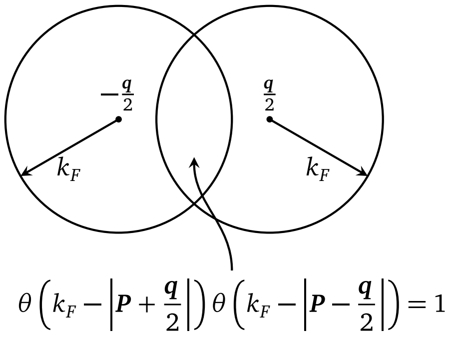
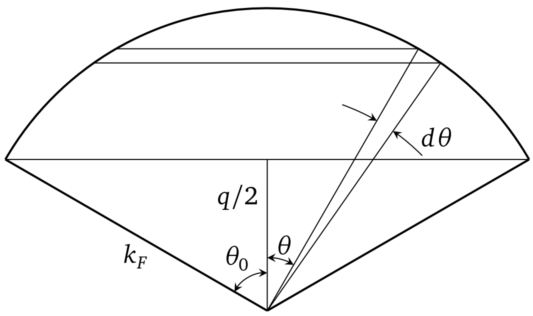
Here we fill the details that lead from equation (7.157) to (7.158).
The inner integral of eq (7.157)

$$I = \int d^3P\,\theta\left(k_F - \left|\boldsymbol{P} + \frac{\boldsymbol{q}}{2}\right|\right)\theta\left(k_F - \left|\boldsymbol{P} - \frac{\boldsymbol{q}}{2}\right|\right) \tag{1}$$

is over the overlapping volume of the two spheres located at $\pm\boldsymbol{q}/2$, with radius k_F.



The integral I is then twice the integration of the volumes of all the infinitesimally thin disks in the range $\theta \in [0, \theta_0]$, where $\cos\theta_0 = (q/2)/k_F = q/2k_F$.



That is,

$$\begin{aligned}
\frac{I}{2} &= \int_0^{\theta_0} \pi\,(k_F\sin\theta)^2\,(k_F d\theta\sin\theta) \\
&= \pi k_F^3 \int_0^{\theta_0} \sin^3\theta\, d\theta \qquad\qquad (\text{define } y = -\cos\theta) \\
&= \pi k_F^3 \int_{q/2k_F}^{1} dy(1-y^2) \\
&= \pi k_F^3\left(\frac{2}{3} - \frac{q}{2k_F} + \frac{q^3}{24k_F^3}\right)
\end{aligned} \tag{2}$$

Then the outer integral of eq (7.157) becomes

$$\begin{aligned}
2\int_0^{2k_F} d^3q\,\frac{1}{q^2}\pi k_F^3\left(\frac{2}{3} - \frac{q}{2k_F} + \frac{q^3}{24k_F^3}\right) &= 2\left\{\frac{2\pi k_F^3}{3}(4\pi\cdot 2k_F) - \frac{\pi k_F^2}{2}\left[4\pi\frac{(2k_F)^2}{2}\right] + \frac{\pi}{24}\left[4\pi\frac{(2k_F)^4}{4}\right]\right\} \\
&= 4\pi^2 k_F^4
\end{aligned} \tag{3}$$

which now gives the first-order energy shift

$$
\begin{aligned}
E^{(1)} &= -e^2 \frac{4\pi V}{(2\pi)^6} 4\pi^2 k_F^4 && \text{(by eq (7.154))} \\
&= -e^2 \frac{4\pi V}{(2\pi)^6} 4\pi^2 \left(\frac{9\pi}{4}\right)\left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_0^4} && \text{(by eq (7.152), } \frac{V}{a_0^3} = \frac{4\pi N}{3}\text{)} \\
&= -e^2 \frac{4\pi \cdot 4\pi^2 \cdot 9\pi}{64\pi^6 \cdot 4} \left(\frac{9\pi}{4}\right)^{1/3} \frac{4\pi N}{3} \frac{1}{r_0} \\
&= -\frac{e^2}{2a_0} N \frac{3}{2\pi} \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s}
\end{aligned}
\tag{4}
$$

There seems to be a typo in eq (7.159), the right formula should be

$$
\begin{aligned}
\frac{E}{N} &= \frac{e^2}{2a_0} \left[\frac{3}{5} \left(\frac{9\pi}{4}\right)^{2/3} \frac{1}{r_s^2} - \frac{3}{2\pi} \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s} \right] \\
&\approx \frac{e^2}{2a_0} \left(\frac{2.21}{r_s^2} - \frac{0.916}{r_s} \right)
\end{aligned}
\tag{5}
$$

which agrees with Fetter and Walecka (2003).

The numerical values in the paragraph after equation (7.159) seem correct though.