For the symmetric stress tensor $\Theta^{\mu\nu}$, the corresponding integrals in (12.106) are

$$\int \Theta^{00} d^3x = \frac{1}{8\pi}\int \left(\mathbf{E}^2 + \mathbf{B}^2\right) d^3x \qquad\qquad \int \Theta^{0i} d^3x = \frac{1}{4\pi}\int (\mathbf{E}\times\mathbf{B})_i\, d^3x \tag{1}$$

This form is not manifestly covariant because of the 3-space integral volume element d^3x. To generalize this for all inertial frames, we will write d^3x as covariant vector $n_\nu d\sigma$, where

$$n^\nu = (\gamma, \gamma\boldsymbol{\beta}) = \frac{U^\nu}{c} \tag{2}$$

is $1/c$ times the 4-velocity and $d\sigma$ is the Lorentz invariant volume element which is equal to $dxdydz$ in the rest frame S. If S' moves with relative velocity $\boldsymbol{\beta}c$ with respect to S, then the hyperplane in S' with constant time $t' = t'_0$ (i.e., the "now" plane at $t' = t'_0$) is represented in S by the equation

$$n_\mu x^\mu - ct'_0 = 0 \tag{3}$$

which is identical to the Lorentz transformation between S and S'. The volume element on this hyperplane, seen from S is $n^\mu d\sigma$.

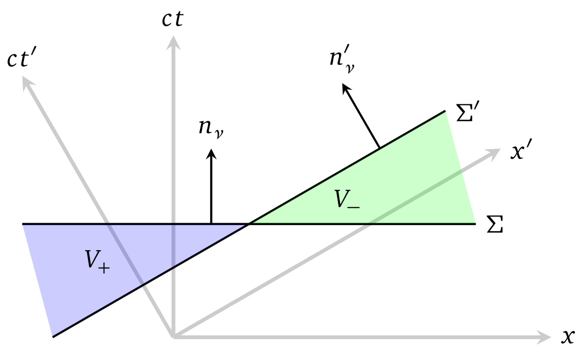
The manifestly covariant form of the integrals in (1) is thus

$$P^\mu = \int \Theta^{\mu\nu} n_\nu d\sigma \tag{4}$$

Note for S, $n^\mu = (1,0,0,0)$, (4) and (1) are equivalent. But as seen from S', the integration domain's hyperplane involves both time component and space component, justifying the sum over μ index in (4). Another way to justify the inner product $\Theta^{\mu\nu}n_\nu$ is to think of the integral (4) as the flux of tensor $\Theta^{\mu\nu}$ through the hyperplane with normal n^ν (see equation (6.122) as an analogy in 3-space).

Because it is covariant, it certainly transforms as a 4-vector between inertial frames, but it may not be a "constant" 4-vector in the sense that it might be dependent on a particular choice of frame (or rather, a hyperplane Σ in the 4-spacetime). To prove its constancy, we must invoke the "source-free" condition

$$\partial_\mu \Theta^{\mu\nu} = 0 \tag{5}$$



Now consider two inertial frames S and S', whose "now" hyperplanes are Σ and Σ' respectively. In S, Σ is a hyperplane orthogonal to the time axis, but Σ' is tilted with respect to the time axis (but still space-like). Let V_4 be the 4-volume enclosed by Σ and Σ' (but divided into two regions $V_\pm$ as indicated in the diagram), and a time-like hypersurface Σ_∞ connecting them at infinity, i.e.,

$$V_4 = V_+ \cup V_- \qquad\qquad \partial V_4 = \Sigma \cup \Sigma' \cup \Sigma_\infty \tag{6}$$

Then by Gauss theorem, we have

$$\int_{V_+} \partial_\mu \Theta^{\mu\nu} dV_4 + \int_{V_-} \left(-\partial_\mu\Theta^{\mu\nu}\right) dV_4 = \int_\Sigma \Theta^{\mu\nu} n_\nu d\sigma - \int_{\Sigma'} \Theta^{\mu\nu} n'_\nu d\sigma' + \int_{\Sigma_\infty} \tag{7}$$

The second term has minus sign because the normal vector must be pointing "outwards". Now with the source-free condition, LHS vanishes. Because of the boundedness of the field, the third term also vanishes, leaving the two surface integrals on Σ and Σ' equal to each other, i.e., the integral (4) is independent of the choice of Σ.

In particular, if Σ and Σ' are parallel to each other (i.e., $V_- = \emptyset$) but separated by infinitesimal dt, the RHS of (7) becomes the differential of the space integral $d\int \Theta^{\mu\nu} d^3x$. Recognizing that $dV_4 = d^3x dt$, we see that the LHS is $dt \int \partial_\mu \Theta^{\mu\nu} d^3x$. This turns (7) into the conservation law

$$\frac{d}{dt}\int \Theta^{\mu\nu} d^3x = 0 \tag{8}$$