In Jackson 14.2, Larmor formula (14.22) was given without rigorous proof. It was tempting to integrate (14.21) over the solid angles to arrive at (14.22). However (14.21) was to be evaluate at the retarded time. This makes the integral untenable. In the next paragraph, Jackson claimed that "P is a Lorentz invariant" by citing *Rohrlich*. It turns out Rohrlich had a very elegant, geometric, but highly non-trivial proof of the Lorentz invariance of P. We will restate the proof here.

1. **Gauss Theorem in 4-Dimensional Spacetime, Motivation of the Definition of Hypersurface Element**

Gauss theorem in 3-dimensional space states that for any vector field $\mathbf{F}$,

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{F})\, d^3x = \oint_{\partial V} \mathbf{F}\cdot\mathbf{n} d^2\sigma \tag{1}$$

where $d^2\sigma$ is the surface element of the boundary surface ∂V. If we write $d^2\boldsymbol{\sigma} \equiv \mathbf{n}d^2\sigma$, (1) can be rewritten as

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{F})\, d^3x = \oint_{\partial V} \mathbf{F}\cdot d^2\boldsymbol{\sigma} \tag{2}$$

Our motivation is to generalize (2) into 4-dimensional spacetime.

- The vector field $\mathbf{F}$ is generalized to a field strength tensor $F^{\mu\nu}$,
- The divergence $\boldsymbol{\nabla}\cdot\mathbf{F}$ is generalized to $\partial_\nu F^{\mu\nu}$,
- The volume element d^3x is generalized to the 4-volume element d^4x.

To keep the tensor form, we need to generatlize $d^2\boldsymbol{\sigma}$ to a covariant hypersurface element $d^3\sigma_\nu$.

Let's consider how $d^2\boldsymbol{\sigma}$ is defined. Locally on the 2-dimensional surface, let it be parameterized by two orthogonal coordinates $\mathbf{x}$ and $\mathbf{y}$, it is easy to see that

$$\mathbf{n} = \hat{\mathbf{x}}\times\hat{\mathbf{y}} \qquad\qquad d^2\sigma = dx\,dy \tag{3}$$

Then it is natural to generalize this to 3-dimensional hypersurface, with (not necessarily orthogonal) local coordinates $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$

$$d^3\sigma_\nu = \epsilon_{\nu\mu\rho\lambda}\frac{\partial x^\mu}{\partial\sigma_1}\frac{\partial x^\rho}{\partial\sigma_2}\frac{\partial x^\lambda}{\partial\sigma_3}d\sigma_1 d\sigma_2 d\sigma_3 \tag{4}$$

where the $\epsilon_{\nu\mu\rho\lambda}$ is the Levi-Civita symbol in 4-dimensions, a natural generalization of the cross product in 3-dimensions. The product of partial derivatives is the Jacobian determinant for the (potentially non-orthogonal) coordinate transformation. Note the subscripts and superscripts are carefully arranged so that the LHS is a covariant vector.

The 4-dimensional Gauss theorem is then

$$\int_V \partial_\nu F^{\mu\nu} d^4x = \oint_{\partial V} F^{\mu\nu} d^3\sigma_\nu \tag{5}$$

We can also define a covariant normal vector n_ν such that

$$d^3\sigma_\nu = n_\nu d^3\sigma \qquad\qquad n_\nu = \epsilon_{\nu\mu\rho\lambda}\frac{\partial x^\mu}{\partial\sigma_1}\frac{\partial x^\rho}{\partial\sigma_2}\frac{\partial x^\lambda}{\partial\sigma_3} \tag{6}$$

The covariant form (4) is fully consistent with Lorentz transformation. We can see this by considering a inertial frame K, where the covariant surface element of the hyperplane of constant time is

$$d^3\sigma_\nu = n_\nu d^3\sigma = n_\nu dx\,dy\,dz \qquad\qquad n_\nu = (1,0,0,0) \tag{7}$$

In a different inertial frame K' moving with velocity βc along the x-axis relative to K, the transformation between coordinates is

$$ct' = \gamma(ct - \beta x) \qquad x' = \gamma(x - \beta ct) \qquad y' = y \qquad z' = z \tag{8}$$

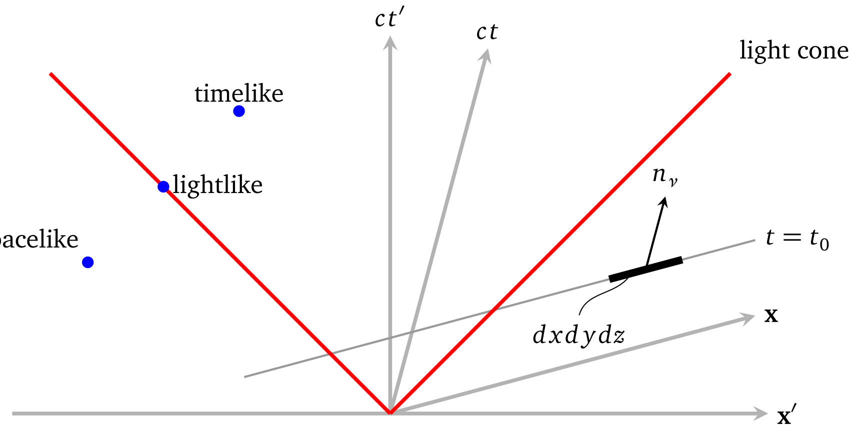
Straightforward calculation of (4) yields

$$d^3\sigma'_0 = \gamma\, dx\,dy\,dz \qquad d^3\sigma'_1 = \gamma\beta\, dx\,dy\,dz \qquad d^3\sigma'_2 = 0 \qquad d^3\sigma'_3 = 0 \tag{9}$$

which, under signature $(+,-,-,-)$, gives the contravariant normal vector $n'^\nu = (\gamma, -\gamma\beta, 0, 0)$, which is the Lorentz transformation of $n^\nu = (1,0,0,0)$.

2. **Spacetime Diagram, Timelike, Spacelike and Lightlike**

Spacetime diagram is drawn so for the rest frame, the time axis and the space axis are orthogonal. The light cone, being a surface where $|\mathbf{x}| = ct$, occupies 45 degree lines in the diagram. For other inertial frames, their time axis and the space axis are bisected by the light cone, and will not be orthogonal in the drawing.



Vectors inside the light cone are "timelike", those outside are "spacelike", and those on the light cone are "lightlike", or "null vector". For signature $(+,-,-,-)$,

$$x^\mu x_\mu = c^2t^2 - |\mathbf{x}|^2 \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{lightlike} \\ < 0 & \text{spacelike} \end{cases} \tag{10}$$

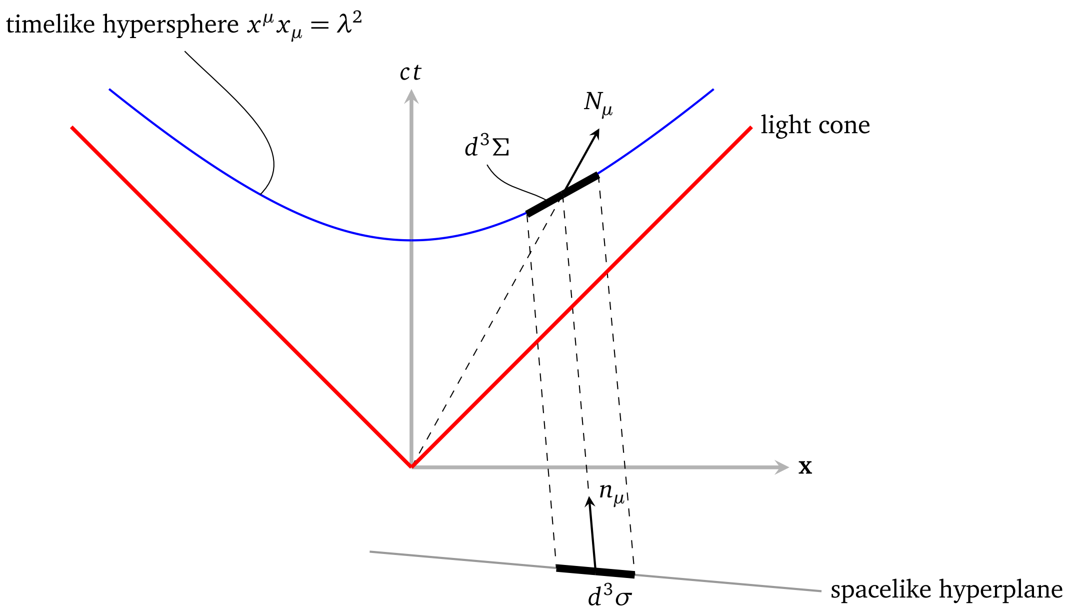
We can speak of a hypersurface being timelike or spacelike according to its normal vector. I.e., if the normal vector is timelike, the hypersurface is spacelike and vice versa.

3. **Surface Element of the Light Cone**

Consider the "timelike" hypersphere

$$x^\mu x_\mu = \lambda^2 \tag{11}$$

where λ is the "radius" of the hypersphere and $\lambda^2 > 0$. The depiction of this hypersphere in a spacetime diagram is a hyperbola $c^2t^2 - |\mathbf{x}|^2 = \lambda^2$.



Let $d^3\Sigma$ be the surface element on the hypersphere, with corresponding normal vector

$$N^\mu = \frac{x^\mu}{\lambda} \tag{12}$$

Then consider an arbitrary spacelike hyperplane with normal n_μ. Let $d^3\sigma$ be the projection of $d^3\Sigma$ onto this hyperplane along n_μ. The projection gives the relation

$$d^3\sigma = \left|N^\mu n_\mu\right| d^3\Sigma \tag{13}$$

Then we have

$$d^3\sigma = \left|x^\mu n_\mu\right| \frac{d^3\Sigma}{\lambda} \qquad \Longrightarrow \qquad d^2\omega \equiv \frac{d^3\sigma}{\left|x^\mu n_\mu\right|} = \frac{d^3\Sigma}{\lambda} \tag{14}$$

It is clear that $d^2\omega$ is independent of the choice of spacelike hyperplane, and is therefore Lorentz invariant. This allows us to write the surface element of the sphere as

$$d^3\Sigma^\mu = N^\mu d^3\Sigma = x^\mu d^2\omega \tag{15}$$

This holds as $\lambda \to 0$, which represents the surface element of the light cone (zero-radius hypersphere).

4. **Geometric Representation of Field Strength Tensor**

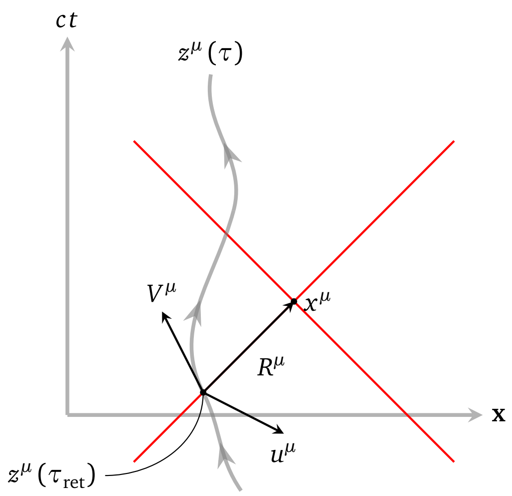
From Jackson equation (14.6), we have the Liénard-Wiechert potential of a moving charge

$$A^\mu = \left. \frac{eV^\mu(\tau)}{V_\nu[x^\nu - z^\nu(\tau)]} \right|_{\tau=\tau_{\text{ret}}} \tag{16}$$

where the retarded time τ_{ret} is defined by the light cone condition

$$R^\mu R_\mu = 0 \qquad \text{where} \qquad R^\mu = x^\mu - z^\mu(\tau) \tag{17}$$

The retarded time can be found in the spacetime diagram as the intersection of the world line of the charge and the past light cone of the observation point x^μ.



In the diagram above, V^μ is the 4-velocity of the charge at the retarded time, a timelike vector. We have also drawn a spacelike unit vector u^μ, orthogonal to V^μ, defined as

$$\rho \equiv \frac{V^\mu R_\mu}{c} \qquad R_\perp^\mu \equiv R^\mu - \frac{\rho V^\mu}{c} \qquad u^\mu \equiv \frac{R_\perp^\mu}{\rho} \tag{18}$$

It is easy to see that

$$u^\mu V_\mu = 0 \qquad u^\mu u_\mu = -1 \qquad u^\mu R_\mu = -\rho \qquad R^\mu = \rho\left(u^\mu + \frac{V^\mu}{c}\right) \tag{19}$$

The Liénard-Wiechert potential can then be rewritten in simple form as

$$A^\mu = \frac{eV^\mu(\tau_{\text{ret}})}{c\rho} \tag{20}$$

The geometric meaning of ρ and u^μ is clear in the rest frame of the charge, where

$$R^\mu = (|R|, |R|\,\hat{\mathbf{u}}) \qquad V^\mu = (c, 0, 0, 0) \tag{21}$$

Putting (21) into (18) gives

$$\rho = |R| \qquad\qquad u^{\mu} = (0, \hat{\mathbf{u}}) \tag{22}$$

i.e., in the rest frame of the charge, ρ represents the spatial separation between the charge's retarded position and the observation point, and u^{μ} represents the unit direction from the charge to the observation point.

The field strength can be obtained from the potential by differentiation, however due to the retarded time dependence, the differentiation is somewhat involved.

Since τ_{ret} is uniquely determined by x^{μ} and $z(\tau)$, this enables us to find the $\partial^{\mu}\tau$ via

$$\begin{aligned}
R^{\mu}R_{\mu} &= 0 && \Longrightarrow \\
\frac{d\left(R^{\mu}R_{\mu}\right)}{d\tau} &= 2R^{\mu}\left(\frac{dx_{\mu}}{d\tau} - V_{\mu}\right) = 0 && \Longrightarrow \\
R^{\mu}\frac{dx_{\mu}}{d\tau} &= R^{\mu}V_{\mu} = c\rho && \Longrightarrow \\
\partial^{\mu}\tau &= \frac{\partial\tau}{\partial x_{\mu}} = \left(\frac{dx_{\mu}}{d\tau}\right)^{-1} = \frac{R^{\mu}}{c\rho} = \frac{1}{c}\left(u^{\mu} + \frac{V^{\mu}}{c}\right)
\end{aligned} \tag{23}$$

and furthermore,

$$\begin{aligned}
\partial^{\mu}\rho = \partial^{\mu}\left(\frac{V^{\nu}R_{\nu}}{c}\right) &= \frac{1}{c}\left[(\partial^{\mu}V^{\nu})R_{\nu} + V^{\nu}(\partial^{\mu}R_{\nu})\right] \\
&= \frac{1}{c}\left[\left(\frac{dV^{\nu}}{d\tau}\partial^{\mu}\tau\right)R_{\nu} + V^{\nu}(\delta^{\mu}{}_{\nu} - V_{\nu}\partial^{\mu}\tau)\right] && \text{use (19), (23)} \\
&= \frac{1}{c}\left[\frac{dV^{\nu}}{d\tau}\cdot\frac{R^{\mu}}{c\rho}\cdot\rho\left(u_{\nu} + \frac{V_{\nu}}{c}\right) + V^{\mu} - c^{2}\cdot\frac{1}{c}\left(u^{\mu} + \frac{V^{\mu}}{c}\right)\right] && \frac{dV_{\nu}}{d\tau}V^{\nu} = 0, a^{\nu} \equiv \frac{dV^{\nu}}{d\tau} \\
&= -u^{\mu} + \frac{a^{\nu}u_{\nu}R^{\mu}}{c^{2}} && a_{u} \equiv a^{\nu}u_{\nu} \\
&= -u^{\mu} + \frac{a_{u}R^{\mu}}{c^{2}}
\end{aligned} \tag{24}$$

Taking the derivative of (20) gives

$$\partial^{\mu}A^{\nu} = \frac{e}{c\rho}a^{\nu}\partial^{\mu}\tau - \frac{e}{c\rho^{2}}V^{\nu}\partial^{\mu}\rho \tag{25}$$

With the help of the simplifying notation

$$A^{[\mu}B^{\nu]} = A^{\mu}B^{\nu} - A^{\nu}B^{\mu} \tag{26}$$

we finally can write the field strength tensor as

$$\begin{aligned}
F^{\mu\nu} = \partial^{[\mu}A^{\nu]} &= \frac{e}{c\rho}a^{[\nu}\partial^{\mu]}\tau - \frac{e}{c\rho^{2}}V^{[\nu}\partial^{\mu]}\rho \\
&= \frac{e}{c\rho}\frac{a^{[\nu}R^{\mu]}}{c\rho} - \frac{e}{c\rho^{2}}V^{[\nu}\left[(-u)^{\mu]} + \frac{a_{u}R^{\mu]}}{c^{2}}\right] \\
&= \frac{e}{c\rho^{2}}u^{[\mu}V^{\nu]} + \frac{e}{c^{2}\rho}\left(a^{[\nu} - \frac{a_{u}V^{[\nu}}{c}\right)\frac{R^{\mu]}}{\rho} && \text{note that } \frac{R^{\mu}}{\rho} \sim O\left(\rho^{0}\right) \\
&= \frac{e}{c\rho^{2}}u^{[\mu}V^{\nu]} + \frac{e}{c^{2}\rho}\frac{\kappa^{[\nu}R^{\mu]}}{\rho} && \text{where } \kappa^{\nu} \equiv a^{\nu} - \frac{a_{u}V^{\nu}}{c}
\end{aligned} \tag{27}$$

Here we introduced another 4-vector κ^{μ}. With the orthogonality relation $a^{\nu}V_{\nu} = 0$ and $u^{\nu}V_{\nu} = 0$, it is easy to see that

$$\kappa^{\nu}\kappa_{\nu} = a^{\nu}a_{\nu} + a_{u}^{2} \tag{28}$$

$$\kappa^{\nu}R_{\nu} = \rho\left(a^{\nu} - \frac{a_{u}V^{\nu}}{c}\right)\left(u_{\nu} + \frac{V_{\nu}}{c}\right) = 0 \tag{29}$$

5. **Geometric Representation of Radiation Power, Main Proof**

The symmetric stress tensor is given by Jackson equation (12.113)

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left(g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{30}$$

Putting (27) into (30), we have

$$\Theta^{\mu\nu} = O\left(\frac{1}{\rho^4}\right) + O\left(\frac{1}{\rho^3}\right) + \frac{e^2}{4\pi c^4 \rho^2}\left[g^{\mu\alpha}\left(\frac{\kappa_{[\alpha}R_{\beta]}}{\rho}\right)\left(\frac{\kappa^{[\beta}R^{\nu]}}{\rho}\right) + \frac{1}{4}g^{\mu\nu}\left(\frac{\kappa_{[\alpha}R_{\beta]}}{\rho}\right)\left(\frac{\kappa^{[\alpha}R^{\beta]}}{\rho}\right)\right] \tag{31}$$

Using (29) and the fact that $R^\mu R_\mu = 0$, we have

$$\kappa_{[\alpha}R_{\beta]}\kappa^{[\beta}R^{\nu]} = \kappa_\alpha R_\beta \kappa^\beta R^\nu - \kappa_\beta R_\alpha \kappa^\beta R^\nu - \kappa_\alpha R_\beta \kappa^\nu R^\beta + \kappa_\beta R_\alpha \kappa^\nu R^\beta = -\left(\kappa_\beta \kappa^\beta\right) R_\alpha R^\nu \tag{32}$$

$$\kappa_{[\alpha}R_{\beta]}\kappa^{[\alpha}R^{\beta]} = \kappa_\alpha R_\beta \kappa^\alpha R^\beta - \kappa_\beta R_\alpha \kappa^\alpha R^\beta - \kappa_\alpha R_\beta \kappa^\beta R^\alpha + \kappa_\beta R_\alpha \kappa^\beta R^\alpha = 0 \tag{33}$$

giving

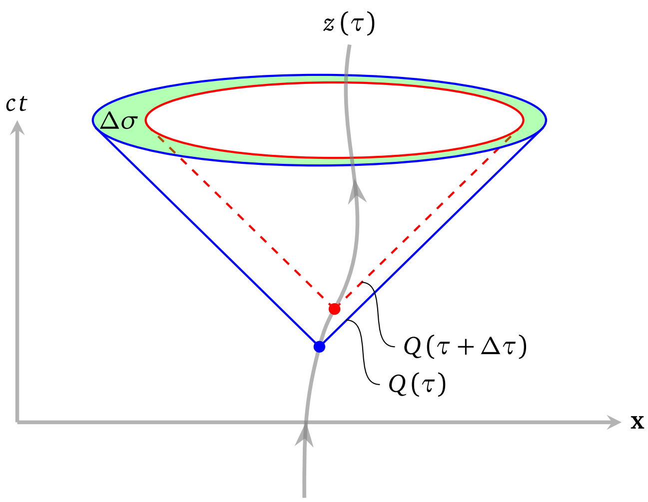
$$\Theta^{\mu\nu} = O\left(\frac{1}{\rho^4}\right) + O\left(\frac{1}{\rho^3}\right) - \frac{e^2}{4\pi c^4 \rho^2}\left(a^\lambda a_\lambda + a_u^2\right)\frac{R^\mu R^\nu}{\rho^2} \tag{34}$$

In Jackson problem 12.18, we proved that

$$P^\mu = \frac{1}{c}\int_{\text{spacelike plane }\sigma} \Theta^{\mu\nu} d^3\sigma_\nu \tag{35}$$

is independent of the choice of spacelike hyperplane σ, therefore P^μ is a 4-vector.

The integral domain in (35) is the entire hyperplane, but the effect on this hyperplane is caused by the moving charge at different past times. Therefore, to consider the effect by a proper time range $[\tau, \tau + \Delta\tau]$, the integral domain should be restricted to the hyper elliptic annulus formed by the intersection of the hyperplane and the light cones emitted at τ and $\tau + \Delta\tau$ respectively.



Let's denote this restricted integral as

$$\Delta P^\mu(\Delta\sigma) = \frac{1}{c}\int_{\Delta\sigma} \Theta^{\mu\nu} d^3\sigma_\nu \tag{36}$$

If we can show that it is independent of the choice of $\Delta\sigma$, then it is a 4-vector. To check this, we take two different spacelike hyperplanes σ and σ', and ask whether

$$\Delta P^\mu(\Delta\sigma) \stackrel{?}{=} \Delta P^\mu\left(\Delta\sigma'\right) \tag{37}$$

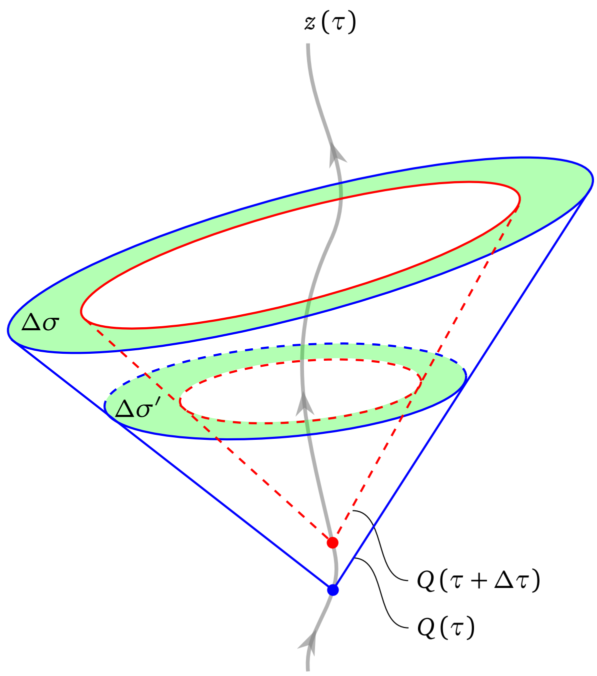
Here $\Delta\sigma$ and $\Delta\sigma'$ are the annuli formed by the intersection of the two hyperplanes with the pair of light cones emitted at τ and $\tau + \Delta\tau$ respectively.

For the 4-volume V enclosed by the two "caps" ($\Delta\sigma$ and $\Delta\sigma'$) as well as the two lightlike "walls", we can apply the Gauss theorem

$$\int_V \partial_\nu \Theta^{\mu\nu} d^4x = \oint_{\partial V} \Theta^{\mu\nu} d^3\sigma_\nu \tag{38}$$

Since $\Theta^{\mu\nu}$ is divergence-free, the surface integral on the RHS vanishes, i.e.,

$$\Delta P^\mu(\Delta\sigma) - \Delta P^\mu\left(\Delta\sigma'\right) = \frac{1}{c}\int_{\text{wall of }Q(\tau+\Delta\tau)} \Theta^{\mu\nu} d^3\sigma_\nu - \frac{1}{c}\int_{\text{wall of }Q(\tau)} \Theta^{\mu\nu} d^3\sigma_\nu \tag{39}$$



This means (37) is equivalent to the statement that the fluxes of $\Theta^{\mu\nu}$ through the two lightlike "walls" are equal.

In fact, the flux can be obtained using (34) and the light cone surface element (15)

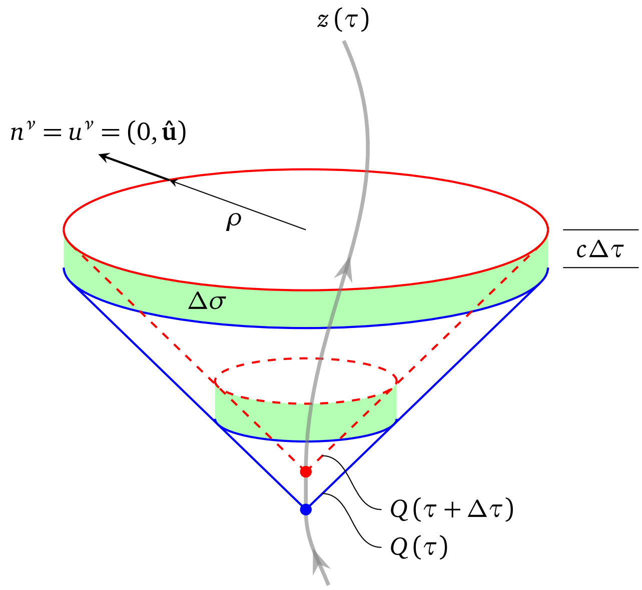
$$\int_{\text{wall of } Q} \Theta^{\mu\nu} d^3\sigma_\nu = \int_{\text{wall of } Q} \left[O\left(\frac{1}{\rho^4}\right) + O\left(\frac{1}{\rho^3}\right) - \frac{e^2}{4\pi c^4 \rho^2} \left(a^\lambda a_\lambda + a_u^2\right) \frac{R^\mu R^\nu}{\rho^2} \right] R_\nu d^2\omega \tag{40}$$

which obviously vanishes as $\rho \to \infty$.

In summary, we have proved that in the limit $\rho \to \infty$, (37) holds and represents a contravariant 4-vector.

Since ρ represents the spatial distance between the charge's retarded position and the observation point, ΔP^μ represents the radiated 4-momentum by the charge during proper time interval $[\tau, \tau + \Delta\tau]$ observed at infinite distance. Rohrlich summarized this well – "*The radiation field detaches itself from the charge which is its source and leads an independent existence; it is endowed with energy, momentum, and, at times also with angular momentum.*"

Looking back, the proof does not use the spacelike nature of the caps $\Delta\sigma$ and $\Delta\sigma'$. It must apply to all orientations of the caps. In calculating the value of (37), we can choose the most convenient orientation. In the rest frame of the charge, we choose σ to be the right cylinder (worldtube), and $\Delta\sigma$ is the intersection of the cylinder with the two light cones (cylindrical band).



From the regularity of the shape, the magnitude of the surface element must be

$$d^3\sigma = \rho^2 d\Omega c d\tau \tag{41}$$

But for the normal n^ν to point "outward" of the volume, and with $(+,-,-,-)$ metric signature, we must have

$$d^3\sigma_\nu = -u_\nu \rho^2 d\Omega c d\tau \tag{42}$$

since $u_\nu = (1, -\hat{\mathbf{u}})$

The seemingly arbitrary negative sign was enforced by the choice of the "outward" direction, which manifests itself in the choice of the parameter ordering $(\sigma_1, \sigma_2, \sigma_3)$ in the definition of the surface element (4). For example, if we use the parameter ordering $(c\tau, \theta, \phi)$, we would have

$$x^\mu(c\tau, \theta, \phi) = (c\tau, \rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta) \tag{43}$$

and by (4),

$$d^3\sigma_\nu = \left(0, -\rho^2\sin^2\theta\cos\phi, -\rho^2\sin^2\theta\sin\phi, -\rho^2\sin\theta\cos\theta\right) d\theta d\phi c d\tau = (0, \hat{\mathbf{u}})\,\rho^2 d\Omega c d\tau \tag{44}$$

Under the metric signature $(+,-,-,-)$, this corresponds to an inward contravariant normal $n^\nu = (0, -\hat{\mathbf{u}})$, opposite to our intended outward direction. Therefore the correct parameter ordering must be $(c\tau, \phi, \theta)$ or their permutations with the same parity.

After establishing the correct form of the surface element (42), we can calculate the value of (37)

$$\begin{aligned}
dP^\mu(\Delta\sigma) &= \lim_{\rho\to\infty} \frac{1}{c}\int_{\Delta\sigma} \Theta^{\mu\nu} d^3\sigma_\nu \\
&= \lim_{\rho\to\infty} \frac{1}{c}\int \left[O\left(\frac{1}{\rho^4}\right) + O\left(\frac{1}{\rho^3}\right) - \frac{e^2}{4\pi c^4\rho^2}\left(a^\lambda a_\lambda + a_u^2\right)\frac{R^\mu R^\nu}{\rho^2}\right](-u_\nu)\,\rho^2 d\Omega c d\tau \qquad \Longrightarrow \\
\frac{dP^\mu_{\text{rad}}}{d\tau} &= \frac{e^2}{4\pi c^4}\int \left(a^\lambda a_\lambda + a_u^2\right)\left(u^\mu + \frac{V^\mu}{c}\right)\overbrace{\left(u^\nu + \frac{V^\nu}{c}\right)u_\nu}^{-1} d\Omega \\
&= -\frac{e^2}{4\pi c^4}\int \left(a^\lambda a_\lambda + a_u^2\right)\left(u^\mu + \frac{V^\mu}{c}\right) d\Omega
\end{aligned} \tag{45}$$

This integral is most easily evaluated in the rest frame of the charge, where

$$u^\mu + \frac{V^\mu}{c} = (1, \hat{\mathbf{u}}) \tag{46}$$

so

$$\begin{aligned}
\left.\frac{dP^\mu_{\text{rad}}}{d\tau}\right|_{\text{rest frame}} &= -\frac{e^2}{4\pi c^4}\left(\int \left[-\mathbf{a}^2 + (\mathbf{a}\cdot\hat{\mathbf{u}})^2\right] d\Omega, \int \left[-\mathbf{a}^2 + (\mathbf{a}\cdot\hat{\mathbf{u}})^2\right]\hat{\mathbf{u}}\, d\Omega\right) \\
&= \left(\frac{2a^2e^2}{3c^4}, 0, 0, 0\right)
\end{aligned} \tag{47}$$

In frame with velocity V^μ, the covariant form must be

$$\frac{dP^\mu_{\text{rad}}}{d\tau} = -\frac{2e^2}{3c^5}\left(a^\lambda a_\lambda\right) V^\mu \tag{48}$$

Further contracting (48) with V_μ gives the Larmor formula for the radiated power

$$\frac{dW_{\text{rad}}}{d\tau} = V_\mu \frac{dP^\mu_{\text{rad}}}{d\tau} = -\frac{2e^2}{3c^3}\left(a^\lambda a_\lambda\right) \tag{49}$$

This quantity has only the time component in the rest frame, which is the rate at which energy is radiated, so its covariant form corresponds to the radiated power in arbitrary frame.

(49) indicates that the charge radiates only when it has acceleration.

Taking derivative of (45) with respect to the solid angle, we have

$$\begin{aligned}
\frac{dP^\mu_{\text{rad}}}{d\tau d\Omega} &= -\frac{e^2}{4\pi c^4}\left(a^\lambda a_\lambda + a_u^2\right)\left(u^\mu + \frac{V^\mu}{c}\right) \qquad \text{let } a_\perp^\lambda \equiv a^\lambda + a_u u^\lambda \text{ so that } a_\perp^\lambda u_\lambda = 0 \\
&= -\frac{e^2}{4\pi c^4} a_\perp^\lambda a_{\perp\lambda}\left(u^\mu + \frac{V^\mu}{c}\right)
\end{aligned} \tag{50}$$

which is a null vector.

In the rest frame where $u^\mu = (0, \hat{\mathbf{u}})$, $a_\perp^\mu = (0, \mathbf{a}_\perp)$, we have the angular distribution of the radiated power

$$\left.\frac{dP^\mu_{\text{rad}}}{d\tau d\Omega}\right|_{\text{rest frame}} = \frac{e^2}{4\pi c^4} a_\perp^2\,(1, \hat{\mathbf{u}}) \tag{51}$$