

Let \mathbf{r} be the observation point, and let $\mathbf{x}(t)$ be the position of the charge at time t . Let $t'(t)$ be the retarded time corresponding to the observation time t , of which the governing equation is

$$t' = t - \frac{R(t')}{c} \quad (1)$$

where $\mathbf{R}(t') = \mathbf{r} - \mathbf{x}(t')$ and $R = |\mathbf{R}| = [(\mathbf{r} - \mathbf{x}) \cdot (\mathbf{r} - \mathbf{x})]^{1/2}$.

We use the "overdot" to denote time derivative evaluated the retarded time t' . An equivalent (informal) notation is d/dt' , i.e., it will be understood that for a time dependent function f ,

$$\dot{f} = \frac{df}{dt'} = \left. \frac{df}{dt} \right|_{t=t'} \quad (2)$$

It is easy to see that

$$\dot{\mathbf{R}} = -c\boldsymbol{\beta} \quad (3)$$

$$\dot{R} = \frac{1}{2} \left[\frac{-2(\mathbf{r} - \mathbf{x}) \cdot c\boldsymbol{\beta}}{R} \right] = -c\boldsymbol{\beta} \cdot \mathbf{n} \quad (4)$$

where it is understood that the RHS are evaluated at the retarded time t' .

Differentiating (1) gives

$$\frac{dt}{dt'} = 1 + \frac{\dot{R}}{c} = 1 - \boldsymbol{\beta} \cdot \mathbf{n} \equiv \kappa \quad \text{or} \quad \frac{d}{dt} = \frac{1}{\kappa} \frac{d}{dt'} \quad (5)$$

Furthermore,

$$\dot{\mathbf{n}} = \frac{d}{dt'} \left(\frac{\mathbf{R}}{R} \right) = \frac{\dot{\mathbf{R}}}{R} - \frac{\mathbf{R}\dot{R}}{R^2} = -\frac{c\boldsymbol{\beta}}{R} + \frac{c\mathbf{n}(\boldsymbol{\beta} \cdot \mathbf{n})}{R} = \frac{c[-\boldsymbol{\beta} + \mathbf{n}(\boldsymbol{\beta} \cdot \mathbf{n})]}{R} = -\frac{c}{R} \overbrace{[\mathbf{n} \times (\boldsymbol{\beta} \times \mathbf{n})]}^{\mathbf{m} \equiv \boldsymbol{\beta} - (\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{n}} \quad (6)$$

$$\dot{\kappa} = \frac{d(1 - \boldsymbol{\beta} \cdot \mathbf{n})}{dt'} = -\dot{\boldsymbol{\beta}} \cdot \mathbf{n} - \boldsymbol{\beta} \cdot \dot{\mathbf{n}} = \underbrace{-\dot{\boldsymbol{\beta}} \cdot \mathbf{n}}_{\kappa_A} + \underbrace{\frac{c}{R} \boldsymbol{\beta} \cdot \mathbf{m}}_{\kappa_B} \quad (7)$$

The Heaviside-Feynman formula is

$$\mathbf{E} = e \left[\frac{\mathbf{n}}{R^2} \right]_{\text{ret}} + e \left[\frac{R}{c} \right]_{\text{ret}} \frac{d}{dt} \left[\frac{\mathbf{n}}{R^2} \right]_{\text{ret}} + \frac{e}{c^2} \frac{d^2[\mathbf{n}]}{dt^2} \Big|_{\text{ret}} \quad (8)$$

where

$$\left[\frac{R}{c} \right]_{\text{ret}} \frac{d}{dt} \left[\frac{\mathbf{n}}{R^2} \right]_{\text{ret}} = \frac{R}{c} \cdot \frac{1}{\kappa} \frac{d}{dt'} \left(\frac{\mathbf{n}}{R^2} \right) = \frac{R}{c\kappa} \left(\frac{\dot{\mathbf{n}}}{R^2} - \frac{2\dot{R}\mathbf{n}}{R^3} \right) = -\frac{\mathbf{m}}{\kappa R^2} + \frac{2(\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{n}}{\kappa R^2} \quad (9)$$

$$\begin{aligned} \frac{d^2[\mathbf{n}]}{dt^2} &= \frac{1}{\kappa} \frac{d}{dt'} \left(\frac{\dot{\mathbf{n}}}{\kappa} \right) = -\frac{c}{\kappa} \frac{d}{dt'} \left(\frac{\mathbf{m}}{\kappa R} \right) \\ &= -\frac{c}{\kappa} \left[\frac{d}{dt'} \left(\frac{1}{\kappa R} \right) \mathbf{m} + \frac{\dot{\mathbf{n}} \times (\boldsymbol{\beta} \times \mathbf{n})}{\kappa R} + \frac{\mathbf{n} \times (\dot{\boldsymbol{\beta}} \times \mathbf{n})}{\kappa R} + \frac{\mathbf{n} \times (\boldsymbol{\beta} \times \dot{\mathbf{n}})}{\kappa R} \right] \\ &= -\frac{c}{\kappa} \left[-\frac{(\kappa_A + \kappa_B)\mathbf{m}}{\kappa^2 R} - \frac{\dot{\mathbf{R}}\mathbf{m}}{\kappa R^2} + \frac{\dot{\mathbf{n}} \times (\boldsymbol{\beta} \times \mathbf{n})}{\kappa R} + \frac{\mathbf{n} \times (\dot{\boldsymbol{\beta}} \times \mathbf{n})}{\kappa R} + \frac{\mathbf{n} \times (\boldsymbol{\beta} \times \dot{\mathbf{n}})}{\kappa R} \right] \end{aligned} \quad (10)$$

Since κ_B and $\dot{\mathbf{n}}$ decay as $1/R$ already, we see that the terms in the H-F formula either decay as $1/R$ or $1/R^2$. Summing up all the $1/R$ terms gives

$$\begin{aligned} \text{HF}_{1/R} &= \frac{e}{c^2} \cdot \left(-\frac{c}{\kappa} \right) \left[-\frac{\kappa_A \mathbf{m}}{\kappa^2 R} + \frac{\mathbf{n} \times (\dot{\boldsymbol{\beta}} \times \mathbf{n})}{\kappa R} \right] \\ &= -\frac{e}{c\kappa^3 R} \{ (\dot{\boldsymbol{\beta}} \cdot \mathbf{n}) [\boldsymbol{\beta} - (\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{n}] + (1 - \boldsymbol{\beta} \cdot \mathbf{n}) [\dot{\boldsymbol{\beta}} - (\dot{\boldsymbol{\beta}} \cdot \mathbf{n})\mathbf{n}] \} \\ &= -\frac{e}{c\kappa^3 R} [(\dot{\boldsymbol{\beta}} \cdot \mathbf{n})\boldsymbol{\beta} + (1 - \boldsymbol{\beta} \cdot \mathbf{n})\dot{\boldsymbol{\beta}} - (\dot{\boldsymbol{\beta}} \cdot \mathbf{n})\mathbf{n}] \\ &= -\frac{e}{c\kappa^3 R} \{ (\dot{\boldsymbol{\beta}} \cdot \mathbf{n})(\boldsymbol{\beta} - \mathbf{n}) - [\mathbf{n} \cdot (\boldsymbol{\beta} - \mathbf{n})]\dot{\boldsymbol{\beta}} \} \\ &= \frac{e}{c} \left\{ \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{\kappa^3 R} \right\} \end{aligned} \quad (11)$$

which matches the acceleration field in the Liénard-Wiechert formula.

Collecting all the $1/R^2$ terms, we have

$$\begin{aligned} \text{HF}_{1/R^2} &= e \left(\frac{\mathbf{n}}{R^2} \right) + e \left[\frac{-\mathbf{m} + 2(\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{n}}{\kappa R^2} \right] - \frac{e}{c\kappa} \left[-\frac{\kappa_B \mathbf{m}}{\kappa^2 R} - \frac{\dot{R}\mathbf{m}}{\kappa R^2} + \frac{\dot{\mathbf{n}} \times (\boldsymbol{\beta} \times \mathbf{n})}{\kappa R} + \frac{\mathbf{n} \times (\boldsymbol{\beta} \times \dot{\mathbf{n}})}{\kappa R} \right] \\ &= \frac{e}{\kappa^3 R^2} \cdot \mathbf{X} \end{aligned} \quad (12)$$

where

$$\mathbf{X} = \overbrace{\kappa^3 \mathbf{n}}^{X_1} + \overbrace{\kappa^2 [-\mathbf{m} + 2(\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{n}]}^{X_2} + \overbrace{(\boldsymbol{\beta} \cdot \mathbf{m})\mathbf{m}}^{X_3} \overbrace{-\kappa(\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{m}}^{X_4} + \overbrace{\kappa \mathbf{m} \times (\boldsymbol{\beta} \times \mathbf{n})}^{X_5} + \overbrace{\kappa \mathbf{n} \times (\boldsymbol{\beta} \times \mathbf{m})}^{X_6} \quad (13)$$

By definition of \mathbf{m} (see (6)),

$$\mathbf{m} \cdot \mathbf{m} = \boldsymbol{\beta} \cdot \mathbf{m} = \beta^2 - (\boldsymbol{\beta} \cdot \mathbf{n})^2 \quad \mathbf{n} \cdot \mathbf{m} = 0 \quad (14)$$

which turns (13) into

$$\begin{aligned} \mathbf{X} &= \overbrace{\kappa^3 \mathbf{n}}^{X_1} \overbrace{-\kappa^2 \boldsymbol{\beta} + 3\kappa^2 (\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{n}}^{X_2} + \overbrace{[\beta^2 - (\boldsymbol{\beta} \cdot \mathbf{n})^2]\boldsymbol{\beta} - [\beta^2 - (\boldsymbol{\beta} \cdot \mathbf{n})^2](\boldsymbol{\beta} \cdot \mathbf{n})\mathbf{n}}^{X_3} \\ &\quad \underbrace{-\kappa(\boldsymbol{\beta} \cdot \mathbf{n})\boldsymbol{\beta} + \kappa(\boldsymbol{\beta} \cdot \mathbf{n})^2 \mathbf{n}}_{X_4} \underbrace{-\kappa[\beta^2 - (\boldsymbol{\beta} \cdot \mathbf{n})^2]\mathbf{n}}_{X_5} \underbrace{-\kappa(\boldsymbol{\beta} \cdot \mathbf{n})\boldsymbol{\beta} + \kappa(\boldsymbol{\beta} \cdot \mathbf{n})^2 \mathbf{n}}_{X_6} \end{aligned} \quad (15)$$

Collecting the coefficients of \mathbf{n} and $\boldsymbol{\beta}$ respectively, we have

$$\begin{aligned} \text{coefficient of } \mathbf{n} &= \kappa^3 + 3\kappa^2 (\boldsymbol{\beta} \cdot \mathbf{n}) - [\beta^2 - (\boldsymbol{\beta} \cdot \mathbf{n})^2](\boldsymbol{\beta} \cdot \mathbf{n}) + \kappa(\boldsymbol{\beta} \cdot \mathbf{n})^2 - \kappa[\beta^2 - (\boldsymbol{\beta} \cdot \mathbf{n})^2] + \kappa(\boldsymbol{\beta} \cdot \mathbf{n})^2 \\ &= \kappa^3 + 3\kappa^2 (\boldsymbol{\beta} \cdot \mathbf{n}) - \beta^2 (\boldsymbol{\beta} \cdot \mathbf{n}) + (\boldsymbol{\beta} \cdot \mathbf{n})^3 + 3\kappa(\boldsymbol{\beta} \cdot \mathbf{n})^2 - \kappa\beta^2 \\ &= [\kappa + (\boldsymbol{\beta} \cdot \mathbf{n})]^3 - \beta^2 [\kappa + (\boldsymbol{\beta} \cdot \mathbf{n})] \\ &= 1 - \beta^2 = \frac{1}{\gamma^2} \end{aligned} \quad (16)$$

$$\begin{aligned} \text{coefficient of } \boldsymbol{\beta} &= -\kappa^2 + [\beta^2 - (\boldsymbol{\beta} \cdot \mathbf{n})^2] - \kappa(\boldsymbol{\beta} \cdot \mathbf{n}) - \kappa(\boldsymbol{\beta} \cdot \mathbf{n}) \\ &= -[\kappa + (\boldsymbol{\beta} \cdot \mathbf{n})]^2 + \beta^2 \\ &= -1 + \beta^2 = -\frac{1}{\gamma^2} \end{aligned} \quad (17)$$

Putting these back to (12) gives

$$\text{HF}_{1/R^2} = e \left(\frac{\mathbf{n} - \boldsymbol{\beta}}{\gamma^2 \kappa^3 R^2} \right) \quad (18)$$

which matches the velocity field in the Liénard-Wiechert formula.