1. From problem 14.13, we have for $k \neq 0$

$$\left\langle \frac{dP_k}{d\Omega} \right\rangle = \frac{e^2 k^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^T e^{ik\omega_0[t - \mathbf{n}\cdot\mathbf{x}(t)/c]} \mathbf{n} \times \mathbf{v} dt \right|^2 \tag{1}$$

Denoting $b \equiv \sqrt{1-\epsilon^2} a$, we have

$$\mathbf{x}(u) = a(\cos u - \epsilon)\hat{\mathbf{x}} + b \sin u \hat{\mathbf{y}} \tag{2}$$

$$\tag{3}$$

It is most convenient to evaluate the integral in variable u. First, the phase factor can be simplified as

$$\begin{aligned} e^{ik\omega_0[t - \mathbf{n}\cdot\mathbf{x}(u)/c]} &= \exp\left[ik\omega_0 t \overbrace{- ik\left(\frac{\omega_0 a}{c}\right)\sin\theta\cos\phi(\cos u - \epsilon) - \left(\frac{\omega_0 b}{c}\right)\sin\theta\sin\phi\sin u}^{\text{ignored for nonrelativistic motion}} \right] \\ &\approx e^{ik\omega_0 t} = e^{ik(u - \epsilon \sin u)} \end{aligned} \tag{4}$$

Next, notice that

$$\begin{aligned} \mathbf{n} \times \mathbf{v} dt = \mathbf{n} \times \frac{d\mathbf{x}}{du} du &= (\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}) \times (-a\sin u\hat{\mathbf{x}} + b\cos u\hat{\mathbf{y}}) du \\ &= [-b\cos\theta\cos u\hat{\mathbf{x}} - a\cos\theta\sin u\hat{\mathbf{y}} + (a\sin\theta\sin\phi\sin u + b\sin\theta\cos\phi\cos u)\hat{\mathbf{z}}] du \end{aligned} \tag{5}$$

turning the integral in (1) into

$$\int_0^T e^{ik\omega_0[t - \mathbf{n}\cdot\mathbf{x}(t)/c]} \mathbf{n} \times \mathbf{v} dt = \mathbf{I}_x + \mathbf{I}_y + \mathbf{I}_z \tag{6}$$

where

$$\mathbf{I}_x = -b\cos\theta\hat{\mathbf{x}} \int_0^{2\pi} e^{ik(u - \epsilon\sin u)} \cos u du \tag{7}$$

$$\mathbf{I}_y = -a\cos\theta\hat{\mathbf{y}} \int_0^{2\pi} e^{ik(u - \epsilon\sin u)} \sin u du \tag{8}$$

$$\mathbf{I}_z = \sin\theta\hat{\mathbf{z}} \left[a\sin\phi \int_0^{2\pi} e^{ik(u - \epsilon\sin u)} \sin u du + b\cos\phi \int_0^{2\pi} e^{ik(u - \epsilon\sin u)} \cos u du \right] \tag{9}$$

With the Jacobi-Anger expansion

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\phi} \tag{10}$$

we can write

$$\begin{aligned} e^{-ik\epsilon\sin u} &= \sum_{n=-\infty}^{\infty} J_n(-k\epsilon) e^{inu} \qquad\qquad J_n(-x) = (-1)^n J_n(x) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n J_n(k\epsilon) e^{inu} \end{aligned} \tag{11}$$

The $\cos u$ integral in (7) becomes

$$\begin{aligned} \int_0^{2\pi} e^{ik(u - \epsilon\sin u)} \cos u du &= \sum_{n=-\infty}^{\infty} (-1)^n J_n(k\epsilon) \overbrace{\frac{1}{2} \int_0^{2\pi} \left[e^{i(k+n+1)u} + e^{i(k+n-1)u} \right] du}^{2\pi(\delta_{k+n+1,0} + \delta_{k+n-1,0})} \\ &= \pi\left[(-1)^{k+1} J_{-(k+1)}(k\epsilon) + (-1)^{k-1} J_{-(k-1)}(k\epsilon) \right] \qquad J_{-n}(x) = (-1)^n J_n(x) \\ &= \pi[J_{k+1}(k\epsilon) + J_{k-1}(k\epsilon)] \qquad J_{k-1}(x) + J_{k+1}(x) = \frac{2k}{x} J_k(x) \\ &= \frac{2\pi}{\epsilon} J_k(k\epsilon) \end{aligned} \tag{12}$$

The $\sin u$ integral in (8) becomes

$$\int_0^{2\pi} e^{ik(u-\epsilon\sin u)}\sin u\,du = \sum_{n=-\infty}^{\infty}(-1)^n J_n(k\epsilon)\frac{1}{2i}\overbrace{\int_0^{2\pi}\left[e^{i(k+n+1)u}-e^{i(k+n-1)u}\right]du}^{2\pi\left(\delta_{k+n+1,0}-\delta_{k+n-1,0}\right)}$$
$$= \frac{\pi}{i}\left[J_{k+1}(k\epsilon)-J_{k-1}(k\epsilon)\right] \qquad J_{k-1}(x)-J_{k+1}(x)=2J_k'(x)$$
$$= 2\pi i J_k'(k\epsilon) \tag{13}$$

Substituting (12) and (13) back into (7)–(9) gives

$$\mathbf{I}_x = -\frac{2\pi b\cos\theta}{\epsilon}J_k(k\epsilon)\hat{\mathbf{x}} \tag{14}$$
$$\mathbf{I}_y = -2\pi i a\cos\theta J_k'(k\epsilon)\hat{\mathbf{y}} \tag{15}$$
$$\mathbf{I}_z = 2\pi\sin\theta\left[ia\sin\phi J_k'(k\epsilon)+\frac{b\cos\phi}{\epsilon}J_k(k\epsilon)\right]\hat{\mathbf{z}} \tag{16}$$

Thus

$$|\mathbf{I}|^2 = 4\pi^2\left\{\frac{b^2\cos^2\theta}{\epsilon^2}J_k^2(k\epsilon)+a^2\cos^2\theta J_k'^2(k\epsilon)+\sin^2\theta\left[a^2\sin^2\phi J_k'^2(k\epsilon)+\frac{b^2\cos^2\phi}{\epsilon^2}J_k^2(k\epsilon)\right]\right\}$$
$$= 4\pi^2\left[\frac{b^2}{\epsilon^2}\left(\cos^2\theta+\sin^2\theta\cos^2\phi\right)J_k^2(k\epsilon)+a^2\left(\cos^2\theta+\sin^2\theta\sin^2\phi\right)J_k'^2(k\epsilon)\right] \tag{17}$$

Recalling that $b^2=\left(1-\epsilon^2\right)a^2$, we can write

$$\left\langle\frac{dP_k}{d\Omega}\right\rangle = \frac{e^2k^2\omega_0^4a^2}{2\pi c^3}\left[\left(\frac{1-\epsilon^2}{\epsilon^2}\right)\left(\cos^2\theta+\sin^2\theta\cos^2\phi\right)J_k^2(k\epsilon)+\left(\cos^2\theta+\sin^2\theta\sin^2\phi\right)J_k'^2(k\epsilon)\right] \tag{18}$$

With the solid angle integration

$$\int\left(\cos^2\theta+\sin^2\theta\cos^2\phi\right)d\Omega = \int\left(\cos^2\theta+\sin^2\theta\sin^2\phi\right)d\Omega = \frac{8\pi}{3} \tag{19}$$

we have the total power for the kth harmonic

$$P_k = \frac{4e^2k^2\omega_0^4a^2}{3c^3}\left[\left(\frac{1-\epsilon^2}{\epsilon^2}\right)J_k^2(k\epsilon)+J_k'^2(k\epsilon)\right] \tag{20}$$

2. For circular orbit, $\epsilon\to 0$. Under this limit, all harmonics but $k=1$ vanishes, and for $k=1$,

$$\lim_{\epsilon\to 0}\left[\left(\frac{1-\epsilon^2}{\epsilon^2}\right)J_1^2(\epsilon)\right] = \lim_{\epsilon\to 0}\left[\left(\frac{1-\epsilon^2}{\epsilon^2}\right)\left(\frac{\epsilon}{2}\right)^2\right] = \frac{1}{4} \qquad \lim_{\epsilon\to 0}J_1'^2(\epsilon) = \frac{1}{4} \tag{21}$$

which gives

$$P_1 = \frac{2e^2\omega_0^4a^2}{3c^3} \tag{22}$$

agreeing with problem 14.21 (a) (under the approximation $\gamma\to 1$).