This is similar to the treatment of photoelectric effect, except the initial state $|i\rangle$ is now the ground state of the 3-dimensional isotropic oscillator. Referencing to eq (2.151), we have

$$\langle \boldsymbol{x}|i\rangle = \left(\frac{1}{\sqrt{\pi}x_0}\right)^{3/2} e^{-r^2/(2x_0^2)} \qquad \text{where } x_0 = \sqrt{\frac{\hbar}{m_e\omega_0}} \tag{1}$$

Eq (5.338) still holds

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{4\pi^2\alpha\hbar}{m_e^2\omega}\left|\overbrace{\left\langle \boldsymbol{k}_f\left|e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{p}\right|i\right\rangle}^{A}\right|^2 \frac{m_e k_f L^3}{\hbar^2(2\pi)^3}\\ &= \frac{\alpha k_f L^3}{2\pi m_e\omega\hbar}|A|^2\end{aligned} \tag{2}$$

And

$$\begin{aligned}A &= \hat{\boldsymbol{\epsilon}}\cdot\int d^3x\frac{e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}}{L^{3/2}}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}(-i\hbar\boldsymbol{\nabla})\left[\left(\frac{1}{\sqrt{\pi}x_0}\right)^{3/2}e^{-r^2/(2x_0^2)}\right]\\ &= \frac{-i\hbar}{L^{3/2}}\left(\frac{1}{\sqrt{\pi}x_0}\right)^{3/2}\underbrace{\hat{\boldsymbol{\epsilon}}\cdot\int d^3x e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\boldsymbol{\nabla}\left[e^{-r^2/(2x_0^2)}\right]}_{B}\end{aligned} \tag{3}$$

Combining with (2), we have

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\alpha k_f L^3}{2\pi m_e\omega\hbar}\left|\frac{-i\hbar}{L^{3/2}}\left(\frac{1}{\sqrt{\pi}x_0}\right)^{3/2}\right|^2|B|^2\\ &= \frac{\alpha k_f\hbar}{2\pi m_e\omega}\left(\frac{1}{\sqrt{\pi}x_0}\right)^3|B|^2\end{aligned} \tag{4}$$

Now we calculate B using integration-by-parts:

$$\begin{aligned}B &= \hat{\boldsymbol{\epsilon}}\cdot\int d^3x e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\boldsymbol{\nabla}\left[e^{-r^2/(2x_0^2)}\right]\\ &= \hat{\boldsymbol{\epsilon}}\cdot\left[\underbrace{e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}e^{-r^2/(2x_0^2)}\Big|_{-\infty}^{+\infty}}_{=0} - \int d^3x\boldsymbol{\nabla}\left(e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\right)e^{-r^2/(2x_0^2)}\right]\\ &= -\hat{\boldsymbol{\epsilon}}\cdot\int d^3x\left[\left(\boldsymbol{\nabla}e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}\right)e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}e^{-r^2/(2x_0^2)} + e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}\left(\boldsymbol{\nabla}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\right)e^{-r^2/(2x_0^2)}\right]\end{aligned} \tag{5}$$

We can drop the second term in the integral since the gradient $\boldsymbol{\nabla}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}$ is along the $\hat{\boldsymbol{n}}$ direction, and $\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{n}} = 0$.

Now (5) becomes

$$\begin{aligned}B &= -\hat{\boldsymbol{\epsilon}}\cdot\int d^3x\left(\boldsymbol{\nabla}e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}\right)e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}e^{-r^2/(2x_0^2)}\\ &= i\left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{k}_f\right)\int d^3x e^{-i(\boldsymbol{k}_f-\omega\hat{\boldsymbol{n}}/c)\cdot\boldsymbol{x}}e^{-r^2/(2x_0^2)} \qquad (\text{define } \boldsymbol{q}\equiv\boldsymbol{k}_f-\left(\frac{\omega}{c}\right)\hat{\boldsymbol{n}})\\ &= i\left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{k}_f\right)\underbrace{\int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}e^{-r^2/(2x_0^2)}}_{C}\end{aligned} \tag{6}$$

Next, we evaluate the Fourier transform integral C in spherical coordinates where $\boldsymbol{q}$ aligns with the z-axis (note, the z-axis in this integral evaluation, together with the corresponding θ,ϕ angles are different from the coordinates in Fig 5.13).

$$
\begin{aligned}
C &= \int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} e^{-r^2/(2x_0^2)} \\
&= \int_0^{2\pi} d\phi \int_0^\infty r^2 e^{-r^2/(2x_0^2)} dr \int_0^\pi \sin\theta d\theta e^{-iqr\cos\theta} \qquad (\text{let } y \equiv -\cos\theta) \\
&= 2\pi \int_0^\infty r^2 e^{-r^2/(2x_0^2)} dr \int_{y=-1}^1 dy e^{iqry} \\
&= \frac{2\pi}{iq} \int_0^\infty r e^{-r^2/(2x_0^2)} dr \left(e^{iqr} - e^{-iqr}\right) \\
&= \frac{2\pi}{iq} \int_0^\infty r e^{-r^2/(2x_0^2)} 2i \sin(qr) dr \\
&= \frac{4\pi}{q} \int_0^\infty r e^{-r^2/(2x_0^2)} \sin(qr) dr
\end{aligned} \tag{7}
$$

There is a clever trick to evaluate the integral in (7) as the following. Define

$$
I(q) \equiv \int_0^\infty e^{-ar^2} \cos(qr) dr \tag{8}
$$

Then

$$
\begin{aligned}
I'(q) &= \int_0^\infty -re^{-ar^2} \sin(qr) dr \\
&= \int_0^\infty d\left(\frac{e^{-ar^2}}{2a}\right) \sin(qr) \\
&= -\frac{q}{2a} \int_0^\infty \cos(qr) e^{-ar^2} dr \\
&= -\frac{q}{2a} I(q)
\end{aligned} \tag{9}
$$

Viewing (9) as an ODE for variable q, we can see the general solution

$$
I(q) = Ae^{-q^2/(4a)} \tag{10}
$$

where A can be determined by the boundary condition

$$
A = I(0) = \int_0^\infty e^{-ar^2} dr = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{11}
$$

Plugging in $a = 1/(2x_0^2)$, we now have

$$
\begin{aligned}
C &= \frac{4\pi}{q}\left[-I'(q)\right] = \frac{4\pi}{2a} I(q) = 4\pi x_0^2 \cdot \frac{1}{2}\sqrt{2\pi} x_0 e^{-q^2x_0^2/2} \\
&= 2\sqrt{2}\left(\sqrt{\pi}x_0\right)^3 e^{-q^2x_0^2/2}
\end{aligned} \tag{12}
$$

Now, together with (4) and (6),

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\alpha k_f \hbar}{2\pi m_e \omega} \left(\frac{1}{\sqrt{\pi}x_0}\right)^3 \left(\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{k}_f\right)^2 \cdot 8\left(\sqrt{\pi}x_0\right)^6 e^{-q^2x_0^2} \\
&= \frac{4\alpha k_f \hbar}{\pi m_e \omega} \left(\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{k}_f\right)^2 \pi x_0^2 \sqrt{\pi x_0} e^{-q^2x_0^2} \\
&= \frac{4\alpha k_f \hbar^2}{m_e^2 \omega \omega_0} \sqrt{\frac{\hbar\pi}{m_e\omega_0}} \left(\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{k}_f\right) e^{-q^2x_0^2}
\end{aligned} \tag{13}
$$

With the coordinate given by Fig 5.13,

$$\left(\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{k}_f\right)^2 = k_f^2 \sin^2\theta \cos^2\phi \tag{14}$$

$$\begin{aligned} q^2 x_0^2 &= \left[\left(k_f \cos\theta - \frac{\omega}{c}\right)^2 + (k_f \sin\theta)^2\right] \frac{\hbar}{m_e \omega_0} \\ &= \frac{\hbar}{m_e\omega_0}\left[k_f^2 + \left(\frac{\omega}{c}\right)^2\right] - \frac{2\hbar k_f \omega \cos\theta}{m_e c \omega_0} \end{aligned} \tag{15}$$

which yields the final form

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha k_f^3 \hbar^2}{m_e^2 \omega \omega_0} \sqrt{\frac{\hbar\pi}{m_e\omega_0}} \sin^2\theta \cos^2\phi \exp\left\{-\frac{\hbar}{m_e\omega_0}\left[k_f^2 + \left(\frac{\omega}{c}\right)^2\right]\right\} \exp\left(\frac{2\hbar k_f \omega\cos\theta}{m_e c\omega_0}\right) \tag{16}$$