

1. Prob 14.13

The Fourier series expansion of $\mathbf{A}(t)$ is

$$\mathbf{A}(t) = \sum_{m=-\infty}^{\infty} \mathbf{A}_m e^{-im\omega_0 t} \quad (1)$$

$$\mathbf{A}_m = \frac{1}{T} \int_0^T \mathbf{A}(t) e^{im\omega_0 t} dt \quad (2)$$

where $T = 2\pi/\omega_0$ is the period.

Then

$$\begin{aligned} \left\langle \frac{dP}{d\Omega} \right\rangle &= \frac{1}{T} \int_{-\infty}^{\infty} |\mathbf{A}(t)|^2 dt = \frac{1}{T} \int_{-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} \mathbf{A}_m e^{-im\omega_0 t} \right] \cdot \left[\sum_{n=-\infty}^{\infty} \mathbf{A}_n^* e^{in\omega_0 t} \right] dt \\ &= \frac{1}{T} \sum_{m,n=-\infty}^{\infty} \mathbf{A}_m \cdot \mathbf{A}_n^* \underbrace{\int_{-\infty}^{\infty} e^{-i(m-n)\omega_0 t} dt}_{2\pi\delta[(m-n)\omega_0]} \\ &= \sum_{m=-\infty}^{\infty} |\mathbf{A}_m|^2 \end{aligned} \quad (3)$$

We thus identify $|\mathbf{A}_m|^2$ as the time-averaged power radiated per unit solid angle attributed to the m -th harmonic.

Using (14.52) and the radiation term of (14.14) for the electric field, we have (following similar steps from 14.61 – 14.67)

$$\begin{aligned} \mathbf{A}_m &= \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} \int_0^T e^{im\omega_0 t} \left\{ \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^3} \right\}_{\text{ret}} dt \\ &= \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} \int_0^T e^{im\omega_0 [t' + R(t')/c]} \left\{ \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^2} \right\} dt' \quad \text{up to an overall phase factor} \\ &= \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} \left\{ \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^2} \right\} dt \quad \text{integration by parts} \\ &= \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} (im\omega_0) \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} [\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})] dt \\ &= \sqrt{\frac{e^2}{4\pi c}} \frac{im\omega_0^2}{2\pi c} \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} [\mathbf{n} \times (\mathbf{n} \times \mathbf{v})] dt \end{aligned} \quad (4)$$

Since $|\mathbf{n} \times (\mathbf{n} \times \mathbf{u})|^2 = |\mathbf{n} \times \mathbf{u}|^2$, we can write

$$|\mathbf{A}_m|^2 = \frac{e^2 m^2 \omega_0^4}{16\pi^3 c^3} \left| \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} (\mathbf{n} \times \mathbf{v}) dt \right|^2 \quad (5)$$

For $m \neq 0$, we can add contributions of $\pm m$ together and write

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = 2 |\mathbf{A}_m|^2 = \frac{e^2 m^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} \mathbf{n} \times \mathbf{v} dt \right|^2 \quad \text{for } m \geq 0 \quad (6)$$

2. Prob 14.14

(a) For harmonic motion $z(t) = a \cos(\omega_0 t)$, the amplitude of the vector integral in (6) becomes

$$\begin{aligned} I_m &= \int_0^T e^{im\omega_0 [t - a \cos \theta \cos(\omega_0 t)/c]} \omega_0 a \sin(\omega_0 t) \sin \theta dt \\ &= \beta c \sin \theta \int_0^T e^{im\omega_0 t} e^{-im\beta \cos \theta \cos(\omega_0 t)} \left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \right) dt \end{aligned} \quad (7)$$

With the Jacobi-Anger expansion (see problem 3.16 c)

$$e^{iz \cos \phi} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\phi} \quad (8)$$

we can write

$$\begin{aligned} e^{-im\beta \cos \theta \cos(\omega_0 t)} &= \sum_{n=-\infty}^{\infty} i^n J_n(-m\beta \cos \theta) e^{in\omega_0 t} & J_n(-x) &= (-1)^n J_n(x) \\ &= \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta \cos \theta) e^{in\omega_0 t} \end{aligned} \quad (9)$$

turning (7) into

$$\begin{aligned} I_m &= \beta c \sin \theta \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta \cos \theta) \frac{1}{2i} \overbrace{\int_0^T [e^{i(m+n+1)\omega_0 t} - e^{i(m+n-1)\omega_0 t}] dt}^{\frac{2\pi}{\omega_0} (\delta_{m+n+1,0} - \delta_{m+n-1,0})} \\ &= \frac{\pi \beta c}{\omega_0 i} \sin \theta [(-i)^{-(m+1)} J_{-(m+1)}(m\beta \cos \theta) - (-i)^{-(m-1)} J_{-(m-1)}(m\beta \cos \theta)] & J_{-n}(x) &= (-1)^n J_n(x) \\ &= \frac{\pi \beta c}{\omega_0 i} \sin \theta [i^{m+1} J_{m+1}(m\beta \cos \theta) - i^{m-1} J_{m-1}(m\beta \cos \theta)] \\ &= \frac{\pi \beta c}{\omega_0} \sin \theta i^m [J_{m-1}(m\beta \cos \theta) + J_{m+1}(m\beta \cos \theta)] & J_{m-1}(x) + J_{m+1}(x) &= \frac{2m}{x} J_m(x) \\ &= \frac{2\pi c}{\omega_0} \tan \theta i^m J_m(m\beta \cos \theta) \end{aligned} \quad (10)$$

Putting this back into (6) gives

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = \frac{e^2 \beta^2 c}{2\pi a^2} m^2 \tan^2 \theta J_m^2(m\beta \cos \theta) \quad (11)$$

(b) For non-relativistic motion $\beta \ll 1$, small argument approximation of Bessel functions gives

$$J_m(m\beta \cos \theta) \approx \frac{1}{m!} \left(\frac{m\beta \cos \theta}{2} \right)^m \quad (12)$$

The radiated power is thus dominated by the $m = 1$ harmonic, giving

$$P \approx 2\pi \int_0^\pi \frac{dP_1}{d\Omega} \sin \theta d\theta = 2\pi \cdot \frac{e^2 \beta^2 c}{2\pi a^2} \int_0^\pi \tan^2 \theta \left(\frac{\beta \cos \theta}{2} \right)^2 \sin \theta d\theta = \frac{e^2 \omega_0^4 a^2}{3c^3} \quad (13)$$

Since

$$\overline{a^2} = \left\langle a^2 \cos^2 \omega_0 t \right\rangle = \frac{a^2}{2} \quad (14)$$

we can also write

$$P \approx \frac{2e^2 \omega_0^4 \overline{a^2}}{3c^3} \quad (15)$$

3. Prob 14.15

(a) The circular motion of the charge is described by

$$\mathbf{x}(t) = R [\cos(\omega_0 t) \hat{\mathbf{x}} + \sin(\omega_0 t) \hat{\mathbf{y}}] \quad \mathbf{v}(t) = \beta c [-\sin(\omega_0 t) \hat{\mathbf{x}} + \cos(\omega_0 t) \hat{\mathbf{y}}] \quad \text{where } \beta = \frac{\omega_0 R}{c} \quad (16)$$

Without loss of generality, let

$$\mathbf{n} = \cos \theta \hat{\mathbf{z}} + \sin \theta \hat{\mathbf{x}} \quad (17)$$

The vector integral in (6) becomes

$$\mathbf{I}_m = -\beta c \int_0^T e^{im\omega_0 t} e^{-im\beta \sin \theta \cos(\omega_0 t)} [\cos \theta \sin(\omega_0 t) \hat{\mathbf{y}} + \cos \theta \cos(\omega_0 t) \hat{\mathbf{x}} - \sin \theta \cos(\omega_0 t) \hat{\mathbf{z}}] dt \quad (18)$$

Similar to problem 14.14, we can evaluate the relevant integrals as

$$\begin{aligned} \int_0^T e^{im\omega_0 t} e^{-im\beta \sin \theta \cos(\omega_0 t)} \sin(\omega_0 t) dt &= \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta \sin \theta) \frac{1}{2i} \overbrace{\int_0^T [e^{i(m+n+1)\omega_0 t} - e^{i(m+n-1)\omega_0 t}] dt}^{\frac{2\pi}{\omega_0} (\delta_{m+n+1,0} - \delta_{m+n-1,0})} \\ &= \frac{\pi}{\omega_0 i} [i^{m+1} J_{m+1}(m\beta \sin \theta) - i^{m-1} J_{m-1}(m\beta \sin \theta)] \\ &= \frac{\pi}{\omega_0} i^m \frac{2}{\beta \sin \theta} J_m(m\beta \sin \theta) \end{aligned} \quad (19)$$

$$\begin{aligned} \int_0^T e^{im\omega_0 t} e^{-im\beta \sin \theta \cos(\omega_0 t)} \cos(\omega_0 t) dt &= \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta \sin \theta) \frac{1}{2} \overbrace{\int_0^T [e^{i(m+n+1)\omega_0 t} + e^{i(m+n-1)\omega_0 t}] dt}^{\frac{2\pi}{\omega_0} (\delta_{m+n+1,0} + \delta_{m+n-1,0})} \\ &= \frac{\pi}{\omega_0} [i^{m+1} J_{m+1}(m\beta \sin \theta) + i^{m-1} J_{m-1}(m\beta \sin \theta)] \\ &= \frac{\pi}{\omega_0} i^{m-1} \cdot 2 J'_m(m\beta \sin \theta) \end{aligned} \quad (20)$$

Putting (19) and (20) into (18) and (6) gives

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = \frac{e^2 \omega_0^4 R^2}{2\pi c^3} m^2 \left[J_m'^2(m\beta \sin \theta) + \frac{\cot^2 \theta J_m^2(m\beta \sin \theta)}{\beta^2} \right] \quad (21)$$

(b) For non-relativistic motion, (21) is again dominated by the $m = 1$ harmonic, in which case

$$J_1(\beta \sin \theta) \approx \frac{\beta \sin \theta}{2} \quad J'_1(\beta \sin \theta) \approx \frac{1}{2} \quad (22)$$

hence

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle \approx \frac{e^2 \omega_0^4 R^2}{2\pi c^3} \left(\frac{1}{4} + \frac{\cos^2 \theta}{4} \right) \implies P \approx \frac{e^2 \omega_0^4 R^2}{2\pi c^3} \cdot \left(\pi + \frac{2\pi}{4} \int_0^\pi \cos^2 \theta \sin \theta d\theta \right) = \frac{2e^2 \omega_0^4 R^2}{3c^3} \quad (23)$$

agreeing with 14.4 (b) since for circular motion $\bar{a^2} = R^2$.

(c) We wish to make the connection from (21) to Jackson (14.79)

$$\begin{aligned} \frac{d^2 I}{d\omega d\Omega} &= \frac{e^2}{3\pi^2 c} \left(\frac{\omega \rho}{c} \right)^2 \left(\frac{1}{\gamma^2} + \theta_J^2 \right)^2 \left[K_{2/3}^2(\xi) + \left(\frac{\theta_J^2}{\frac{1}{\gamma^2} + \theta_J^2} \right) K_{1/3}^2(\xi) \right] \quad \text{where} \\ \xi &= \frac{\omega \rho}{3c} \left(\frac{1}{\gamma^2} + \theta_J^2 \right)^{3/2} \end{aligned} \quad (24)$$

Here we have changed the θ symbol in (14.79) to θ_J since it represents the angle between \mathbf{n} and the x -axis, and our θ symbol represents the angle between \mathbf{n} and the z -axis. In other words, $\theta_J = \pi/2 - \theta$.

From the hint of the problem, we reference the following asymptotic form, see Watson pp. 249–(1).

$$J_n(x) \rightarrow \frac{1}{\pi} \left[\frac{2(n-x)}{3x} \right]^{1/2} K_{1/3} \left[\frac{2^{3/2} (n-x)^{3/2}}{3x^{1/2}} \right] \quad \text{for } x < n \quad (25)$$

In extreme relativistic motion, where $\omega_0 R/c = \beta \approx 1$ and $\gamma \gg 1$, only small θ_J is appreciable, so the argument of J_m in (21) is

$$x = m\beta \sin \theta = m\beta \cos \theta_J \approx m - \frac{m\theta_J^2}{2} \quad (26)$$

Also by (25) we have

$$J_m(m\beta \sin \theta) \rightarrow \frac{1}{\pi} \left[\frac{2 \cdot \frac{m\theta_J^2}{2}}{3 \left(m - \frac{m\theta_J^2}{2} \right)} \right]^{1/2} K_{1/3} \left[\frac{2^{3/2} \left(\frac{m\theta_J^2}{2} \right)^{3/2}}{3 \left(m - \frac{m\theta_J^2}{2} \right)^{1/2}} \right] \approx \frac{1}{\pi} \sqrt{\frac{\theta_J^2}{3}} K_{1/3} \left(\frac{m\theta_J^3}{3} \right) \quad (27)$$

With this, the second term in (21) becomes

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle_{II} \rightarrow \frac{e^2 m^2 c}{2\pi^3 R^2} \frac{\theta_J^4}{3} K_{1/3}^2 \left(\frac{m\theta_J^3}{3} \right) \quad (28)$$

while the second term in (24), under the limit $\omega\rho/c \rightarrow m, \gamma \rightarrow \infty$, becomes

$$\left. \frac{d^2 I}{d\omega d\Omega} \right|_{II} \rightarrow \frac{e^2 m^2}{3\pi^2 c} \theta_J^4 K_{1/3}^2(\xi) \quad \xi \approx \frac{m\theta_J^3}{3} \quad (29)$$

The equivalence of (28) and (29) is then clear since they both imply the same total energy radiated per unit solid angle in the frequency interval $[m\omega_0, (m+1)\omega_0]$:

$$\begin{aligned} \frac{dW}{d\Omega} &= \left\langle \frac{dP_m}{d\Omega} \right\rangle_{II} \cdot T = \frac{e^2 m^2 \omega_0}{3\pi^2 c} \theta_J^4 K_{1/3}^2 \left(\frac{m\theta_J^3}{3} \right) \\ \frac{dW}{d\Omega} &= \left. \frac{d^2 I}{d\omega d\Omega} \right|_{II} \cdot \omega_0 = \frac{e^2 m^2 \omega_0}{3\pi^2 c} \theta_J^4 K_{1/3}^2 \left(\frac{m\theta_J^3}{3} \right) \end{aligned} \quad (30)$$

To connect the J'_m term in (21) with the $K_{2/3}$ term in (24), let

$$\delta = m - x \quad \rho = \left(\frac{2\delta}{3x} \right)^{1/2} \quad \eta = \frac{2^{3/2} \delta^{3/2}}{3x^{1/2}} \quad (31)$$

then the derivative of (25) gives

$$J'_m(x) \approx \frac{1}{\pi} \left[\rho' K_{1/3}(\eta) + \rho \eta' K'_{1/3}(\eta) \right] = \frac{1}{\pi} \left[\left(\rho' - \frac{\rho \eta'}{3\eta} \right) K_{1/3}(\eta) - \rho \eta' K_{2/3}(\eta) \right] \quad (32)$$

where we have used the recurrence relation of modified Bessel functions

$$K'_\nu(z) = -\frac{\nu}{z} K_\nu(z) - K_{\nu-1}(z) \quad (33)$$

Direct calculation yields

$$\rho' - \frac{\rho \eta'}{3\eta} = -\frac{\rho}{3x} = -\frac{\sqrt{2}}{3\sqrt{3}} \frac{\delta^{1/2}}{x^{3/2}} \quad \rho \eta' = -\frac{2}{\sqrt{3}} \left(\frac{\delta}{x} + \frac{\delta^2}{3x^2} \right) \quad (34)$$

Assigning $n = m, x \approx m - m\theta_J^2/2$, we see that for large m ,

$$\delta \approx \frac{m\theta_J^2}{2} \quad \eta \approx \frac{m\theta_J^3}{3} \quad (35)$$

hence (32) can be approximated by the leading order

$$J'_m(m\beta \sin \theta) = J'_m(x) \approx \left(\frac{2}{\sqrt{3}\pi} \frac{\delta}{x} \right) K_{2/3}(\eta) = \frac{\theta_J^2}{\sqrt{3}\pi} K_{2/3} \left(\frac{m\theta_J^3}{3} \right) \quad (36)$$

With this, the equivalence of the J'_m term of (21) and the $K_{2/3}$ term of (24) can be shown similarly as before.