We explore two methods for constructing n-dimensional representation of the J_i operators.

1. **base case** $n=2$. This is a single spin-1/2 particle, whose J_i operators have representations $J_i = \frac{\hbar}{2}\sigma_i$, i.e.,

$$J_x^{(2)} = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad J_y^{(2)} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad J_z^{(2)} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2. $n=3$. In this case $j=1$, and we consider the transformation among the triplet states $|j=1, m=\pm 1, 0\rangle$.

   (a) **method 1**. Consider the actions of raising/lowering operator

$$J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}\hbar|j,m+1\rangle \tag{1}$$
$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}\hbar|j,m-1\rangle \tag{2}$$

   Applying on the triplet states, we have (writing $|j,m\rangle$ as $|m\rangle$)

$$J_+|0\rangle = \sqrt{2}\hbar|1\rangle \qquad J_-|1\rangle = \sqrt{2}\hbar|0\rangle$$
$$J_+|-1\rangle = \sqrt{2}\hbar|0\rangle \qquad J_-|0\rangle = \sqrt{2}\hbar|-1\rangle$$

   Therefore, the $n=3$ representation for $J_\pm$ is

$$J_+ = \hbar \begin{array}{c c} \begin{array}{ccc} |1\rangle & |0\rangle & |-1\rangle \end{array} & \\ \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} |1\rangle \\ |0\rangle \\ |-1\rangle \end{array} \end{array} \qquad J_- = \hbar \begin{array}{c c} \begin{array}{ccc} |1\rangle & |0\rangle & |-1\rangle \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} & \begin{array}{c} |1\rangle \\ |0\rangle \\ |-1\rangle \end{array} \end{array}$$

   Then using

$$J_x = \frac{J_+ + J_-}{2} \tag{3}$$
$$J_y = -i\frac{J_+ - J_-}{2} \tag{4}$$

   we have

$$J_x^{(3)} = \hbar \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \qquad J_y^{(3)} = \hbar \begin{bmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{bmatrix}$$

   (b) **method 2**. Method 1 gives the representation in the spherical tensor basis $\{|-1\rangle, |0\rangle, |1\rangle\}$, but we know this basis can be obtained from the Cartesian basis of a pair of spin-1/2 particles.
   In the Cartesian tensor basis, J_x, J_y have the form

$$J_x^C = J_x^{(2)} \otimes I + I \otimes J_x^{(2)} = \frac{\hbar}{2} \begin{array}{c c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{array} \end{array}$$

$$J_y^C = J_y^{(2)} \otimes I + I \otimes J_y^{(2)} = \frac{\hbar}{2} \begin{array}{c c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{bmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{bmatrix} & \begin{array}{c} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{array} \end{array}$$

Since $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$, our goal is to perform a basis transform so the J_x, J_y in the above form will be decomposed into $1 \oplus 3$ blocks, where the 3×3 block represents $J_x^{(3)}, J_y^{(3)}$ matrix in the spherical tensor basis.
Using CG coefficients, we know

$$\begin{aligned}
|0,0\rangle &= \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \\
|1,1\rangle &= |++\rangle \\
|1,0\rangle &= \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} \\
|1,-1\rangle &= |--\rangle
\end{aligned}$$

which yields the desired transformation matrix

$$U = \begin{array}{c} \begin{array}{cccc} |0,0\rangle & |1,1\rangle & |1,0\rangle & |1,-1\rangle \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{array}$$

which expresses the spherical tensor basis as Cartesian basis. This means $U^\dagger J_x^C U$ and $U^\dagger J_y^C U$ are the matrix representations in spherical tensor basis.
It is straightforward to obtain

$$J_x^S = U^\dagger J_x^C U = \frac{\hbar}{2} = \begin{array}{c} \begin{array}{cccc} |0,0\rangle & |1,1\rangle & |1,0\rangle & |1,-1\rangle \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix} \end{array} \begin{array}{c} |0,0\rangle \\ |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{array}$$

$$J_y^S = U^\dagger J_y^C U = \frac{\hbar}{2} = \begin{array}{c} \begin{array}{cccc} |0,0\rangle & |1,1\rangle & |1,0\rangle & |1,-1\rangle \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}i & 0 \\ 0 & \sqrt{2}i & 0 & -\sqrt{2}i \\ 0 & 0 & \sqrt{2}i & 0 \end{bmatrix} \end{array} \begin{array}{c} |0,0\rangle \\ |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{array}$$

These matrices break into 1×1 block (northwest corner) and 3×3 block (southeast corner), with the 3×3 block agreeing with method 1.

3. $n = 4$. In this case $j = 3/2$.

   (a) **method 1**. Applying (1) and (2) to the $j = 3/2$ states:

$$J_+\left|\frac{1}{2}\right\rangle = \sqrt{3}\hbar\left|\frac{3}{2}\right\rangle \qquad J_-\left|\frac{3}{2}\right\rangle = \sqrt{3}\hbar\left|\frac{1}{2}\right\rangle$$

$$J_+\left|-\frac{1}{2}\right\rangle = 2\hbar\left|\frac{1}{2}\right\rangle \qquad J_-\left|\frac{1}{2}\right\rangle = 2\hbar\left|-\frac{1}{2}\right\rangle$$

$$J_+\left|-\frac{3}{2}\right\rangle = \sqrt{3}\hbar\left|-\frac{1}{2}\right\rangle \qquad J_-\left|-\frac{1}{2}\right\rangle = \sqrt{3}\hbar\left|-\frac{3}{2}\right\rangle$$

Then $n = 4$ representation for $J_\pm$ is

$$J_+ = \hbar \begin{array}{c} \begin{array}{cccc} |3/2\rangle & |1/2\rangle & |-1/2\rangle & |-3/2\rangle \end{array} \\ \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} |3/2\rangle \\ |1/2\rangle \\ |-1/2\rangle \\ |-3/2\rangle \end{array}$$

$$
\begin{array}{c|cccc|c}
 & |3/2\rangle & |1/2\rangle & |-1/2\rangle & |-3/2\rangle & \\
J_- = \hbar & \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{array}\right] & & & & \begin{array}{c} |3/2\rangle \\ |1/2\rangle \\ |-1/2\rangle \\ |-3/2\rangle \end{array}
\end{array}
$$

which gives

$$
J_x^{(4)} = \hbar \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{bmatrix} \qquad J_y^{(4)} = \hbar \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2}i & 0 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 0 & \frac{\sqrt{3}}{2}i & 0 \end{bmatrix}
$$

(b) **method 2**. Similar to the $n = 3$ case, we will express the spherical tensor $|j = 3/2, m\rangle$ as linear combinations of Cartesian tensor basis $\{|+++\rangle, |++-\rangle, \cdots, |---\rangle\}$.
Again, using CG coefficients, we have

$$
\begin{aligned}
\left|\frac{3}{2}\right\rangle &= |1\rangle \otimes |+\rangle = |+++\rangle \\
\left|\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{3}}|1\rangle \otimes |-\rangle + \sqrt{\frac{2}{3}}|0\rangle \otimes |+\rangle \\
&= \sqrt{\frac{1}{3}}|++-\rangle + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}(|+-+\rangle + |-++\rangle) \\
\left|-\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{3}}|-1\rangle \otimes |+\rangle + \sqrt{\frac{2}{3}}|0\rangle \otimes |-\rangle \\
&= \sqrt{\frac{1}{3}}|--+\rangle + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}(|+--\rangle + |-+-\rangle) \\
\left|-\frac{3}{2}\right\rangle &= |-1\rangle \otimes |-\rangle = |---\rangle
\end{aligned}
$$

which gives the transformation matrix

$$
\begin{array}{cc}
 & \begin{array}{cccc} |3/2\rangle & |1/2\rangle & |-1/2\rangle & |-3/2\rangle \end{array} \\
U = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1/3} & 0 & 0 \\ 0 & \sqrt{1/3} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 \\ 0 & \sqrt{1/3} & 0 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 \\ 0 & 0 & \sqrt{1/3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} |+++\rangle \\ |++-\rangle \\ |+-+\rangle \\ |+--\rangle \\ |-++\rangle \\ |-+-\rangle \\ |--+\rangle \\ |---\rangle \end{array}
\end{array}
$$

by which, we obtain $J_x^{(4)}$ and $J_y^{(4)}$ as

$$
J_x^{(4)} = U^\dagger \left[J_x^{(2)} \otimes I \otimes I + I \otimes J_x^{(2)} \otimes I + I \otimes I \otimes J_x^{(2)}\right] U = \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}
$$

$$
J_y^{(4)} = U^\dagger \left[J_y^{(2)} \otimes I \otimes I + I \otimes J_y^{(2)} \otimes I + I \otimes I \otimes J_y^{(2)}\right] U = \frac{\hbar}{2} \begin{bmatrix} 0 & -\sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{bmatrix}
$$