1. **Prob 13.7**

   Let θ be the small deflection angle from each collision in the multiple scattering process. Let θ_x and θ_y be the projected angle, then

$$\theta_x^2 + \theta_y^2 = \theta^2 \tag{1}$$

   The symmetry in x and y implies

$$\left\langle \theta_x^2 \right\rangle = \left\langle \theta_y^2 \right\rangle = \frac{\left\langle \theta^2 \right\rangle}{2} \tag{2}$$

   Let z be the incident direction and let t be the thickness of the material, and let n be the number of collisions before the particle exits the material. Also let the i-th collision happen at z_i, thus the total y displacement can be written as

$$y = \int_0^t \left(\sum_{i:z_i<z} \theta_{y,i} \right) dz \tag{3}$$

   where the sum in the integrand is the cumulative deflection angle projected in y direction before reaching z. Exchanging the order of summation and integration, we will change the lower limit of the integration to z_i (to indicate that only the collisions before z_i will contribute to the y offset at z_i), and then sum over all collisions, i.e.,

$$y = \sum_i \theta_{y,i} \int_{z_i}^t dz = \sum_i \theta_{y,i}(t - z_i) \tag{4}$$

   The mean square of y can be calculated as

$$\left\langle y^2 \right\rangle = \left\langle \sum_{i,j} \theta_{y,i}\theta_{y,j}\,(t - z_i)\left(t - z_j\right) \right\rangle \tag{5}$$

   When $i \neq j$, $\theta_{y,i}$ and $\theta_{y,j}$ are uncorrelated hence have no contribution to the mean, the double sum can be reduced to a single sum,

$$\left\langle y^2 \right\rangle = \left\langle \sum_i \theta_{y,i}^2\,(t - z_i)^2 \right\rangle = n\left\langle \theta_{y,i}^2 \right\rangle \left\langle (t - z_i)^2 \right\rangle \tag{6}$$

   where the latter equality is due to the independence of $\theta_{y,i}$ and z_i.
   For any single collision, z_i is uniformly distributed in $(0, t)$ so it has a probability density of $1/t$, giving

$$\left\langle (t - z_i)^2 \right\rangle = \int_0^t \frac{1}{t}\,(t - z_i)^2\, dz_i = \frac{t^2}{3} \tag{7}$$

   Together with (2) and (6), we have

$$\left\langle y^2 \right\rangle = \frac{nt^2 \left\langle \theta^2 \right\rangle}{6} = \frac{\left\langle \Theta^2 \right\rangle t^2}{6} \tag{8}$$

   and by central limit theorem, the distribution of y is Gaussian

$$P(y)\,dy = \frac{1}{\sqrt{2\pi \left\langle y^2 \right\rangle}} \exp\left[-\frac{y^2}{2\left\langle y^2 \right\rangle} \right] dy \tag{9}$$

2. **Prob 13.8**

   (a) Per the last paragraph of section 13.6, the single scattering tail will be absent if $\left\langle \Theta^2 \right\rangle$ is comparable to $\theta_{\max}^2$ – the limit of the single scattering deflection angle (see equation (13.57)). Thus we define the critical thickness x_c to be the thickness that makes $\left\langle \Theta^2 \right\rangle = \theta_{\max}^2$. Using (13.57) for $\theta_{\max}$ and (13.66) for $\left\langle \Theta^2 \right\rangle$, we have

$$4\pi N \left(\frac{2zZe^2}{pv} \right)^2 \ln\left(204Z^{-1/3}\right) x_c = \left(\frac{274}{A^{1/3}} \frac{mc}{p} \right)^2 \tag{10}$$

If the incident particle is relativistic (and $z = 1$), we can approximate $pv \approx pc$, then canceling $1/(pc)^2$ from both sides gives

$$4\pi N \cdot 4Z^2 e^4 \ln\left(204 Z^{-1/3}\right) x_c = \frac{274^2}{A^{2/3}} \left(mc^2\right)^2 \quad \Longrightarrow \quad x_c = \frac{274^2}{A^{2/3}} \cdot \frac{1}{16\pi N Z^2 \ln(204 Z^{-1/3})} \cdot \left(\frac{mc^2}{e^2}\right)^2 \tag{11}$$

Note that

$$\frac{e^2}{1.44 \times 10^{-7}\text{cm}} = 1\text{eV} \qquad \Longrightarrow \qquad \frac{mc^2}{e^2} = \frac{0.511 \times 10^6 \text{eV}}{1.44 \times 10^{-7} \text{eV} \cdot \text{cm}} \approx 3.55 \times 10^{12} \text{cm}^{-1} \tag{12}$$

Thus for

- aluminum: $Z = 13, A = 27, N = 6.02 \times 10^{22}$atoms/cm^3, we have

$$x_c \approx 46\text{cm} \tag{13}$$

- lead: $Z = 82, A = 207, N = 3.3 \times 10^{22}$atoms/cm^3,

$$x_c \approx 0.63\text{cm} \tag{14}$$

(b) By (13.64)

$$\begin{aligned}
n = N\sigma t &= \pi N \cdot \frac{4Z^2 e^4}{p^2 c^2} \frac{x_c}{\theta_{\min}^2} && \text{by (13.55)} \\
&= \frac{4\pi N Z^2 e^4}{p^2 c^2} \frac{192^2 p^2 c^2}{Z^{2/3} m^2 c^4} \cdot x_c \\
&= 192^2 \cdot 4\pi N Z^{4/3} \left(\frac{e^2}{mc^2}\right)^2 \cdot x_c \\
&\approx \begin{cases} 3.1 \times 10^6 & \text{for aluminum} \\ 2.7 \times 10^5 & \text{for lead} \end{cases}
\end{aligned} \tag{15}$$

These are very large numbers, justifying the Gaussian approximation.