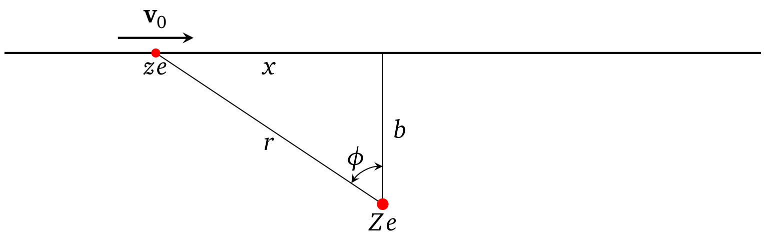
1. **Prob 14.7**



(a) At the location indicated by the diagram, the acceleartion of the incoming charge is

$$|\mathbf{a}| = \frac{zZe^2}{mr^2} \tag{1}$$

Larmor formula gives the power radiated at this location

$$P = \frac{2}{3}\frac{z^2e^2}{c^3}|\mathbf{a}|^2 = \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^3r^4} \tag{2}$$

With the (somewhat inconsistent) assumption that the charge is moving with uniform speed v_0 on a straight line, we have

$$r^2 = b^2 + x^2 \qquad \Longrightarrow \qquad xdx = rdr \qquad \Longrightarrow \qquad dt = \frac{dx}{v_0} = \frac{rdr}{xv_0} = \frac{dr}{v_0 \sin\phi} \tag{3}$$

The total radiated energy can be found by

$$\Delta W = \int_{-\pi/2}^{\pi/2} \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^3v_0}\frac{1}{(b/\cos\phi)^4}\frac{d\left(\dfrac{b}{\cos\phi}\right)}{\sin\phi}d\phi = \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^3v_0b^3}\int_{-\pi/2}^{\pi/2}\cos^2\phi\, d\phi = \frac{\pi z^4Z^2e^6}{3m^2c^3v_0b^3} \tag{4}$$

(b) Replacing b by

$$r_c = \frac{2zZe^2}{mv_0^2} \tag{5}$$

in (4) gives

$$\Delta W = \frac{\pi z^4Z^2e^6}{3m^2c^3v_0}\left(\frac{mv_0^2}{2zZe^2}\right)^3 = \frac{\pi}{24}\frac{zmv_0^5}{Zc^3} \tag{6}$$

Comparing this with the calculation of the exact non-relativistic head-on collision in problem 14.5, we see the coefficient changes from $8/45 \approx 0.18$ to $\pi/24 \approx 0.13$. The straight-line approximation error is about 30%.

(c) The radiation cross section can be calculated via

$$\chi = \int_{b_{\min}}^{\infty} \Delta W \cdot 2\pi b db = \frac{2\pi^2z^4Z^2e^6}{3m^2c^3v_0}\int_{b_{\min}}^{\infty}\frac{db}{b^2} = \frac{2\pi^2z^4Z^2e^6}{3m^2c^2v_0}\cdot\frac{1}{b_{\min}} \tag{7}$$

Using uncertainty principle as in chapter 13, we set $b_{\min} = \hbar/mv_0$, then

$$\chi = \frac{2\pi^2z^4Z^2e^6}{3\hbar c^3m} \tag{8}$$

where the result given by (15.30) is

$$\chi = \frac{16z^4Z^2e^6}{3\hbar c^3m} \tag{9}$$

The numerical coefficient has moderate difference: $2\pi^2/3 \approx 6.58$ from (10) versus $16/3 \approx 5.33$ from (11).

2. **Prob 14.8**

First recall the invariant form of Larmor formula (14.24), (14.25) for total radiated power

$$P = \frac{2}{3}\frac{z^2e^2}{m^2c^3}\left[\left(\frac{d\mathbf{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right] \tag{10}$$

Let $\mathbf{F}$ be the force acting on the particle, we have

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \quad \Longrightarrow \quad \frac{d\mathbf{p}}{d\tau} = \gamma\mathbf{F} \tag{11}$$

$$E = \sqrt{p^2c^2 + m^2c^4} \quad \Longrightarrow \quad \frac{dE}{d\tau} = \left(\boldsymbol{\nabla}_{\mathbf{p}}E\right)\cdot\frac{d\mathbf{p}}{d\tau} = \left(\frac{\mathbf{p}c^2}{E}\right)\cdot\gamma\mathbf{F} = \gamma\mathbf{v}\cdot\mathbf{F} \tag{12}$$

turning (10) into

$$P = \frac{2}{3}\frac{z^2e^2}{m^2c^3}\gamma^2\left[F^2 - (\boldsymbol{\beta}\cdot\mathbf{F})^2\right] \tag{13}$$

Invoking the straight-line, constant-speed assumption and referring to the diagram in problem 14.7, we know

$$F^2 - (\boldsymbol{\beta}\cdot\mathbf{F})^2 = F^2 - \beta_0^2F^2\sin^2\phi = \left(\frac{zZe^2}{r^2}\right)^2\left(1 - \beta_0^2\sin^2\phi\right) \tag{14}$$

With (3), the total radiated energy is the time integral

$$\begin{aligned}
\Delta W = \int_{-\infty}^{\infty} P\,dt &= \frac{2}{3}\frac{z^2e^2}{m^2c^3}\gamma_0^2\left(zZe^2\right)^2\int_{-\pi/2}^{\pi/2}\left[\frac{1}{(b/\cos\phi)^4}\right]\left(1-\beta_0^2\sin^2\phi\right)\frac{d\left(\dfrac{b}{\cos\phi}\right)}{v_0\sin\phi}d\phi \\
&= \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^3}\frac{1}{b^3v_0}\gamma_0^2\underbrace{\int_{-\pi/2}^{\pi/2}\left(1-\beta_0^2\sin^2\phi\right)\cos^2\phi\,d\phi}_{\frac{\pi}{8}\left(\frac{1}{\gamma_0^2}+3\right)} \\
&= \frac{\pi z^4Z^2e^6}{4m^2c^4\beta_0}\frac{1}{b^3}\left(\gamma_0^2 + \frac{1}{3}\right)
\end{aligned} \tag{15}$$

3. **Prob 14.9**

(a) Using (13) and the fact that the magnetic force is perpendicular to the velocity, we have the radiation power

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2(q\beta B)^2 = \frac{2}{3}\frac{q^4B^2}{m^2c^3}\gamma^2\beta^2 = \frac{2}{3}\frac{q^4B^2}{m^2c^3}\left(\gamma^2 - 1\right) \tag{16}$$

(b) With $\gamma \gg 1$, (16) becomes

$$-\frac{dE}{dt} = -mc^2\frac{d\gamma}{dt} \approx \frac{2}{3}\frac{q^4B^2}{m^2c^3}\gamma^2 \quad \Longrightarrow \quad -\frac{d\gamma}{\gamma^2} \approx \frac{2}{3}\frac{q^4B^2}{m^3c^5}dt \quad \Longrightarrow \quad t \approx \frac{3}{2}\frac{m^3c^5}{q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right) \tag{17}$$

(c) Without the approximation $\gamma \gg 1$, we would have

$$-\left(\frac{d\gamma}{\gamma^2-1}\right) = \frac{2}{3}\frac{q^4B^2}{m^3c^5}dt \quad \Longrightarrow \quad t = \frac{3}{2}\frac{m^3c^5}{q^4B^2}\cdot\frac{1}{2}\ln\left(\frac{\gamma+1}{\gamma-1}\right)\Bigg|_{\gamma_0}^{\gamma_t} \tag{18}$$

Define

$$\lambda \equiv \frac{4}{3}\frac{q^4B^2}{m^3c^5} \tag{19}$$

then from (18), we have

$$\ln\left(\frac{\gamma_t+1}{\gamma_t-1}\right) = \lambda t + \ln\left(\frac{\gamma_0+1}{\gamma_0-1}\right) \quad \Longrightarrow \quad \gamma_t = \frac{e^{\lambda t}\left(\dfrac{\gamma_0+1}{\gamma_0-1}\right)+1}{e^{\lambda t}\left(\dfrac{\gamma_0+1}{\gamma_0-1}\right)-1} \tag{20}$$

The kinetic energy at time t is

$$T_t = T_0\cdot\left(\frac{\gamma_t-1}{\gamma_0-1}\right) = \frac{2T_0}{e^{\lambda t}(\gamma_0+1)-(\gamma_0-1)} \quad \longrightarrow \quad T_0e^{-\lambda t} \quad \text{as} \quad \gamma_0 \to 1 \tag{21}$$

(d) For magnetic dipole $\mathbf{m}$, the magnetic induction is

$$\mathbf{B}=\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}}{r^3} \tag{22}$$

Let $\mathbf{m}$ be along the z-axis, and with $\hat{\mathbf{z}}=\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}$, the above can be written as

$$\mathbf{B}=\frac{m}{r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right) \qquad\Longrightarrow\qquad B^2=\frac{m^2}{r^6}\left(1+3\cos^2\theta\right) \tag{23}$$

By (16), the radiated power is proportional to B^2, which is greater at the turning point $\theta\neq\pi/2$ than at the equator $\theta=\pi/2$.

4. **Prob 14.10**

(a) For linear motion, the radiation power's angular distribution is given by (14.39)

$$\frac{dP}{d\Omega}=\frac{e^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{24}$$

where

$$\dot{v}=-\frac{\beta_0 c}{\Delta t} \qquad \beta=\beta_0\left(1-\frac{t}{\Delta t}\right) \qquad \text{for}\quad 0\le t\le\Delta t \tag{25}$$

The radiated energy per unit solid angle is the time integral

$$\begin{aligned}
\frac{dE}{d\Omega}&=\int_0^{\Delta t}\frac{dP}{d\Omega}dt=\frac{e^2}{4\pi c}\left(\frac{\beta_0}{\Delta t}\right)^2\sin^2\theta\int_0^{\Delta t}\frac{dt}{\left[\underbrace{1-\beta_0\left(1-\frac{t}{\Delta t}\right)\cos\theta}_{\equiv u}\right]^5}\\
&=\frac{e^2}{4\pi c}\left(\frac{\beta_0}{\Delta t}\right)^2\sin^2\theta\cdot\frac{\Delta t}{\beta_0\cos\theta}\int_{1-\beta_0\cos\theta}^{1}\frac{du}{u^5}\\
&=\frac{e^2\beta_0}{4\pi c\Delta t}\frac{\sin^2\theta}{\cos\theta}\cdot\frac{1}{4}\left[\frac{1}{(1-\beta_0\cos\theta)^4}-1\right]\\
&=\frac{e^2\beta_0}{16\pi c\Delta t}\frac{\sin^2\theta}{\cos\theta}\left\{\frac{\left[1+(1-\beta_0\cos\theta)^2\right]\left[1-(1-\beta_0\cos\theta)^2\right]}{(1-\beta_0\cos\theta)^4}\right\}\\
&=\frac{e^2\beta_0^2}{16\pi c\Delta t}\frac{(2-\beta_0\cos\theta)\left[1+(1-\beta_0\cos\theta)^2\right]\sin^2\theta}{(1-\beta_0\cos\theta)^4}
\end{aligned} \tag{26}$$

(b) Up to order $O\left(\gamma^{-2},\theta^2\right)$, we have

$$\beta=\left(1-\frac{1}{\gamma^2}\right)^{1/2}\approx 1-\frac{1}{2\gamma^2} \qquad \cos\theta\approx 1-\frac{\theta^2}{2} \qquad \sin\theta\approx\theta \tag{27}$$

then (26) can be approximated as

$$\frac{dE}{d\Omega}\approx\frac{e^2\beta^2}{16\pi c\Delta t}\frac{\overbrace{\left(1+\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)\left[1+\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^2\right]\theta^2}^{\approx\theta^2}}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^4}\approx\frac{e^2\beta^2}{\pi c\Delta t}\frac{\gamma^6(\gamma\theta)^2}{(1+\gamma^2\theta^2)^4}=\frac{e^2\beta^2\gamma^6}{\pi c\Delta t}\frac{\xi}{(1+\xi)^4} \tag{28}$$

From $\xi=\gamma^2\theta^2$, we have $d\xi=2\gamma^2\theta d\theta$, therefore

$$\frac{dE}{d\xi}=\frac{dE}{2\gamma^2\theta d\theta}=\frac{1}{2\gamma^2}\cdot 2\pi\frac{dE}{d\Omega}=\frac{e^2\beta^2\gamma^4}{c\Delta t}\frac{\xi}{(1+\xi)^4} \tag{29}$$

By definition (see 13.60),

$$\left\langle\theta^2\right\rangle=\frac{\displaystyle\int_{-\infty}^{\infty}\frac{\xi\theta^2\theta d\theta}{(1+\xi)^4}}{\displaystyle\int_{-\infty}^{\infty}\frac{\xi\theta d\theta}{(1+\xi)^4}}=\frac{\displaystyle\frac{1}{2\gamma^4}\cdot 2\int_0^{\infty}\frac{\xi^2 d\xi}{(1+\xi)^4}}{\displaystyle\frac{1}{2\gamma^2}\cdot 2\int_0^{\infty}\frac{\xi d\xi}{(1+\xi)^4}}=\frac{2}{\gamma^2} \qquad\Longrightarrow\qquad \sqrt{\langle\theta^2\rangle}=\frac{\sqrt{2}}{\gamma} \tag{30}$$