) dt \tag{8}$$
$$I_{jy} = -\beta c\cos\theta \int_0^T e^{im\omega_0 t} e^{-im\beta\sin\theta\cos\left(\omega_0 t+\theta_j\right)} \sin\left(\omega_0 t + \theta_j\right) dt \tag{9}$$
$$I_{jz} = \beta c\sin\theta \int_0^T e^{im\omega_0 t} e^{-im\beta\sin\theta\cos\left(\omega_0 t+\theta_j\right)} \cos\left(\omega_0 t + \theta_j\right) dt \tag{10}$$

Substituting $u = \omega_0 t + \theta_j$, we have

$$I_{jx} = -e^{-im\theta_j}\left(\frac{\beta c}{\omega_0}\right)\cos\theta \cdot C \qquad I_{jy} = -e^{-im\theta_j}\left(\frac{\beta c}{\omega_0}\right)\cos\theta \cdot S \qquad I_{jz} = e^{-im\theta_j}\left(\frac{\beta c}{\omega_0}\right)\sin\theta \cdot C \tag{11}$$

where

$$S = \int_0^{2\pi} e^{imu} e^{-im\beta\sin\theta\cos u} \sin u\, du = \frac{2\pi i^m}{\beta\sin\theta} J_m(m\beta\sin\theta) \tag{12}$$
$$C = \int_0^{2\pi} e^{imu} e^{-im\beta\sin\theta\cos u} \cos u\, du = 2\pi i^{m-1} J'_m(m\beta\sin\theta) \tag{13}$$

The evaluation of S and C are done via the Jacobi-Anger expansion, similar to problem 14.15. Putting these back to (4), we have

$$\begin{aligned}\left\langle \frac{dP_m(N)}{d\Omega} \right\rangle &= \frac{m^2 \omega_0^4}{(2\pi c)^3} \left(\frac{\beta c}{\omega_0}\right)^2 \left(|C|^2 + \cos^2\theta\,|S|^2\right) \left| \sum_{j=1}^N q_j e^{-im\theta_j} \right|^2 \\ &= \underbrace{\frac{m^2 \omega_0^4 R^2}{2\pi c^3} \left[J_m'^2(m\beta\sin\theta) + \frac{\cot^2\theta}{\beta^2} J_m^2(m\beta\sin\theta) \right]}_{dP_m(1)/d\Omega} \underbrace{\left| \sum_{j=1}^N q_j e^{im\theta_j} \right|^2}_{F_m(N)}\end{aligned} \tag{14}$$

2. For N equal charges $q_j = q$ uniformly spaced around the circle, $\theta_j = 2\pi(j-1)/N$. The form factor becomes

$$F_m(N) = q^2 \left| \overbrace{\sum_{k=0}^{N-1} e^{i2\pi mk/N}}^{N\delta_{m,pN}} \right|^2 = N^2 q^2 \delta_{m,pN} \qquad \text{for } p = 1,2,3\ldots \tag{15}$$

so radiation is emitted only into multiples of $N\omega_0$, with intensity N^2 times that of a single charge. Qualitatively: at $m \neq pN$, the N equally spaced phases sum to zero (destructive interference); at $m = pN$, all phases coincide, giving coherent N^2 enhancement.

3. In this setting, the first non-vanishing harmonic is $m = N$. If $\beta \ll 1$, we can use the small argument approximation for the Bessel functions $J_n(x) \approx (x/2)^n/n!$ and see that the square bracket of (14) is proportional to β^{2N-2}. Considering $\omega_0 R = \beta c$, one can say (14) is proportional to β^{2N}, or β^{2N+2} (depending whether we leave an extra ω_0^2 factor on the outside). In either case the radiation goes to zero as $N \to \infty$.

4. Again, the lowest contributing harmonic is $m = N$. For relativistic motion, the synchrotron spectrum (see 14.79) is controlled by $K_{2/3}(\xi)$ and $K_{1/3}(\xi)$ with $\xi \approx N\omega_0 R/\left(3c\gamma^3\right) \approx N/3\gamma^2$ at small angles. For $N \gg \gamma^3$, the asymptotic form $K_\nu(\xi) \sim e^{-\xi}$ gives

$$\frac{dP_N}{d\Omega} \propto e^{-2N/3\gamma^3} \tag{16}$$

5. A steady current in a loop is the continuum limit $N \to \infty$ of equally spaced charges with Nq and q/Δ held constant. Parts (c) and (d) show that the radiated power vanishes in this limit regardless of the speed of the charges, consistent with the well-known fact that a DC current loop produces only static magnetic fields and does not radiate.