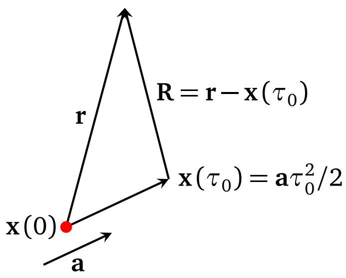

1. Let $\mathbf{a}$ be the acceleration of the particle at time $t = 0$ in the said frame. First let's find the retarded time τ_0 for the observation point at $\mathbf{r}$. Since at $t = 0$, the particle is instantaneously at rest, then at the retarded time τ_0, the particle was at position

$$\mathbf{x}(\tau_0) = \frac{1}{2}\mathbf{a}\tau_0^2 \tag{1}$$

where we have ignored higher order derivatives.

The equation of τ_0 is thus

$$c^2\tau_0^2 = |\mathbf{R}|^2 = \left|\mathbf{r} - \frac{1}{2}\mathbf{a}\tau_0^2\right|^2 = r^2 + \frac{1}{4}a^2\tau_0^4 - (\mathbf{a}\cdot\mathbf{r})\,\tau_0^2 \tag{2}$$

Solving the quadratic equation for τ_0^2, we have

$$\tau_0^2 = \frac{\left(c^2 + \mathbf{a}\cdot\mathbf{r}\right) \pm \sqrt{\left(c^2 + \mathbf{a}\cdot\mathbf{r}\right)^2 - a^2r^2}}{\frac{1}{2}a^2} \tag{3}$$

Using $\sqrt{1+x} = 1 + x/2 - x^2/8 + x^3/16 + \cdots$, we can approximate the square root term as

$$\begin{aligned}
\sqrt{\Delta} &= c^2\sqrt{1 + \frac{2(\mathbf{a}\cdot\mathbf{r})}{c^2} + \frac{(\mathbf{a}\cdot\mathbf{r})^2 - a^2r^2}{c^4}} \\
&= c^2\left\{1 + \frac{(\mathbf{a}\cdot\mathbf{r})}{c^2} + \frac{(\mathbf{a}\cdot\mathbf{r})^2 - a^2r^2}{2c^4} - \frac{1}{8}\left[\frac{4(\mathbf{a}\cdot\mathbf{r})^2}{c^4} + \frac{4(\mathbf{a}\cdot\mathbf{r})^3 - 4(\mathbf{a}\cdot\mathbf{r})\,a^2r^2}{c^6}\right] + \frac{1}{16}\frac{8(\mathbf{a}\cdot\mathbf{r})^3}{c^6} + O\left(\frac{1}{c^8}\right)\right\} \\
&\approx c^2 + \mathbf{a}\cdot\mathbf{r} - \frac{a^2r^2}{2c^2} + \frac{(\mathbf{a}\cdot\mathbf{r})\,a^2r^2}{2c^4}
\end{aligned} \tag{4}$$

We take the negative sign in (3) for the most recent retarded time (because approximation of linear acceleration only works within small time interval), giving

$$\tau_0^2 \approx \frac{r^2}{c^2} - \frac{(\mathbf{a}\cdot\mathbf{r})\,r^2}{c^4} = \frac{r^2}{c^2}\left(1 - \frac{\mathbf{a}\cdot\mathbf{r}}{c^2}\right) \qquad \Longrightarrow \qquad \tau_0 \approx -\frac{r}{c}\left(1 - \frac{\mathbf{a}\cdot\mathbf{r}}{2c^2}\right) \tag{5}$$

where negative root was kept for retarded solution.

From (14.14), the velocity and acceleration fields are

$$\mathbf{E}_v = e\left[\frac{\mathbf{n} - \boldsymbol{\beta}}{\gamma^2(1 - \boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{\text{ret}} \qquad \mathbf{E}_a = \frac{e}{c}\left\{\frac{\mathbf{n}\times\left[(\mathbf{n} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})^3R}\right\}_{\text{ret}} \tag{6}$$

With (5), we can evaluate the various quantities in $\mathbf{E}_v$ and $\mathbf{E}_a$ at the retarded time up to $O\left(1/c^2\right)$.

$$\mathbf{R} = \mathbf{r} - \frac{1}{2}\mathbf{a}\tau_0^2 \approx \mathbf{r} - \frac{\mathbf{a}r^2}{2c^2} \tag{7}$$

$$R = |c\tau_0| \approx r\left(1 - \frac{\mathbf{a}\cdot\mathbf{r}}{2c^2}\right) \tag{8}$$

$$\mathbf{n} = \frac{\mathbf{R}}{R} \approx \left[\frac{\mathbf{r} - \frac{\mathbf{a}r^2}{2c^2}}{r\left(1 - \frac{\mathbf{a}\cdot\mathbf{r}}{2c^2}\right)}\right] = \hat{\mathbf{r}} - \frac{\left[\mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\right]r}{2c^2} \tag{9}$$

$$\boldsymbol{\beta} = \frac{\mathbf{a}\tau_0}{c} \approx -\frac{\mathbf{a}r}{c^2} \qquad \gamma^2 = \frac{1}{1 - \beta^2} \approx 1 \tag{10}$$

by which

$$\mathbf{n}-\boldsymbol{\beta} \approx \hat{\mathbf{r}}+\frac{[\mathbf{a}+(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\,r}{2c^2} \tag{11}$$

$$1-\boldsymbol{\beta}\cdot\mathbf{n} \approx 1+\frac{\mathbf{a}\cdot\mathbf{r}}{c^2} \tag{12}$$

The velocity field is thus

$$\mathbf{E}_v \approx e\left\{\frac{\hat{\mathbf{r}}+\dfrac{[\mathbf{a}+(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\,r}{2c^2}}{\left(1+\dfrac{\mathbf{a}\cdot\mathbf{r}}{c^2}\right)^3 r^2\left(1-\dfrac{\mathbf{a}\cdot\mathbf{r}}{2c^2}\right)^2}\right\} = e\left\{\frac{\hat{\mathbf{r}}}{r^2}+\frac{[\mathbf{a}+(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{2c^2 r}\right\}\left[1-\frac{2\,(\mathbf{a}\cdot\mathbf{r})}{c^2}\right] = \frac{e\hat{\mathbf{r}}}{r^2}+\frac{e\,[\mathbf{a}-3\,(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{2c^2 r} \tag{13}$$

For acceleration field, notice that

$$(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}} = (\mathbf{n}-\boldsymbol{\beta})\times\frac{\mathbf{a}}{c} \approx \frac{\hat{\mathbf{r}}\times\mathbf{a}}{c} \qquad\Longrightarrow\qquad \mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right] \approx \frac{\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{a})}{c} \tag{14}$$

To keep $\mathbf{E}_a$ at order of $1/c^2$, we can approximate its denominator to the 0-th order, giving

$$\mathbf{E}_a \approx \frac{e\,[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{a})]}{c^2 r} = -\frac{e\,[\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{c^2 r} \tag{15}$$

Adding (13) and (15) gives the total electric field

$$\mathbf{E} = \frac{e\hat{\mathbf{r}}}{r^2}-\frac{e\,[\mathbf{a}+(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{2c^2 r} \tag{16}$$

The approximation used here is valid when $ar \ll c^2$, i.e., the near-zone field.

2. To the order of $1/c^2$ inclusive, the instantaneous magnetic induction $\mathbf{B}$ is

$$\mathbf{B} = \mathbf{n}\times\mathbf{E} = \left\{\hat{\mathbf{r}}-\frac{[\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\,r}{2c^2}\right\}\times\left\{\frac{e\hat{\mathbf{r}}}{r^2}-\frac{e\,[\mathbf{a}+(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]}{2c^2 r}\right\} \approx 0 \tag{17}$$

This is expected since at $t=0$, the particle is instantaneously at rest, thus no magnetic field should be observed.

3. The total electric field $\mathbf{E}$ is the sum of regular Coulomb field of a charge e and the $1/c^2$ correction term. The divergence of the Coulomb field is zero everywhere except at the origin. So it remains to show that the divergence of the correction term vanishes. First, we see that

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{a}}{r}\right) = \mathbf{a}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{r^2} \tag{18}$$

Let $g=(\mathbf{a}\cdot\hat{\mathbf{r}})/r=(\mathbf{a}\cdot\mathbf{r})/r^2$, then

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(g\hat{\mathbf{r}}) &= g(\boldsymbol{\nabla}\cdot\hat{\mathbf{r}})+\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}g \\
&= g\cdot\frac{2}{r}+\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^2}\right) \\
&= \frac{2\,(\mathbf{a}\cdot\mathbf{r})}{r^3}+\hat{\mathbf{r}}\cdot\left[\frac{1}{r^2}\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{r})+(\mathbf{a}\cdot\mathbf{r})\,\boldsymbol{\nabla}\left(\frac{1}{r^2}\right)\right] \\
&= \frac{2\,(\mathbf{a}\cdot\mathbf{r})}{r^3}+\hat{\mathbf{r}}\cdot\left[\frac{\mathbf{a}}{r^2}-\frac{2\,(\mathbf{a}\cdot\mathbf{r})\hat{\mathbf{r}}}{r^3}\right] = \frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{r^2}
\end{aligned} \tag{19}$$

Combining (18) and (19), we see that the divergence of the correction term also vanishes, as desired.

The contribution of the Coulomb field to the curl is obviously zero. For the two parts of the correction term, we have

$$\boldsymbol{\nabla}\times\left(\frac{\mathbf{a}}{r}\right) = \boldsymbol{\nabla}\left(\frac{1}{r}\right)\times\mathbf{a} = \frac{\mathbf{a}\times\hat{\mathbf{r}}}{r^2} \tag{20}$$

$$\boldsymbol{\nabla}\times(g\hat{\mathbf{r}}) = \boldsymbol{\nabla}g\times\hat{\mathbf{r}} = \frac{\mathbf{a}\times\hat{\mathbf{r}}}{r^2} \qquad \text{use } \boldsymbol{\nabla}g \text{ from (19)} \tag{21}$$

Putting (20) and (21) into the total curl gives

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{e\,(\mathbf{a}\times\hat{\mathbf{r}})}{c^2 r^2} \tag{22}$$

By Faraday's law,

$$\frac{\partial\mathbf{B}}{\partial t} = -c\boldsymbol{\nabla}\times\mathbf{E} = \frac{e\,(\mathbf{a}\times\hat{\mathbf{r}})}{c r^2} \qquad\Longrightarrow\qquad \mathbf{B} = \frac{e\,(\mathbf{v}\times\hat{\mathbf{r}})}{c r^2} \tag{23}$$

which is the Biot-Savart law for a moving charge.