1. From problem 14.13, we have for $k \neq 0$

$$\left\langle \frac{dP_k}{d\Omega} \right\rangle = \frac{e^2 k^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^T e^{ik\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]} \mathbf{n} \times \mathbf{v} dt \right|^2 \tag{1}$$

Denoting $b \equiv \sqrt{1-\epsilon^2}a$, we have

$$\mathbf{x}(u) = a(\cos u - \epsilon)\hat{\mathbf{x}} + b\sin u\hat{\mathbf{y}} \tag{2}$$

$$\tag{3}$$

It is most convenient to evaluate the integral in variable $u$. First, the phase factor can be simplified as

$$e^{ik\omega_0[t-\mathbf{n}\cdot\mathbf{x}(u)/c]} = \exp\left[ ik\omega_0 t \overbrace{-ik\left(\frac{\omega_0 a}{c}\right)\sin\theta\cos\phi\,(\cos u - \epsilon) - \left(\frac{\omega_0 b}{c}\right)\sin\theta\sin\phi\sin u}^{\text{ignored for nonrelativistic motion}} \right]$$

$$\approx e^{ik\omega_0 t} = e^{ik(u-\epsilon\sin u)} \tag{4}$$

Next, notice that

$$\mathbf{n} \times \mathbf{v} dt = \mathbf{n} \times \frac{d\mathbf{x}}{du} du = (\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}) \times (-a\sin u\hat{\mathbf{x}} + b\cos u\hat{\mathbf{y}})\, du$$

$$= -b\cos\theta\cos u\hat{\mathbf{x}} - a\cos\theta\sin u\hat{\mathbf{y}} + (a\sin\theta\sin\phi\sin u + b\sin\theta\cos\phi\cos u)\hat{\mathbf{z}}\, du \tag{5}$$

turning the integral in (1) into

$$\int_0^T e^{ik\omega_0[t-\mathbf{n}\cdot\mathbf{x}(t)/c]} \mathbf{n} \times \mathbf{v} dt = \mathbf{I}_x + \mathbf{I}_y + \mathbf{I}_z \tag{6}$$

where

$$\mathbf{I}_x = -b\cos\theta\hat{\mathbf{x}} \int_0^{2\pi} e^{ik(u-\epsilon\sin u)}\cos u\, du \tag{7}$$

$$\mathbf{I}_y = -a\cos\theta\hat{\mathbf{y}} \int_0^{2\pi} e^{ik(u-\epsilon\sin u)}\sin u\, du \tag{8}$$

$$\mathbf{I}_z = \sin\theta\hat{\mathbf{z}} \int_0^{2\pi} e^{ik(u-\epsilon\sin u)}(a\sin\phi\sin u + b\cos\phi\cos u)\, du \tag{9}$$

With the Jacobi-Anger expansion

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\phi} \tag{10}$$

we can write

$$e^{-ik\epsilon\sin u} = \sum_{n=-\infty}^{\infty} J_n(-k\epsilon) e^{inu} \qquad\qquad J_n(-x) = (-1)^n J_n(x)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n J_n(k\epsilon) e^{inu} \tag{11}$$

The $\cos u$ integral in (7) becomes

$$\int_0^{2\pi} e^{ik(u-\epsilon\sin u)}\cos u\, du = \sum_{n=-\infty}^{\infty} (-1)^n J_n(k\epsilon) \frac{1}{2} \overbrace{\int_0^{2\pi} \left[ e^{i(k+n+1)u} + e^{i(k+n-1)u} \right] du}^{2\pi\left(\delta_{k+n+1,0} + \delta_{k+n-1,0}\right)}$$

$$= \pi\left[ (-1)^{k+1} J_{-(k+1)}(k\epsilon) + (-1)^{k-1} J_{-(k-1)}(k\epsilon) \right] \qquad\qquad J_{-n}(x) = (-1)^n J_n(x)$$

$$= \pi\left[ J_{k+1}(k\epsilon) + J_{k-1}(k\epsilon) \right] \qquad\qquad J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x}J_k(x)$$

$$= \frac{2\pi}{\epsilon}J_k(k\epsilon) \tag{12}$$

The $\sin u$ integral in (8) becomes

$$\int_0^{2\pi} e^{ik(u-\epsilon \sin u)} \sin u \, du = \sum_{n=-\infty}^{\infty} (-1)^n J_n(k\epsilon) \frac{1}{2i} \overbrace{\int_0^{2\pi} \left[ e^{i(k+n+1)u} - e^{i(k+n-1)u} \right] du}^{2\pi \left( \delta_{k+n+1,0} - \delta_{k+n-1,0} \right)}$$

$$= \frac{\pi}{i} \left[ J_{k+1}(k\epsilon) - J_{k-1}(k\epsilon) \right] \qquad\qquad J_{k-1}(x) - J_{k+1}(x) = 2J_k'(x)$$

$$= 2\pi i J_k'(k\epsilon) \tag{13}$$

Substituting (12) and (13) back into (7)–(9) gives

$$\mathbf{I}_x = -\frac{2\pi b \cos \theta}{\epsilon} J_k(k\epsilon) \hat{\mathbf{x}} \tag{14}$$

$$\mathbf{I}_y = -2\pi i \, a \cos \theta \, J_k'(k\epsilon) \hat{\mathbf{y}} \tag{15}$$

$$\mathbf{I}_z = 2\pi \sin \theta \left[ ia \sin \phi \, J_k'(k\epsilon) + \frac{b \cos \phi}{\epsilon} J_k(k\epsilon) \right] \hat{\mathbf{z}} \tag{16}$$

Thus

$$|\mathbf{I}|^2 = 4\pi^2 \left\{ \frac{b^2 \cos^2 \theta}{\epsilon^2} J_k^2(k\epsilon) + a^2 \cos^2 \theta J_k'^2(k\epsilon) + \sin^2 \theta \left[ a^2 \sin^2 \phi \, J_k'^2(k\epsilon) + \frac{b^2 \cos^2 \phi}{\epsilon^2} J_k^2(k\epsilon) \right] \right\}$$

$$= 4\pi^2 \left[ \frac{b^2}{\epsilon^2} \left( \cos^2 \theta + \sin^2 \theta \cos^2 \phi \right) J_k^2(k\epsilon) + a^2 \left( \cos^2 \theta + \sin^2 \theta \sin^2 \phi \right) J_k'^2(k\epsilon) \right] \tag{17}$$

Recalling that $b^2 = \left( 1 - \epsilon^2 \right) a^2$, we can write

$$\left\langle \frac{dP_k}{d\Omega} \right\rangle = \frac{e^2 k^2 \omega_0^4 a^2}{2\pi c^3} \left[ \left( \frac{1-\epsilon^2}{\epsilon^2} \right) \left( \cos^2 \theta + \sin^2 \theta \cos^2 \phi \right) J_k^2(k\epsilon) + \left( \cos^2 \theta + \sin^2 \theta \sin^2 \phi \right) J_k'^2(k\epsilon) \right] \tag{18}$$

With the solid angle integration

$$\int \left( \cos^2 \theta + \sin^2 \theta \cos^2 \phi \right) d\Omega = \int \left( \cos^2 \theta + \sin^2 \theta \sin^2 \phi \right) d\Omega = \frac{8\pi}{3} \tag{19}$$

we have the total power for the $k$th harmonic

$$P_k = \frac{4e^2 k^2 \omega_0^4 a^2}{3c^3} \left[ \left( \frac{1-\epsilon^2}{\epsilon^2} \right) J_k^2(k\epsilon) + J_k'^2(k\epsilon) \right] \tag{20}$$

2. For circular orbit, $\epsilon \to 0$. Only $k = 1$ harmonic survives

$$\lim_{\epsilon \to 0} \left[ \left( \frac{1-\epsilon^2}{\epsilon^2} \right) J_1^2(\epsilon) \right] = \lim_{\epsilon \to 0} \left[ \left( \frac{1-\epsilon^2}{\epsilon^2} \right) \left( \frac{\epsilon}{2} \right)^2 \right] = \frac{1}{4} \qquad\qquad \lim_{\epsilon \to 0} J_1'^2(\epsilon) = \frac{1}{4} \tag{21}$$

which gives

$$P_1 = \frac{2e^2 \omega_0^4 a^2}{3c^3} \tag{22}$$

agreeing with problem 14.21 (a) (under the approximation $\gamma \to 1$).