1. The radiated field is the coherent sum of the radiated fields from both media. Similar to (13.78), we have

$$\mathbf{E}_1 = \frac{e^{ik_1 r}}{r}\left(\frac{-\omega_1^2}{4\pi c^2}\right)\overbrace{\int_{z'<0}\left(\hat{\mathbf{k}}\times\mathbf{E}_i\right)\times\hat{\mathbf{k}}e^{-i\mathbf{k}_1\cdot\mathbf{x}'}dx'^3}^{\mathbf{F}_1} \tag{1}$$

$$\mathbf{E}_2 = \frac{e^{ik_2 r}}{r}\left(\frac{-\omega_2^2}{4\pi c^2}\right)\underbrace{\int_{z'>0}\left(\hat{\mathbf{k}}\times\mathbf{E}_i\right)\times\hat{\mathbf{k}}e^{-i\mathbf{k}_2\cdot\mathbf{x}'}dx'^3}_{\mathbf{F}_2} \tag{2}$$

Notice that $\mathbf{F}_1$ is integrated over the lower half-space, while $\mathbf{F}_2$ is integrated over the upper half-space.

The upper half-space integral is already given after (13.80), and the lower-space integral can be obtained similarly except the z integral is from $-\infty$ to 0. If we ignore the contribution when $|Z| \gg D$, we have (note the minus sign in $\mathbf{F}_1$)

$$\mathbf{F}_2 = \frac{i}{\left(\dfrac{\omega}{v} - k_2\cos\theta\right)}\iint dxdy\left(\hat{\mathbf{k}}\times\mathbf{E}_i\right)_{z=0}\times\hat{\mathbf{k}}e^{-ik_1 x\sin\theta} \tag{3}$$

$$\mathbf{F}_1 = -\frac{i}{\left(\dfrac{\omega}{v} - k_1\cos\theta\right)}\iint dxdy\left(\hat{\mathbf{k}}\times\mathbf{E}_i\right)_{z=0}\times\hat{\mathbf{k}}e^{-ik_2 x\sin\theta} \tag{4}$$

The results can be directly obtained from (13.83)

$$\mathbf{F}_2 = \epsilon_a 4\sqrt{2\pi}\frac{ze}{c}\left(\frac{c}{\omega_2}\right)^2\frac{\gamma}{\nu_2^2}\frac{\gamma\theta}{\left(1+\dfrac{1}{\nu_2^2}+\gamma^2\theta^2\right)\left(1+\gamma^2\theta^2\right)} \tag{5}$$

$$\mathbf{F}_1 = -\epsilon_a 4\sqrt{2\pi}\frac{ze}{c}\left(\frac{c}{\omega_1}\right)^2\frac{\gamma}{\nu_1^2}\frac{\gamma\theta}{\left(1+\dfrac{1}{\nu_1^2}+\gamma^2\theta^2\right)\left(1+\gamma^2\theta^2\right)} \tag{6}$$

If we ignore the difference in the phase factor in the spherical wave, the total radiatied field is

$$\begin{aligned}
\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 &= \frac{e^{ikr}}{r}\frac{\epsilon_a}{4\pi}4\sqrt{2\pi}\frac{ze}{c}\gamma^2\theta\left[\frac{\dfrac{1}{\nu_1^2}}{\left(1+\dfrac{1}{\nu_1^2}+\gamma^2\theta^2\right)\left(1+\gamma^2\theta^2\right)} - \frac{\dfrac{1}{\nu_2^2}}{\left(1+\dfrac{1}{\nu_2^2}+\gamma^2\theta^2\right)\left(1+\gamma^2\theta^2\right)}\right] \\
&= \frac{e^{ikr}}{r}\epsilon_a\sqrt{\frac{2}{\pi}}\frac{ze}{c}\gamma^2\theta\left[\left(\frac{1}{1+\gamma^2\theta^2} - \frac{1}{1+\dfrac{1}{\nu_1^2}+\gamma^2\theta^2}\right) - \left(\frac{1}{1+\gamma^2\theta^2} - \frac{1}{1+\dfrac{1}{\nu_2^2}+\gamma^2\theta^2}\right)\right] \\
&= \frac{e^{ikr}}{r}\epsilon_a\sqrt{\frac{2}{\pi}}\frac{ze}{c}\gamma^2\theta\left(\frac{1}{1+\dfrac{1}{\nu_2^2}+\gamma^2\theta^2} - \frac{1}{1+\dfrac{1}{\nu_1^2}+\gamma^2\theta^2}\right) \qquad \text{recall } \frac{1}{\nu_i} = \frac{\gamma\omega_i}{\omega} \\
&= \frac{e^{ikr}}{r}\epsilon_a\sqrt{\frac{2}{\pi}}\frac{ze}{c}\theta\left(\frac{1}{\dfrac{1}{\gamma^2}+\dfrac{\omega_2^2}{\omega^2}+\theta^2} - \frac{1}{\dfrac{1}{\gamma^2}+\dfrac{\omega_1^2}{\omega^2}+\theta^2}\right)
\end{aligned} \tag{7}$$

By (14.52) and (14.60), we have the differential spectrum in angle

$$\frac{d^2 I}{d\omega d\Omega} = \underbrace{2\cdot\frac{c}{4\pi}\cdot\left(\frac{2}{\pi}\right)\left(\frac{ze}{c}\right)^2\theta^2}_{z^2e^2\theta^2/\pi^2 c}\left|\frac{1}{\dfrac{1}{\gamma^2}+\dfrac{\omega_2^2}{\omega^2}+\theta^2} - \frac{1}{\dfrac{1}{\gamma^2}+\dfrac{\omega_1^2}{\omega^2}+\theta^2}\right|^2 \tag{8}$$

2. The total radiation is obtained by integrating (8) over all frequencies and solid angles

$$I = \int_0^\infty d\omega \int d\Omega \frac{d^2 I}{d\omega d\Omega} \qquad \text{(small angle approximation)}$$

$$\approx \frac{z^2 e^2}{\pi^2 c} \cdot 2\pi \int_0^\infty \theta^3 d\theta \underbrace{\int_0^\infty d\omega \left| \frac{1}{\frac{1}{\gamma^2} + \frac{\omega_2^2}{\omega^2} + \theta^2} - \frac{1}{\frac{1}{\gamma^2} + \frac{\omega_1^2}{\omega^2} + \theta^2} \right|^2}_{J(\theta)} \tag{9}$$

Denoting $\alpha^2 = 1/\gamma^2 + \theta^2$, the inner integral can be written as

$$J(\theta) = \left(\omega_1^2 - \omega_2^2\right)^2 \int_0^\infty \frac{\omega^4 d\omega}{\left(\omega^2\alpha^2 + \omega_1^2\right)^2 \left(\omega^2\alpha^2 + \omega_2^2\right)^2} \tag{10}$$

To evaluate this integral, we use Cauchy's theorem. Consider the function over the complex plane

$$\begin{aligned} f(z) &= \frac{z^4}{\left(z^2\alpha^2 + \omega_1^2\right)^2 \left(z^2\alpha^2 + \omega_2^2\right)^2} \\ &= \frac{z^4}{\alpha^8 (z - z_1)^2 (z + z_1)^2 (z - z_2)^2 (z + z_2)^2} \qquad \text{where } z_{1,2} = \frac{i\omega_{1,2}}{\alpha} \end{aligned} \tag{11}$$

In the upper plane, we have two poles of order 2 at $z_{1,2}$, the residues of which are

$$\operatorname{Res}\left(f, z_j\right) = \lim_{z \to z_j} \frac{d}{dz}\left[\left(z - z_j\right)^2 f(z)\right] \tag{12}$$

For $j = 1$, let

$$h(z) = (z - z_1)^2 f(z) = \frac{z^4}{\alpha^8 (z + z_1)^2 (z - z_2)^2 (z + z_2)^2} \tag{13}$$

We can find the derivative of h at z_1 via the logarithmic derivative

$$h'(z_1) = h(z_1) \frac{h'(z_1)}{h(z_1)} = h(z_1) \cdot \left.\frac{d \ln h}{dz}\right|_{z=z_1} \tag{14}$$

where

$$h(z_1) = \frac{z_1^4}{\alpha^8 (2z_1)^2 (z_1 - z_2)^2 (z_1 + z_2)^2} = -\frac{\omega_1^2}{4\alpha^6 \left(\omega_1^2 - \omega_2^2\right)^2} \tag{15}$$

$$\left.\frac{d\ln h}{dz}\right|_{z=z_1} = \frac{4}{z_1} - \frac{2}{z_1 + z_1} - \frac{2}{z_1 - z_2} - \frac{2}{z_1 + z_2} = \frac{3}{z_1} - \frac{4z_1}{z_1^2 - z_2^2} = \frac{\alpha}{i}\left(\frac{3}{\omega_1} - \frac{4\omega_1}{\omega_1^2 - \omega_2^2}\right) = i\alpha\left[\frac{\omega_1^2 + 3\omega_2^2}{\omega_1\left(\omega_1^2 - \omega_2^2\right)}\right] \tag{16}$$

giving the first residue

$$\operatorname{Res}(f, z_1) = -\frac{i\omega_1\left(\omega_1^2 + 3\omega_2^2\right)}{4\alpha^5\left(\omega_1^2 - \omega_2^2\right)^3} \tag{17}$$

The other residue is obtained by exchanging the indices

$$\operatorname{Res}(f, z_2) = \frac{i\omega_2\left(\omega_2^2 + 3\omega_1^2\right)}{4\alpha^5\left(\omega_1^2 - \omega_2^2\right)^3} \tag{18}$$

By Cauchy's theorem while noticing that the integral (10) is half of the integral over the entire real axis, we have

$$\begin{aligned} J(\theta) &= \left(\omega_1^2 - \omega_2^2\right)^2 i\pi \cdot \left[\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right] \\ &= \frac{\pi(\omega_1 - \omega_2)^2}{4\left(\frac{1}{\gamma^2} + \theta^2\right)^{5/2}(\omega_1 + \omega_2)} \end{aligned} \tag{19}$$

Putting this back to (9) yields

$$
\begin{aligned}
I &= \frac{z^2e^2}{\pi^2 c} 2\pi \frac{\pi}{4} \frac{(\omega_1-\omega_2)^2}{(\omega_1+\omega_2)} \int_0^\infty \frac{\theta^3 d\theta}{\left(\frac{1}{\gamma^2}+\theta^2\right)^{5/2}} && \text{let } \eta = \gamma^2\theta^2 \\
&= \frac{z^2e^2}{4c} \frac{(\omega_1-\omega_2)^2}{(\omega_1+\omega_2)} \gamma \int_0^\infty \frac{\eta d\eta}{(1+\eta)^{5/2}} && \text{let } u = 1+\eta \\
&= \frac{z^2e^2}{4c} \frac{(\omega_1-\omega_2)^2}{(\omega_1+\omega_2)} \gamma \int_1^\infty \left(\frac{du}{u^{3/2}} - \frac{du}{u^{5/2}}\right) \\
&= \frac{z^2e^2}{4c} \frac{(\omega_1-\omega_2)^2}{(\omega_1+\omega_2)} \gamma \left(-2u^{-1/2} + \frac{2}{3}u^{-3/2}\right)\Bigg|_1^\infty \\
&= \frac{z^2e^2}{3c} \frac{(\omega_1-\omega_2)^2}{(\omega_1+\omega_2)} \gamma
\end{aligned} \tag{20}
$$