1. The goal is to prove

$$\frac{\hbar^2}{2m}\left\langle \boldsymbol{x}\left|\frac{1}{E-H_0+i\epsilon}\right|\boldsymbol{x}'\right\rangle = -ik\sum_{l,m} Y_l^m(\hat{\boldsymbol{r}})Y_l^{m*}(\hat{\boldsymbol{r}}')j_l(kr_<)h_l^{(1)}(kr_>) \tag{1}$$

Insert two complete sets of $|Elm\rangle$ bases into the LHS,

$$\begin{aligned}
\text{LHS}_{(1)} &= \frac{\hbar^2}{2m}\int dE'\int dE''\sum_{l,m,l',m'}\langle \boldsymbol{x}|E'lm\rangle\left\langle E'lm\left|\frac{1}{E-H_0+i\epsilon}\right|E''l'm'\right\rangle\langle E''l'm'|\boldsymbol{x}'\rangle\\
&= \frac{\hbar^2}{2m}\int dE'\int dE''\sum_{l,m,l',m'}\langle \boldsymbol{x}|E'lm\rangle\left[\frac{1}{E-E''+i\epsilon}\delta(E''-E')\delta_{ll'}\delta_{mm'}\right]\langle E''l'm'|\boldsymbol{x}'\rangle\\
&= \frac{\hbar^2}{2m}\int dE'\left(\frac{1}{E-E'+i\epsilon}\right)\sum_{l,m}\langle \boldsymbol{x}|E'lm\rangle\langle E'lm|\boldsymbol{x}'\rangle
\end{aligned} \tag{2}$$

Recall the wave function of $|E'lm\rangle$ is (equation 6.107b)

$$\langle \boldsymbol{x}|E'lm\rangle = \frac{i^l}{\hbar}\sqrt{\frac{2mk'}{\pi}}j_l(k'r)Y_l^m(\hat{\boldsymbol{r}}) \qquad (\text{where } E'=\frac{\hbar^2k'^2}{2m}) \tag{3}$$

then (2) becomes

$$\begin{aligned}
\text{LHS}_{(1)} &= \frac{\hbar^2}{2m}\int dE'\left(\frac{1}{E-E'+i\epsilon}\right)\sum_{l,m}\left[\frac{i^l}{\hbar}\sqrt{\frac{2mk'}{\pi}}j_l(k'r)Y_l^m(\hat{\boldsymbol{r}})\right]\left[\frac{(-i)^l}{\hbar}\sqrt{\frac{2mk'}{\pi}}j_l(k'r')Y_l^{m*}(\hat{\boldsymbol{r}}')\right]\\
&= \frac{1}{\pi}\sum_{l,m}Y_l^m(\hat{\boldsymbol{r}})Y_l^{m*}(\hat{\boldsymbol{r}}')\int dE'\frac{k'j_l(k'r)j_l(k'r')}{E-E'+i\epsilon}\\
&= \frac{1}{\pi}\sum_{l,m}Y_l^m(\hat{\boldsymbol{r}})Y_l^{m*}(\hat{\boldsymbol{r}}')\int_0^\infty \frac{\hbar^2k'dk'}{m}\frac{k'j_l(k'r)j_l(k'r')}{E-E'+i\epsilon}\\
&= \frac{1}{\pi}\sum_{l,m}Y_l^m(\hat{\boldsymbol{r}})Y_l^{m*}(\hat{\boldsymbol{r}}')\int_0^\infty 2dk'\cdot\frac{k'^2j_l(k'r)j_l(k'r')}{k^2-k'^2+i\epsilon} \qquad (\text{integrand is even in } k')\\
&= \frac{1}{\pi}\sum_{l,m}Y_l^m(\hat{\boldsymbol{r}})Y_l^{m*}(\hat{\boldsymbol{r}}')\int_{-\infty}^\infty dk'\cdot\frac{k'^2j_l(k'r)j_l(k'r')}{k^2-k'^2+i\epsilon}
\end{aligned} \tag{4}$$

Compare (1) with (4), we can see that the desired claim is proved if we can prove

$$\int_{-\infty}^\infty dk'\cdot\frac{k'^2j_l(k'r)j_l(k'r')}{k^2-k'^2+i\epsilon} = -i\pi k\cdot j_l(kr_<)h_l^{(1)}(kr_>) \tag{5}$$

Now we will use contour integral to prove (5). In the following, we can assume $r\geq r'$ without loss of generality. Recall from previous notes that $j_l(x), n_l(x)$ have the following forms

$$j_l(x) = A_l(x)\frac{\sin x}{x}+B_l(x)\cos x \tag{6}$$

$$n_l(x) = S_l(x)\sin x + T_l(x)\frac{\cos x}{x} \tag{7}$$

where A_l, B_l, S_l, T_l are polynomials of $1/x$, subject to the following recurrence relation and initial values

$$A_0 = 1 \qquad B_0 = 0 \tag{8}$$

$$A_1 = \frac{1}{x} \qquad B_1 = -\frac{1}{x} \tag{9}$$

$$S_0 = 0 \qquad T_0 = -1 \tag{10}$$

$$S_1 = -\frac{1}{x} \qquad T_1 = -\frac{1}{x} \tag{11}$$

$$F_{l+1} = \frac{2l+1}{x}F_l(x) - F_{l-1}(x) \qquad (F = A, B, S, T) \tag{12}$$

From these, it is easy to see that

$$S_l = B_l \qquad\qquad T_l = -A_l \qquad\qquad \Longrightarrow \tag{13}$$

$$n_l(x) = B_l(x)\sin x - A_l(x)\frac{\cos x}{x} \tag{14}$$

Now rewrite (6) to make it work in the complex domain,

$$\begin{aligned} j_l(z) &= A_l \cdot \left(\frac{e^{iz}-e^{-iz}}{2iz}\right) + B_l \cdot \left(\frac{e^{iz}+e^{-iz}}{2}\right) \\ &= \left(\frac{A_l}{2iz} + \frac{B_l}{2}\right)\cdot e^{iz} + \left(\frac{-A_l}{2iz} + \frac{B_l}{2}\right)\cdot e^{-iz} \\ &= \underbrace{(A_l + izB_l)}_{\equiv C_l(z)} \cdot \frac{e^{iz}}{2iz} + \underbrace{(-A_l + izB_l)}_{\equiv D_l(z)} \cdot \frac{e^{-iz}}{2iz} \end{aligned} \tag{15}$$

where

$$C_l(z) = A_l(z) + izB_l(z) \tag{16}$$

$$D_l(z) = -A_l(z) + izB_l(z) \tag{17}$$

Moreover $h_l^{(1)} = j_l + in_l$, so

$$\begin{aligned} h_l^{(1)}(z) &= \left(A_l\frac{\sin z}{z} + B_l\cos z\right) + i\left(B_l\sin z - A_l\frac{\cos z}{z}\right) \\ &= -iA_l\frac{\cos z + i\sin z}{z} + B_l(\cos z + i\sin z) \\ &= (A_l + izB_l)\frac{e^{iz}}{iz} \\ &= C_l\frac{e^{iz}}{iz} \end{aligned} \tag{18}$$

Plug (15) and (18) into the RHS of (5),

$$\begin{aligned} \mathrm{RHS}_{(5)} &= -i\pi k j_l(kr')h_l^{(1)}(kr) \\ &= -i\pi k\left[C_l(kr')\frac{e^{ikr'}}{2ikr'} + D_l(kr')\frac{e^{-ikr'}}{2ikr'}\right]\cdot C_l(kr)\frac{e^{ikr}}{ikr} \\ &= \frac{i\pi}{2krr'}\left[C_l(kr)C_l(kr')e^{ik(r+r')} + C_l(kr)D_l(kr')e^{ik(r-r')}\right] \end{aligned} \tag{19}$$

Now focus on the LHS of (5),

$$\begin{aligned} j_l(k'r)j_l(k'r') &= \left[C_l(k'r)\frac{e^{ik'r}}{2ik'r} + D_l(k'r)\frac{e^{-ik'r}}{2ik'r}\right]\left[C_l(k'r')\frac{e^{ik'r'}}{2ik'r'} + D_l(k'r')\frac{e^{-ik'r'}}{2ik'r'}\right] \\ &\equiv -\frac{1}{4k'^2rr'}\left[I_+(k') + I_-(k')\right] \end{aligned} \tag{20}$$

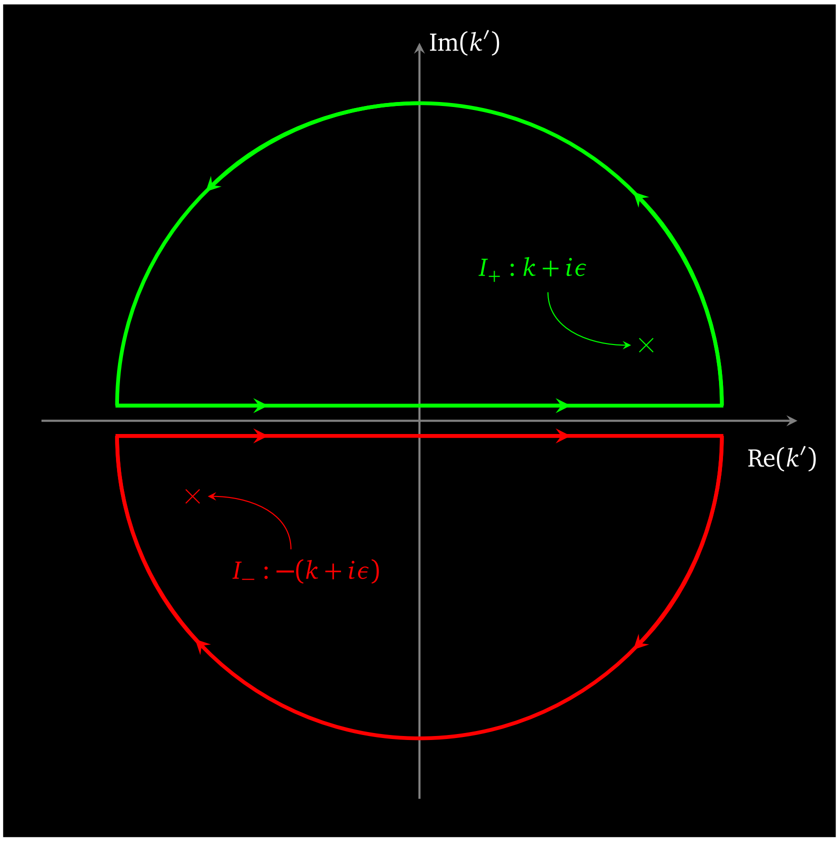
where

$$I_+(k') = C_l(k'r)C_l(k'r')e^{ik'(r+r')} + C_l(k'r)D_l(k'r')e^{ik'(r-r')} \tag{21}$$

$$I_-(k') = D_l(k'r)C_l(k'r')e^{-ik'(r-r')} + D_l(k'r)D_l(k'r')e^{-ik'(r+r')} \tag{22}$$

Now the line integral on the LHS of (5) becomes

$$\begin{aligned} \mathrm{LHS}_{(5)} &= -\frac{1}{4rr'}\int_{-\infty}^{\infty}\frac{I_+(k') + I_-(k')}{-[k'-(k+i\epsilon)][k'+(k+i\epsilon)]}dk' \\ &= \frac{1}{4rr'}\int_{-\infty}^{\infty}\frac{I_+(k') + I_-(k')}{[k'-(k+i\epsilon)][k'+(k+i\epsilon)]}dk' \end{aligned} \tag{23}$$



Since as $\operatorname{Im} k' \to +\infty$, $I_+ \to 0$, and as $\operatorname{Im} k' \to -\infty$, $I_- \to 0$, the I_+ line integral and the I_- line integral are equal to the upper and lower contour integral respectively (by taking the radius to infinity), i.e.,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{I_+(k')}{[k'-(k+i\epsilon)][k'+(k+i\epsilon)]} dk' &= \oint_{\text{upper}} \frac{I_+(k')}{[k'-(k+i\epsilon)][k'+(k+i\epsilon)]} dk' \\
&= 2i\pi \cdot \frac{I_+(k)}{2k} \\
&= \frac{i\pi}{k}\left[C_l(kr)C_l(kr')e^{ik(r+r')} + C_l(kr)D_l(kr')e^{ik(r-r')}\right] \qquad (24)
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{I_-(k')}{[k'-(k+i\epsilon)][k'+(k+i\epsilon)]} dk' &= \oint_{\text{lower}} \frac{I_-(k')}{[k'-(k+i\epsilon)][k'+(k+i\epsilon)]} dk' \\
&= -2i\pi \cdot \frac{I_-(-k)}{-2k} \\
&= \frac{i\pi}{k}\left[D_l(-kr)C_l(-kr')e^{ik(r-r')} + D_l(-kr)D_l(-kr')e^{ik(r+r')}\right] \qquad (25)
\end{aligned}$$

Plug (24),(25) into (23), we have

$$\begin{aligned}
\text{LHS}_{(5)} = \frac{1}{4rr'} \cdot \frac{i\pi}{k}\Bigg\{&\Big[C_l(kr)C_l(kr') + D_l(-kr)D_l(-kr')\Big]e^{ik(r+r')} \\
&+ \Big[C_l(kr)D_l(kr') + D_l(-kr)C_l(-kr')\Big]e^{ik(r-r')}\Bigg\} \qquad (26)
\end{aligned}$$

Now compare (26) with (19), it's sufficient to prove

$$D_l(-kr)D_l(-kr') = C_l(kr)C_l(kr') \qquad (27)$$
$$D_l(-kr)C_l(-kr') = C_l(kr)D_l(kr') \qquad (28)$$

Indeed, these can be shown by noting that when l is even (odd), both A_l and B_l are even (odd), therefore

$$\begin{aligned}
D_l(-z)D_l(-z') &= [-A_l(-z) - izB_l(-z)][-A_l(-z') - iz'B_l(-z')] \\
&= \underbrace{A_l(-z)A_l(-z')}_{A_l(z)A_l(z')} + \underbrace{izB_l(-z)A_l(-z')}_{izB_l(z)A_l(z')} + \underbrace{iz'A_l(-z)B_l(-z')}_{iz'A_l(z)B_l(z')} - \underbrace{zz'B_l(-z)B_l(-z')}_{zz'B_l(z)B_l(z')} \\
&= [A_l(z) + izB_l(z)][A_l(z') + iz'B_l(z')] = C_l(z)C_l(z') \qquad (29) \\
D_l(-z)C_l(-z') &= [-A_l(-z) - izB_l(-z)][A_l(-z') - iz'B_l(-z')] \\
&= -\underbrace{A_l(-z)A_l(-z')}_{A_l(z)A_l(z')} - \underbrace{izB_l(-z)A_l(-z')}_{izB_l(z)A_l(z')} + \underbrace{iz'A_l(-z)B_l(-z')}_{iz'A_l(z)B_l(z')} - \underbrace{zz'B_l(-z)B_l(-z')}_{zz'B_l(z)B_l(z')} \\
&= [A_l(z) + izB_l(z)][-A_l(z') + iz'B_l(z')] = C_l(z)D_l(z') \qquad (30)
\end{aligned}$$

2. For this part, assume the wavefunction of $|Elm^{(+)}\rangle$ has the form

$$\langle \boldsymbol{x}|Elm^{(+)}\rangle = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}A_l(k;r)Y_l^m(\hat{\boldsymbol{r}}) \qquad (31)$$

where the factor in front of A_l is there so when $V=0$, $A_l(k;r)$ becomes $j_l(kr)$.

Then the $\boldsymbol{x}$ representation of the Lippmann-Schwinger equation for the spherical wave can be written as

$$\begin{aligned}
\langle \boldsymbol{x}|Elm^{(+)}\rangle = \langle \boldsymbol{x}|Elm\rangle + \left\langle \boldsymbol{x}\left|\frac{1}{E-H_0+i\epsilon}V\right|Elm^{(+)}\right\rangle \qquad &\Longrightarrow \\
\frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}A_l(k;r)Y_l^m(\hat{\boldsymbol{r}}) = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}j_l(kr)Y_l^m(\hat{\boldsymbol{r}}) + \int d^3x'\left\langle \boldsymbol{x}\left|\frac{1}{E-H_0+i\epsilon}\right|\boldsymbol{x}'\right\rangle\langle \boldsymbol{x}'|V|Elm^{(+)}\rangle \qquad &\Longrightarrow \\
A_l(k;r)Y_l^m(\hat{\boldsymbol{r}}) = j_l(kr)Y_l^m(\hat{\boldsymbol{r}}) + \underbrace{\int d^3x'\left\langle \boldsymbol{x}\left|\frac{1}{E-H_0+i\epsilon}\right|\boldsymbol{x}'\right\rangle V(r')A_l(k;r')Y_l^m(\hat{\boldsymbol{r}}')}_{C} \qquad & (32)
\end{aligned}$$

where from the first part of the problem, we have

$$\begin{aligned}
C &= \int r'^2dr'\int d\Omega_{\hat{\boldsymbol{r}}'}\left[\frac{-2imk}{\hbar^2}\sum_{l',m'}Y_{l'}^{m'}(\hat{\boldsymbol{r}})Y_{l'}^{m'*}(\hat{\boldsymbol{r}}')j_{l'}(kr_<)h_{l'}^{(1)}(kr_>)\right]V(r')A_l(k;r')Y_l^m(\hat{\boldsymbol{r}}') \\
&= \frac{-2imk}{\hbar^2}\int r'^2dr'V(r')A_l(k;r')\sum_{l',m'}j_{l'}(kr_<)h_{l'}^{(1)}(kr_>)Y_{l'}^{m'}(\hat{\boldsymbol{r}})\underbrace{\int d\Omega_{\hat{\boldsymbol{r}}'}Y_{l'}^{m'*}(\hat{\boldsymbol{r}}')Y_l^m(\hat{\boldsymbol{r}}')}_{\delta_{ll'}\delta_{mm'}} \\
&= \frac{-2imk}{\hbar^2}Y_l^m(\hat{\boldsymbol{r}})\int r'^2dr'V(r')A_l(k;r')j_l(kr_<)h_l^{(1)}(kr_>) \qquad (33)
\end{aligned}$$

Plug (33) back into (32) and cancel $Y_l^m(\hat{\boldsymbol{r}})$, we get what we wanted

$$A_l(k;r) = j_l(kr) - \frac{2imk}{\hbar^2}\int r'^2dr'V(r')A_l(k;r')j_l(kr_<)h_l^{(1)}(kr_>) \qquad (34)$$

3. For this, we will look at equation 6.117

$$\langle \boldsymbol{x}|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}^3}\sum_l(2l+1)\frac{P_l}{2ik}\left\{[1+2ikf_l(k)]\frac{e^{ikr}}{r} - \frac{e^{-ik}(-1)^l}{r}\right\} \qquad (35)$$

Here the radial function of the l-th partial wave has an outgoing wave and an incoming wave, whose relative phase depends on $f_l(k)$.

In part (2), we have already established the radial equation (34), now let's rewrite it in the "large-r" asymptotic form. Recall that

$$j_l(kr) \xrightarrow{\text{large } r} \frac{e^{ikr}(-i)^l - e^{-ikr}i^l}{2ikr} \qquad (36)$$

$$h_l^{(1)}(kr) \xrightarrow{\text{large } r} \frac{e^{ikr}(-i)^l}{ikr} \qquad (37)$$

Plug (36),(37) into (34),

$$A_l(k;r) \xrightarrow{\text{large } r} \frac{e^{ikr}(-i)^l - e^{-ikr} i^l}{2ikr} - \frac{2imk}{\hbar^2} \frac{e^{ikr}(-i)^l}{ikr} \int r'^2 dr' V(r') A_l(k;r') j_l(kr')$$
$$= \frac{(-i)^l}{2ik} \left\{ \left[1 - \frac{4imk}{\hbar^2} \int r'^2 dr' V(r') A_l(k;r') j_l(kr') \right] \frac{e^{ikr}}{r} - \frac{e^{-ikr}(-1)^l}{r} \right\} \tag{38}$$

Now compare the relative phase of the outgoing vs incoming wave in (35) and (38), we can match

$$2ikf_l(k) \qquad \longleftrightarrow \qquad -\frac{4imk}{\hbar^2} \int r'^2 dr' V(r') A_l(k;r') j_l(kr') \tag{39}$$

which gives

$$f_l(k) = -\frac{2m}{\hbar^2} \int r'^2 dr' V(r') A_l(k;r') j_l(kr') \tag{40}$$