These notes are basically the same treatment as Sakurai, but hopefully will clarify some vague places in the original texts.

1. **Lippmann-Schwinger Equation**

In time-dependent potential theory, let the Hamiltonian of the particle be $H_0 + V(t)$, and let t_0 be the initial time of the particle. Let $|\alpha, t\rangle_S$ be any state at time t in the Schrödinger picture. The interaction picture state ket is defined as

$$|\alpha, t\rangle_I = e^{iH_0 t/\hbar}|\alpha, t\rangle_S \tag{1}$$

Rewriting the Schrödinger equation in the interaction picture

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\alpha, t\rangle_S &= H|\alpha, t\rangle_S && \Longrightarrow\\
i\hbar\frac{\partial}{\partial t}\left(e^{-iH_0 t/\hbar}|\alpha, t\rangle_I\right) &= [H_0 + V(t)]e^{-iH_0 t/\hbar}|\alpha, t\rangle_I && \Longrightarrow\\
i\hbar\left(-\frac{iH_0}{\hbar}e^{-iH_0 t/\hbar}|\alpha, t\rangle_I + e^{-iH_0 t/\hbar}\frac{\partial}{\partial t}|\alpha, t\rangle_I\right) &= [H_0 + V(t)]e^{-iH_0 t/\hbar}|\alpha, t\rangle_I && \Longrightarrow\\
i\hbar\frac{\partial}{\partial t}|\alpha, t\rangle_I &= \overbrace{\left[e^{iH_0 t/\hbar}V(t)e^{-iH_0 t/\hbar}\right]}^{\equiv V_I(t)}|\alpha, t\rangle_I && (2)
\end{aligned}$$

Let $U_I(t, t_0)$ be the time evolution operator from $t_0 \to t$ in the interaction picture, where

$$|\alpha, t\rangle_I = U_I(t, t_0)|\alpha, t_0\rangle_I \tag{3}$$

Then (2) can be rewritten as

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left[U_I(t, t_0)|\alpha, t_0\rangle_I\right] &= V_I(t)U_I(t, t_0)|\alpha, t_0\rangle_I && \Longrightarrow\\
i\hbar\frac{d}{dt}U_I(t, t_0) &= V_I(t)U_I(t, t_0) && (4)
\end{aligned}$$

Integrating both sides, we have

$$\begin{aligned}
U_I(t, t_0) &= U_I(t_0, t_0) - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t', t_0)dt'\\
&= 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')U_I(t', t_0)dt' && (5)
\end{aligned}$$

In the scattering process, the scatterer potential is time independent, but we treat the potential as being turned on at some time when incident particle is in the vicinity of the potential, and we take H_0 to be the free-partical Hamiltonian $\boldsymbol{p}^2/2m$. If the incident (free) particle was originally (at t_0) in some energy eigenstate $|i\rangle_S$, we would like to determine the probability that at t, the particle is in the energy eigenstate $|n\rangle_S$ after the scatterer.

By definition of $U_I(t, t_0)$, this amplitude is easily expressed in the interaction picture

$$\langle n|U_I(t, t_0)|i\rangle_I = \left\langle n\middle|e^{-iH_0 t/\hbar}U_I(t, t_0)e^{iH_0 t_0/\hbar}\middle|i\right\rangle_S = e^{i(\omega_i t_0 - \omega_n t)}\langle n|U_I(t, t_0)|i\rangle_S \tag{6}$$

which shows that for the fixed t_0, t, the transition amplitude from $|i, t_0\rangle_I$ to $|n, t\rangle_I$ in the interaction picture is equivalent, up to a global phase factor $e^{i(\omega_i t_0 - \omega_n t)}$, to the matrix element of $U_I(t, t_0)$ between Schrödinger-picture energy eigenstates $|n\rangle_S, |i\rangle_S$. In scattering process, we will generally take $t_0 \to -\infty$ and $t \to \infty$. Therefore from now on, we will make $|n\rangle, |i\rangle$ etc. represent the Schrödinger picture energy eigenstates, without loss of generality for purpose of probability calculation.

Then (5) gives

$$\begin{aligned}
\langle n|U_I(t, t_0)|i\rangle &= \delta_{ni} - \frac{i}{\hbar}\int_{t_0}^{t}\sum_m \langle n|V_I(t')|m\rangle\langle m|U_I(t', t_0)|i\rangle dt'\\
&= \delta_{ni} - \frac{i}{\hbar}\sum_m \int_{t_0}^{t}\left\langle n\middle|e^{iH_0 t'/\hbar}V(t')e^{-iH_0 t'/\hbar}\middle|m\right\rangle\langle m|U_I(t', t_0)|i\rangle dt'\\
&= \delta_{ni} - \frac{i}{\hbar}\sum_m V_{nm}\int_{t_0}^{t} e^{i\omega_{nm}t'}\langle m|U_I(t', t_0)|i\rangle dt' && (7)
\end{aligned}$$

(7) is not a solution, but an integral equation. The idea is we can substitute $\langle m|U_I(t',t_0)|i\rangle$ by (7) to certain order to obtain the next order solution. For example, if we use the 0th order solution

$$\langle n|U_I(t,t_0)|i\rangle\Big|_0 = \delta_{ni}$$

in the integrand, we obtain the 1st order solution

$$\langle n|U_I(t,t_0)|i\rangle\Big|_1 = \delta_{ni} - \frac{i}{\hbar}V_{ni}\int_{t_0}^{t} e^{i\omega_{ni}t'}dt' = \delta_{ni} - \frac{i}{\hbar}V_{ni}\frac{e^{i\omega_{ni}t} - e^{i\omega_{ni}t_0}}{i\omega_{ni}}$$

We can keep iteratively getting to the higher orders in this fashion, but now we hope the fact that we are only interested in the asymptotic behavior where t_0 and t are in the far past or future can give us simpler forms directly.

In other words, we are interested in the limit

$$S^{(+)}_{ni} \equiv \lim_{t\to\infty}\lim_{t_0\to-\infty}\langle n|U_I(t,t_0)|i\rangle \tag{8}$$

But when we play the scattering process backwards, i.e., $t_0\to\infty, t\to-\infty$, the corresponding limit will be

$$S^{(-)}_{ni} \equiv \lim_{t\to-\infty}\lim_{t_0\to\infty}\langle n|U_I(t,t_0)|i\rangle \tag{9}$$

For this, we introduce a guessed solution of (7), denoted $\langle n|\widetilde{U}_I(t,t_0)|i\rangle$, which has the form

$$\langle n|\widetilde{U}_I(t,t_0)|i\rangle = \delta_{ni} - \frac{i}{\hbar}T_{ni}\int_{t_0}^{t} e^{i\omega_{ni}t'+\epsilon t'}dt' \tag{10}$$

where T_{ni} is to be determined. We hope to obtain the requirements for ϵ and T_{ni} by making (10) compatible with (7) in the two limiting situations $t_0\to-\infty, t\to\infty$ and $t_0\to\infty, t\to-\infty$ respectively.

Without assuming the sign of ϵ yet, we can integrate (10) to obtain

$$\langle n|\widetilde{U}_I(t,t_0)|i\rangle = \delta_{ni} - \frac{i}{\hbar}T_{ni}\frac{e^{(i\omega_{ni}+\epsilon)t} - e^{(i\omega_{ni}+\epsilon)t_0}}{i\omega_{ni}+\epsilon} \tag{11}$$

For the two limiting situations:

- $\lim_{t\to\infty}\lim_{t_0\to-\infty}$, (10) will not blow up only if $\epsilon>0$. But before taking the limit $t\to\infty$, we must first let $\epsilon\to 0^+$ from the right, so (8) will be well defined.
- $\lim_{t\to-\infty}\lim_{t_0\to\infty}$, (10) will not blow up only if $\epsilon<0$. But before taking the limit $t\to-\infty$, we must first let $\epsilon\to 0^-$ from the left, so (9) will be well defined.

To summarize, we have two forms of the guessed solution (10) (below, we set $\epsilon>0$ in both cases, but use signs $\pm$ to distinguish):

$$\langle n|\widetilde{U}^{(+)}_I(t,t_0)|i\rangle = \delta_{ni} - \frac{i}{\hbar}T^{(+)}_{ni}\int_{t_0}^{t} e^{i\omega_{ni}t'+\epsilon t'}dt' \qquad\Longrightarrow\qquad S^{(+)}_{ni} = \lim_{t\to\infty}\lim_{\epsilon\to0^+}\lim_{t_0\to-\infty}\langle n|\widetilde{U}^{(+)}_I(t,t_0)|i\rangle \tag{12}$$

$$\langle n|\widetilde{U}^{(-)}_I(t,t_0)|i\rangle = \delta_{ni} - \frac{i}{\hbar}T^{(-)}_{ni}\int_{t_0}^{t} e^{i\omega_{ni}t'-\epsilon t'}dt' \qquad\Longrightarrow\qquad S^{(-)}_{ni} = \lim_{t\to-\infty}\lim_{\epsilon\to0^+}\lim_{t_0\to\infty}\langle n|\widetilde{U}^{(-)}_I(t,t_0)|i\rangle \tag{13}$$

The guessed solution will now be written in general as

$$\begin{aligned}\langle n|\widetilde{U}^{(\pm)}_I(t,t_0)|i\rangle &= \delta_{ni} - \frac{i}{\hbar}T^{(\pm)}_{ni}\int_{t_0}^{t} e^{i\omega_{ni}t'\pm\epsilon t'}dt' \\ &= \delta_{ni} - \frac{i}{\hbar}T^{(\pm)}_{ni}\frac{e^{(i\omega_{ni}\pm\epsilon)t} - e^{(i\omega_{ni}\pm\epsilon)t_0}}{i\omega_{ni}\pm\epsilon}\end{aligned} \tag{14}$$

Now recall that (7) is an exact equation, and (14) is our guessed solution of (7) in the two limiting situations $t_0\to-\infty, t\to\infty$ and $t_0\to\infty, t\to-\infty$. Let's find the restriction on $T^{(\pm)}_{ni}$ if (14) and (7) are compatible.

- For the $+\epsilon$ case, take the limit (12) of (14) directly,

$$\begin{aligned}S_{ni}^{(+)} &= \lim_{t\to\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to-\infty}\langle n|\widetilde{U}_I^{(+)}(t,t_0)|i\rangle \\ &= \lim_{t\to\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to-\infty}\left[\delta_{ni}-\frac{i}{\hbar}T_{ni}^{(+)}\frac{e^{(i\omega_{ni}+\epsilon)t}-e^{(i\omega_{ni}+\epsilon)t_0}}{i\omega_{ni}+\epsilon}\right] \\ &= \lim_{t\to\infty}\lim_{\epsilon\to 0^+}\left[\delta_{ni}-\frac{i}{\hbar}T_{ni}^{(+)}\frac{e^{(i\omega_{ni}+\epsilon)t}}{i\omega_{ni}+\epsilon}\right]\end{aligned} \tag{15}$$

  On the other hand, if we plug (14) into the integrand of (7) and then take the limit (12),

$$\begin{aligned}S_{ni}^{(+)} &= \lim_{t\to\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to-\infty}\langle n|\widetilde{U}_I^{(+)}(t,t_0)|i\rangle \\ &= \lim_{t\to\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to-\infty}\left\{\delta_{ni}-\frac{i}{\hbar}\sum_m V_{nm}\int_{t_0}^{t}e^{i\omega_{nm}t'}\left[\delta_{mi}-\frac{i}{\hbar}T_{mi}^{(+)}\frac{e^{(i\omega_{mi}+\epsilon)t'}-e^{(i\omega_{mi}+\epsilon)t_0}}{i\omega_{mi}+\epsilon}\right]dt'\right\} \\ &= \lim_{t\to\infty}\lim_{\epsilon\to 0^+}\left[\delta_{ni}-\frac{i}{\hbar}V_{ni}\int_{-\infty}^{t}e^{i\omega_{ni}t'}dt'+\left(\frac{i}{\hbar}\right)^2\sum_m\frac{V_{nm}T_{mi}^{(+)}}{i\omega_{mi}+\epsilon}\int_{-\infty}^{t}e^{i\omega_{nm}t'}e^{(i\omega_{mi}+\epsilon)t'}dt'\right] \\ &= \lim_{t\to\infty}\lim_{\epsilon\to 0^+}\left[\delta_{ni}-\frac{i}{\hbar}V_{ni}\frac{e^{(i\omega_{ni}+\epsilon)t}}{i\omega_{ni}+\epsilon}+\left(\frac{i}{\hbar}\right)^2\sum_m\frac{V_{nm}T_{mi}^{(+)}}{i\omega_{mi}+\epsilon}\frac{e^{(i\omega_{ni}+\epsilon)t}}{i\omega_{ni}+\epsilon}\right]\end{aligned} \tag{16}$$

  where in the first integral of the last step, we have silently multiplied the integral by $e^{\epsilon t'}$ which makes no difference after the $\lim_{\epsilon\to 0^+}$.
  Compare (15) and (16), we can identify

$$T_{ni}^{(+)} = V_{ni}-\frac{i}{\hbar}\sum_m\frac{V_{nm}T_{mi}^{(+)}}{i\omega_{mi}+\epsilon} = V_{ni}+\sum_m\frac{V_{nm}T_{mi}^{(+)}}{E_i-E_m+i\hbar\epsilon} \tag{17}$$

- For the $-\epsilon$ case, take the limit (13) of (14) directly,

$$\begin{aligned}S_{ni}^{(-)} &= \lim_{t\to-\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to\infty}\langle n|\widetilde{U}_I^{(-)}(t,t_0)|i\rangle \\ &= \lim_{t\to-\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to\infty}\left[\delta_{ni}-\frac{i}{\hbar}T_{ni}^{(-)}\frac{e^{(i\omega_{ni}-\epsilon)t}-e^{(i\omega_{ni}-\epsilon)t_0}}{i\omega_{ni}-\epsilon}\right] \\ &= \lim_{t\to-\infty}\lim_{\epsilon\to 0^+}\left[\delta_{ni}-\frac{i}{\hbar}T_{ni}^{(-)}\frac{e^{(i\omega_{ni}-\epsilon)t}}{i\omega_{ni}-\epsilon}\right]\end{aligned} \tag{18}$$

  On the other hand, if we plug (14) into the integrand of (7) and then take the limit (13),

$$\begin{aligned}S_{ni}^{(-)} &= \lim_{t\to-\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to\infty}\langle n|\widetilde{U}_I^{(-)}(t,t_0)|i\rangle \\ &= \lim_{t\to-\infty}\lim_{\epsilon\to 0^+}\lim_{t_0\to\infty}\left\{\delta_{ni}-\frac{i}{\hbar}\sum_m V_{nm}\int_{t_0}^{t}e^{i\omega_{nm}t'}\left[\delta_{mi}-\frac{i}{\hbar}T_{mi}^{(-)}\frac{e^{(i\omega_{mi}-\epsilon)t'}-e^{(i\omega_{mi}-\epsilon)t_0}}{i\omega_{mi}-\epsilon}\right]dt'\right\} \\ &= \lim_{t\to-\infty}\lim_{\epsilon\to 0^+}\left[\delta_{ni}-\frac{i}{\hbar}V_{ni}\int_{\infty}^{t}e^{i\omega_{ni}t'}dt'+\left(\frac{i}{\hbar}\right)^2\sum_m\frac{V_{nm}T_{mi}^{(-)}}{i\omega_{mi}-\epsilon}\int_{\infty}^{t}e^{i\omega_{nm}t'}e^{(i\omega_{mi}-\epsilon)t'}dt'\right] \\ &= \lim_{t\to-\infty}\lim_{\epsilon\to 0^+}\left[\delta_{ni}-\frac{i}{\hbar}V_{ni}\frac{e^{(i\omega_{ni}-\epsilon)t}}{i\omega_{ni}-\epsilon}+\left(\frac{i}{\hbar}\right)^2\sum_m\frac{V_{nm}T_{mi}^{(-)}}{i\omega_{mi}-\epsilon}\frac{e^{(i\omega_{ni}-\epsilon)t}}{i\omega_{ni}-\epsilon}\right]\end{aligned} \tag{19}$$

  where in the first integral of the last step, we have silently multiplied the integral by $e^{-\epsilon t'}$ which makes no difference after the $\lim_{\epsilon\to 0^+}$.
  Compare (18) and (19), we can identify

$$T_{ni}^{(-)} = V_{ni}-\frac{i}{\hbar}\sum_m\frac{V_{nm}T_{mi}^{(-)}}{i\omega_{mi}-\epsilon} = V_{ni}+\sum_m\frac{V_{nm}T_{mi}^{(-)}}{E_i-E_m-i\hbar\epsilon} \tag{20}$$

  Or, combining (17) and(20) together,

$$T_{ni}^{(\pm)} = V_{ni}+\sum_m\frac{V_{nm}T_{mi}^{(\pm)}}{E_i-E_m\pm i\hbar\epsilon} \tag{21}$$

Now if we define

$$|\phi^{(\pm)}\rangle \equiv \sum_n T_{ni}^{(\pm)}|n\rangle \qquad \Longrightarrow \qquad T_{ni}^{(\pm)} = \langle n|\phi^{(\pm)}\rangle \tag{22}$$

then (21) gives

$$\begin{aligned}\langle n|\phi^{(\pm)}\rangle &= \langle n|V|i\rangle + \sum_m \frac{\langle n|V|m\rangle\langle m|\phi^{(\pm)}\rangle}{E_i - E_m \pm i\hbar\epsilon} \qquad\qquad \Longrightarrow \\ |\phi^{(\pm)}\rangle &= V|i\rangle + \sum_m \frac{V|m\rangle\langle m|\phi^{(\pm)}\rangle}{E_i - E_m \pm i\hbar\epsilon} \\ &= V|i\rangle + V\frac{1}{E_i - H_0 \pm i\hbar\epsilon}|\phi^{(\pm)}\rangle \end{aligned}\tag{23}$$

Now if somehow we can introduce $|\psi^{(\pm)}\rangle$ to satisfy

$$|\phi^{(\pm)}\rangle = V|\psi^{(\pm)}\rangle \tag{24}$$

then (23) gives rise to the Lippmann-Schwinger equation for $|\psi^{(\pm)}\rangle$:

$$|\psi^{(\pm)}\rangle = |i\rangle + \frac{1}{E_i - H_0 \pm i\hbar\epsilon}V|\psi^{(\pm)}\rangle \tag{25}$$

(Note here while $|\phi^{(\pm)}\rangle$ can be explicitly constructed from (22) given $T_{ni}^{(\pm)}$, (24) does not uniquely define what $|\psi^{(\pm)}\rangle$ is since V, a Hermitian operator, may not be invertible.)

2. **Scattering Amplitude**

In coordinate basis, (25) has the form (rescaling $\hbar\epsilon \to \epsilon$)

$$\begin{aligned}\langle \boldsymbol{x}|\psi^{(\pm)}\rangle &= \langle \boldsymbol{x}|i\rangle + \left\langle \boldsymbol{x}\left|\frac{1}{E_i - H_0 \pm i\epsilon}V\right|\psi^{(\pm)}\right\rangle \\ &= \langle \boldsymbol{x}|i\rangle + \int_{\boldsymbol{x}'} d^3\boldsymbol{x}' \underbrace{\left\langle \boldsymbol{x}\left|\frac{1}{E_i - H_0 \pm i\epsilon}\right|\boldsymbol{x}'\right\rangle}_{\equiv G_\pm(\boldsymbol{x},\boldsymbol{x}')}\langle \boldsymbol{x}'|V|\psi^{(\pm)}\rangle \end{aligned}\tag{26}$$

where

$$\begin{aligned}G_\pm(\boldsymbol{x},\boldsymbol{x}') &= \left\langle \boldsymbol{x}\left|\frac{1}{E_i - H_0 \pm i\epsilon}\right|\boldsymbol{x}'\right\rangle \\ &= \sum_{\boldsymbol{k}',\boldsymbol{k}''}\langle \boldsymbol{x}|\boldsymbol{k}'\rangle\left\langle \boldsymbol{k}'\left|\frac{1}{E_i - H_0 \pm i\epsilon}\right|\boldsymbol{k}''\right\rangle\langle \boldsymbol{k}''|\boldsymbol{x}'\rangle \qquad (\text{denote } E_i = \hbar^2k^2/2m) \\ &= \sum_{\boldsymbol{k}',\boldsymbol{k}''}\langle \boldsymbol{x}|\boldsymbol{k}'\rangle\left\langle \boldsymbol{k}'\left|\frac{2m/\hbar^2}{k^2 - k''^2 \pm i\epsilon}\right|\boldsymbol{k}''\right\rangle\langle \boldsymbol{k}''|\boldsymbol{x}'\rangle \\ &= \frac{2m}{\hbar^2}\sum_{\boldsymbol{k}'}\frac{e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}}{\sqrt{L^3}}\cdot\frac{1}{k^2 - k'^2 \pm i\epsilon}\cdot\frac{e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}'}}{\sqrt{L^3}} \\ &= \frac{2m}{\hbar^2L^3}\sum_{\boldsymbol{k}'}\frac{e^{i\boldsymbol{k}'\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{k^2 - k'^2 \pm i\epsilon}\end{aligned}\tag{27}$$

Since

$$\boldsymbol{k}' = \frac{2\pi}{L}\left(n_x\hat{\boldsymbol{x}} + n_y\hat{\boldsymbol{y}} + n_y\hat{\boldsymbol{y}}\right)$$

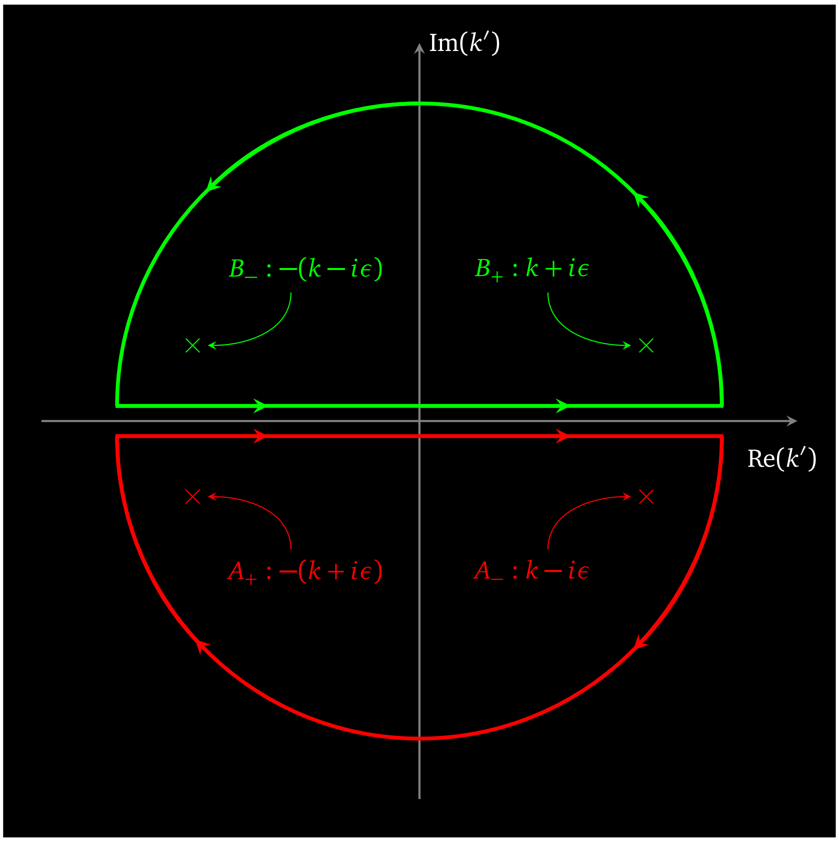
The sum in (27) can be seen to be done over all the grid points (i.e., $\Delta n_x = \Delta n_y = \Delta n_z = 1$)

$$\sum_{\boldsymbol{k}'} = \sum_{n_x,n_y,n_z}\Delta n_x\Delta n_y\Delta n_z = \sum_{k'_x,k'_y,k'_z}\Delta k'_x\Delta k'_y\Delta k'_z\frac{L^3}{(2\pi)^3}$$

when L is large enough, $\boldsymbol{k}'$ becomes continuous, where the sum in (27) becomes an integral over $\boldsymbol{k}'$ space, where we now integrate using spherical coordinates (with z axis aligned with $\boldsymbol{x}-\boldsymbol{x}'$):

$$\begin{aligned}
G_{\pm}(\boldsymbol{x},\boldsymbol{x}') &= \frac{2m}{\hbar^2 L^3}\int_{k'}\frac{L^3}{(2\pi)^3}d^3\boldsymbol{k}'\frac{e^{i\boldsymbol{k}'\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{k^2-k'^2\pm i\epsilon} && (d\equiv|\boldsymbol{x}-\boldsymbol{x}'|)\\
&= \frac{m}{4\pi^3\hbar^2}\int_0^{2\pi}d\phi\int_0^{\infty}\frac{k'^2dk'}{k^2-k'^2\pm i\epsilon}\int_0^{\pi}\sin\theta\, d\theta\, e^{ik'd\cos\theta}\\
&= \frac{m}{2\pi^2\hbar^2}\int_0^{\infty}\frac{k'^2dk'}{k^2-k'^2\pm i\epsilon}\frac{e^{ik'd}-e^{-ik'd}}{ik'd} && \text{(even integrand)}\\
&= \frac{mi}{2\pi^2\hbar^2 d}\cdot\frac{1}{2}\left[\overbrace{\int_{-\infty}^{\infty}\frac{e^{-ik'd}k'dk'}{k^2-k'^2\pm i\epsilon}}^{A_{\pm}}-\overbrace{\int_{-\infty}^{\infty}\frac{e^{ik'd}k'dk'}{k^2-k'^2\pm i\epsilon}}^{B_{\pm}}\right] && (28)
\end{aligned}$$

With $\epsilon\to 0$, for $A_{\pm},B_{\pm}$, the denominator can be written as $-[k'-(k\pm i\epsilon)][k'+(k\pm i\epsilon)]$. The line integral $\int_{-\infty}^{\infty}$ can be obtained via complex contour integrals.



For the $A_{\pm}$ integrals, because of its $e^{-ik'd}$ factor, we choose the lower semicircle since then the arc's contribution vanishes as $\mathrm{Im}(k')\to-\infty$. Similarly, for the $B_{\pm}$ integrals, we choose the upper semicircle. The poles for the four cases $\{A,B\}\times\{+,-\}$ are labeled in the diagram above.

Now as $\epsilon\to 0$, we have

$$A_+ = \oint_{\text{lower}}dk' = -2\pi i f(z_0) = -2\pi i\left.\frac{e^{-iz_0 d}z_0}{-[z_0-(k+i\epsilon)]}\right|_{z_0=-(k+i\epsilon)} = -2\pi i\frac{-ke^{ikd}}{-(-2k)} = i\pi e^{ikd} \tag{29}$$

$$B_+ = \oint_{\text{upper}}dk' = 2\pi i f(z_0) = 2\pi i\left.\frac{e^{iz_0 d}z_0}{-[z_0+(k+i\epsilon)]}\right|_{z_0=(k+i\epsilon)} = 2\pi i\frac{ke^{ikd}}{-2k} = -i\pi e^{ikd} \tag{30}$$

$$A_- = \oint_{\text{lower}}dk' = -2\pi i f(z_0) = -2\pi i\left.\frac{e^{-iz_0 d}z_0}{-[z_0+(k-i\epsilon)]}\right|_{z_0=k-i\epsilon} = -2\pi i\frac{ke^{-ikd}}{-2k} = i\pi e^{-ikd} \tag{31}$$

$$B_- = \oint_{\text{upper}}dk' = 2\pi i f(z_0) = 2\pi i\left.\frac{e^{iz_0 d}z_0}{-[z_0-(k-i\epsilon)]}\right|_{z_0=-(k-i\epsilon)} = 2\pi i\frac{-ke^{-ikd}}{-(-2k)} = -i\pi e^{-ikd} \tag{32}$$

Going back to (28):

$$G_{\pm}(\boldsymbol{x},\boldsymbol{x}') = \frac{mi}{2\pi^2\hbar^2 d}\cdot\frac{1}{2}\cdot 2\pi i e^{\pm ikd} = \frac{-m}{2\pi\hbar^2}\frac{e^{\pm ikd}}{d} \tag{33}$$

Now (26) becomes

$$\begin{aligned}
\langle \boldsymbol{x}|\psi^{(\pm)}\rangle &= \langle \boldsymbol{x}|i\rangle + \int_{x'} d^3\boldsymbol{x}' G_{\pm}(\boldsymbol{x},\boldsymbol{x}')\left\langle \boldsymbol{x}'|V|\psi^{(\pm)}\right\rangle \\
&= \langle \boldsymbol{x}|i\rangle - \frac{m}{2\pi\hbar^2}\int_{x'} d^3\boldsymbol{x}' \frac{e^{\pm ik|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}\left\langle \boldsymbol{x}'|V|\psi^{(\pm)}\right\rangle
\end{aligned} \tag{34}$$

So far until this point, (34) is exact (of course under the asymptotic limits $\lim_{t\to\infty}\lim_{t_0\to-\infty}$ or $\lim_{t\to-\infty}\lim_{t_0\to\infty}$ which gave the Lippmann-Schwinger equation in the first place). In particular, we have not made any assumptions about the range of V, or the relative position of $\boldsymbol{x}$ and $\boldsymbol{x}'$ yet.

To further simplify the problem, let's make the following practical assumptions

(a) V depends only on position, and has a finite range, i.e., $\langle \boldsymbol{x}'|V|\boldsymbol{x}''\rangle = V(\boldsymbol{x}')\delta(\boldsymbol{x}'-\boldsymbol{x}'')$ and $V(\boldsymbol{x}') = 0$ when $|\boldsymbol{x}'| \gg a$, for V's range size a.

(b) the point $\boldsymbol{x}$ where we measure wave function, is far from the range of the potential, i.e., in (34) we will have $|\boldsymbol{x}| \gg a > |\boldsymbol{x}'|$.

This allows us to make the following approximations in the integrand of (34)

$$k|\boldsymbol{x}-\boldsymbol{x}'| \approx k\left(|\boldsymbol{x}| - \frac{\boldsymbol{x}'\cdot\boldsymbol{x}}{|\boldsymbol{x}|}\right) = k|\boldsymbol{x}| - \left(k\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\cdot\boldsymbol{x}' \tag{35}$$

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \approx \frac{1}{|\boldsymbol{x}|} \tag{36}$$

Now define $r \equiv |\boldsymbol{x}|$ and $\boldsymbol{k}_s = k\boldsymbol{x}/|\boldsymbol{x}|$ (scattered wave number aligning with $\boldsymbol{x}$), (34) eventually becomes

$$\begin{aligned}
\langle \boldsymbol{x}|\psi^{(\pm)}\rangle &\approx \langle \boldsymbol{x}|\boldsymbol{k}\rangle - \frac{m}{2\pi\hbar^2}\int_{x'} d^3\boldsymbol{x}' \frac{e^{\pm ikr}}{r} e^{\mp i\boldsymbol{k}_s\cdot\boldsymbol{x}'} V(\boldsymbol{x}')\left\langle \boldsymbol{x}'|\psi^{(\pm)}\right\rangle \\
&= \frac{1}{\sqrt{L^3}}\left\{ e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \frac{e^{\pm ikr}}{r}\left[-\frac{mL^3}{2\pi\hbar^2}\int_{x'} d^3\boldsymbol{x}' \frac{e^{\mp i\boldsymbol{k}_s\cdot\boldsymbol{x}'}}{\sqrt{L^3}} V(\boldsymbol{x}')\left\langle \boldsymbol{x}'|\psi^{(\pm)}\right\rangle\right]\right\} \\
&= \frac{1}{\sqrt{L^3}}\left[e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \frac{e^{\pm ikr}}{r}\left(-\frac{mL^3}{2\pi\hbar^2}\left\langle \pm\boldsymbol{k}_s\left|V\right|\psi^{(\pm)}\right\rangle\right)\right]
\end{aligned} \tag{37}$$

Thus we call

$$f(\boldsymbol{k},\boldsymbol{k}_s) \equiv -\frac{mL^3}{2\pi\hbar^2}\left\langle \pm\boldsymbol{k}_s\left|V\right|\psi^{(\pm)}\right\rangle \tag{38}$$

the scattering amplitude for the $|\psi^{(\pm)}\rangle$ solution, which modulates the outgoing (incoming) spherical wave $e^{\pm ikr}/r$ in addition to the incident wave.