1. We are solving for the PDE

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\rho,\theta,z) = E\psi(\rho,\theta,z) \tag{1}$$

subject to the boundary condition

$$\psi(\rho,\theta,0) = \psi(\rho,\theta,L) = 0$$
$$\psi(\rho_a,\theta,z) = \psi(\rho_b,\theta,z) = 0$$

First note that the Laplacian in cylindrical coordinates is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \tag{2}$$

Using separation of variables, write

$$\psi(\rho,\theta,z) = G(\rho,\theta)H(z) \tag{3}$$

(1) becomes

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)H + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\theta^2}H + G\frac{\partial^2 H}{\partial z^2} &= -\frac{2mE}{\hbar^2}GH \qquad \Longrightarrow \\
\frac{1}{\rho G}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2 G}\frac{\partial^2 G}{\partial\theta^2} + \frac{1}{H}\frac{\partial^2 H}{\partial z^2} &= -\frac{2mE}{\hbar^2} \equiv -a \qquad \Longrightarrow \\
\frac{1}{\rho G}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2 G}\frac{\partial^2 G}{\partial\theta^2} + a &= -\frac{1}{H}\frac{\partial^2 H}{\partial z^2} \equiv A
\end{aligned} \tag{4}$$

For the PDE for $H(z)$ with the boundary condition at $z = 0$ and $z = L$, we can easily solve (ignoring normalization factor)

$$H(z) = e^{il\pi z/L} \tag{5}$$

where A has to take values

$$A = \frac{l^2\pi^2}{L^2}$$

Now further separating G into $G(\rho,\theta) = R(\rho)K(\theta)$, (4) will become

$$\begin{aligned}
\frac{1}{\rho RK}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)K + \frac{1}{\rho^2 RK}\frac{\partial^2 K}{\partial\theta^2}R + a - \frac{l^2\pi^2}{L^2} &= 0 \qquad \Longrightarrow \\
\frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + a - \frac{l^2\pi^2}{L^2} &= -\frac{1}{\rho^2 K}\frac{\partial^2 K}{\partial\theta^2} \qquad \Longrightarrow \\
\frac{\rho}{R}\left(\rho\frac{\partial^2 R}{\partial\rho^2} + \frac{\partial R}{\partial\rho}\right) + \left(a - \frac{l^2\pi^2}{L^2}\right)\rho^2 &= -\frac{1}{K}\frac{\partial^2 K}{\partial\theta^2} \equiv B
\end{aligned} \tag{6}$$

For K to be a singled-value function in θ (i.e., $K(\theta) = K(2\pi + \theta)$), we must have

$$K(\theta) = Me^{im\theta} + Ne^{-im\theta}$$

for $m = 0, 1, 2, \cdots$ (hence $B = m^2$).

Defining $\gamma = \sqrt{a - \frac{l^2\pi^2}{L^2}}$ the PDE for $R(\rho)$ in (6) becomes

$$\rho^2\frac{\partial^2 R}{\partial\rho^2} + \rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R - m^2 R = 0 \qquad (\text{let } r \equiv \gamma\rho)$$

$$r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} + (r^2 - m^2)R = 0 \tag{7}$$

which is in the canonical PDE form whose general solution is linear combination of Bessel functions

$$R(\gamma\rho) = CJ_m(\gamma\rho) + DN_m(\gamma\rho)$$

For the boundary condition to hold, γ must satisfy

$$\begin{aligned} R(\gamma\rho_a) = CJ_m(\gamma\rho_a) + DN_m(\gamma\rho_a) = 0 \\ R(\gamma\rho_b) = CJ_m(\gamma\rho_b) + DN_m(\gamma\rho_b) = 0 \\ J_m(\gamma\rho_a)N_m(\gamma\rho_b) - J_m(\gamma\rho_b)N_m(\gamma\rho_a) = 0 \end{aligned} \quad \Longrightarrow \tag{8}$$

When γ is the n-th root k_{mn} of (8), we have

$$E_{mnl} = \frac{\hbar^2}{2m}\left(k_{mn}^2 + \frac{l^2\pi^2}{L^2}\right) \tag{9}$$

2. The cylindrically-symmetric vector potential $\mathbf{A}$ can be obtained as below

$$A \cdot 2\pi\rho = B \cdot \pi\rho_a^2 \qquad \Longrightarrow \qquad A = \frac{B\rho_a^2}{2\rho} \qquad \Longrightarrow \qquad \mathbf{A} = \frac{B\rho_a^2}{2\rho^2}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} \tag{10}$$

In the Schrödinger equation, with $\mathbf{A}$ present, we have to replace ∇ with $\nabla - ie\mathbf{A}/\hbar c$, i.e., we are now solving for PDE

$$\left(\nabla - \frac{ie\mathbf{A}}{\hbar c}\right)^2 \psi(\rho,\theta,z) = -\frac{2mE}{\hbar^2}\psi(\rho,\theta,z) \tag{11}$$

where the LHS is

$$\begin{aligned} \left(\nabla - \frac{ie\mathbf{A}}{\hbar c}\right)\left(\nabla\psi - \frac{ie\mathbf{A}\psi}{\hbar c}\right) &= \nabla^2\psi - \frac{ie}{\hbar c}\nabla\cdot(\mathbf{A}\psi) - \frac{ie}{\hbar c}\mathbf{A}\cdot\nabla\psi - \frac{e^2}{\hbar^2 c^2}A^2\psi \\ &= \nabla^2\psi - \frac{ie}{\hbar c}\left[(\nabla\cdot\mathbf{A})\psi + 2\mathbf{A}\cdot\nabla\psi\right] - \frac{e^2B^2\rho_a^4}{4\hbar^2c^2\rho^2}\psi \\ &= \nabla^2\psi - \frac{ie}{\hbar c}\cdot\frac{B\rho_a^2}{\rho^2}\left(-y\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial y}\right) - \frac{e^2B^2\rho_a^4}{4\hbar^2c^2\rho^2}\psi \end{aligned}$$

where the term in the parenthesis gives

$$\begin{aligned} -y\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial y} &= -y\left(\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x}\right) + x\left(\frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial y} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y}\right) \\ &= -y\left(\frac{\partial\psi}{\partial\rho}\frac{x}{\rho} + \frac{\partial\psi}{\partial\theta}\frac{-y}{\rho^2}\right) + x\left(\frac{\partial\psi}{\partial\rho}\frac{y}{\rho} + \frac{\partial\psi}{\partial\theta}\frac{x}{\rho^2}\right) = \frac{\partial\psi}{\partial\theta} \end{aligned}$$

Therefore (11) becomes

$$\begin{aligned} \nabla^2\psi - \frac{ieB\rho_a^2}{\hbar c\rho^2}\frac{\partial\psi}{\partial\theta} - \frac{e^2B^2\rho_a^4}{4\hbar^2c^2\rho^2}\psi &= -\frac{2mE}{\hbar^2}\psi \qquad (\text{let } W = \tfrac{eB\rho_a^2}{2\hbar c}) \qquad \Longrightarrow \\ \nabla^2\psi - \frac{2iW}{\rho^2}\frac{\partial\psi}{\partial\theta} - \frac{W^2}{\rho^2}\psi &= -a\psi \end{aligned} \tag{12}$$

Again, writing $\psi(\rho,\theta,z) = G(\rho,\theta)H(z)$, we have

$$\begin{aligned} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)H + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\theta^2}H + G\frac{\partial^2 H}{\partial z^2} - \frac{2iW}{\rho^2}\frac{\partial G}{\partial\theta}H - \frac{W^2}{\rho^2}GH = -aGH \qquad \Longrightarrow \\ \frac{1}{\rho G}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2 G}\frac{\partial^2 G}{\partial\theta^2} - \frac{2iW}{\rho^2 G}\frac{\partial G}{\partial\theta} - \frac{W^2}{\rho^2} + a = -\frac{1}{H}\frac{\partial^2 H}{\partial z^2} = \frac{l^2\pi^2}{L^2} \end{aligned} \tag{13}$$

Writing $G(\rho,\theta) = R(\rho)K(\theta)$, we have

$$\begin{aligned} \frac{1}{\rho RK}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)K + \frac{1}{\rho^2 RK}\frac{\partial^2 K}{\partial\theta^2}R - \frac{2iW}{\rho^2 RK}\frac{\partial K}{\partial\theta}R - \frac{W^2}{\rho^2} + \left(a - \frac{l^2\pi^2}{L^2}\right) = 0 \qquad \Longrightarrow \\ \frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \left(a - \frac{l^2\pi^2}{L^2}\right)\rho^2 = -\frac{1}{K}\frac{\partial^2 K}{\partial\theta^2} + \frac{2iW}{K}\frac{\partial K}{\partial\theta} + W^2 \end{aligned} \tag{14}$$

3. Now compare the LHS of (14) with LHS of (6). In (6), the ground state is achieved when its LHS is 0. If we set (14) to 0, we have a solution for the RHS

$$K(\theta) = e^{\pm iW\theta} \tag{15}$$

which is a good solution only when $W = N$ with $N = 0, 1, 2\cdots$, which is equivalent to requiring

$$W = \frac{eB\rho_a^2}{2\hbar c} = N \qquad \Longrightarrow \qquad B\pi\rho_a^2 = \frac{2\pi N\hbar c}{e} \tag{16}$$