

We will prove the referenced *addition theorem of spherical harmonics*

$$\sum_m Y_l^m(\hat{r}) Y_l^{m*}(\hat{k}) = \frac{2l+1}{4\pi} P_l(\hat{r} \cdot \hat{k}) \quad (1)$$

By Sakurai equation (3.260)

$$Y_l^m(\hat{r}) = \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}_{m0}^{(l)*}(\phi_{\hat{r}}, \theta_{\hat{r}}, 0) \quad (2)$$

$$Y_l^{m*}(\hat{k}) = \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}_{m0}^{(l)}(\phi_{\hat{k}}, \theta_{\hat{k}}, 0) \quad (3)$$

Then we have

$$\begin{aligned} \sum_m Y_l^m(\hat{r}) Y_l^{m*}(\hat{k}) &= \frac{2l+1}{4\pi} \sum_m \mathcal{D}_{m0}^{(l)*}(\phi_{\hat{r}}, \theta_{\hat{r}}, 0) \mathcal{D}_{m0}^{(l)}(\phi_{\hat{k}}, \theta_{\hat{k}}, 0) \\ &= \frac{2l+1}{4\pi} \sum_m [\mathcal{D}^{(l)\dagger}(\phi_{\hat{r}}, \theta_{\hat{r}}, 0)]_{m0}^T \mathcal{D}_{m0}^{(l)}(\phi_{\hat{k}}, \theta_{\hat{k}}, 0) \\ &= \frac{2l+1}{4\pi} \sum_m [\mathcal{D}^{(l)\dagger}(\phi_{\hat{r}}, \theta_{\hat{r}}, 0)]_{0m} \mathcal{D}_{m0}^{(l)}(\phi_{\hat{k}}, \theta_{\hat{k}}, 0) \\ &= \frac{2l+1}{4\pi} [\mathcal{D}^{(l)\dagger}(\phi_{\hat{r}}, \theta_{\hat{r}}, 0) \cdot \mathcal{D}^{(l)}(\phi_{\hat{k}}, \theta_{\hat{k}}, 0)]_{00} \end{aligned} \quad (4)$$

Recall that $\mathcal{D}^{(l)}(\alpha, \beta, 0)$ is just the l -dimensional representation of the rotation operator $R_z(\alpha)R_y(\beta)$ (Sakurai Fig 3.3), so

$$\mathcal{D}^{(l)\dagger}(\phi_{\hat{r}}, \theta_{\hat{r}}, 0) \cdot \mathcal{D}^{(l)}(\phi_{\hat{k}}, \theta_{\hat{k}}, 0)$$

is just the l -dimensional representation of the rotation operator

$$R_y^\dagger(\theta_{\hat{r}}) R_z^\dagger(\phi_{\hat{r}}) R_z(\phi_{\hat{k}}) R_y(\theta_{\hat{k}}) = R_y(-\theta_{\hat{r}}) R_z(\phi_{\hat{k}} - \phi_{\hat{r}}) R_y(\theta_{\hat{k}}) \quad (5)$$

Also recall Sakurai equation (3.262)

$$\mathcal{D}_{00}^{(l)}(\phi, \theta, 0) = P_l(\cos \theta) \quad (6)$$

Compare (4)-(6) with (1), it remains to prove that

$$R_y(-\theta_{\hat{r}}) R_z(\phi_{\hat{k}} - \phi_{\hat{r}}) R_y(\theta_{\hat{k}}) = R_z(\phi) R_y(\angle_{\hat{k}, \hat{r}}) \quad (7)$$

which is a rotation with polar angle equal to the angle between \hat{k} and \hat{r} , and some azimuthal angle ϕ for which we don't care about.

To see the LHS of (7) has an effective polar angle of $\angle_{\hat{k}, \hat{r}}$, it's sufficient to prove that it rotates the \hat{z} vector into a vector whose z component equals to $\cos \angle_{\hat{k}, \hat{r}} = \hat{k} \cdot \hat{r}$.

Indeed, on the one hand

$$\begin{aligned} R_y(-\theta_{\hat{r}}) R_z(\phi_{\hat{k}} - \phi_{\hat{r}}) R_y(\theta_{\hat{k}}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= R_y(-\theta_{\hat{r}}) R_z(\phi_{\hat{k}} - \phi_{\hat{r}}) \begin{bmatrix} \cos \theta_{\hat{k}} & 0 & \sin \theta_{\hat{k}} \\ 0 & 1 & 0 \\ -\sin \theta_{\hat{k}} & 0 & \cos \theta_{\hat{k}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= R_y(-\theta_{\hat{r}}) \begin{bmatrix} \cos(\phi_{\hat{k}} - \phi_{\hat{r}}) & -\sin(\phi_{\hat{k}} - \phi_{\hat{r}}) & 0 \\ \sin(\phi_{\hat{k}} - \phi_{\hat{r}}) & \cos(\phi_{\hat{k}} - \phi_{\hat{r}}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \theta_{\hat{k}} \\ 0 \\ \cos \theta_{\hat{k}} \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta_{\hat{r}} & 0 & -\sin \theta_{\hat{r}} \\ 0 & 1 & 0 \\ \sin \theta_{\hat{r}} & 0 & \cos \theta_{\hat{r}} \end{bmatrix} \begin{bmatrix} \sin \theta_{\hat{k}} \cos(\phi_{\hat{k}} - \phi_{\hat{r}}) \\ \sin \theta_{\hat{k}} \sin(\phi_{\hat{k}} - \phi_{\hat{r}}) \\ \cos \theta_{\hat{k}} \end{bmatrix} \\ &= \begin{bmatrix} \vdots \\ \vdots \\ \sin \theta_{\hat{r}} \sin \theta_{\hat{k}} \cos(\phi_{\hat{k}} - \phi_{\hat{r}}) + \cos \theta_{\hat{r}} \cos \theta_{\hat{k}} \end{bmatrix} \end{aligned} \quad (8)$$

and on the other hand

$$\begin{aligned}\cos \angle_{\hat{\mathbf{k}}, \hat{\mathbf{r}}} &= \hat{\mathbf{k}} \cdot \hat{\mathbf{r}} \\ &= \begin{bmatrix} \sin \theta_{\hat{\mathbf{k}}} \cos \phi_{\hat{\mathbf{k}}} \\ \sin \theta_{\hat{\mathbf{k}}} \sin \phi_{\hat{\mathbf{k}}} \\ \cos \theta_{\hat{\mathbf{k}}} \end{bmatrix} \cdot \begin{bmatrix} \sin \theta_{\hat{\mathbf{r}}} \cos \phi_{\hat{\mathbf{r}}} \\ \sin \theta_{\hat{\mathbf{r}}} \sin \phi_{\hat{\mathbf{r}}} \\ \cos \theta_{\hat{\mathbf{r}}} \end{bmatrix} \\ &= \sin \theta_{\hat{\mathbf{k}}} \sin \theta_{\hat{\mathbf{r}}} (\cos \phi_{\hat{\mathbf{k}}} \cos \phi_{\hat{\mathbf{r}}} + \sin \phi_{\hat{\mathbf{k}}} \sin \phi_{\hat{\mathbf{r}}}) + \cos \theta_{\hat{\mathbf{k}}} \cos \theta_{\hat{\mathbf{r}}} \\ &= \sin \theta_{\hat{\mathbf{k}}} \sin \theta_{\hat{\mathbf{r}}} \cos(\phi_{\hat{\mathbf{k}}} - \phi_{\hat{\mathbf{r}}}) + \cos \theta_{\hat{\mathbf{k}}} \cos \theta_{\hat{\mathbf{r}}}\end{aligned}\tag{9}$$