1. **Prob 14.13**

   The Fourier series expansion of $\mathbf{A}(t)$ is

   $$\mathbf{A}(t) = \sum_{m=-\infty}^{\infty} \mathbf{A}_m e^{-im\omega_0 t} \tag{1}$$

   $$\mathbf{A}_m = \frac{1}{T} \int_0^T \mathbf{A}(t) e^{im\omega_0 t} dt \tag{2}$$

   where $T = 2\pi/\omega_0$ is the period.

   Then

   $$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{T} \int_{-\infty}^{\infty} |\mathbf{A}(t)|^2 dt = \frac{1}{T} \int_{-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} \mathbf{A}_m e^{-im\omega_0 t} \right] \cdot \left[\sum_{n=-\infty}^{\infty} \mathbf{A}_n^* e^{in\omega_0 t} \right] dt$$

   $$= \frac{1}{T} \sum_{m,n=-\infty}^{\infty} \mathbf{A}_m \cdot \mathbf{A}_n^* \underbrace{\int_{-\infty}^{\infty} e^{-i(m-n)\omega_0 t} dt}_{2\pi\delta[(m-n)\omega_0]}$$

   $$= \sum_{m=\infty}^{\infty} |\mathbf{A}_m|^2 \tag{3}$$

   We thus identify $|\mathbf{A}_m|^2$ as the time-averaged power radiated per unit solid angle attributed to the m-th harmonic.

   Using (14.52) and the radiation term of (14.14) for the electric field, we have (following similar steps from 14.61 – 14.67)

   $$\mathbf{A}_m = \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} \int_0^T e^{im\omega_0 t} \left\{ \frac{\mathbf{n} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^3} \right\}_{\text{ret}} dt$$

   $$= \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} \int_0^T e^{im\omega_0 [t' + R(t')/c]} \left\{ \frac{\mathbf{n} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^2} \right\} dt' \qquad \text{up to an overall phase factor}$$

   $$= \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} \left\{ \frac{\mathbf{n} \times \left[(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^2} \right\} dt \qquad \text{integration by parts}$$

   $$= \frac{1}{T} \sqrt{\frac{e^2}{4\pi c}} (im\omega_0) \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} [\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta})] dt$$

   $$= \sqrt{\frac{e^2}{4\pi c}} \frac{im\omega_0^2}{2\pi c} \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} [\mathbf{n} \times (\mathbf{n} \times \mathbf{v})] dt \tag{4}$$

   Since $|\mathbf{n} \times (\mathbf{n} \times \mathbf{u})|^2 = |\mathbf{n} \times \mathbf{u}|^2$, we can write

   $$|\mathbf{A}_m|^2 = \frac{e^2 m^2 \omega_0^4}{16\pi^3 c^3} \left| \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} (\mathbf{n} \times \mathbf{v}) dt \right|^2 \tag{5}$$

   For $m \neq 0$, we can add contributions of $\pm m$ together and write

   $$\left\langle \frac{dP_m}{d\Omega} \right\rangle = 2 |\mathbf{A}_m|^2 = \frac{e^2 m^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^T e^{im\omega_0 [t - \mathbf{n} \cdot \mathbf{x}(t)/c]} \mathbf{n} \times \mathbf{v} dt \right|^2 \qquad \text{for } m \geq 0 \tag{6}$$

2. **Prob 14.14**

   (a) For harmonic motion $z(t) = a \cos(\omega_0 t)$, the amplitude of the vector integral in (6) becomes

   $$I_m = \int_0^T e^{im\omega_0 [t - a \cos\theta \cos(\omega_0 t)/c]} \omega_0 a \sin(\omega_0 t) \sin\theta \, dt$$

   $$= \beta c \sin\theta \int_0^T e^{im\omega_0 t} e^{-im\beta \cos\theta \cos(\omega_0 t)} \left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \right) dt \tag{7}$$

With the Jacobi-Anger expansion (see problem 3.16 c)

$$e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\phi} \tag{8}$$

we can write

$$e^{-im\beta\cos\theta\cos(\omega_0 t)} = \sum_{n=-\infty}^{\infty} i^n J_n(-m\beta\cos\theta) e^{in\omega_0 t} \qquad J_n(-x) = (-1)^n J_n(x)$$

$$= \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta\cos\theta) e^{in\omega_0 t} \tag{9}$$

turning (7) into

$$I_m = \beta c \sin\theta \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta\cos\theta) \frac{1}{2i} \overbrace{\int_0^T \left[e^{i(m+n+1)\omega_0 t} - e^{i(m+n-1)\omega_0 t} \right] dt}^{\frac{2\pi}{\omega_0}\left(\delta_{m+n+1,0} - \delta_{m+n-1,0}\right)}$$

$$= \frac{\pi\beta c}{\omega_0 i} \sin\theta \left[(-i)^{-(m+1)} J_{-(m+1)}(m\beta\cos\theta) - (-i)^{-(m-1)} J_{-(m-1)}(m\beta\cos\theta) \right] \qquad J_{-n}(x) = (-1)^n J_n(x)$$

$$= \frac{\pi\beta c}{\omega_0 i} \sin\theta \left[i^{m+1} J_{m+1}(m\beta\cos\theta) - i^{m-1} J_{m-1}(m\beta\cos\theta) \right]$$

$$= \frac{\pi\beta c}{\omega_0} \sin\theta\, i^m \left[J_{m-1}(m\beta\cos\theta) + J_{m+1}(m\beta\cos\theta) \right] \qquad J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$$

$$= \frac{2\pi c}{\omega_0} \tan\theta\, i^m J_m(m\beta\cos\theta) \tag{10}$$

Putting this back into (6) gives

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = \frac{e^2\beta^2 c}{2\pi a^2} m^2 \tan^2\theta\, J_m^2(m\beta\cos\theta) \tag{11}$$

(b) For non-relativistic motion $\beta \ll 1$, small argument approximation of Bessel functions gives

$$J_m(m\beta\cos\theta) \approx \frac{1}{m!} \left(\frac{m\beta\cos\theta}{2} \right)^m \tag{12}$$

The radiated power is thus dominated by the $m = 1$ harmonic, giving

$$P \approx 2\pi \int_0^\pi \frac{dP_1}{d\Omega} \sin\theta\, d\theta = 2\pi \cdot \frac{e^2\beta^2 c}{2\pi a^2} \int_0^\pi \tan^2\theta \left(\frac{\beta\cos\theta}{2} \right)^2 \sin\theta\, d\theta = \frac{e^2\omega_0^4 a^2}{3c^3} \tag{13}$$

Since

$$\overline{a^2} = \left\langle a^2\cos^2\omega_0 t \right\rangle = \frac{a^2}{2} \tag{14}$$

we can also write

$$P \approx \frac{2e^2\omega_0^4 \overline{a^2}}{3c^3} \tag{15}$$

3. **Prob 14.15**

   (a) The circular motion of the charge is described by

   $$\mathbf{x}(t) = R\left[\cos(\omega_0 t)\hat{\mathbf{x}} + \sin(\omega_0 t)\hat{\mathbf{y}}\right] \qquad \mathbf{v}(t) = \beta c\left[-\sin(\omega_0 t)\hat{\mathbf{x}} + \cos(\omega_0 t)\hat{\mathbf{y}}\right] \qquad \text{where } \beta = \frac{\omega_0 R}{c} \tag{16}$$

   Without loss of generality, let

   $$\mathbf{n} = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}} \tag{17}$$

The vector integral in (6) becomes

$$\mathbf{I}_m = -\beta c \int_0^T e^{im\omega_0 t} e^{-im\beta \sin\theta \cos(\omega_0 t)} \left[\cos\theta \sin(\omega_0 t)\hat{\mathbf{y}} + \cos\theta \cos(\omega_0 t)\hat{\mathbf{x}} - \sin\theta \cos(\omega_0 t)\hat{\mathbf{z}}\right] \qquad (18)$$

Similar to problem 14.14, we can evaluate the relevant integrals as

$$\int_0^T e^{im\omega_0 t} e^{-im\beta \sin\theta \cos(\omega_0 t)} \sin(\omega_0 t) = \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta \sin\theta) \frac{1}{2i} \overbrace{\int_0^T \left[e^{i(m+n+1)\omega_0 t} - e^{i(m+n-1)\omega_0 t}\right]dt}^{\frac{2\pi}{\omega_0}\left(\delta_{m+n+1,0} - \delta_{m+n-1,0}\right)}$$

$$= \frac{\pi}{\omega_0 i}\left[i^{m+1}J_{m+1}(m\beta \sin\theta) - i^{m-1}J_{m-1}(m\beta \sin\theta)\right]$$

$$= \frac{\pi}{\omega_0}i^m \frac{2}{\beta \sin\theta}J_m(m\beta \sin\theta) \qquad (19)$$

$$\int_0^T e^{im\omega_0 t} e^{-im\beta \sin\theta \cos(\omega_0 t)} \cos(\omega_0 t) = \sum_{n=-\infty}^{\infty} (-i)^n J_n(m\beta \sin\theta) \frac{1}{2} \overbrace{\int_0^T \left[e^{i(m+n+1)\omega_0 t} + e^{i(m+n-1)\omega_0 t}\right]dt}^{\frac{2\pi}{\omega_0}\left(\delta_{m+n+1,0} + \delta_{m+n-1,0}\right)}$$

$$= \frac{\pi}{\omega_0}\left[i^{m+1}J_{m+1}(m\beta \sin\theta) + i^{m-1}J_{m-1}(m\beta \sin\theta)\right]$$

$$= \frac{\pi}{\omega_0}i^{m-1} \cdot 2J'_m(m\beta \sin\theta) \qquad (20)$$

Putting (19) and (20) into (18) and (6) gives

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle = \frac{e^2 \omega_0^4 R^2}{2\pi c^3}m^2 \left[J'^2_m(m\beta \sin\theta) + \frac{\cot^2\theta J^2_m(m\beta \sin\theta)}{\beta^2}\right] \qquad (21)$$

(b) For non-relativistic motion, (21) is again dominated by the $m=1$ harmonic, in which case

$$J_1(\beta \sin\theta) \approx \frac{\beta \sin\theta}{2} \qquad\qquad J'_1(\beta \sin\theta) \approx \frac{1}{2} \qquad (22)$$

hence

$$\left\langle \frac{dP_m}{d\Omega} \right\rangle \approx \frac{e^2 \omega_0^4 R^2}{2\pi c^3}\left(\frac{1}{4} + \frac{\cos^2\theta}{4}\right) \implies P \approx \frac{e^2 \omega_0^4 R^2}{2\pi c^3}\cdot\left(\pi + \frac{2\pi}{4}\int_0^\pi \cos^2\theta \sin\theta \, d\theta\right) = \frac{2e^2 \omega_0^4 R^2}{3c^3} \qquad (23)$$

agreeing with 14.4 (b) since for circular motion $\overline{a^2} = R^2$.

(c) We wish to make the connection from (21) to Jackson (14.79)

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2} + \theta_J^2\right)^2\left[K^2_{2/3}(\xi) + \left(\frac{\theta_J^2}{\frac{1}{\gamma^2} + \theta_J^2}\right)K^2_{1/3}(\xi)\right] \qquad\qquad \text{where}$$

$$\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2} + \theta_J^2\right)^{3/2} \qquad (24)$$

Here we have changed the θ symbol in (14.79) to θ_J since it represents the angle between $\mathbf{n}$ and the x-axis, and our θ symbol represents the angle between $\mathbf{n}$ and the z-axis. In other words, $\theta_J = \pi/2 - \theta$.

From the hint of the problem, we reference the following asymptotic form, see *Watson* pp. 249–(1).

$$J_n(x) \to \frac{1}{\pi}\left[\frac{2(n-x)}{3x}\right]^{1/2}K_{1/3}\left[\frac{2^{3/2}(n-x)^{3/2}}{3x^{1/2}}\right] \qquad\qquad \text{for } x < n \qquad (25)$$

In extreme relativistic motion, where $\omega_0 R/c = \beta \approx 1$ and $\gamma \gg 1$, only small θ_J is appreciable, so the argument of J_m in (21) is

$$x = m\beta \sin\theta = m\beta \cos\theta_J \approx m - \frac{m\theta_J^2}{2} \qquad (26)$$

Byso by (25) we have

$$J_m\left(m\beta\sin\theta\right) \to \frac{1}{\pi}\left[\frac{2\cdot\dfrac{m\theta_J^2}{2}}{3\left(m-\dfrac{m\theta_J^2}{2}\right)}\right]^{1/2} K_{1/3}\left[\frac{2^{3/2}\left(\dfrac{m\theta_J^2}{2}\right)^{3/2}}{3\left(m-\dfrac{m\theta_J^2}{2}\right)^{1/2}}\right] \approx \frac{1}{\pi}\sqrt{\frac{\theta_J^2}{3}}K_{1/3}\left(\frac{m\theta_J^3}{3}\right) \qquad (27)$$

With this, the second term in (21) becomes

$$\left\langle\frac{dP_m}{d\Omega}\right\rangle_{II} \to \frac{e^2m^2c}{2\pi^3R^2}\frac{\theta_J^4}{3}K_{1/3}^2\left(\frac{m\theta_J^3}{3}\right) \qquad (28)$$

while the second term in (24), under the limit $\omega\rho/c \to m$, $\gamma \to \infty$, becomes

$$\frac{d^2I}{d\omega d\Omega}\bigg|_{II} \to \frac{e^2m^2}{3\pi^2c}\theta_J^4K_{1/3}^2\left(\xi\right) \qquad\qquad \xi \approx \frac{m\theta_J^3}{3} \qquad (29)$$

The equivalence of (28) and (29) is then clear since they both imply the same total energy radiated per unit solid angle in the frequency interval $[m\omega_0, (m+1)\omega_0]$:

$$\frac{dW}{d\Omega} = \left\langle\frac{dP_m}{d\Omega}\right\rangle_{II}\cdot T = \frac{e^2m^2\omega_0}{3\pi^2c}\theta_J^4K_{1/3}^2\left(\frac{m\theta_J^3}{3}\right)$$

$$\frac{dW}{d\Omega} = \frac{d^2I}{d\omega d\Omega}\bigg|_{II}\cdot\omega_0 = \frac{e^2m^2\omega_0}{3\pi^2c}\theta_J^4K_{1/3}^2\left(\frac{m\theta_J^3}{3}\right) \qquad (30)$$

To connect the J_m' term, assume the prefactor of (25) varies slowly compared to $K_{1/3}$, allowing us to write

$$J_n'\left(x\right) \approx \frac{1}{\pi}\left[\frac{2\left(n-x\right)}{3x}\right]^{1/2}\frac{d\eta}{dx}K_{1/3}'\left(\eta\right) \qquad\qquad \text{where } \eta = \frac{2^{3/2}\left(n-x\right)^{3/2}}{3x^{1/2}} \qquad (31)$$

Assigning $n = m$, $x \approx m - m\theta_J^2/2$, we have

$$\eta \approx \frac{m\theta_J^3}{3} \qquad\qquad \Longrightarrow \qquad\qquad \frac{d\eta}{dx} = \frac{d\eta}{d\theta_J}\cdot\frac{d\theta_J}{dx} = -\theta_J \qquad (32)$$

thus

$$J_m'\left(m\beta\sin\theta\right) \approx -\frac{1}{\pi}\sqrt{\frac{\theta_J^2}{3}}\theta_J K_{1/3}'\left(\eta\right) \qquad (33)$$

On the other hand, by the recurrence relation of modified Bessel functions,

$$K_\nu'\left(z\right) = -\frac{\nu}{z}K_\nu\left(z\right) - K_{\nu-1}\left(z\right) \quad\Longrightarrow\quad K_{1/3}'\left(z\right) = -\frac{1}{3z}K_{1/3}\left(z\right) - K_{2/3}\left(z\right) \approx -K_{2/3}\left(z\right) \qquad \text{for large } z \qquad (34)$$

we can finally write

$$J_m'\left(m\beta\sin\theta\right) \approx \frac{1}{\pi}\frac{\theta_J^2}{\sqrt{3}}K_{2/3}\left(\frac{m\theta_J^3}{3}\right) \qquad (35)$$

With this, the equivalence of the J_m' term of (21) and the $K_{2/3}$ term of (24) can be shown similarly as before.