1. **Prob 13.13**

   With the equation after (13.80), the amplitude $\mathbf{F}$ in the thin foil case is (using u for velocity since v and ν are easily confused)

$$\mathbf{F} = \frac{i\left\{1-\exp\left[i\left(\frac{\omega}{u}-k\cos\theta\right)a\right]\right\}}{\frac{\omega}{u}-k\cos\theta}\int dx\int dy\left(\hat{\mathbf{k}}\times\mathbf{E}_i\right)|_{z=0}\times\hat{\mathbf{k}}e^{-ikx\sin\theta} \tag{1}$$

   The semi infinite case in the text just treats the exponential in the first factor as zero, so clearly the result of thin foil case gains an extra factor of

$$1-\exp\left[i\overbrace{\left(\frac{\omega}{u}-k\cos\theta\right)a}^{2\Theta}\right]$$

   whose square is

$$\mathscr{F} = 4\sin^2\Theta \tag{2}$$

   Now with

$$\eta = \gamma^2\theta^2 \qquad\qquad \nu = \frac{\omega}{\gamma\omega_p} \tag{3}$$

   and

$$k = \frac{\omega n(\omega)}{c} = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{\omega}{c}\sqrt{1-\frac{1}{\gamma^2\nu^2}} \approx \frac{\omega}{c}\left(1-\frac{1}{2\gamma^2\nu^2}\right) \tag{4}$$

$$u = \beta c = c\sqrt{1-\frac{1}{\gamma^2}} \approx c\left(1-\frac{1}{2\gamma^2}\right) \tag{5}$$

   we have

$$\begin{aligned}
\Theta = \frac{a}{2}\left(\frac{\omega}{u}-k\cos\theta\right) &\approx \frac{\omega a}{2c}\left[\frac{1}{\beta}-\left(1-\frac{1}{2\gamma^2\nu^2}\right)\cos\theta\right] \\
&\approx \frac{\omega a}{2c}\left[\left(1+\frac{1}{2\gamma^2}\right)-\left(1-\frac{1}{2\gamma^2\nu^2}\right)\left(1-\frac{\theta^2}{2}\right)\right] \\
&\approx \frac{\omega a}{4\gamma^2 c}\left(1+\frac{1}{\nu^2}+\overbrace{\gamma^2\theta^2}^{\eta}\right) \\
&= \frac{a}{4}\left(\frac{\gamma c}{\omega_p}\right)^{-1}\cdot\left(\frac{\omega}{\gamma\omega_p}\right)\left(1+\frac{1}{\nu^2}+\eta\right) \qquad \text{recall } D = \frac{\gamma c}{\omega_p} \\
&= \nu\left(1+\frac{1}{\nu^2}+\eta\right)\frac{a}{4D}
\end{aligned} \tag{6}$$

2. **Prob 13.14**

(a) For dielectric constant varying with z as

$$\epsilon(\omega,z) = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \rho(z) \tag{7}$$

we can view this medium as having a z-dependent plasma frequency, whose square is weighted by $\rho(z)$,

$$\omega_p^2(z) = \omega_p^2(0)\rho(z) \tag{8}$$

This weighting function $\rho(z)$ enables us to rewrite (13.78) as

$$\mathbf{E}_{\text{rad}} \approx \frac{e^{ikr}}{r}\left[\frac{-\omega_p^2(0)}{4\pi c^2}\right]\int_{z'>0} \rho(z)\left(\hat{\mathbf{k}}\times\mathbf{E}_i\right)\times\hat{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}'}d^3x' \tag{9}$$

Correspondingly, the z-integral in the equation after (13.80) gains an additional weight $\rho(z)$ in the integrand, i.e.,

$$I_z = \int_0^Z dz\rho(z)\exp\left\{i\left[\frac{\omega}{u} - k(z)\cos\theta\right]z\right\} \tag{10}$$

With the reasonable assumption that $\rho(z)$, hence $k(z)$, varies slowly with z, we can approximate

$$k(z)z = \int_0^z k(z)dz' \approx \int_0^z k(z')dz' \tag{11}$$

hence

$$I_z \approx \int_0^Z dz\rho(z)e^{i\omega z/u}\exp\left[-i\cos\theta\int_0^z k(z')dz'\right] \tag{12}$$

The same z integral for the semi infinite case in the text yields $[\omega/u - k(0)\cos\theta]^{-1} = \mu^{-1}$, so overall there will be an additional factor of

$$\mathscr{F} = \left|\mu\int_0^Z dz\rho(z)e^{i\omega z/u}\exp\left[-i\cos\theta\int_0^z k(z')dz'\right]\right|^2 \tag{13}$$

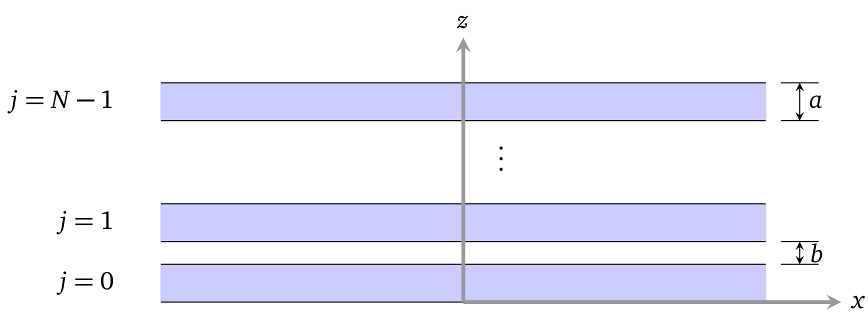
for the differential spectrum of transition radiation.

In the passing, we note that the approximation (11) is better than the alternative approximation $k(z)z \approx k(0)z$ since the respective errors are

$$\begin{aligned}
\text{Err}_{(11)} &= \left|\int_0^z k(z')dz' - k(z)z\right| \\
&= \left|\int_0^z \left[k(z) - k(z')\right]dz'\right| \\
&= \left|\int_0^z dz'\int_{z'}^z k'(s)ds\right| && \text{exchange integral order} \\
&= \left|\int_0^z ds\int_0^s k'(s)dz'\right| = \left|\int_0^z k'(s)sds\right| && (14) \\
\text{Err}_{\text{alt}} &= |k(z) - k(0)|z = \left|\int_0^z k'(s)zds\right| \geq \text{Err}_{(11)} && (15)
\end{aligned}$$

The supieriority of (11) will be more pronounced considering the piece-wise constant $k(z)$ in the stacked foils case below.

(b) For the stacked foils, $\rho = 0$ in the air and $\rho = 1$ in the foil.



The integral $\int_0^Z$ in (12) becomes a discrete sum over the N foils, i.e.,

$$I_z = \sum_{j=0}^{N-1} \int_{j(a+b)}^{j(a+b)+a} dz e^{i\omega z/u} \exp\left[-i \cos\theta \int_0^z k\left(z'\right) dz'\right] \tag{16}$$

For z in the j-th foil, i.e., $z \in [j(a+b), j(a+b)+a]$, the integral in the exponential becomes

$$\int_0^z k\left(z'\right) dz' = j\left(k_{\text{foil}} a + k_{\text{air}} b\right) + k_{\text{foil}} \left[z - j(a+b)\right] \tag{17}$$

turning (16) into

$$\begin{aligned} I_z &= \sum_{j=0}^{N-1} \int_{j(a+b)}^{j(a+b)+a} dz e^{i\omega z/u} \exp\left\{-i\cos\theta \left[j\left(k_{\text{foil}} a + k_{\text{air}} b\right)\right]\right\} \exp\left\{-i\cos\theta k_{\text{foil}} \left[z - j(a+b)\right]\right\} \\ &= \sum_{j=0}^{N-1} \exp\left\{-i\cos\theta \left[j\left(k_{\text{foil}} a + k_{\text{air}} b\right)\right]\right\} \exp\left[ij(a+b) \cdot \frac{\omega}{u}\right] \underbrace{\int_0^a dz \exp\left[i\left(\frac{\omega}{u} - k_{\text{foil}}\cos\theta\right) z\right]}_{A} \\ &= A \sum_{j=0}^{N-1} B^j = A \cdot \left(\frac{1-B^N}{1-B}\right) \end{aligned} \tag{18}$$

where

$$B = \exp\left[i\, \overbrace{a\left(\frac{\omega}{u} - k_{\text{foil}}\cos\theta\right)}^{2\Theta}\right] \cdot \exp\left[i\, \overbrace{b\left(\frac{\omega}{u} - k_{\text{air}}\cos\theta\right)}^{2\Psi}\right] \tag{19}$$

From part (1), the integral A is just

$$A = \frac{i\left\{1 - \exp\left[i\left(\frac{\omega}{u} - k_{\text{foil}}\cos\theta\right) a\right]\right\}}{\frac{\omega}{u} - k_{\text{foil}}\cos\theta} \tag{20}$$

whose square, compared to the semi infinite case in the text, contributes a factor $4\sin^2\Theta$ to the differential spectrum of transition radiation. However the stacked foils case has an additional factor coming from

$$\left|\frac{1-B^N}{1-B}\right|^2 = \frac{\sin^2\left[N(\Theta+\Psi)\right]}{\sin^2(\Theta+\Psi)} \tag{21}$$

where Θ is given in (6). Similarly, Ψ can be approximated

$$\Psi = \frac{b}{2}\left(\frac{\omega}{u} - k_{\text{air}}\cos\theta\right) = \frac{\omega b}{2c}\left(\frac{1}{\beta} - \cos\theta\right) \approx \frac{\omega b}{2c}\left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right) = \nu(1+\eta)\frac{b}{4D} \tag{22}$$

Combining contributions from both A and B, the overall factor gained by the differential spectrum of transition radiation in the stacked foils case, compared to the semi infinite case in the text, is

$$\mathscr{F} = 4\sin^2\Theta \cdot \frac{\sin^2\left[N(\Theta+\Psi)\right]}{\sin^2(\Theta+\Psi)} \tag{23}$$