1. **Prob 14.5**

   (a) We can use (14.28) for the total radiated power for a linearly accelerated charge

$$P = \frac{2}{3}\frac{z^2e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2 = \frac{2}{3}\frac{z^2e^2}{m^2c^3}\left(\frac{dV}{dr}\right)^2 \tag{1}$$

   Thus the total radiated energy is the integral

$$\Delta W = \int_{-\infty}^{\infty} P\,dt = 2\int_{r_{\min}}^{\infty} P\frac{dr}{v} \tag{2}$$

   The speed can be found from energy conservation

$$\frac{1}{2}mv^2 + V(r) = \frac{1}{2}mv_0^2 = V(r_{\min}) \qquad \Longrightarrow \qquad v = \sqrt{\frac{2}{m}\left[V(r_{\min}) - V(r)\right]} \tag{3}$$

   Plugging this into (2) gives

$$\Delta W = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)^2\frac{dr}{\sqrt{V(r_{\min}) - V(r)}} \tag{4}$$

   (b) For the Coulomb potential $V(r) = zZe^2/r$, we have

$$\frac{dV}{dr} = -\frac{zZe^2}{r^2} \tag{5}$$

   turning (4) into

$$\begin{aligned}
\Delta W &= \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\frac{\left(zZe^2\right)^2}{r^4}\frac{dr}{\sqrt{\frac{1}{2}mv_0^2 - \frac{zZe^2}{r}}} && \text{let } w \equiv \frac{1}{r}\\
&= \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(zZe^2\right)^2\int_0^{1/r_{\min}}\frac{w^2dw}{\sqrt{\frac{1}{2}mv_0^2 - zZe^2w}} && \text{let } u \equiv \sqrt{\frac{1}{2}mv_0^2 - zZe^2w}\\
&= \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\left(zZe^2\right)^2\frac{2}{\left(zZe^2\right)^3}\int_0^{\sqrt{mv_0^2/2}}\left(u^2 - \frac{1}{2}mv_0^2\right)^2du\\
&= \frac{8}{3}\frac{z}{Zm^2c^3}\sqrt{\frac{m}{2}}\int_0^{\sqrt{mv_0^2/2}}\left(u^4 - mv_0^2u^2 + \frac{1}{4}m^2v_0^4\right)du\\
&= \frac{8}{3}\frac{z}{Zm^2c^3}\sqrt{\frac{m}{2}}\left(\frac{u^5}{5} - \frac{mv_0^2u^3}{3} + \frac{1}{4}m^2v_0^4u\right)\Bigg|_0^{\sqrt{mv_0^2/2}}\\
&= \frac{8}{45}\frac{zmv_0^5}{Zc^3}
\end{aligned} \tag{6}$$

2. **Prob 14.6**

   (a) Similar to Prob 14.5, we need to find the expression for the differential time dt. Angular momentum conservation requires

$$L = mr^2\dot{\phi} \tag{7}$$

   Plugging this into the energy conservation equation, we have

$$\begin{aligned}
&E = V(r) + \frac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\phi}\right)^2\right] = V(r) + \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} \qquad \Longrightarrow\\
&\frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - V(r) - \frac{L^2}{2mr^2}\right]}
\end{aligned} \tag{8}$$

giving

$$\Delta W = 2\int_{r_{\min}}^{\infty} P\,dt = \int_{r_{\min}}^{\infty} \frac{P}{dr/dt}dr = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\overbrace{\int_{r_{\min}}^{\infty}\left(\frac{dV}{dr}\right)^2\frac{dr}{\sqrt{E-V(r)-\dfrac{L^2}{2mr^2}}}}^{I} \tag{9}$$

(b) For the Coulomb potential, define the length scale

$$a = \frac{zZe^2}{mv_0^2} \tag{10}$$

then we have

$$\begin{aligned} E-V(r)-\frac{L^2}{2mr^2} &= \frac{mv_0^2}{2} - \frac{zZe^2}{r} - \frac{(mbv_0)^2}{2mr^2} = \frac{mv_0^2}{2}\left(1-\frac{2a}{r}-\frac{b^2}{r^2}\right) \qquad \text{let } u \equiv \frac{1}{r} \\ &= \frac{mv_0^2}{2}\left(1-2au-b^2u^2\right) \end{aligned} \tag{11}$$

The integral I in (9) is

$$\begin{aligned} I &= \int_0^{u_{\max}} \left(zZe^2u^2\right)^2 \frac{du/u^2}{\sqrt{\frac{1}{2}mv_0^2\left(1-2au-b^2u^2\right)}} \\ &= \frac{\left(zZe^2\right)^2}{\sqrt{mv_0^2/2}}\int_0^{u_{\max}} \frac{u^2du}{\sqrt{1-2au-b^2u^2}} \\ &= \frac{\left(zZe^2\right)^2}{\sqrt{mv_0^2/2}}\int_0^{u_{\max}} \frac{u^2du}{\sqrt{\left(\dfrac{a^2+b^2}{b^2}\right)-b^2\left(u+\dfrac{a}{b^2}\right)^2}} \qquad \text{let } h^2 = a^2+b^2 \\ &= \frac{\left(zZe^2\right)^2}{\sqrt{mv_0^2/2}}\int_0^{u_{\max}} \frac{u^2du}{\sqrt{\dfrac{h^2}{b^2}\left[1-\dfrac{b^4}{h^2}\left(u+\dfrac{a}{b^2}\right)^2\right]}} \qquad \text{let } u+\frac{a}{b^2} = \frac{h}{b^2}\sin\theta \\ &= \frac{\left(zZe^2\right)^2}{\sqrt{mv_0^2/2}}\int_{\theta_0}^{\pi/2} \frac{\left(\dfrac{h}{b^2}\sin\theta-\dfrac{a}{b^2}\right)^2\dfrac{h}{b^2}\cos\theta\,d\theta}{\dfrac{h}{b}\cos\theta} \\ &= \frac{\left(zZe^2\right)^2}{\sqrt{mv_0^2/2}}\frac{1}{b^5}\underbrace{\int_{\theta_0}^{\pi/2}(h\sin\theta-a)^2\,d\theta}_{J} \end{aligned} \tag{12}$$

Denote $t = b/a$, then we see that $\theta_0 = \sin^{-1}(a/h) = \pi/2 - \tan^{-1} t$. Thus the J integral in (12) can be calculated as

$$\begin{aligned} J &= \int_{\theta_0}^{\pi/2}\left[h^2\left(\frac{1-\cos 2\theta}{2}\right) - 2ah\sin\theta + a^2\right]d\theta \\ &= \frac{h^2}{2}\left(\frac{\pi}{2}-\theta_0+\sin\theta_0\cos\theta_0\right) - 2ah\cos\theta_0 + a^2\left(\frac{\pi}{2}-\theta_0\right) \\ &= \left(\frac{3a^2+b^2}{2}\right)\left(\frac{\pi}{2}-\theta_0\right) - \frac{3ab}{2} \\ &= \frac{3b^2}{2}\left[-\frac{1}{t}+\left(\frac{1}{t^2}+\frac{1}{3}\right)\tan^{-1} t\right] \end{aligned} \tag{13}$$

Plugging (13) into (12) and (9), we have

$$\Delta W = \frac{2zmv_0^5}{Zc^3}\left[-\frac{1}{t^4}+\frac{1}{t^5}\left(1+\frac{t^2}{3}\right)\tan^{-1} t\right] \tag{14}$$

When $t \to 0$,

$$\left(1+\frac{t^2}{3}\right)\tan^{-1}t = \left(1+\frac{t^2}{3}\right)\left[t-\frac{t^3}{3}+\frac{t^5}{5}+O\left(t^7\right)\right]$$
$$= t+\frac{4}{45}t^5+O\left(t^7\right) \tag{15}$$

thus

$$-\frac{1}{t^4}+\frac{1}{t^5}\left(1+\frac{t^2}{3}\right)\tan^{-1}t = \frac{4}{45}+O\left(t^2\right) \qquad \Longrightarrow \qquad \Delta W \to \frac{8}{45}\frac{zmv_0^5}{Zc^3} \tag{16}$$

agreeing with problem 14.5b.

On the other hand when $t \to \infty$, the dominating contribution of ΔW is

$$\Delta W \to \frac{2zmv_0^5}{Zc^3}\frac{1}{3t^3}\frac{\pi}{2} = \frac{\pi z^4Z^2e^6}{3m^2c^3v_0}\frac{1}{b^3} \tag{17}$$

agreeing with problem 14.7a.

(c) When we write $t = \cot(\theta/2)$, (14) is easily seen to have the form

$$\Delta W = \frac{2zmv_0^5}{Zc^3}\tan^3\left(\frac{\theta}{2}\right)\left\{\left(\frac{\pi-\theta}{6}\right)\left[1+3\tan^2\left(\frac{\theta}{2}\right)\right]-\tan\left(\frac{\theta}{2}\right)\right\} \tag{18}$$

(d) With an attractive Coulomb potential, whether the incident charge can escape depends on its initial kinetic energy. If the total energy is positive, the whole scattering process is similar to the repulsive case (except $zZ < 0$). If the total energy is negative, the incident charge will be captured and go into a bound orbit around the central charge, never escaping to infinity. In this case, the charge will keep radiating energy until it spirals into the central charge.