In these notes, we provide the proofs of various referenced properties of Bessel functions in Sakurai. Also refer to the previous notes for their definitions and basic properties.

1. "Large-x" asymptotic forms:

$$j_l(x) \xrightarrow{\text{large } x} \frac{e^{i[x-(l\pi/2)]} - e^{-i[x-(l\pi/2)]}}{2ix} \qquad \text{(eq 6.116)} \tag{1}$$

$$j_l(x) \xrightarrow{\text{large } x} \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) \qquad \text{(eq 6.175)} \tag{2}$$

$$n_l(x) \xrightarrow{\text{large } x} -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right) \qquad \text{(eq 6.175)} \tag{3}$$

$$h_l^{(1)} \xrightarrow{\text{large } x} \frac{e^{i[x-(l\pi/2)]}}{ix} \qquad \text{(eq 6.133)} \tag{4}$$

$$h_l^{(2)} \xrightarrow{\text{large } x} \frac{e^{-i[x-(l\pi/2)]}}{ix} \qquad \text{(eq 6.133)} \tag{5}$$

Proof. We will prove (2) and (3), the rest will follow trivially. To see this, recall the closed-form expression of spherical Bessel functions and their recurrence relations (proved in earlier notes):

$$j_l(x) = A_l(x)\frac{\sin x}{x} + B_l(x)\cos x \tag{6}$$

$$n_l(x) = S_l(x)\sin x + T_l(x)\frac{\cos x}{x} \tag{7}$$

$$A_0 = 1 \qquad B_0 = 0 \tag{8}$$

$$A_1 = \frac{1}{x} \qquad B_1 = -\frac{1}{x} \tag{9}$$

$$S_0 = 0 \qquad T_0 = -1 \tag{10}$$

$$S_1 = -\frac{1}{x} \qquad T_1 = -\frac{1}{x} \tag{11}$$

$$F_{l+1} = \frac{2l+1}{x}F_l(x) - F_{l-1}(x) \qquad (F = A, B, S, T) \tag{12}$$

The A_l, B_l, S_l, T_l are all degree-l polynomials in $1/x$, which at large x, are dominated by the lowest power term. From the recurrence relations (8)-(12), we can see that

- when $l = 2k$,

$$\begin{aligned} j_l(x) &\approx (-1)^k\frac{\sin x}{x} + O\left(\frac{1}{x^2}\right)\cos x \\ &\approx (-1)^k\frac{\sin x}{x} \\ &= \frac{\sin(x-k\pi)}{x} = \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) \end{aligned} \tag{13}$$

$$\begin{aligned} n_l(x) &\approx O\left(\frac{1}{x^2}\right)\sin x - (-1)^k\frac{\cos x}{x} \\ &\approx -(-1)^k\frac{\cos x}{x} \\ &= -\frac{\cos(x-k\pi)}{x} = -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right) \end{aligned} \tag{14}$$

- when $l = 2k+1$,

$$\begin{aligned}
j_l(x) &\approx O\left(\frac{1}{x^2}\right)\sin x - (-1)^k \frac{\cos x}{x} \\
&\approx -(-1)^k \frac{\cos x}{x} \\
&= \frac{1}{x}\sin\left[x - \frac{(2k+1)\pi}{2}\right] = \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) &(15)\\
n_l(x) &\approx -(-1)^k \frac{\sin x}{x} + O\left(\frac{1}{x^2}\right)\cos x \\
&\approx -(-1)^k \frac{\sin x}{x} \\
&= -\frac{1}{x}\cos\left[x - \frac{(2k+1)\pi}{2}\right] = -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right) &(16)
\end{aligned}$$

□

2. "Small-x" asymptotic forms (eq 6.151):

$$j_l(x) \xrightarrow{\text{small } x} \frac{x^l}{(2l+1)!!} \tag{17}$$

$$n_l(x) \xrightarrow{\text{small } x} -\frac{(2l-1)!!}{x^{l+1}} \tag{18}$$

Proof. For this we use the definition of the spherical Bessel functions

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) \tag{19}$$

$$n_l(x) = (-1)^{l+1}\sqrt{\frac{\pi}{2x}} J_{-(l+1/2)}(x) \tag{20}$$

Recall that

$$J_\nu(x) = \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} \tag{21}$$

Then at small x, (19) and (20) are dominated by the lowest power term ($q = 0$)

$$\begin{aligned}
j_l(x) &= \sqrt{\frac{\pi}{2x}} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+l+1/2}}{q!\Gamma(q+l+3/2)} \\
&\xrightarrow{\text{small } x} \sqrt{\frac{\pi}{2x}} \frac{\left(\frac{x}{2}\right)^{l+1/2}}{\Gamma(l+3/2)} \qquad \text{(recall } \Gamma(n+1/2) = \sqrt{\pi}(2n)!/(4^n n!)\text{)} \\
&= \sqrt{\frac{\pi}{2x}} \sqrt{\frac{x}{2}} \frac{x^l}{2^l} \frac{4^{(l+1)}(l+1)!}{\sqrt{\pi}(2l+2)!} \\
&= x^l \cdot \frac{2^{l+1}(l+1)!}{(2l+2)!} = \frac{x^l}{(2l+1)!!} &(22)\\
n_l(x) &= (-1)^{l+1}\sqrt{\frac{\pi}{2x}} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q-l-1/2}}{q!\Gamma(q-l+1/2)} \\
&\xrightarrow{\text{small } x} (-1)^{l+1}\sqrt{\frac{\pi}{2x}} \frac{\left(\frac{2}{x}\right)^{l+1/2}}{\Gamma(-l+1/2)} \qquad \text{(recall } \Gamma(-n+1/2) = \sqrt{\pi}(-4)^n n!/(2n)!\text{)} \\
&= (-1)^{l+1}\sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{x}} \frac{2^l}{x^l} \frac{(2l)!}{\sqrt{\pi}(-4)^l l!} \\
&= -\frac{1}{x^{l+1}} \cdot \frac{(2l)!}{2^l l!} = -\frac{(2l-1)!!}{x^{l+1}} &(23)
\end{aligned}$$

□

3. In section 6.5.1, there is a claim

$$P_l(\cos\theta) \xrightarrow[\text{small } \theta]{\text{large } l} J_0(l\theta) \qquad \text{(eq 6.170)} \tag{24}$$

Here we give a more general proof

$$P_l(\cos\theta) \xrightarrow{\text{small } \theta} J_0\left(\sqrt{2l(l+1)(1-\cos\theta)}\right) \tag{25}$$

which apparently implies (24) when l is large, but (25) only requires θ to be small without assumptions about l.

Proof. We are going to show equality of (25) up to θ^2 order. Since $\cos\theta \approx 1-\theta^2/2$, we only need to verify that the expansion of both sides of (25) have matching coefficients for θ^0 and θ^2. Note for the RHS,

$$\begin{aligned} J_0\left(\sqrt{2l(l+1)(1-\cos\theta)}\right) &\approx J_0(\sqrt{l(l+1)}\theta) \\ &= \sum_q \frac{(-1)^q \left[\frac{\sqrt{l(l+1)}\theta}{2}\right]^{2q}}{q!q!} \\ &\approx 1 - \frac{l(l+1)\theta^2}{4} \end{aligned} \tag{26}$$

Let the Legendre polynomials be expressed as

$$P_l(x) = \sum_{k=0}^{l} c_{l,k} x^k \tag{27}$$

then the LHS of (25) is (up to θ^2 order):

$$\begin{aligned} P_l(\cos\theta) &= \sum_{k=0}^{l} c_{l,k}\left(1-\frac{\theta^2}{2}\right)^k \\ &\approx \left(\sum_{k=0}^{l} c_{l,k}\right) - \left(\sum_{k=0}^{l} \frac{k}{2} c_{l,k}\theta^2\right) \end{aligned} \tag{28}$$

Compare (26) and (28), it remains to show

$$\sum_{k=0}^{l} c_{l,k} = 1 \tag{29}$$

$$\sum_{k=0}^{l} \frac{k}{2} c_{l,k} = \frac{l(l+1)}{4} \tag{30}$$

(29) is obvious since $\sum c_{l,k}$ is just $P_l(1)$ which is well known to be 1. We now use induction to prove (30). The $l=0,1$ cases are trivial by noting $P_0(x)=1$ and $P_1(x)=x$. Now at $l+1$, recall the recurrence relation of Legendre polynomials

$$P_{l+1} = \frac{2l+1}{l+1} x P_l - \frac{l}{l+1} P_{l-1} \qquad \Longrightarrow \tag{31}$$

$$c_{l+1,k} = \frac{2l+1}{l+2} c_{l,k-1} - \frac{l}{l+1} c_{l-1,k} \tag{32}$$

In (32), index k loops from 0 to $l+1$, with any "out-of-bound" c's set to zero. Now,

$$\begin{aligned}
\sum_{k=0}^{l+1} \frac{k}{2} c_{l+1,k} &= \frac{2l+1}{l+1} \sum_{k=0}^{l+1} \frac{k}{2} c_{l,k-1} - \frac{l}{l+1} \sum_{k=0}^{l+1} \frac{k}{2} c_{l-1,k} \\
&= \frac{2l+1}{l+1} \sum_{k'=-1}^{l} \frac{k'+1}{2} c_{l,k'} - \frac{l}{l+1} \sum_{k=0}^{l-1} \frac{k}{2} c_{l-1,k} && \text{(by induction and equation (29))} \\
&= \frac{2l+1}{l+1} \left[\frac{l(l+1)}{4} + \frac{1}{2} \right] - \frac{l}{l+1} \frac{l(l-1)}{4} \\
&= \frac{(2l+1)(l^2+l+2) - l^2(l-1)}{4(l+1)} \\
&= \frac{2l^3+3l^2+5l+2-l^3+l^2}{4(l+1)} \\
&= \frac{l^3+4l^2+5l+2}{4(l+1)} \\
&= \frac{(l+1)(l^2+3l+2)}{4(l+1)} \\
&= \frac{(l+1)(l+2)}{4} && (33)
\end{aligned}$$

which proved (30) for order $l+1$. □

4. Next, we prove two differential recurrence relationships

$$\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] = x^{\nu} J_{\nu-1}(x) \tag{34}$$

$$\frac{d}{dx}\left[x^{-\nu} J_{\nu}(x)\right] = -x^{-\nu} J_{\nu+1}(x) \tag{35}$$

Proof. These are straightforward to see by applying the definition of J_{ν}.

$$\begin{aligned}
\frac{d}{dx}\left[x^{\nu} J_{\nu}(x)\right] &= \frac{d}{dx}\left[x^{\nu} \sum_{q} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} \right] \\
&= \sum_{q} \frac{(-1)^q 2^{\nu}}{q!\Gamma(q+\nu+1)} \frac{d}{dx} \left(\frac{x}{2}\right)^{2q+2\nu} \\
&= \sum_{q} \frac{(-1)^q 2^{\nu}}{q!\Gamma(q+\nu+1)} \frac{1}{2} (2q+2\nu) \left(\frac{x}{2}\right)^{2q+2\nu-1} \\
&= \sum_{q} \frac{(-1)^q}{q!} \frac{q+\nu}{\Gamma(q+\nu+1)} x^{\nu} \left(\frac{x}{2}\right)^{2q+\nu-1} \\
&= x^{\nu} \sum_{q} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu-1}}{q!\Gamma(q+\nu)} \\
&= x^{\nu} J_{\nu-1}(x) && (36)
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] &= \frac{d}{dx}\left[x^{-\nu}\sum_q \frac{(-1)^q\left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)}\right] \\
&= \sum_q \frac{(-1)^q 2^{-\nu}}{q!\Gamma(q+\nu+1)}\frac{d}{dx}\left(\frac{x}{2}\right)^{2q} \\
&= \sum_{q=0}^{\infty} \frac{(-1)^q 2^{-\nu}}{q!\Gamma(q+\nu+1)}\frac{1}{2}(2q)\left(\frac{x}{2}\right)^{2q-1} && \text{(drop } q=0 \text{ term and relabel } q'=q-1\text{)} \\
&= \sum_{q'=0}^{\infty} \frac{(-1)(-1)^{q'}}{q'!\Gamma[q'+(\nu+1)+1]}x^{-\nu}\left(\frac{x}{2}\right)^{2(q'+1)+\nu-1} \\
&= -x^{-\nu}\sum_{q'=0}^{\infty} \frac{(-1)^{q'}\left(\frac{x}{2}\right)^{2q'+\nu+1}}{q'!\Gamma[q'+(\nu+1)+1]} \\
&= -x^{-\nu}J_{\nu+1}(x) \qquad (37)
\end{aligned}
$$

□

In particular setting $\nu = 1$ in (34) and $\nu = 0$ in (35), we have

$$
(xJ_1)' = xJ_0 \iff xJ_1' + J_1 = xJ_0 \qquad (38)
$$

$$
J_0' = -J_1 \qquad (39)
$$

5. With (38) and (39), we can prove the unproved relation in Sakurai (6.179) and (6.180). Equation (6.179) is a direct application of (38), while for (6.180) we need to show

$$
\int_0^\infty J_1^2(x)\frac{dx}{x} = \frac{1}{2} \qquad (40)
$$

Proof. By (38),

$$
\begin{aligned}
\int_0^\infty \frac{J_1^2(x)}{x}dx &= \int_0^\infty J_1(J_0 - J_1')dx && \text{(by (39))} \\
&= \int_0^\infty (-J_0'J_0 - J_1'J_1)dx \\
&= -\frac{J_0^2+J_1^2}{2}\bigg|_0^\infty \\
&= \frac{1}{2} \qquad (41)
\end{aligned}
$$

□