1. **Prob 14.19**

(a) From (14.70)

$$\frac{d^2I}{d\omega d\Omega} = \frac{\omega^2}{4\pi^2c^3}\left|\int dt \int d^3x \mathbf{n}\times[\mathbf{n}\times\mathbf{J}(\mathbf{x},t)]\, e^{i\omega[t-\mathbf{n}\cdot\mathbf{x}(t)/c]}\right|^2 \tag{1}$$

Recall this formula was derived with the far field approximation, where $\mathbf{n}$ is treated as constant (see (14.63)). The relevant space integral is thus

$$\begin{aligned}
\mathbf{I}(t) &= \int d^3x \mathbf{J}(\mathbf{x},t)\, e^{i\omega[t-\mathbf{n}\cdot\mathbf{x}(t)/c]} \\
&= c\int d^3x \nabla\times\{\boldsymbol{\mu}(t)\,\delta[\mathbf{x}-\mathbf{r}(t)]\} e^{i\omega[t-\mathbf{n}\cdot\mathbf{x}(t)/c]} \\
&= c\left\{\int d^3x \nabla\delta[\mathbf{x}-\mathbf{r}(t)]\, e^{i\omega[t-\mathbf{n}\cdot\mathbf{x}(t)/c]}\right\}\times\boldsymbol{\mu}(t)
\end{aligned} \tag{2}$$

With integration by parts, we have

$$\int d^3x \nabla\delta(\mathbf{x}-\mathbf{a}) f(\mathbf{x}) = -\int d^3x \delta(\mathbf{x}-\mathbf{a}) \nabla f(\mathbf{x}) = -\nabla f(\mathbf{a}) \tag{3}$$

Thus (2) becomes

$$\mathbf{I}(t) = i\omega\mathbf{n}\times\boldsymbol{\mu}(t)\, e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}(t)/c]} \tag{4}$$

and (1) becomes

$$\frac{d^2I}{d\omega d\Omega} = \frac{\omega^4}{4\pi^2c^3}\left|\int dt \mathbf{n}\times\boldsymbol{\mu}(t)\, e^{i\omega[t-\mathbf{n}\cdot\mathbf{r}(t)/c]}\right|^2 \tag{5}$$

(b) For this part, $\mathbf{r}(t) = 0$, and

$$\boldsymbol{\mu}(t) = \mu_0[\sin(\omega_0 t)\hat{\mathbf{x}} + \cos(\omega_0 t)\hat{\mathbf{z}}] \tag{6}$$

We need to consider the time integrals (Fourier transform)

$$\widetilde{\mu}_x(\omega) = \mu_0\int_{-T/2}^{T/2}\sin(\omega_0 t)\, e^{i\omega t}dt = \frac{\mu_0}{2i}\int_{-T/2}^{T/2}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)e^{i\omega t}dt \tag{7}$$

$$\widetilde{\mu}_z(\omega) = \mu_0\int_{-T/2}^{T/2}\cos(\omega_0 t)\, e^{i\omega t}dt = \frac{\mu_0}{2}\int_{-T/2}^{T/2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)e^{i\omega t}dt \tag{8}$$

With

$$\int_{-T/2}^{T/2} e^{i\lambda t}dt = T\frac{\sin(\lambda T/2)}{\lambda T/2} = T\,\mathrm{sinc}\left(\frac{\lambda T}{2}\right) \tag{9}$$

we have

$$\widetilde{\mu}_x(\omega) = \frac{\mu_0 T}{2i}\left\{\mathrm{sinc}\left[\frac{(\omega+\omega_0)T}{2}\right] - \mathrm{sinc}\left[\frac{(\omega-\omega_0)T}{2}\right]\right\} \tag{10}$$

$$\widetilde{\mu}_z(\omega) = \frac{\mu_0 T}{2}\left\{\mathrm{sinc}\left[\frac{(\omega+\omega_0)T}{2}\right] + \mathrm{sinc}\left[\frac{(\omega-\omega_0)T}{2}\right]\right\} \tag{11}$$

Due to the sinc function, we see that both $\widetilde{\mu}_x(\omega)$ and $\widetilde{\mu}_z(\omega)$ are sharply peaked at $\omega = \pm\omega_0$. But because we only consider positive frequencies, and because $\omega_0 T/2\pi \gg 1$, the positive and negative peaks are rather separated, the second term in (10) and (11) will dominate.

With this approximation, we can rewrite (5) as

$$\frac{d^2 I}{d\omega d\Omega} = \frac{\omega^4}{4\pi^2 c^3}\left|(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \times (\widetilde{\mu}_x\hat{\mathbf{x}} + \widetilde{\mu}_z\hat{\mathbf{z}})\right|^2$$
$$\approx \frac{\omega^4\mu_0^2 T^2}{16\pi^2 c^3}\left(1+\sin^2\theta\sin^2\phi\right)\operatorname{sinc}^2\left[\frac{(\omega-\omega_0)\,T}{2}\right] \tag{12}$$

Obviously after integrating over solid angles, $dI/d\omega$ will also be sharply peaked at $\omega = \omega_0$. Alternatively, if we integrate over frequencies, the angular distribution is proportional to $1+\sin^2\theta\sin^2\phi$, as claimed.

(c) The total average power radiated is the total intensity I (energy) divided by T, i.e.,

$$\langle P\rangle = \frac{1}{T}\int d\omega \int d\Omega \frac{d^2 I}{d\omega d\Omega}$$
$$= \frac{\mu_0^2 T}{16\pi^2 c^3}\int_0^\infty d\omega\,\omega^4 \operatorname{sinc}^2\left[\frac{(\omega-\omega_0)\,T}{2}\right]\overbrace{\int d\Omega\left(1+\sin^2\theta\sin^2\phi\right)}^{16\pi/3}$$
$$= \frac{\mu_0^2 T}{3\pi c^3}\int_0^\infty d\omega\,\omega^4 \operatorname{sinc}^2\left[\frac{(\omega-\omega_0)\,T}{2}\right] \tag{13}$$

To evaluate the last integral, notice that the integrand is sharply peaked at $\omega = \omega_0$, so under the limit $T\to\infty$, we can approximate ω^4 by ω_0^4, and extend the lower limit to $-\infty$, i.e.,

$$\int_0^\infty d\omega\,\omega^4 \operatorname{sinc}^2\left[\frac{(\omega-\omega_0)\,T}{2}\right] \approx \omega_0^4\int_{-\infty}^\infty d\omega \operatorname{sinc}^2\left[\frac{(\omega-\omega_0)\,T}{2}\right] = \frac{2\pi\omega_0^4}{T} \tag{14}$$

where we have used the well known result

$$\int_{-\infty}^\infty \operatorname{sinc}^2(t)\,dt = \pi \tag{15}$$

Plugging (14) into (13) gives

$$\langle P\rangle \approx \frac{2\mu_0^2\omega_0^4}{3c^3} \tag{16}$$

In problem 9.7, we have derived the power distribution of an electric dipole

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\left|\left[\mathbf{n}\times\ddot{\mathbf{p}}\left(t'\right)\right]\times\mathbf{n}\right|^2 \tag{17}$$

For magnetic dipole, we substitute $\ddot{\boldsymbol{\mu}}\times\mathbf{n}/c$ for $(\mathbf{n}\times\ddot{\mathbf{p}})\times\mathbf{n}$. In Gaussian units, we have

$$\frac{dP(t)}{d\Omega} = \frac{1}{4\pi c^3}\left|\mathbf{n}\times\ddot{\boldsymbol{\mu}}\right|^2 \tag{18}$$

The instantaneous power is the integration of (18) over solid angles,

$$P(t) = \frac{1}{4\pi c^3}\int \left|\mathbf{n}\times\ddot{\boldsymbol{\mu}}\right|^2 d\Omega = \frac{1}{4\pi c^3}\left(\int \left|\ddot{\boldsymbol{\mu}}\right|^2 d\Omega - \int \left|\mathbf{n}\cdot\ddot{\boldsymbol{\mu}}\right|^2 d\Omega\right) = \frac{2}{3c^3}\left|\ddot{\boldsymbol{\mu}}\right|^2 \tag{19}$$

In this problem, $|\ddot{\boldsymbol{\mu}}|^2 = \mu_0^2\omega_0^4$, so the instantaneous power is

$$P(t) = \frac{2\mu_0^2\omega_0^4}{3c^3} \tag{20}$$

matching (16).

2. **Prob 14.20**

   (a) Similar to problem 14.19, we need to calculate the Fourier transform $\widetilde{\mu}_x(\omega), \widetilde{\mu}_z(\omega)$ of

$$\mu_z(t) = \mu_0 \tanh(\nu t) \qquad\qquad \mu_x(t) = \mu_0 \operatorname{sech}(\nu t) \tag{21}$$

Since sech is even, we have

$$\widetilde{\mu}_x(\omega) = \mu_0 \int_{-\infty}^{\infty} \operatorname{sech}(\nu t) e^{i\omega t} dt = 2\mu_0 \overbrace{\int_0^{\infty} \left[\frac{\cos(\omega t)}{\cosh(\nu t)}\right] dt}^{\substack{\text{by Gradshteyn \& Ryzhik (8th ed.) 3.981.3} \\ =(\pi/2\nu)\operatorname{sech}(\omega\pi/2\nu)}} = \frac{\pi\mu_0}{\nu} \operatorname{sech}\left(\frac{\omega\pi}{2\nu}\right) \tag{22}$$

On the other hand

$$\frac{d\tanh(\nu t)}{dt} = \nu \operatorname{sech}^2(\nu t) \tag{23}$$

the Fourier derivative theorem gives

$$-i\omega\widetilde{\mu}_z(\omega) = \mu_0 \nu \int_{-\infty}^{\infty} \operatorname{sech}^2(\nu t) e^{i\omega t} dt = 2\mu_0 \nu \overbrace{\int_0^{\infty} \left[\frac{\cos(\omega t)}{\cosh^2(\nu t)}\right] dt}^{\substack{\text{by Gradshteyn \& Ryzhik (8th ed.) 3.982.1} \\ =\left(\omega\pi/2\nu^2\right)\operatorname{csch}(\omega\pi/2\nu)}} = \frac{\omega\pi\mu_0}{\nu} \operatorname{csch}\left(\frac{\omega\pi}{2\nu}\right) \tag{24}$$

or

$$\widetilde{\mu}_z(\omega) = \frac{i\pi\mu_0}{\nu} \operatorname{csch}\left(\frac{\omega\pi}{2\nu}\right) \tag{25}$$

(*Note that strictly speaking, the Fourier derivative theorem cannot apply here since $\mu_z(t)$ does not vanish at infinity. The "Fourier transform" of $\mu_z(t)$ is only in the distributional sense, which is appropriate since it will later be used in the integrand to calculate power or total radiated energy.*)

Thus using (5) we have

$$\begin{aligned} \frac{d^2 I}{d\omega d\Omega} &= \frac{\omega^4}{4\pi^2 c^3} \left|(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \times (\widetilde{\mu}_x \hat{\mathbf{x}} + \widetilde{\mu}_z \hat{\mathbf{z}})\right|^2 \\ &= \frac{\omega^4 \mu_0^2}{4c^3\nu^2} \left[\sin^2\theta \operatorname{csch}^2\left(\frac{\omega\pi}{2\nu}\right) + \left(1 - \sin^2\theta\cos^2\phi\right) \operatorname{sech}^2\left(\frac{\omega\pi}{2\nu}\right)\right] \end{aligned} \tag{26}$$

Integrating over solid angles gives

$$\begin{aligned} \frac{dI}{d\omega} &= \frac{\omega^4\mu_0^2}{4c^3\nu^2} \left[\operatorname{csch}^2\left(\frac{\omega\pi}{2\nu}\right) \overbrace{\int \sin^2\theta\, d\Omega}^{8\pi/3} + \operatorname{sech}^2\left(\frac{\omega\pi}{2\nu}\right) \overbrace{\int \left(1 - \sin^2\theta\cos^2\phi\right) d\Omega}^{8\pi/3}\right] \\ &= \frac{2\pi\omega^4\mu_0^2}{3c^3\nu^2} \left[\operatorname{csch}^2\left(\frac{\omega\pi}{2\nu}\right) + \operatorname{sech}^2\left(\frac{\omega\pi}{2\nu}\right)\right] \end{aligned} \tag{27}$$

Writing in dimensionless frequency variable $x = \pi\omega/2\nu$ yields

$$\frac{dI}{dx} = \frac{4}{3}\left(\frac{\nu}{c}\right)^3 \mu_0^2 \cdot 16 \left(\frac{x}{\pi}\right)^4 \left[\operatorname{csch}^2(x) + \operatorname{sech}^2(x)\right] \tag{28}$$

   (b) We use (19) to calculate the instantaneous power from problem 9.7, where we need $|\ddot{\mu}|^2$. For this, let's introduce a complex variable

$$\zeta(t) = \mu_x(t) + i\mu_z(t) = \mu_0\left[\operatorname{sech}(\nu t) + i\tanh(\nu t)\right] \tag{29}$$

so we have $|\ddot{\mu}|^2 = \left|\ddot{\zeta}\right|^2$. Since $\zeta(t)$ has constant modulus

$$|\zeta(t)| = \mu_0\sqrt{\tanh^2(\nu t) + \operatorname{sech}^2(\nu t)} = \mu_0 \tag{30}$$

we can write

$$\zeta(t) = \mu_0 e^{i\phi(t)} \tag{31}$$

Therefore

$$\dot{\zeta}(t) = i\mu_0\dot{\phi}(t)e^{i\phi(t)} = i\dot{\phi}(t)\zeta(t) \qquad \Longrightarrow \qquad \dot{\phi}(t) = -i\frac{\dot{\zeta}(t)}{\zeta(t)} = \nu\,\mathrm{sech}(\nu t) \tag{32}$$

$$\ddot{\zeta}(t) = \left(i\ddot{\phi} - \dot{\phi}^2\right)\zeta \qquad \Longrightarrow \qquad |\ddot{\boldsymbol{\mu}}|^2 = \left|\ddot{\zeta}\right|^2 = \mu_0^2\left(\dot{\phi}^4 + \ddot{\phi}^2\right) = \mu_0^2\nu^4\,\mathrm{sech}^2(\nu t) \tag{33}$$

From (19), the instantaneous power is thus

$$P(t) = \frac{2}{3c^3}|\ddot{\boldsymbol{\mu}}|^2 = \frac{2\mu_0^2\nu^4}{3c^3}\mathrm{sech}^2(\nu t) \tag{34}$$

giving a total radiated energy

$$W = \int_{-\infty}^{\infty} P(t)\,dt = \frac{2\mu_0^2\nu^4}{3c^3}\overbrace{\int_{-\infty}^{\infty}\mathrm{sech}^2(\nu t)\,dt}^{2/\nu} = \frac{4\mu_0^2\nu^3}{3c^3} \tag{35}$$

On the other hand, from (28), we can calculate the total radiated energy by integrating over frequencies,

$$W' = \int_0^{\infty}\frac{dI}{dx}dx = \frac{64\nu^3\mu_0^2}{3\pi^4c^3}\left[\int_0^{\infty}x^4\,\mathrm{csch}^2(x)\,dx + \int_0^{\infty}x^4\,\mathrm{sech}^2(x)\,dx\right] \tag{36}$$

From Gradshteyn & Ryzhik (8th ed.) 3.527.2 and 3.527.5,

$$\int_0^{\infty}x^{2n}\,\mathrm{csch}^2(x)\,dx = \pi^{2n}|B_{2n}| \tag{37}$$

$$\int_0^{\infty}x^{2n}\,\mathrm{sech}^2(x)\,dx = \left(\frac{2^{2n}-2}{2^{2n}}\right)\pi^{2n}|B_{2n}| \tag{38}$$

where B_{2n} is the Bernoulli number. In particular, $\left|B_4\right| = 1/30$. Putting everything together, we get

$$W' = \frac{64\nu^3\mu_0^2}{3\pi^4c^3}\cdot\left(\frac{\pi^4}{30} + \frac{14}{16}\frac{\pi^4}{30}\right) = \frac{4\nu^3\mu_0^2}{3c^3} \tag{39}$$

exactly matching (35).