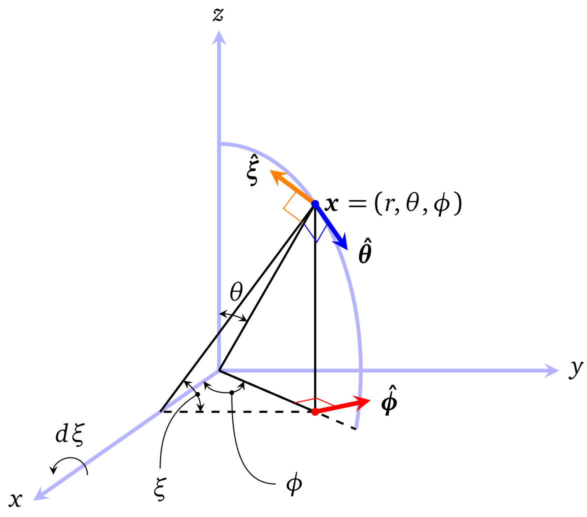

We shall derive equation (3.220), referring to the diagram above.

Let $\boldsymbol{x} = (r = 1, \theta, \phi)$ be a point in spherical coordinates. Consider an infinitesimal rotation around x-axis by angle $d\xi$. We would like to find the expression of the corresponding change in $d\theta$ and $d\phi$ by such an infinitesimal rotation.

In order to do this, we shall consider the infinitesimal arc sweeped by $d\xi$, and project this arc onto the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ direction.

Referring to the diagram, we know the infinitesimal sweeped arc vector is

$$d\boldsymbol{s} = \sqrt{\cos^2\theta + \sin^2\theta \sin^2\phi}\, d\xi \hat{\xi} \tag{1}$$

where the unit vector

$$\hat{\xi} = 0\hat{\mathbf{x}} - \sin\xi\hat{\mathbf{y}} + \cos\xi\hat{\mathbf{z}} \tag{2}$$

We also know that

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \tag{3}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \tag{4}$$

From which we obtain the projection coefficients

$$\hat{\xi}\cdot\hat{\boldsymbol{\theta}} = -\sin\xi\cos\theta\sin\phi - \sin\theta\cos\xi \tag{5}$$

$$\hat{\xi}\cdot\hat{\boldsymbol{\phi}} = -\sin\xi\cos\phi \tag{6}$$

But by geometry, we know

$$\sin\xi = \frac{\cos\theta}{\sqrt{\cos^2\theta + \sin^2\theta\sin^2\phi}} \tag{7}$$

$$\cos\xi = \frac{\sin\theta\sin\phi}{\sqrt{\cos^2\theta + \sin^2\theta\sin^2\phi}} \tag{8}$$

Thus (5) and (6) become

$$\hat{\xi}\cdot\hat{\boldsymbol{\theta}} = -\frac{\cos^2\theta\sin\phi + \sin^2\theta\sin\phi}{\sqrt{\cos^2\theta + \sin^2\theta\sin^2\phi}} = -\frac{\sin\phi}{\sqrt{\cos^2\theta + \sin^2\theta\sin^2\phi}} \tag{9}$$

$$\hat{\xi}\cdot\hat{\boldsymbol{\phi}} = -\frac{\cos\theta\cos\phi}{\sqrt{\cos^2\theta + \sin^2\theta\sin^2\phi}} \tag{10}$$

It is straightforward to verify that the projection is complete in the sense that $(\hat{\xi}\cdot\hat{\boldsymbol{\theta}})^2 + (\hat{\xi}\cdot\hat{\boldsymbol{\phi}})^2 = 1$, as it should since $\hat{\xi}$ does not have a radial component.

Using (1), (9) and (10) to project the arc onto $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ direction, the projected arc lengths are

$$d\boldsymbol{s} \cdot \hat{\boldsymbol{\theta}} = -\sin\phi\, d\xi \tag{11}$$

$$d\boldsymbol{s} \cdot \hat{\boldsymbol{\phi}} = -\cos\theta\cos\phi\, d\xi \tag{12}$$

Then $d\theta$ and $d\phi$ can be obtained by dividing (11) and (12) by the corresponding radius 1 and $\sin\theta$, which gives the relation

$$d\theta = -\sin\phi\, d\xi \tag{13}$$

$$d\phi = -\cot\theta\cos\phi\, d\xi \tag{14}$$

To summarize, when an infinitesimal rotation $d\xi$ around x-axis is applied to position eigenket $|r,\theta,\phi\rangle$, the state changes to $|r,\theta+d\theta,\phi+d\phi\rangle$ according to (13) and (14), i.e.,

$$\left(1-\frac{id\xi L_x}{\hbar}\right)|r,\theta,\phi\rangle = |r,\theta+d\theta,\phi+d\phi\rangle \tag{15}$$

Equivalently

$$\begin{aligned}\langle r,\theta,\phi|\left(1-\frac{id\xi L_x}{\hbar}\right) &= \left[\left(1+\frac{id\xi L_x}{\hbar}\right)|r,\theta,\phi\rangle\right]^\dagger \\ &= \left[\left(1-\frac{i(-d\xi)L_x}{\hbar}\right)|r,\theta,\phi\rangle\right]^\dagger \\ &= (|r,\theta-d\theta,\phi-d\phi\rangle)^\dagger = \langle r,\theta-d\theta,\phi-d\phi| \end{aligned} \tag{16}$$

Therefore

$$\begin{aligned}\left\langle r,\theta,\phi\left|1-\frac{id\xi L_x}{\hbar}\right|\alpha\right\rangle &= \langle r,\theta-d\theta,\phi-d\phi\,|\alpha\rangle \\ &= \langle r,\theta,\phi|\alpha\rangle - \left(\frac{\partial}{\partial\theta}d\theta + \frac{\partial}{\partial\phi}d\phi\right)\langle r,\theta,\phi|\alpha\rangle\end{aligned}$$

From which we identify

$$\begin{aligned}\langle r,\theta,\phi|L_x|\alpha\rangle &= \frac{\hbar}{id\xi}\left(\frac{\partial}{\partial\theta}d\theta + \frac{\partial}{\partial\phi}\phi\right)\langle r,\theta,\phi|\alpha\rangle \\ &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\langle r,\theta,\phi|\alpha\rangle\end{aligned} \tag{17}$$