In these notes, we give the full treatment of Runge-Lenz operator

$$\boldsymbol{M} = \frac{1}{2m}(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}) - \frac{Ze^2}{r}\boldsymbol{x} \tag{1}$$

in a central Coulomb potential $V(r) = -Ze^2/r$, i.e., with Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{Ze^2}{r} \tag{2}$$

We shall show the following relationships

$$[M_i, L_j] = i\hbar\epsilon_{ijk}M_k \tag{3}$$
$$[\boldsymbol{M}, H] = 0 \tag{4}$$
$$\boldsymbol{L} \cdot \boldsymbol{M} = 0 \tag{5}$$
$$\boldsymbol{M} \cdot \boldsymbol{L} = 0 \tag{6}$$
$$\boldsymbol{M}^2 = \frac{2}{m}H\left(\boldsymbol{L}^2 + \hbar^2\right) + Z^2e^4 \tag{7}$$
$$[M_i, M_j] = -i\hbar\epsilon_{ijk}\frac{2}{m}HL_k \tag{8}$$

First we recall a few results that are used throughout the proofs.

1. From excercise 1.31,

$$[p_k, F(\boldsymbol{x})] = -i\hbar\frac{\partial F}{\partial x_k} \tag{9}$$

2. Both $\boldsymbol{p}$ and $\boldsymbol{x}$ are vector operators:

$$[p_i, L_j] = \epsilon_{lkj}[p_i, x_l p_k] = \epsilon_{lkj}[p_i, x_l]p_k = -i\hbar\epsilon_{lkj}\delta_{il}p_k = i\hbar\epsilon_{ijk}p_k \tag{10}$$
$$[x_i, L_j] = \epsilon_{lkj}[x_i, x_l p_k] = \epsilon_{lkj}x_l[x_i, p_k] = i\hbar\epsilon_{lkj}x_l\delta_{ik} = i\hbar\epsilon_{ijl}x_l \tag{11}$$

3. For two vector operators $\boldsymbol{u}, \boldsymbol{v}$, $\boldsymbol{u} \cdot \boldsymbol{v}$ is a scalar operator, i.e., it commutes with L_j. Since

$$\begin{aligned}\left[\boldsymbol{u} \cdot \boldsymbol{v}, L_j\right] &= \left[u_i v_i, L_j\right] = \left[u_i, L_j\right]v_i + u_i\left[v_i, L_j\right] \\ &= i\hbar\epsilon_{ijk}u_k v_i + i\hbar\epsilon_{ijk}u_i v_k = u_k\left[L_j, v_k\right] + \left[L_j, u_k\right]v_k = \left[L_j, u_k v_k\right] = \left[L_j, \boldsymbol{u} \cdot \boldsymbol{v}\right]\end{aligned} \tag{12}$$

4. Any spherically symmetric function of position $F(\boldsymbol{x}) = f(r)$ is a scalar operator, i.e., it commutes with the generator of rotation, i.e.,

$$[L_k, f(r)] = 0 \tag{13}$$

   Indeed, with the antisymmetry of ϵ_{ijk} symbol,

$$[L_k, f(r)] = \epsilon_{ijk}[x_i p_j, f(r)] = \epsilon_{ijk}x_i[p_j, f(r)] = -i\hbar\epsilon_{ijk}x_i\frac{\partial f(r)}{\partial x_j} = -i\hbar\epsilon_{ijk}x_i f'(r)\frac{x_j}{r} = 0$$

5. For a scalar operator s and a vector operator $\boldsymbol{v}$, both $s\boldsymbol{v}$ and $\boldsymbol{v}s$ are vector operators:

$$\left[sv_i, L_j\right] = s\left[v_i, L_j\right] = i\hbar\epsilon_{ijk}sv_k \tag{14}$$
$$\left[v_i s, L_j\right] = \left[v_i, L_j\right]s = i\hbar\epsilon_{ijk}v_k s \tag{15}$$

6.

$$\boldsymbol{L} \cdot \boldsymbol{p} = L_k p_k = \epsilon_{ijk}x_i p_j p_k = 0 \tag{16}$$
$$\boldsymbol{p} \cdot \boldsymbol{L} = p_k L_k = \epsilon_{ijk}p_k x_i p_j = \epsilon_{ijk}p_k\left(p_j x_i + \delta_{ij}i\hbar\right) = \epsilon_{ijk}p_k p_j x_i + \epsilon_{ijk}\delta_{ij}i\hbar = 0 \tag{17}$$

7. For any given $i,j,l,m,k \in \{1,2,3\}$,

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{18}$$

Note here we temporarily suspend the repeated-index summation convention, i.e., the LHS is to be treated as a single term.

The proof is straightforward by fixing, say $k = 3$, and enumerating possible combinations of i, j, l and m.

8. For vector operators $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{u} \times \boldsymbol{v}$ is also a vector operator.

Indeed

$$\begin{aligned}
\left[(\boldsymbol{u}\times\boldsymbol{v})_i, L_j\right] &= \epsilon_{lmi}\left[u_l v_m, L_j\right] = \epsilon_{lmi}\left(\left[u_l, L_j\right]v_m + u_l\left[v_m, L_j\right]\right) \\
&= \epsilon_{lmi}\left(i\hbar\epsilon_{ljk}u_k v_m + i\hbar\epsilon_{mjk}u_l v_k\right) \\
&= i\hbar\left(\epsilon_{lmi}\epsilon_{ljk}u_k v_m + \epsilon_{lmi}\epsilon_{mjk}u_l v_k\right) \\
&= i\hbar\left[\left(\delta_{jm}\delta_{ik} - \delta_{mk}\delta_{ij}\right)u_k v_m - \left(\delta_{ik}\delta_{jl} - \delta_{lk}\delta_{ij}\right)u_l v_k\right] \\
&= i\hbar\left(u_i v_j - \delta_{ij}u_k v_k - u_j v_i + \delta_{ij}u_k v_k\right) \\
&= i\hbar\left(u_i v_j - u_j v_i\right) \\
&= i\hbar\epsilon_{ijk}\left(\boldsymbol{u}\times\boldsymbol{v}\right)_k
\end{aligned} \tag{19}$$

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1. **Proof of (3)**.

This is obvious since due to (19), both $\boldsymbol{p} \times \boldsymbol{L}$ and $\boldsymbol{L} \times \boldsymbol{p}$ are vector operators. Furthermore, $\boldsymbol{x}/r$ is the product of a vector operator and a scalar (i.e., setting $f(r) = 1/r$ in (13)), so $\boldsymbol{M}$ is indeed a vector operator.

2. **Proof of (4)**.

By definition of $\boldsymbol{M}$, we have

$$[M_k, H] = \overbrace{\frac{1}{2m}\epsilon_{ijk}\left[p_i L_j - L_i p_j, H\right]}^{A} - \overbrace{\left[Ze^2\frac{x_k}{r}, H\right]}^{B}$$

where

$$\begin{aligned}
B = Ze^2\left[\frac{x_k}{r}, \frac{p_i p_i}{2m}\right] &= \frac{Ze^2}{2m}\left(p_i\left[\frac{x_k}{r}, p_i\right] + \left[\frac{x_k}{r}, p_i\right]p_i\right) \\
&= \frac{i\hbar Ze^2}{2m}\left[p_i\left(\frac{\delta_{ik}}{r} - \frac{x_i x_k}{r^3}\right) + \left(\frac{\delta_{ik}}{r} - \frac{x_i x_k}{r^3}\right)p_i\right] \\
&= \frac{i\hbar Ze^2}{2m}\underbrace{\left[\left(p_k\frac{1}{r} - p_i\frac{x_i x_k}{r^3}\right) + \left(\frac{1}{r}p_k - \frac{x_i x_k}{r^3}p_i\right)\right]}_{B'}
\end{aligned} \tag{20}$$

On the other hand, recall that both $\boldsymbol{p}^2$ and $1/r$ in H are scalars, so $[L_i, H] = 0$ for all i, which gives

$$\begin{aligned}
A &= \frac{1}{2m}\epsilon_{ijk}\left([p_i, H]L_j - L_i[p_j, H]\right) \\
&= -\frac{Ze^2}{2m}\epsilon_{ijk}\left(\left[p_i, \frac{1}{r}\right]L_j - L_i\left[p_j, \frac{1}{r}\right]\right) \\
&= -\frac{i\hbar Ze^2}{2m}\epsilon_{ijk}\left(\frac{x_i}{r^3}L_j - L_i\frac{x_j}{r^3}\right) \\
&= \frac{i\hbar Ze^2}{2m}\underbrace{\epsilon_{ijk}\left(L_i\frac{x_j}{r^3} - \frac{x_i}{r^3}L_j\right)}_{A'}
\end{aligned} \tag{21}$$

Now compare (20) with (21), indeed they are equal because

$$\begin{aligned}
A' &= \epsilon_{ijk}\left(L_i\frac{x_j}{r^3}-\frac{x_i}{r^3}L_j\right)\\
&= \epsilon_{ijk}\left(\epsilon_{lmi}x_lp_m\frac{x_j}{r^3}-\epsilon_{lmj}\frac{x_i}{r^3}x_lp_m\right)\\
&= \left(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}\right)x_lp_m\frac{x_j}{r^3}+\left(\delta_{il}\delta_{km}-\delta_{im}\delta_{kl}\right)\frac{x_i}{r^3}x_lp_m\\
&= x_jp_k\frac{x_j}{r^3}-x_kp_j\frac{x_j}{r^3}+\frac{x_ix_i}{r^3}p_k-\frac{x_ix_k}{r^3}p_i\\
&= \left(p_kx_j+\left[x_j,p_k\right]\right)\frac{x_j}{r^3}-\left(p_jx_k+\left[x_k,p_j\right]\right)\frac{x_j}{r^3}+\frac{x_ix_i}{r^3}p_k-\frac{x_ix_k}{r^3}p_i\\
&= p_k\frac{x_jx_j}{r^3}+i\hbar\delta_{jk}\frac{x_j}{r^3}-p_j\frac{x_jx_k}{r^3}-i\hbar\delta_{jk}\frac{x_j}{r^3}+\frac{x_ix_i}{r^3}p_k-\frac{x_ix_k}{r^3}p_i\\
&= B'
\end{aligned}$$

3. **Proof of (5).**

$$\boldsymbol{L}\cdot\boldsymbol{M}=L_kM_k=\frac{1}{2m}\overbrace{\epsilon_{ijk}\left[L_k\left(p_iL_j-L_ip_j\right)\right]}^{A}-Ze^2\overbrace{L_k\frac{x_k}{r}}^{B} \tag{22}$$

where

$$\begin{aligned}
A &= \epsilon_{ijk}\left(L_kp_iL_j-L_kL_ip_j\right)=\epsilon_{ijk}L_kp_iL_j-\epsilon_{ijk}L_kL_ip_j\\
&= \epsilon_{ijk}\left(p_iL_k-\left[p_i,L_k\right]\right)L_j-\epsilon_{ijk}L_kL_ip_j\\
&= \epsilon_{ijk}p_iL_kL_j-\epsilon_{ijk}\left(i\hbar\epsilon_{ikj}p_j\right)L_j-\epsilon_{ijk}L_kL_ip_j\\
\text{(use } \epsilon_{ijk}L_iL_j=(\boldsymbol{L}\times\boldsymbol{L})_k=i\hbar L_k \text{ etc.)}\quad &= -i\hbar p_iL_i+i\hbar\epsilon_{ijk}^2p_jL_j-i\hbar L_jp_j\\
\text{(use (16), (17))}\quad &= 0\\
B &= L_k\frac{x_k}{r}=\epsilon_{ijk}x_ip_j\frac{x_k}{r}=\epsilon_{ijk}x_i\left(\frac{x_k}{r}p_j+\left[p_j,\frac{x_k}{r}\right]\right)\\
&= \epsilon_{ijk}x_i\frac{x_k}{r}p_j-i\hbar\epsilon_{ijk}x_i\left(\frac{\delta_{jk}}{r}-\frac{x_jx_k}{r^3}\right)\\
&= 0
\end{aligned}$$

4. **Proof of (6).**

This trivially follows from (3) (by setting $i=j$) and (5).

5. **Proof of (7).**

Expanding $\boldsymbol{M}^2$ on the LHS and H on the RHS, we get

$$\begin{aligned}
\text{LHS} &= M_kM_k=\left[\frac{1}{2m}\epsilon_{ijk}\left(p_iL_j-L_ip_j\right)-Ze^2\frac{x_k}{r}\right]\cdot\left[\frac{1}{2m}\epsilon_{lmk}\left(p_lL_m-L_lp_m\right)-Ze^2\frac{x_k}{r}\right]\\
&= \overbrace{\frac{1}{4m^2}\epsilon_{ijk}\epsilon_{lmk}\left(p_iL_j-L_ip_j\right)\left(p_lL_m-L_lp_m\right)}^{A}-\overbrace{\frac{Ze^2}{2m}\epsilon_{ijk}\left[\left(p_iL_j-L_ip_j\right)\frac{x_k}{r}+\frac{x_k}{r}\left(p_iL_j-L_ip_j\right)\right]}^{B}+\overbrace{Z^2e^4\frac{x_kx_k}{r^2}}^{C}\\
\text{RHS} &= \frac{2}{m}\left(\frac{\boldsymbol{p}^2}{2m}-Ze^2\frac{1}{r}\right)\left(\boldsymbol{L}^2+\hbar^2\right)+Z^2e^4\\
&= \overbrace{\frac{\boldsymbol{p}^2}{m^2}\left(\boldsymbol{L}^2+\hbar^2\right)}^{A'}-\overbrace{\frac{2Ze^2}{m}\frac{1}{r}\left(\boldsymbol{L}^2+\hbar^2\right)}^{B'}+\overbrace{Z^2e^4}^{C'}
\end{aligned}$$

For dimensions to match, we must have $A=A', B=B', C=C'$, where $C=C'$ is already trivial to see.

To prove $B=B'$, it's equivalent to prove

$$\epsilon_{ijk}\left[\left(p_iL_j-L_ip_j\right)\frac{x_k}{r}+\frac{x_k}{r}\left(p_iL_j-L_ip_j\right)\right]=\frac{4}{r}\left(\boldsymbol{L}^2+\hbar^2\right) \tag{23}$$

Let's break the LHS of (23) into

$$\overbrace{\epsilon_{ijk}p_iL_j\frac{x_k}{r}}^{B_1}-\overbrace{\epsilon_{ijk}L_ip_j\frac{x_k}{r}}^{B_2}+\overbrace{\epsilon_{ijk}\frac{x_k}{r}p_iL_j}^{B_3}-\overbrace{\epsilon_{ijk}\frac{x_k}{r}L_ip_j}^{B_4}$$

where we immediately see that

$$B_3=\frac{1}{r}L_jL_j=\frac{1}{r}\boldsymbol{L}^2$$

Furthermore

$$B_2=\epsilon_{ijk}L_i\left(\frac{x_k}{r}p_j+\left[p_j,\frac{x_k}{r}\right]\right)=\epsilon_{ijk}L_i\left[\frac{x_k}{r}p_j-i\hbar\left(\frac{\delta_{jk}}{r}-\frac{x_jx_k}{r^3}\right)\right]=L_i\left(-\frac{1}{r}L_i\right)=-\frac{1}{r}\boldsymbol{L}^2$$

where in the last step, we have used (13).

Lastly

$$\begin{aligned}
B_1&=\epsilon_{ijk}p_i\left(\frac{x_k}{r}L_j-\left[\frac{x_k}{r},L_j\right]\right)\\
&=\epsilon_{ijk}\left(\frac{x_k}{r}p_i+\left[p_i,\frac{x_k}{r}\right]\right)L_j-\epsilon_{ijk}p_i\left(i\hbar\epsilon_{kji}\frac{x_i}{r}\right)\\
&=\frac{1}{r}L_jL_j-i\hbar\epsilon_{ijk}\left(\frac{\delta_{ik}}{r}-\frac{x_ix_k}{r^3}\right)L_j+i\hbar\epsilon_{ijk}^2p_i\frac{x_i}{r}\\
&=\frac{1}{r}\boldsymbol{L}^2+i\hbar\epsilon_{ijk}^2p_i\frac{x_i}{r}\\
B_4&=\epsilon_{ijk}\left(L_i\frac{x_k}{r}+\left[\frac{x_k}{r},L_i\right]\right)p_j\\
&=-L_i\frac{1}{r}L_i+\epsilon_{ijk}\left(i\hbar\epsilon_{kij}\frac{x_j}{r}\right)p_j\\
&=-\frac{1}{r}\boldsymbol{L}^2+i\hbar\epsilon_{ijk}^2\frac{x_j}{r}p_j
\end{aligned}$$

Thus

$$B_1-B_2+B_3-B_4=\frac{4}{r}\boldsymbol{L}^2+i\hbar\epsilon_{ijk}^2\left[p_i,\frac{x_i}{r}\right]=\frac{4}{r}\boldsymbol{L}^2+\hbar^2\epsilon_{ijk}^2\left(\frac{1}{r}-\frac{x_ix_i}{r^3}\right)=\frac{4}{r}\left(\boldsymbol{L}^2+\hbar^2\right)\tag{24}$$

It still remains to show $A=A'$, which is equivalent to

$$\epsilon_{ijk}\epsilon_{lmk}\left(p_iL_j-L_ip_j\right)\left(p_lL_m-L_lp_m\right)=4\boldsymbol{p}^2\left(\boldsymbol{L}^2+\hbar^2\right)\tag{25}$$

Using (18), the LHS of (25) can be broken into

$$\begin{aligned}
\text{LHS}&=\left(p_iL_j-L_ip_j\right)\left(p_iL_j-L_ip_j\right)-\left(p_iL_j-L_ip_j\right)\left(p_jL_i-L_jp_i\right)\\
&=\overbrace{p_iL_jp_iL_j}^{A_1}-\overbrace{p_iL_jL_ip_j}^{A_2}-\overbrace{L_ip_jp_iL_j}^{A_3}+\overbrace{L_ip_jL_ip_j}^{A_4}-\overbrace{p_iL_jp_jL_i}^{A_5}+\overbrace{p_iL_jL_jp_i}^{A_6}+\overbrace{L_ip_jp_jL_i}^{A_7}-\overbrace{L_ip_jL_jp_i}^{A_8}
\end{aligned}$$

Attacking them one by one,

$$\begin{aligned}
A_1&=p_iL_jp_iL_j=p_i\left(p_iL_j-\left[p_i,L_j\right]\right)L_j=p_i^2L_j^2-i\hbar\epsilon_{ijk}p_ip_kL_j=\boldsymbol{p}^2\boldsymbol{L}^2\\
A_2&=p_iL_jL_ip_j=p_i\left(L_iL_j-\left[L_i,L_j\right]\right)p_j=(\boldsymbol{p}\cdot\boldsymbol{L})\cdot(\boldsymbol{L}\cdot\boldsymbol{p})-i\hbar\epsilon_{ijk}p_iL_kp_j=-i\hbar\epsilon_{ijk}p_iL_kp_j\\
&=-i\hbar\epsilon_{ijk}\left(L_kp_i+[p_i,L_k]\right)p_j=-i\hbar\epsilon_{ijk}L_kp_ip_j-i\hbar\epsilon_{ijk}(i\hbar\epsilon_{ikj}p_j)p_j\\
&=-\hbar^2\epsilon_{ijk}^2p_j^2=-2\hbar^2\boldsymbol{p}^2\\
A_3&=L_ip_jp_iL_j=L_ip_ip_jL_j=(\boldsymbol{L}\cdot\boldsymbol{p})\cdot(\boldsymbol{p}\cdot\boldsymbol{L})=0\\
A_4&=L_ip_jL_ip_j=L_i\left(L_ip_j+[p_j,L_i]\right)p_j=L_i^2p_j^2-i\hbar\epsilon_{ijk}L_ip_kp_j=\boldsymbol{L}^2\boldsymbol{p}^2\\
A_5&=p_iL_jp_jL_i=p_i\left(\boldsymbol{L}\cdot\boldsymbol{p}\right)L_i=0\\
A_6&=p_iL_jL_jp_i=\left(L_jp_i+\left[p_i,L_j\right]\right)\left(p_iL_j-\left[p_i,L_j\right]\right)=\left(L_jp_i+i\hbar\epsilon_{ijk}p_k\right)\left(p_iL_j-i\hbar\epsilon_{ijk}p_k\right)\\
&=L_j\boldsymbol{p}^2L_j+i\hbar\epsilon_{ijk}\left(p_kp_iL_j-L_jp_ip_k\right)+\hbar^2\epsilon_{ijk}^2p_k^2\qquad(\boldsymbol{p}^2\text{ is scalar})\\
&=\boldsymbol{p}^2\boldsymbol{L}^2+2\hbar^2\boldsymbol{p}^2\\
A_7&=L_ip_jp_jL_i=L_i\left(\boldsymbol{p}^2\right)L_i=\boldsymbol{p}^2\boldsymbol{L}^2\\
A_8&=L_ip_jL_jp_i=L_i\left(\boldsymbol{p}\cdot\boldsymbol{L}\right)L_i=0
\end{aligned}$$

Noting that $\boldsymbol{L}^2\boldsymbol{p}^2=\boldsymbol{p}^2\boldsymbol{L}^2$ since $\boldsymbol{p}^2$ is a scalar, thus we have proved (25), hence (7).

6. **Proof of (8)**.

First, before going into details, we should note that for $\epsilon_{ijk}=1$, $[M_i,M_j]$ is the k-component of the vector operator $\boldsymbol{M}\times\boldsymbol{M}$. Let

$$\boldsymbol{W}=\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{L}\times\boldsymbol{p}$$

be a vector operator, then

$$M_i=\frac{1}{2m}W_i-Ze^2\frac{x_i}{r}$$
$$M_j=\frac{1}{2m}W_j-Ze^2\frac{x_j}{r}$$

Thus

$$[M_i,M_j]=\frac{1}{4m^2}[W_i,W_j]-\frac{Ze^2}{2m}\left(\left[W_i,\frac{x_j}{r}\right]+\left[\frac{x_i}{r},W_j\right]\right) \tag{26}$$

But the RHS of (8) is

$$-i\hbar\epsilon_{ijk}\frac{2}{m}\left(\frac{\boldsymbol{p}^2}{2m}-\frac{Ze^2}{r}\right)L_k=-i\hbar\epsilon_{ijk}\frac{\boldsymbol{p}^2}{m^2}L_k+i\hbar\epsilon_{ijk}\frac{2Ze^2}{m}\frac{1}{r}L_k \tag{27}$$

Comparing (26) with (27), it's clear that we should eventually prove

$$[W_i,W_j]=-4i\hbar\epsilon_{ijk}\boldsymbol{p}^2L_k \tag{28}$$
$$\left[W_i,\frac{x_j}{r}\right]+\left[\frac{x_i}{r},W_j\right]=-4i\hbar\epsilon_{ijk}\frac{1}{r}L_k \tag{29}$$

(a) **Proof of (28)**.

By virtue of being a vector operator, $\boldsymbol{W}$ satisfies

$$[W_i,L_j]=i\hbar\epsilon_{ijk}W_k \tag{30}$$

Also

$$\begin{aligned}
[W_i,p_j]&=\epsilon_{lmi}\left[p_lL_m-L_lp_m,p_j\right]=\epsilon_{lmi}\left(p_l[L_m,p_j]-[L_l,p_j]p_m\right)\\
&=i\hbar\epsilon_{lmi}\left(\epsilon_{mjk}p_lp_k-\epsilon_{ljk}p_kp_m\right)\\
&=i\hbar\epsilon_{lmi}\epsilon_{mjk}p_lp_k-i\hbar\epsilon_{lmi}\epsilon_{ljk}p_kp_m \qquad (l\longleftrightarrow m \text{ in 2nd term})\\
&=i\hbar\epsilon_{lmi}\epsilon_{mjk}p_lp_k-i\hbar\epsilon_{mli}\epsilon_{mjk}p_kp_l\\
&=i\hbar\epsilon_{lmi}\epsilon_{mjk}p_lp_k+i\hbar\epsilon_{lmi}\epsilon_{mjk}p_kp_l\\
&=2i\hbar\left(\delta_{lk}\delta_{ij}-\delta_{lj}\delta_{ik}\right)p_lp_k\\
&=2i\hbar\left(\delta_{ij}\boldsymbol{p}^2-p_ip_j\right)
\end{aligned} \tag{31}$$

It follows that

$$\begin{aligned}
[W_i,W_j]&=\epsilon_{lmj}\left[W_i,p_lL_m-L_lp_m\right]\\
&=\epsilon_{lmj}\left([W_i,p_l]L_m+p_l[W_i,L_m]-[W_i,L_l]p_m-L_l[W_i,p_m]\right)\\
&=\overbrace{\epsilon_{lmj}[W_i,p_l]L_m}^{A}+\overbrace{\epsilon_{lmj}p_l[W_i,L_m]}^{B}-\overbrace{\epsilon_{lmj}[W_i,L_l]p_m}^{C}-\overbrace{\epsilon_{lmj}L_l[W_i,p_m]}^{D}
\end{aligned}$$

Using (30) and (31), we have

$$\begin{aligned}
A&=2i\hbar\epsilon_{lmj}\left(\delta_{il}\boldsymbol{p}^2-p_ip_l\right)L_m=2i\hbar\epsilon_{imj}\boldsymbol{p}^2L_m-2i\hbar\epsilon_{lmj}p_ip_lL_m\\
B&=i\hbar\epsilon_{lmj}p_l\epsilon_{imk}W_k=i\hbar\left(\delta_{il}\delta_{jk}-\delta_{ij}\delta_{lk}\right)p_lW_k=i\hbar p_iW_j-i\hbar\delta_{ij}(\boldsymbol{p}\cdot\boldsymbol{W})\\
C&=i\hbar\epsilon_{lmj}\epsilon_{ilk}W_kp_m=i\hbar\left(\delta_{ij}\delta_{mk}-\delta_{im}\delta_{jk}\right)W_kp_m=i\hbar\delta_{ij}(\boldsymbol{W}\cdot\boldsymbol{p})-i\hbar W_jp_i\\
D&=2i\hbar\epsilon_{lmj}L_l\left(\delta_{im}\boldsymbol{p}^2-p_ip_m\right)=2i\hbar\epsilon_{lij}L_l\boldsymbol{p}^2-2i\hbar\epsilon_{lmj}L_lp_ip_m
\end{aligned}$$

Compare $A+B-C-D$ with RHS of (28), all that remains to show is

$$-2\epsilon_{lmj}p_ip_lL_m+p_iW_j-\delta_{ij}\left(\boldsymbol{p}\cdot\boldsymbol{W}+\boldsymbol{W}\cdot\boldsymbol{p}\right)+W_jp_i+2\epsilon_{lmj}L_lp_ip_m=0$$

which is easily proved considering the following

$$\begin{aligned}
\boldsymbol{p}\cdot\boldsymbol{W}+\boldsymbol{W}\cdot\boldsymbol{p} &= \epsilon_{lmk}\left[p_k\left(p_lL_m-L_lp_m\right)+\left(p_lL_m-L_lp_m\right)p_k\right]\\
&= \epsilon_{lmk}p_kp_lL_m-\epsilon_{lmk}p_kL_lp_m+\epsilon_{lmk}p_lL_mp_k-\epsilon_{lmk}L_lp_mp_k \qquad (lmk\to mkl \text{ in 2nd term})\\
&= 0\\
p_iW_j+W_jp_i &= \epsilon_{lmj}\left[p_i\left(p_lL_m-L_lp_m\right)+\left(p_lL_m-L_lp_m\right)p_i\right]\\
&= \epsilon_{lmj}p_ip_lL_m-\epsilon_{lmj}p_iL_lp_m+\epsilon_{lmj}p_lL_mp_i-\epsilon_{lmj}L_lp_mp_i\\
&= \epsilon_{lmj}p_ip_lL_m-\epsilon_{lmj}p_i\left(p_mL_l+[L_l,p_m]\right)+\epsilon_{lmj}\left(L_mp_l+[p_l,L_m]\right)p_i-\epsilon_{lmj}L_lp_mp_i\\
&= \underbrace{\epsilon_{lmj}p_ip_lL_m-\epsilon_{lmj}p_ip_mL_l}_{2\epsilon_{lmj}p_ip_lL_m}-\overbrace{\epsilon_{lmj}p_i[L_l,p_m]}^{i\hbar\epsilon_{lmj}^2p_ip_j}+\overbrace{\epsilon_{lmj}[p_l,L_m]p_i}^{i\hbar\epsilon_{lmj}^2p_jp_i}+\underbrace{\epsilon_{lmj}L_mp_lp_i-\epsilon_{lmj}L_lp_mp_i}_{-2\epsilon_{lmj}L_lp_mp_i}\\
&= 2\epsilon_{lmj}p_ip_lL_m-2\epsilon_{lmj}L_lp_mp_i
\end{aligned}$$

(b) **Proof of (29)**.
Rewriting LHS of (29), we have

$$\begin{aligned}
\text{LHS} &= \epsilon_{lmi}\left[p_lL_m-L_lp_m,\frac{x_j}{r}\right]+\epsilon_{stj}\left[\frac{x_i}{r},p_sL_t-L_sp_t\right]\\
&= \overbrace{\epsilon_{lmi}\left[p_lL_m,\frac{x_j}{r}\right]}^{A}-\overbrace{\epsilon_{stj}\left[p_sL_t,\frac{x_i}{r}\right]}^{B}+\overbrace{\epsilon_{stj}\left[L_sp_t,\frac{x_i}{r}\right]}^{C}-\overbrace{\epsilon_{lmi}\left[L_lp_m,\frac{x_j}{r}\right]}^{D}
\end{aligned}$$

where

$$\begin{aligned}
A &= \epsilon_{lmi}\left(p_l\left[L_m,\frac{x_j}{r}\right]+\left[p_l,\frac{x_j}{r}\right]L_m\right)\\
&= \epsilon_{lmi}\left[i\hbar\epsilon_{mjk}p_l\frac{x_k}{r}-i\hbar\left(\frac{\delta_{lj}}{r}-\frac{x_lx_j}{r^3}\right)L_m\right]\\
&= i\hbar\epsilon_{lmi}\epsilon_{mjk}p_l\frac{x_k}{r}-i\hbar\epsilon_{jmi}\frac{1}{r}L_m+i\hbar\epsilon_{lmi}\frac{x_lx_j}{r^3}L_m\\
&= i\hbar\left(\delta_{ij}\delta_{kl}-\delta_{jl}\delta_{ik}\right)p_l\frac{x_k}{r}-i\hbar\epsilon_{jmi}\frac{1}{r}L_m+i\hbar\epsilon_{lmi}\frac{x_lx_j}{r^3}L_m\\
&= i\hbar\delta_{ij}p_k\frac{x_k}{r}-i\hbar p_j\frac{x_i}{r}-i\hbar\epsilon_{ijm}\frac{1}{r}L_m+i\hbar\epsilon_{lmi}\frac{x_lx_j}{r^3}L_m\\
C &= \epsilon_{stj}\left(L_s\left[p_t,\frac{x_i}{r}\right]+\left[L_s,\frac{x_i}{r}\right]p_t\right)\\
&= \epsilon_{stj}\left[-i\hbar L_s\left(\frac{\delta_{ti}}{r}-\frac{x_tx_i}{r^3}\right)+i\hbar\epsilon_{sik}\frac{x_k}{r}p_t\right]\\
&= -i\hbar\epsilon_{sij}L_s\frac{1}{r}+i\hbar\epsilon_{stj}L_s\frac{x_tx_i}{r^3}+i\hbar\epsilon_{stj}\epsilon_{sik}\frac{x_k}{r}p_t\\
&= -i\hbar\epsilon_{sij}L_s\frac{1}{r}+i\hbar\epsilon_{stj}L_s\frac{x_tx_i}{r^3}+i\hbar\left(\delta_{it}\delta_{jk}-\delta_{ij}\delta_{tk}\right)\frac{x_k}{r}p_t\\
&= -i\hbar\epsilon_{ijs}L_s\frac{1}{r}+i\hbar\epsilon_{stj}L_s\frac{x_tx_i}{r^3}+i\hbar\frac{x_j}{r}p_i-i\hbar\delta_{ij}\frac{x_k}{r}p_k
\end{aligned}$$

With mapping of indices $i\longleftrightarrow j, l\longleftrightarrow s, m\longleftrightarrow t$, we obtain the form of B, D from A, C:

$$\begin{aligned}
B &= i\hbar\delta_{ij}p_k\frac{x_k}{r}-i\hbar p_i\frac{x_j}{r}-i\hbar\epsilon_{jit}\frac{1}{r}L_t+i\hbar\epsilon_{stj}\frac{x_sx_i}{r^3}L_t\\
D &= -i\hbar\epsilon_{jil}L_l\frac{1}{r}+i\hbar\epsilon_{lmi}L_l\frac{x_mx_j}{r^3}+i\hbar\frac{x_i}{r}p_j-i\hbar\delta_{ij}\frac{x_k}{r}p_k
\end{aligned}$$

Now combine $A-B+C-D$, notice all terms with δ_{ij} cancel out. Also since $1/r$ commutes with L_i, we have also accounted for the $-4i\hbar\epsilon_{ijk}L_k/r$ term on the RHS of (29), all it remains to prove is

$$-p_j\frac{x_i}{r}+\overbrace{\epsilon_{lmi}\frac{x_lx_j}{r^3}L_m}^{A_1}+p_i\frac{x_j}{r}-\overbrace{\epsilon_{stj}\frac{x_sx_i}{r^3}L_t}^{B_1}+\overbrace{\epsilon_{stj}L_s\frac{x_tx_i}{r^3}}^{C_1}+\frac{x_j}{r}p_i-\overbrace{\epsilon_{lmi}L_l\frac{x_mx_j}{r^3}}^{D_1}-\frac{x_i}{r}p_j=0 \tag{32}$$

Next notice

$$
\begin{aligned}
A_1 &= \epsilon_{lmi}\frac{x_l x_j}{r^3}\epsilon_{uvm}x_u p_v = -(\delta_{lu}\delta_{iv} - \delta_{lv}\delta_{iu})\frac{x_l x_j}{r^3}x_u p_v \\
&= -\frac{x_l x_j x_l}{r^3}p_i + \frac{x_l x_j x_i}{r^3}p_l = -\frac{x_j}{r}p_i + \frac{x_i x_j}{r^3}(\boldsymbol{x}\cdot\boldsymbol{p}) \\
C_1 &= \epsilon_{stj}\epsilon_{uvs}x_u p_v \frac{x_t x_i}{r^3} = \left(\delta_{tu}\delta_{jv} - \delta_{tv}\delta_{ju}\right)x_u p_v \frac{x_t x_i}{r^3} \\
&= \left(x_t p_j - x_j p_t\right)\frac{x_t x_i}{r^3} = \left(p_j x_t - p_t x_j\right)\frac{x_t x_i}{r^3} \\
&= p_j \frac{x_t x_t x_i}{r^3} - p_t \frac{x_j x_t x_i}{r^3} = p_j \frac{x_i}{r} - (\boldsymbol{p}\cdot\boldsymbol{x})\frac{x_i x_j}{r^3}
\end{aligned}
$$

Lastly, with the usual index mapping $i \longleftrightarrow j$, we obtain B_1, D_1 as

$$
\begin{aligned}
B_1 &= -\frac{x_i}{r}p_j + \frac{x_i x_j}{r^3}(\boldsymbol{x}\cdot\boldsymbol{p}) \\
D_1 &= p_i \frac{x_j}{r} - (\boldsymbol{p}\cdot\boldsymbol{x})\frac{x_i x_j}{r^3}
\end{aligned}
$$

We see indeed all terms on the LHS of (32) cancel out.