By eq (5.276), with $|i\rangle = |0\rangle$,

$$\begin{aligned}
c_n^{(1)}(t) &= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{n0}t'}V_{n0}(t')dt' \\
&= \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{n0}t'}\langle n|F_0 x\cos\omega t'|0\rangle dt' \\
&= \frac{-iF_0}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\int_{t_0}^{t} e^{i\omega_{n0}t'}\cos\omega t'\langle n|a+a^\dagger|0\rangle dt' \qquad \text{(only } n=1 \text{ survives, with } \omega_{10}=\omega_0\text{)} \\
&= \frac{-iF_0}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\int_{t_0}^{t} e^{i\omega_0 t'}\cos\omega t' dt' \\
&= \frac{-iF_0}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\int_{t_0}^{t} e^{i\omega_0 t'}\frac{1}{2}\left(e^{i\omega t'}+e^{-i\omega t'}\right)dt' \\
&= \frac{-iF_0}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\frac{1}{2}\left[\frac{e^{i(\omega_0+\omega)t}-1}{i(\omega_0+\omega)}+\frac{e^{i(\omega_0-\omega)t}-1}{i(\omega_0-\omega)}\right] \\
&= -\frac{F_0}{2\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega}+\frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right]
\end{aligned} \tag{1}$$

To evaluate $\langle x\rangle$ at t, we will have to obtain the Schrödinger-picture final state by applying $e^{-iH_0t/\hbar}$ to $|\alpha,t_0;t\rangle_I = |0\rangle + c_1^{(1)}(t)|1\rangle$:

$$|\alpha,t_0;t\rangle_S = e^{-iH_0t/\hbar}|0\rangle + e^{-iH_0t/\hbar}c_1^{(1)}(t)|1\rangle = e^{-i\omega_0t/2}|0\rangle + e^{-i3\omega_0t/2}c_1^{(1)}(t)|1\rangle \tag{2}$$

which gives

$$\begin{aligned}
\langle x\rangle_S &= \sqrt{\frac{\hbar}{2m\omega_0}}\langle\alpha,t_0;t|a+a^\dagger|\alpha,t_0;t\rangle_S \\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\left[e^{i\omega_0t/2}e^{-i3\omega_0t/2}c_1^{(1)}(t)+e^{i3\omega_0t/2}e^{-i\omega_0t/2}c_1^{(1)*}(t)\right] \\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\left[e^{-i\omega_0t}c_1^{(1)}(t)+e^{i\omega_0t}c_1^{(1)*}(t)\right] \\
&= -\frac{F_0}{4m\omega_0}\left[\frac{e^{i\omega t}-e^{-i\omega_0t}}{\omega_0+\omega}+\frac{e^{-i\omega t}-e^{-i\omega_0t}}{\omega_0-\omega}+\frac{e^{-i\omega t}-e^{i\omega_0t}}{\omega_0+\omega}+\frac{e^{i\omega t}-e^{i\omega_0t}}{\omega_0-\omega}\right] \\
&= -\frac{F_0}{4m\omega_0}\left[\frac{2\cos\omega t-2\cos\omega_0t}{\omega_0+\omega}+\frac{2\cos\omega t-2\cos\omega_0t}{\omega_0-\omega}\right] \\
&= -\frac{F_0}{2m\omega_0}(\cos\omega t-\cos\omega_0t)\left(\frac{1}{\omega_0+\omega}+\frac{1}{\omega_0-\omega}\right) \\
&= -\frac{F_0}{m}\frac{\cos\omega t-\cos\omega_0t}{\omega_0^2-\omega^2}
\end{aligned} \tag{3}$$

Note the expected value $\langle x\rangle$ as a function of t agrees with classical solution to the forced oscillator

$$m\frac{d^2x}{dt^2}+m\omega_0^2x+F_0\cos\omega t=0 \tag{4}$$

whose solution, with boundary condition $x(0)=0, \dot{x}(0)=0$, is exactly (3).