We can write the solution $\Upsilon(\mathbf{x},t)$ as

$$\Upsilon(\mathbf{x},t) = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})} \tag{1}$$

Plug this into equation (8.18)

$$i\frac{\partial}{\partial t}\Upsilon(\mathbf{x},t) = \left\{-\frac{1}{2m}\nabla^2\begin{bmatrix}1 & 1 \\ -1 & -1\end{bmatrix} + m\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\right\}\Upsilon(\mathbf{x},t) \tag{2}$$

we end up with

$$\begin{aligned} i(-iE)e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} &= \left\{-\frac{1}{2m}(-p^2)\begin{bmatrix}1 & 1 \\ -1 & -1\end{bmatrix} + \begin{bmatrix}m & 0 \\ 0 & -m\end{bmatrix}\right\}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})} && \Longrightarrow \\ E\begin{bmatrix}\alpha \\ \beta\end{bmatrix} &= \begin{bmatrix}\dfrac{p^2}{2m}+m & \dfrac{p^2}{2m} \\ -\dfrac{p^2}{2m} & -\dfrac{p^2}{2m}-m\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} && (3) \end{aligned}$$

for which we can solve for the eigenvalues:

$$\begin{aligned} \left[E-\left(\frac{p^2}{2m}+m\right)\right]\left[E+\left(\frac{p^2}{2m}+m\right)\right]+\left(\frac{p^2}{2m}\right)^2 &= 0 && \Longrightarrow \\ E^2-\left(p^2+m^2\right) &= 0 && \Longrightarrow \\ E = \pm\sqrt{p^2+m^2} &= \pm E_p && (4) \end{aligned}$$

To solve for α,β, it's easier to work with E_p and m, instead of p^2 and m, we take note that

$$p^2 = E_p^2 - m^2 \tag{5}$$

Then (3) becomes

$$\pm E_p\alpha = \left(\frac{E_p^2-m^2}{2m}+m\right)\alpha + \left(\frac{E_p^2-m^2}{2m}\right)\beta \tag{6}$$

$$\pm E_p\beta = -\left(\frac{E_p^2-m^2}{2m}\right)\alpha - \left(\frac{E_p^2-m^2}{2m}+m\right)\beta \tag{7}$$

Since $\pm E_p$ are the eigenvalues of (3), (6) and (7) are no longer independent. We will use the normalization condition (see 8.20)

$$|\alpha|^2 - |\beta|^2 = \pm 1 \qquad \text{for } E = \pm E_p \tag{8}$$

From (6), we have

$$\begin{aligned} \left(\frac{E_p^2-m^2}{2m}+m\mp E_p\right)\alpha + \left(\frac{E_p^2-m^2}{2m}\right)\beta &= 0 && \Longrightarrow \\ \left(E_p\mp m\right)^2\alpha + \left(E_p^2-m^2\right)\beta &= 0 && \Longrightarrow \\ \beta = -\frac{E_p\mp m}{E_p\pm m}\alpha &= \frac{-E_p\pm m}{E_p\pm m}\alpha && (9) \end{aligned}$$

Together with (8):

$$\begin{aligned} \left[1-\left(\frac{-E_p\pm m}{E_p\pm m}\right)^2\right]\alpha^2 &= \pm 1 && \Longrightarrow \\ \frac{\pm 4E_p m}{(E_p\pm m)^2}\alpha^2 &= \pm 1 && (10) \end{aligned}$$

So we take

$$\alpha = \frac{E_p + m}{2\sqrt{E_p m}}, \beta = \frac{m - E_p}{2\sqrt{E_p m}} \qquad \text{for } E = +E_p \tag{11}$$

$$\alpha = \frac{m - E_p}{2\sqrt{E_p m}}, \beta = \frac{E_p + m}{2\sqrt{E_p m}} \qquad \text{for } E = -E_p \tag{12}$$