The charge's position $\mathbf{r}(t)$ can be expanded at $t=0$ as

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}(0)\,t + \frac{\mathbf{a}(0)}{2}t^2 + \frac{\mathbf{j}(0)}{6}t^3 + \cdots \tag{1}$$

where $\mathbf{v}, \mathbf{a}, \mathbf{j}$ are the velocity, acceleration, and jerk of the charge respectively.

When the charge moves ultrarelativistcially, we can approximately treat its speed as constant, therefore

$$0 = \frac{d(\mathbf{v}\cdot\mathbf{v})}{dt} = 2\mathbf{v}\cdot\mathbf{a} \qquad \Longrightarrow \qquad 0 = \frac{d^2(\mathbf{v}\cdot\mathbf{v})}{dt^2} = 2a^2 + 2\mathbf{v}\cdot\mathbf{j} \qquad \Longrightarrow \qquad \mathbf{v}\cdot\mathbf{j} = -a^2 = -\left(\frac{v^2}{\rho}\right)^2 = -\frac{v^4}{\rho^2} \tag{2}$$

In ultrarelativistic beaming, the photon's direction $\mathbf{k}$ is approximately along $\mathbf{v}$, so we have

$$\begin{aligned}
\mathbf{k}\cdot\mathbf{r}(t) &\approx \mathbf{k}\cdot\mathbf{r}(0) + \mathbf{k}\cdot\mathbf{v}(0)\,t + \frac{\mathbf{k}\cdot\mathbf{a}(0)}{2}t^2 + \frac{\mathbf{k}\cdot\mathbf{j}(0)}{6}t^3 && \text{up to global phase factor}\\
&\approx \mathbf{k}\cdot\mathbf{v}(0)\,t + \frac{k\mathbf{v}(0)\cdot\mathbf{j}(0)}{6v(0)}t^3 && \text{by (2)}\\
&= \mathbf{k}\cdot\mathbf{v}(0)\,t - \frac{k[v(0)]^3}{6\rho^2}t^3 && \text{take } v(0)\approx c \text{ in higher order term}\\
&\approx \mathbf{k}\cdot\mathbf{v}(0)\,t - \frac{\omega c^2}{6\rho^2}t^3
\end{aligned} \tag{3}$$

The phase (14.72) becomes

$$\omega\left[t - \frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}\right] = \omega\left[t - \frac{\mathbf{k}\cdot\mathbf{r}(t)}{kc}\right] = \omega t - \mathbf{k}\cdot\mathbf{r}(t) \approx [\omega - \mathbf{k}\cdot\mathbf{v}(0)]\,t + \frac{\omega c^2}{6\rho^2}t^3 = \frac{p^\mu k_\mu}{\gamma m}t + \frac{\omega c^2}{6\rho^2}t^3 \tag{4}$$

where we can see the identical coefficient of the t^3 term as (14.73).

For this problem, the more useful form is

$$\omega\left[t - \frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}\right] \approx \frac{p^\mu k_\mu}{\gamma m}t - \frac{\mathbf{k}\cdot\mathbf{j}(0)}{6}t^3 \tag{5}$$

because we can express $\mathbf{k}\cdot\mathbf{j}$ in terms of $p^\mu k_\mu$ as follows:

$$\frac{d^2\left(p^\mu k_\mu\right)}{d\tau^2} = \frac{d^2}{d\tau^2}\left[\gamma m(\omega - \mathbf{k}\cdot\mathbf{v})\right] \qquad \Longrightarrow \qquad \frac{1}{\gamma^3 m}\frac{d^2\left(p^\mu k_\mu\right)}{d\tau^2} = -\mathbf{k}\cdot\mathbf{j} = \frac{\omega c^2}{\rho^2} \tag{6}$$

where we have used the approximation that γ is constant and that $d\tau = dt/\gamma$.

From (6), we get

$$\rho^2 = \frac{\omega\gamma^3 mc^2}{d^2\left(p^\mu k_\mu\right)/d\tau^2} = \frac{\omega E^3}{m^2c^4\left[d^2\left(p^\mu k_\mu\right)/d\tau^2\right]} \tag{7}$$

and the phase can be written explicitly as

$$\omega\left[t - \frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}\right] = \overbrace{\left(\frac{p^\mu k_\mu}{\gamma m}\right)}^{a}t + \frac{1}{6}\overbrace{\left[\frac{1}{\gamma^3 m}\frac{d^2\left(p^\mu k_\mu\right)}{d\tau^2}\right]}^{b}t^3 \tag{8}$$

Referencing figure 14.9, we see that the polarization vectors of this problem are $\boldsymbol{\epsilon}_\parallel = \boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_\perp = \mathbf{n}\times\boldsymbol{\epsilon}_1 = -\boldsymbol{\epsilon}_2$, and

$$\mathbf{p}\cdot\boldsymbol{\epsilon}_2 = \mathbf{p}\cdot(\boldsymbol{\epsilon}_1\times\mathbf{n}) = \mathbf{n}\cdot(\mathbf{p}\times\boldsymbol{\epsilon}_1) = \gamma m v\cos\left(\frac{vt}{\rho}\right)\sin\theta \tag{9}$$

$$\frac{d(\boldsymbol{\epsilon}_1\cdot\mathbf{p})}{d\tau} \approx \gamma^2 m\boldsymbol{\epsilon}_1\cdot\mathbf{a} = \gamma^2 m\frac{v^2}{\rho} \tag{10}$$

Thus from (14.71)

$$\begin{aligned}
\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) &= \beta\left[-\boldsymbol{\epsilon}_\parallel\sin\left(\frac{vt}{\rho}\right) + \boldsymbol{\epsilon}_\perp\cos\left(\frac{vt}{\rho}\right)\sin\theta\right]\\
&\approx \frac{1}{c}\left[-\boldsymbol{\epsilon}_1\left(\frac{v^2}{\rho}\right)t - \boldsymbol{\epsilon}_2 v\cos\left(\frac{vt}{\rho}\right)\sin\theta\right]\\
&= -\boldsymbol{\epsilon}_1\left[\frac{d(\boldsymbol{\epsilon}_1\cdot\mathbf{p})/d\tau}{\gamma^2 mc}\right]t - \boldsymbol{\epsilon}_2\left(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}_2}{\gamma mc}\right)
\end{aligned} \tag{11}$$

Putting (11) and (8) into (14.67), we get

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|-\boldsymbol{\epsilon}_1 A_1(\omega) - \boldsymbol{\epsilon}_2 A_2(\omega)\right|^2 \tag{12}$$

where

$$A_1(\omega) \approx \left[\frac{d(\boldsymbol{\epsilon}_1\cdot\mathbf{p})/d\tau}{\gamma^2 mc}\right]\overbrace{\int_{-\infty}^{\infty} t\exp\left[i\left(at+\frac{bt^3}{6}\right)\right]dt}^{I_1} \tag{13}$$

$$A_2(\omega) \approx \left(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}_2}{\gamma mc}\right)\underbrace{\int_{-\infty}^{\infty} \exp\left[i\left(at+\frac{bt^3}{6}\right)\right]dt}_{I_2} \tag{14}$$

Taking parity into consideration, the two integrals are

$$I_1 = 2i\int_0^{\infty} t\sin\left(at+\frac{bt^3}{6}\right)dt \tag{15}$$

$$I_2 = 2\int_0^{\infty} \cos\left(at+\frac{bt^3}{6}\right)dt \tag{16}$$

With variable change

$$t = \left(\frac{3\xi}{2a}\right)x \qquad \text{where} \qquad \xi = \frac{2\sqrt{2}a^{3/2}}{3b^{1/2}} = \frac{2\sqrt{2}}{3m}\frac{\left(p^\mu k_\mu\right)^{3/2}}{\left[d^2\left(p^\mu k_\mu\right)/d\tau^2\right]^{1/2}} \tag{17}$$

the two integrals become

$$I_1 = 2i\left(\frac{3\xi}{2a}\right)^2\int_0^{\infty} x\sin\left[\frac{3}{2}\xi\left(x+\frac{x^3}{3}\right)\right]dx = 2i\left(\frac{3\xi}{2a}\right)^2\cdot\frac{1}{\sqrt{3}}K_{2/3}(\xi) \tag{18}$$

$$I_2 = 2\left(\frac{3\xi}{2a}\right)\int_0^{\infty} \cos\left[\frac{3}{2}\xi\left(x+\frac{x^3}{3}\right)\right]dx = 2\left(\frac{3\xi}{2a}\right)\cdot\frac{1}{\sqrt{3}}K_{1/3}(\xi) \tag{19}$$

Putting these into (12) yields

$$\begin{aligned}\frac{d^2 I}{d\omega d\Omega} &= \frac{e^2\omega^2}{4\pi^2 c}\left[A_1^2(\omega)+A_2^2(\omega)\right]\\ &= \frac{e^2\omega^2}{4\pi^2 c}\left\{\left[\frac{d(\boldsymbol{\epsilon}_1\cdot\mathbf{p})/d\tau}{\gamma^2 mc}\right]^2\left(\frac{3\sqrt{3}\xi^2}{2a^2}\right)^2 K_{2/3}^2(\xi) + \left(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}_2}{\gamma mc}\right)^2\left(\frac{\sqrt{3}\xi}{a}\right)^2 K_{1/3}^2(\xi)\right\}\end{aligned} \tag{20}$$

By (17) and (8), we have

$$\frac{\xi}{a} = \frac{2\sqrt{2}}{3}\left(\frac{a}{b}\right)^{1/2} = \frac{2\sqrt{2}\gamma}{3}\left[\frac{p^\mu k_\mu}{d^2\left(p^\mu k_\mu\right)/d\tau^2}\right]^{1/2} \tag{21}$$

which turns (20) into

$$\frac{d^2 I}{d\omega d\Omega} = \frac{4e^2\omega^2}{3\pi^2 m^2 c^3}\left\{\left[\frac{d(\boldsymbol{\epsilon}_1\cdot\mathbf{p})}{d\tau}\right]^2\left[\frac{p^\mu k_\mu}{d^2\left(p^\mu k_\mu\right)/d\tau^2}\right]^2 K_{2/3}^2(\xi) + \frac{(\mathbf{p}\cdot\boldsymbol{\epsilon}_2)^2}{2}\left[\frac{p^\mu k_\mu}{d^2\left(p^\mu k_\mu\right)/d\tau^2}\right]K_{1/3}^2(\xi)\right\} \tag{22}$$

Substituting the phase space volume element transformation

$$d^3k = k^2 dk d\Omega = \frac{\omega^2}{c^3}d\omega d\Omega \tag{23}$$

into (22) gives the desired result

$$\begin{aligned}\hbar\omega\frac{d^3N}{d^3k} &= \frac{c^3}{\omega^2}\cdot\frac{d^2 I}{d\omega d\Omega}\\ &= \frac{4e^2}{3\pi^2 m^2}\left\{\left[\frac{d(\boldsymbol{\epsilon}_1\cdot\mathbf{p})}{d\tau}\right]^2\left[\frac{p^\mu k_\mu}{d^2\left(p^\mu k_\mu\right)/d\tau^2}\right]^2 K_{2/3}^2(\xi) + \frac{(\mathbf{p}\cdot\boldsymbol{\epsilon}_2)^2}{2}\left[\frac{p^\mu k_\mu}{d^2\left(p^\mu k_\mu\right)/d\tau^2}\right]K_{1/3}^2(\xi)\right\}\end{aligned} \tag{24}$$