1. **Prob 14.13**

The Fourier series expansion of $\mathbf{A}(t)$ is

$$\mathbf{A}(t) = \sum_{m=-\infty}^{\infty} \mathbf{A}_m e^{-im\omega_0 t} \tag{1}$$

$$\mathbf{A}_m = \frac{1}{T}\int_0^T \mathbf{A}(t) e^{im\omega_0 t} dt \tag{2}$$

where $T = 2\pi/\omega_0$ is the period.

Then

$$\begin{aligned}\left\langle \frac{dP}{d\Omega}\right\rangle &= \frac{1}{T}\int_{-\infty}^{\infty} |\mathbf{A}(t)|^2 dt = \frac{1}{T}\int_{-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty}\mathbf{A}_m e^{-im\omega_0 t}\right]\cdot\left[\sum_{n=-\infty}^{\infty}\mathbf{A}_n^* e^{in\omega_0 t}\right]dt \\ &= \frac{1}{T}\sum_{m,n=-\infty}^{\infty}\mathbf{A}_m\cdot\mathbf{A}_n^*\underbrace{\int_{-\infty}^{\infty} e^{-i(m-n)\omega_0 t}dt}_{2\pi\delta[(m-n)\omega_0]} \\ &= \sum_{m=\infty}^{\infty}|\mathbf{A}_m|^2 \end{aligned} \tag{3}$$

We thus identify $|\mathbf{A}_m|^2$ as the time-averaged power radiated per unit solid angle attributed to the m-th harmonic.

Using (14.52) and the radiation term of (14.14) for the electric field, we have (following similar steps from 14.61 – 14.67)

$$\begin{aligned}\mathbf{A}_m &= \frac{1}{T}\sqrt{\frac{e^2}{4\pi c}}\int_0^T e^{im\omega_0 t}\left\{\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right\}_{\text{ret}} dt \\ &= \frac{1}{T}\sqrt{\frac{e^2}{4\pi c}}\int_0^T e^{im\omega_0\left[t'+R\left(t'\right)/c\right]}\left\{\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\right\}dt' && \text{up to an overall phase factor} \\ &= \frac{1}{T}\sqrt{\frac{e^2}{4\pi c}}\int_0^T e^{im\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]}\left\{\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\right\}dt && \text{integration by parts} \\ &= \frac{1}{T}\sqrt{\frac{e^2}{4\pi c}}(im\omega_0)\int_0^T e^{im\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]}[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})]dt \\ &= \sqrt{\frac{e^2}{4\pi c}}\frac{im\omega_0^2}{2\pi c}\int_0^T e^{im\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]}[\mathbf{n}\times(\mathbf{n}\times\mathbf{v})]dt \end{aligned} \tag{4}$$

Since $|\mathbf{n}\times(\mathbf{n}\times\mathbf{u})|^2 = |\mathbf{n}\times\mathbf{u}|^2$, we can write

$$|\mathbf{A}_m|^2 = \frac{e^2m^2\omega_0^4}{16\pi^3c^3}\left|\int_0^T e^{im\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]}(\mathbf{n}\times\mathbf{v})dt\right|^2 \tag{5}$$

For $m \neq 0$, we can add contributions of $\pm m$ together and write

$$\left\langle\frac{dP_m}{d\Omega}\right\rangle = 2|\mathbf{A}_m|^2 = \frac{e^2m^2\omega_0^4}{(2\pi c)^3}\left|\int_0^T e^{im\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]}\mathbf{n}\times\mathbf{v}dt\right|^2 \tag{6}$$

2. **Prob 14.14**

(a) For harmonic motion $z(t) = a\cos(\omega_0 t)$, the amplitude of the vector integral in (6) becomes

$$\begin{aligned} I_m &= \int_0^T e^{im\omega_0[t-a\cos\theta\cos(\omega_0 t)/c]}\omega_0 a\sin(\omega_0 t)\sin\theta\, dt \\ &= \beta c\sin\theta\int_0^T e^{im\omega_0 t}e^{-im\beta\cos\theta\cos(\omega_0 t)}\left(\frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}\right)dt \end{aligned} \tag{7}$$

With the Jacobi-Anger expansion (see problem 3.16 c)

$$e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n J_n(z)\, e^{in\phi} \tag{8}$$

we can write

$$\begin{aligned} e^{-im\beta\cos\theta\cos(\omega_0 t)} &= \sum_{n=-\infty}^{\infty} i^n J_n(-m\beta\cos\theta)\, e^{in\omega_0 t} \qquad\qquad J_n(-x) = (-1)^n J_n(x) \\ &= \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta\cos\theta)\, e^{in\omega_0 t} \end{aligned} \tag{9}$$

turning (7) into

$$\begin{aligned} I_m &= \beta c\sin\theta \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta\cos\theta)\, \frac{1}{2i} \overbrace{\int_0^T \left[e^{i(m+n+1)\omega_0 t} - e^{i(m+n-1)\omega_0 t}\right] dt}^{(2\pi/\omega_0)\left(\delta_{m+n+1,0}-\delta_{m+n-1,0}\right)} \\ &= \frac{\pi\beta c}{\omega_0 i}\sin\theta\left[(-i)^{-(m+1)} J_{-(m+1)}(m\beta\cos\theta) - (-i)^{-(m-1)} J_{-(m-1)}(m\beta\cos\theta)\right] \qquad J_{-n}(x) = (-1)^n J_n(x) \\ &= \frac{\pi\beta c}{\omega_0 i}\sin\theta\left[i^{m+1} J_{m+1}(m\beta\cos\theta) - i^{m-1} J_{m-1}(m\beta\cos\theta)\right] \\ &= \frac{\pi\beta c}{\omega_0}\sin\theta\, i^m \left[J_{m-1}(m\beta\cos\theta) + J_{m+1}(m\beta\cos\theta)\right] \qquad J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x) \\ &= \frac{2\pi c}{\omega_0}\tan\theta\, i^m J_m(m\beta\cos\theta) \end{aligned} \tag{10}$$

Putting this back into (6) gives

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = \frac{e^2\beta^2 c}{2\pi a^2} m^2 \tan^2\theta\, J_m^2(m\beta\cos\theta) \tag{11}$$

(b) For non-relativistic motion $\beta \ll 1$, small argument approximation of Bessel functions gives

$$J_m(m\beta\cos\theta) \approx \frac{1}{m!}\left(\frac{m\beta\cos\theta}{2}\right)^m \tag{12}$$

The radiated power is thus dominated by the $m = 1$ harmonic, giving

$$P \approx 2\pi \int_0^\pi \frac{dP_1}{d\Omega}\sin\theta\, d\theta = 2\pi \cdot \frac{e^2\beta^2 c}{2\pi a^2} \int_0^\pi \tan^2\theta \left(\frac{\beta\cos\theta}{2}\right)^2 \sin\theta\, d\theta = \frac{e^2\omega_0^4 a^2}{3c^3} \tag{13}$$

Since

$$\overline{a^2} = \left\langle a^2\cos^2\omega_0 t \right\rangle = \frac{a^2}{2} \tag{14}$$

we can also write

$$P \approx \frac{2e^2\omega_0^4 \overline{a^2}}{3c^3} \tag{15}$$

3. **Prob 14.15**

(a) The circular motion of the charge is described by

$$\mathbf{x}(t) = R\left[\cos(\omega_0 t)\hat{\mathbf{x}} + \sin(\omega_0 t)\hat{\mathbf{y}}\right] \qquad \mathbf{v}(t) = \beta c\left[-\sin(\omega_0 t)\hat{\mathbf{x}} + \cos(\omega_0 t)\hat{\mathbf{y}}\right] \qquad \text{where } \beta = \frac{\omega_0 R}{c} \tag{16}$$

Without loss of generality, let

$$\mathbf{n} = \cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\mathbf{x}} \tag{17}$$

The vector integral in (6) becomes

$$\mathbf{I}_m = -\beta c \int_0^T e^{im\omega_0 t} e^{-im\beta\sin\theta\cos(\omega_0 t)} \left[\cos\theta\sin(\omega_0 t)\hat{\mathbf{y}} + \cos\theta\cos(\omega_0 t)\hat{\mathbf{x}} - \sin\theta\cos(\omega_0 t)\hat{\mathbf{z}}\right] \tag{18}$$

Similar to problem 14.14, we can evaluate the relevant integrals as

$$\begin{aligned}
\int_0^T e^{im\omega_0 t} e^{-im\beta\sin\theta\cos(\omega_0 t)} \sin(\omega_0 t) &= \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta\sin\theta) \frac{1}{2i} \overbrace{\int_0^T \left[e^{i(m+n+1)\omega_0 t} - e^{i(m+n-1)\omega_0 t}\right] dt}^{\frac{2\pi}{\omega_0}\left(\delta_{m+n+1,0} - \delta_{m+n-1,0}\right)} \\
&= \frac{\pi}{\omega_0 i}\left[i^{m+1} J_{m+1}(m\beta\sin\theta) - i^{m-1} J_{m-1}(m\beta\sin\theta)\right] \\
&= \frac{\pi}{\omega_0} i^m \frac{2}{\beta\sin\theta} J_m(m\beta\sin\theta)
\end{aligned} \tag{19}$$

$$\begin{aligned}
\int_0^T e^{im\omega_0 t} e^{-im\beta\sin\theta\cos(\omega_0 t)} \cos(\omega_0 t) &= \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta\sin\theta) \frac{1}{2} \overbrace{\int_0^T \left[e^{i(m+n+1)\omega_0 t} + e^{i(m+n-1)\omega_0 t}\right] dt}^{\frac{2\pi}{\omega_0}\left(\delta_{m+n+1,0} + \delta_{m+n-1,0}\right)} \\
&= \frac{\pi}{\omega_0}\left[i^{m+1} J_{m+1}(m\beta\sin\theta) + i^{m-1} J_{m-1}(m\beta\sin\theta)\right] \\
&= \frac{\pi}{\omega_0} i^{m-1} \cdot 2J_m'(m\beta\sin\theta)
\end{aligned} \tag{20}$$

Putting (19) and (20) into (18) and (6) gives

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = \frac{e^2\omega_0^4 R^2}{2\pi c^3} m^2 \left[J_m'^2(m\beta\sin\theta) + \frac{\cot^2\theta J_m^2(m\beta\sin\theta)}{\beta^2}\right] \tag{21}$$

(b) For non-relativistic motion, (21) is again dominated by the $m = 1$ harmonic, in which case

$$J_1(\beta\sin\theta) \approx \frac{\beta\sin\theta}{2} \qquad\qquad J_1'(\beta\sin\theta) \approx \frac{1}{2} \tag{22}$$

hence

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle \approx \frac{e^2\omega_0^4 R^2}{2\pi c^3}\left(\frac{1}{4} + \frac{\cos^2\theta}{4}\right) \quad \Longrightarrow \quad P \approx \frac{e^2\omega_0^4 R^2}{2\pi c^3} \cdot \left(\pi + \frac{2\pi}{4}\int_0^\pi \cos^2\theta\sin\theta\, d\theta\right) = \frac{2e^2\omega_0^4 R^2}{3c^3} \tag{23}$$

agreeing with 14.4 (b) since for circular motion $\overline{a^2} = R^2$.

(c) We wish to make the connection from (21) to Jackson (14.79)

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2 \left(\frac{1}{\gamma^2} + \theta_J^2\right)^2 \left[K_{2/3}^2(\xi) + \left(\frac{\theta_J^2}{\frac{1}{\gamma^2} + \theta_J^2}\right) K_{1/3}^2(\xi)\right] \qquad \text{where}$$

$$\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2} + \theta_J^2\right)^{3/2} \tag{24}$$

Here we have changed the θ symbol in (14.79) to θ_J since it represents the angle between $\mathbf{n}$ and the x-axis, and our θ symbol represents the angle between $\mathbf{n}$ and the z-axis. In other words, $\theta_J = \pi/2 - \theta$.

From the hint of the problem, we reference the following asymptotic form, see *Watson* pp. 249–(1).

$$J_n(x) \to \frac{1}{\pi}\left[\frac{2(n-x)}{3x}\right]^{1/2} K_{1/3}\left[\frac{2^{3/2}(n-x)^{3/2}}{3x^{1/2}}\right] \qquad \text{for } x < n \tag{25}$$

In extreme relativistic motion, where $\omega_0 R/c = \beta \approx 1$ and $\gamma \gg 1$, only small θ_J is appreciable, so the argument of J_m in (21) is

$$x = m\beta\sin\theta = m\beta\cos\theta_J \approx m - \frac{m\theta_J^2}{2} \tag{26}$$

Also by (25) we have

$$J_m(m\beta\sin\theta) \to \frac{1}{\pi}\left[\frac{2\cdot\frac{m\theta_J^2}{2}}{3\left(m-\frac{m\theta_J^2}{2}\right)}\right]^{1/2} K_{1/3}\left[\frac{2^{3/2}\left(\frac{m\theta_J^2}{2}\right)^{3/2}}{3\left(m-\frac{m\theta_J^2}{2}\right)^{1/2}}\right] \approx \frac{1}{\pi}\sqrt{\frac{\theta_J^2}{3}}K_{1/3}\left(\frac{m\theta_J^3}{3}\right) \tag{27}$$

With this, the second term in (21) becomes

$$\left\langle\frac{dP_m}{d\Omega}\right\rangle_{II} \to \frac{e^2m^2c}{2\pi^3R^2}\frac{\theta_J^4}{3}K_{1/3}^2\left(\frac{m\theta_J^3}{3}\right) \tag{28}$$

while the second term in (24), under the limit $\omega\rho/c \to m$, $\gamma\to\infty$, becomes

$$\left.\frac{d^2I}{d\omega d\Omega}\right|_{II} \to \frac{e^2m^2}{3\pi^2c}\theta_J^4K_{1/3}^2(\xi) \qquad\qquad \xi\approx\frac{m\theta_J^3}{3} \tag{29}$$

The equivalence of (28) and (29) is then clear since they both imply the same total energy radiated per unit solid angle in the frequency interval $[m\omega_0,(m+1)\omega_0]$:

$$\begin{aligned}
\frac{dW}{d\Omega} &= \left\langle\frac{dP_m}{d\Omega}\right\rangle_{II}\cdot T = \frac{e^2m^2\omega_0}{3\pi^2c}\theta_J^4K_{1/3}^2\left(\frac{m\theta_J^3}{3}\right)\\
\frac{dW}{d\Omega} &= \left.\frac{d^2I}{d\omega d\Omega}\right|_{II}\cdot\omega_0 = \frac{e^2m^2\omega_0}{3\pi^2c}\theta_J^4K_{1/3}^2\left(\frac{m\theta_J^3}{3}\right)
\end{aligned} \tag{30}$$

To connect the J_m' term in (21) with the $K_{2/3}$ term in (24), let

$$\delta = m-x \qquad\qquad \rho=\left(\frac{2\delta}{3x}\right)^{1/2} \qquad\qquad \eta=\frac{2^{3/2}\delta^{3/2}}{3x^{1/2}} \tag{31}$$

then the derivative of (25) gives

$$J_m'(x)\approx\frac{1}{\pi}\left[\rho'K_{1/3}(\eta)+\rho\eta'K_{1/3}'(\eta)\right]=\frac{1}{\pi}\left[\left(\rho'-\frac{\rho\eta'}{3\eta}\right)K_{1/3}(\eta)-\rho\eta'K_{2/3}(\eta)\right] \tag{32}$$

where we have used the recurrence relation of modified Bessel functions

$$K_\nu'(z) = -\frac{\nu}{z}K_\nu(z)-K_{\nu-1}(z) \tag{33}$$

Direct calculation yields

$$\rho'-\frac{\rho\eta'}{3\eta}=-\frac{\rho}{3x}=-\frac{\sqrt{2}}{3\sqrt{3}}\frac{\delta^{1/2}}{x^{3/2}} \qquad\qquad \rho\eta'=-\frac{2}{\sqrt{3}}\left(\frac{\delta}{x}+\frac{\delta^2}{3x^2}\right) \tag{34}$$

Assigning $n=m$, $x\approx m-m\theta_J^2/2$, we see that for large m,

$$\delta\approx\frac{m\theta_J^2}{2} \qquad\qquad \eta\approx\frac{m\theta_J^3}{3} \tag{35}$$

hence (32) can be approximated by the leading order

$$J_m'(m\beta\sin\theta)=J_m'(x)\approx\left(\frac{2}{\sqrt{3}\pi}\frac{\delta}{x}\right)K_{2/3}(\eta)=\frac{\theta_J^2}{\sqrt{3}\pi}K_{2/3}\left(\frac{m\theta_J^3}{3}\right) \tag{36}$$

With this, the equivalence of the J_m' term of (21) and the $K_{2/3}$ term of (24) can be shown similarly as before.