1. Referring to figure 14.9, we see that for the photon propagation direction $\mathbf{n}$, the polarization direction of the positive and negative helicity are correspondingly

$$\boldsymbol{\epsilon}_\pm = \frac{\boldsymbol{\epsilon}_\parallel \pm i\boldsymbol{\epsilon}_\perp}{\sqrt{2}} \tag{1}$$

By (14.74), the vectorial amplitude inside the absolute sign is

$$\mathbf{A}(\omega) = -\boldsymbol{\epsilon}_\parallel A_\parallel(\omega) + \boldsymbol{\epsilon}_\perp A_\perp(\omega) \tag{2}$$

therefore to get the frequency-angle spectra of radiation with positive/negative helicity, we take the scalar product of $\mathbf{A}$ and $\boldsymbol{\epsilon}_\pm^*$ before taking the absolute square, i.e.,

$$\frac{d^2 I_\pm}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\mathbf{A}(\omega)\cdot\boldsymbol{\epsilon}_\pm^*\right|^2 \tag{3}$$

The scalar product can be calculated as

$$\mathbf{A}(\omega)\cdot\boldsymbol{\epsilon}_\pm^* = \left[-\boldsymbol{\epsilon}_\parallel A_\parallel(\omega) + \boldsymbol{\epsilon}_\perp A_\perp(\omega)\right]\left(\frac{\boldsymbol{\epsilon}_\parallel \mp i\boldsymbol{\epsilon}_\perp}{\sqrt{2}}\right) = \frac{-A_\parallel(\omega) \mp iA_\perp(\omega)}{\sqrt{2}} \tag{4}$$

From (14.77), (14.78) we have

$$A_\parallel(\omega) = \frac{\rho}{c}\left(\frac{1}{\gamma^2}+\theta^2\right)\cdot\frac{2i}{\sqrt{3}}K_{2/3}(\xi) \tag{5}$$

$$A_\perp(\omega) = \frac{\rho}{c}\theta\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\cdot\frac{2}{\sqrt{3}}K_{1/3}(\xi) \tag{6}$$

giving

$$\frac{d^2 I_\pm}{d\omega d\Omega} = \frac{e^2}{6\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left|K_{2/3}(\xi) \pm \frac{\theta}{\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}}K_{1/3}(\xi)\right|^2 \tag{7}$$

2. Similar to (10.8), we can define the degree of circular polarization as

$$\Pi_c \equiv \frac{d^2I_+ - d^2I_-}{d^2I_+ + d^2I_-} = \frac{2\alpha K_{2/3}(\xi)K_{1/3}(\xi)}{K_{2/3}^2(\xi)+\alpha^2K_{1/3}^2(\xi)} \qquad \text{where } \alpha \equiv \frac{\theta}{\sqrt{1/\gamma^2+\theta^2}} \tag{8}$$

(a) High frequencies ($\omega \gg \omega_c$), all angles.
The discussion around (14.86) and (14.87) is relevant. From (14.87), the critical angle is

$$\theta_c = \frac{1}{\gamma}\left(\frac{2\omega_c}{3\omega}\right)^{1/2} \tag{9}$$

so all appreciable radiation has $\gamma\theta \ll 1$, hence $\alpha \approx \gamma\theta \ll 1$. Moreover when $\xi \gg 1$, $K_{2/3}(\xi) \approx \sqrt{\pi/2\xi}e^{-\xi} \approx K_{1/3}(\xi)$ so $K_{2/3}(\xi) \gg \alpha K_{1/3}(\xi)$ due to small α. Therefore $\Pi_c \approx 0$ and the radiation is linearly polarized in the orbital plane (in the direction of $\boldsymbol{\epsilon}_\parallel$).

(b) Low/intermediate frequencies ($\omega \ll \omega_c$), large angles ($|\theta| \gg 1/\gamma$).
Here $\alpha \approx \mathrm{sgn}(\theta)$. Since $K_{2/3}$ and $K_{1/3}$ are both positive and of comparable magnitude for ξ of order unity (where the bulk of the emission occurs, near $\theta \sim \theta_c$), one helicity strongly dominates:

$$\Pi_c \approx \frac{2K_{2/3}(\xi)K_{1/3}(\xi)}{K_{2/3}^2(\xi)+K_{1/3}^2(\xi)}\,\mathrm{sgn}(\theta) \approx \mathrm{sgn}(\theta) \tag{10}$$

The radiation is nearly circularly polarized, with the sign of helicity matching the sign of θ.

(c) Low/intermediate frequencies ($\omega \ll \omega_c$), very small angles ($|\theta| \ll 1/\gamma$).
Here $\alpha \approx \gamma\theta \ll 1$ and $\xi \approx \omega\rho/(3c\gamma^3) = \omega/2\omega_c \ll 1$, where $K_{2/3}(\xi)$ dominates $K_{1/3}(\xi)$. So the circular polarization is doubly suppressed (by α and by the smallness of $K_{1/3}$ relative to $K_{2/3}$). The radiation is again linearly polarized in the orbital plane.

The overall picture: in the orbital plane, the radiation is always linearly polarized (by symmetry, the perpendicular component vanishes at $\theta = 0$). Moving off-plane, the ellipticity grows. At low frequencies the critical angle $\theta_c \sim (3c/\omega\rho)^{1/3} \gg 1/\gamma$ is large, so there is a wide angular range with $|\theta| \gg 1/\gamma$ where circular polarization dominates. At high frequencies $\theta_c \ll 1/\gamma$, and the radiation is essentially linearly polarized at all angles of appreciable intensity.