Here we prove the integral representation of $j_l(x)$, i.e., Sakurai equation (6.105).

$$j_l(x) = \frac{1}{2i^l}\int_0^\pi e^{ix\cos\theta}P_l(\cos\theta)\sin\theta d\theta \tag{1}$$

Proof. Let $y = \cos\theta$, and define

$$u_l(x) = \frac{1}{2i^l}\int_{-1}^1 e^{ixy}P_l(y)dy \tag{2}$$

It's straightforward to see that $j_l(x) = u_l(x)$ for $l = 0, 1$:

$$u_0(x) = \frac{1}{2}\int_{-1}^1 e^{ixy}dy = \frac{1}{2ix}(e^{ix} - e^{-ix}) = \frac{\sin x}{x} = j_0(x) \tag{3}$$

$$\begin{aligned} u_1(x) &= \frac{1}{2i}\int_{-1}^1 e^{ixy}ydy = \frac{1}{2i}\frac{1}{ix}\int_{-1}^1 \frac{de^{ixy}}{dy}ydy \\ &= -\frac{1}{2x}\left[e^{ixy}y\Big|_{-1}^1 - \int_{-1}^1 e^{ixy}dy\right] \\ &= -\frac{\cos x}{x} + \frac{\sin x}{x^2} = j_1(x) \end{aligned} \tag{4}$$

To prove $u_l(x) = j_l(x)$ for any $l >= 2$, we will use the recurrence relation of $j_l(x)$ (which will be proved shortly)

$$(l+1)j_{l+1}(x) = lj_{l-1}(x) - (2l+1)j_l'(x) \tag{5}$$

and show that $u_l(x)$ satisfies the same relation

$$(l+1)u_{l+1}(x) = lu_{l-1}(x) - (2l+1)u_l'(x) \tag{6}$$

Once both (5) and (6) are proved, induction argument can be used to conclude that $u_l(x) = j_l(x)$ for any l.

First let's prove (5). Recall from the last notes, we have proved (equation (49))

$$g_l(x) \equiv \frac{j_l(x)}{(-x)^l} = \frac{1}{x}g_{l-1}'(x) \tag{7}$$

for any l.

Then (5) is equivalent to

$$\begin{aligned} (l+1)(-x)^{l+1}g_{l+1} &= l(-x)^{l-1}g_{l-1} - (2l+1)\left[(-x)^l g_l\right]' && \iff \\ (l+1)(-x)^{l+1}\left(\frac{1}{x}g_l'\right) &= l(-x)^{l-1}g_{l-1} - (2l+1)\left[(-x)^l g_l' - l(-x)^{l-1}g_l\right] \quad \text{(drop factor } (-x)^{l-1}) && \iff \\ (l+1)xg_l' &= lg_{l-1} - (2l+1)\left[-xg_l' - lg_l\right] && \iff \\ 0 &= lg_{l-1} + lxg_l' + l(2l+1)g_l \quad \text{(drop factor } l) && \iff \\ 0 &= g_{l-1} + \underbrace{(xg_l' + g_l)}_{(xg_l)' = g_{l-1}''} + 2lg_l && \iff \\ 0 &= g_{l-1} + g_{l-1}'' + 2l\left(\frac{1}{x}g_{l-1}'\right) \end{aligned} \tag{8}$$

which is exactly what we have proved in the last notes (equation (53)), so (5) is proved.

On the other hand (6) is equivalent to

$$\begin{aligned} \int_{-1}^1 e^{ixy}\left[\frac{l+1}{2i^{l+1}}P_{l+1}(y) - \frac{l}{2i^{l-1}}P_{l-1}(y) + \frac{2l+1}{2i^l}(iy)P_l(y)\right]dy = 0 \quad &\iff \\ \int_{-1}^1 e^{ixy}\frac{1}{2i^{l-1}}\left[-(l+1)P_{l+1}(y) - lP_{l-1}(y) + (2l+1)yP_l(y)\right]dy = 0 \end{aligned} \tag{9}$$

which is obviously true given the recurrence relation of Legendre polynomials (proved in earlier notes)

$$(l+1)P_{l+1}(y) = (2l+1)yP_l(y) - lP_{l-1}(y) \tag{10}$$

□