1. We begin with the Klein-Gordon equation with potential $V = -Ze^2/r^2 = -Z\alpha/r$:

$$\left(i\frac{\partial}{\partial t} + \frac{Z\alpha}{r}\right)^2 \Psi(\mathbf{x},t) = \left(-\boldsymbol{\nabla}^2 + m^2\right)\Psi(\mathbf{x},t) \qquad \text{where} \qquad \Psi(\mathbf{x},t) = e^{-iEt}\frac{u_l(r)}{r}Y_l^m(\theta,\phi) \tag{1}$$

which is

$$\left[\boldsymbol{\nabla}^2 - \left(\frac{\partial}{\partial t}\right)^2 + \frac{2iZ\alpha}{r}\left(\frac{\partial}{\partial t}\right) + \frac{(Z\alpha)^2}{r^2} - m^2\right]\left[e^{-iEt}\frac{u_l(r)}{r}Y_l^m(\theta,\phi)\right] = 0 \tag{2}$$

Work on the time derivative first, and then cancel the factor e^{-iEt},

$$\left[\boldsymbol{\nabla}^2 + E^2 + \frac{2Z\alpha E}{r} + \frac{(Z\alpha)^2}{r^2} - m^2\right]\left[\frac{u_l(r)}{r}Y_l^m(\theta,\phi)\right] = 0 \tag{3}$$

The Laplacian in the spherical coordinates is

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{4}$$

Also recall $Y_l^m(\theta,\phi)$ satisfies the differential equation (see eq (3.237))

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi) \tag{5}$$

This gives

$$\begin{aligned}
\boldsymbol{\nabla}^2\left[\frac{u_l(r)}{r}Y_l^m(\theta,\phi)\right] &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\left[\frac{u_l(r)}{r}\right]Y_l^m(\theta,\phi) - \frac{l(l+1)}{r^2}\frac{u_l(r)}{r}Y_l^m(\theta,\phi) \\
&= \left[\frac{1}{r^2}\frac{d}{dr}\left(ru_l' - u_l\right) - \frac{l(l+1)}{r^2}\frac{u_l}{r}\right]Y_l^m \\
&= \left[\frac{u_l''}{r} - \frac{l(l+1)}{r^2}\frac{u_l}{r}\right]Y_l^m
\end{aligned} \tag{6}$$

Plug (6) back into (3) and cancel $Y_l^m(\theta,\phi)/r$, we obtain

$$u_l'' + \left[E^2 - m^2 + \frac{2Z\alpha E}{r} + \frac{(Z\alpha)^2}{r^2} - \frac{l(l+1)}{r^2}\right]u_l = 0 \tag{7}$$

Divide by $\gamma^2 = 4(m^2 - E^2)$,

$$\frac{u_l''}{\gamma^2} + \left[-\frac{1}{4} + \frac{2Z\alpha E}{\gamma^2 r} - \frac{l(l+1) - (Z\alpha)^2}{\gamma^2 r^2}\right]u_l = 0 \tag{8}$$

With change of variable $\rho = \gamma r$, we have $u_l'' = d^2u_l/dr^2 = \gamma^2 d^2u_l/d\rho^2$, we finally get

$$\frac{d^2u_l}{d\rho^2} + \left[-\frac{1}{4} + \frac{2Z\alpha E}{\gamma\rho} - \frac{l(l+1) - (Z\alpha)^2}{\rho^2}\right]u_l = 0 \tag{9}$$

2. If $u_l(\rho)$ has the assumed form

$$u_l(\rho) = \rho^k\left(1 + c_1\rho + c_2\rho^2 + \cdots\right)e^{-\rho/2} \tag{10}$$

we can insert (10) back to (9) and match the ρ^{k-2} term, which will produce

$$\begin{aligned}
k(k-1) - l(l+1) + (Z\alpha)^2 = 0 & \qquad \Longrightarrow \\
\left(k - \frac{1}{2}\right)^2 - \left[\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2\right] = 0 & \qquad \Longrightarrow \\
k = k_\pm = \frac{1}{2} \pm \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2} &
\end{aligned} \tag{11}$$

In a physical situation, $u_l(\rho)$ is dominated by the leading term ρ^k, which makes the radial function $u_l/\rho \propto \rho^{k-1}$. Then the kinetic energy

$$K \propto \int \rho^2 d\rho \rho^{k-1} \frac{d^2 \rho^{k-1}}{d\rho^2} \propto \int \rho^{2k-2} d\rho \propto \rho^{2k-1} \tag{12}$$

which is divergent near origin for $k_- < 1/2$. For $Z\alpha \ll 1$, $k_+ \approx l+1$, so

$$R_l(\rho) \propto \frac{u_l(\rho)}{\rho} \approx \rho^l \tag{13}$$

agreeing with the Schrödinger solution (see eq (3.320)).

3. From (10),

$$\begin{aligned}
u_l(\rho) &= \sum_i e^{-\rho/2} c_i \rho^{k+i} && \Longrightarrow \\
\frac{du_l}{d\rho} &= \sum_i \left[e^{-\rho/2} c_i (k+i) \rho^{k+i-1} - \frac{1}{2} e^{-\rho/2} c_i \rho^{k+i} \right] && \Longrightarrow \\
\frac{d^2 u_l}{d\rho^2} &= \sum_i \left[e^{-\rho/2} c_i (k+i)(k+i-1) \rho^{k+i-2} - e^{-\rho/2} c_i (k+i) \rho^{k+i-1} + \frac{1}{4} e^{-\rho/2} c_i \rho^{k+i} \right]
\end{aligned} \tag{14}$$

Put this back into (9) and match the ρ^{k+i} term

$$c_{i+2}(k+i+2)(k+i+1) - c_{i+1}(k+i+1) + \frac{1}{4}c_i - \frac{1}{4}c_i + \frac{2Z\alpha E}{\gamma} c_{i+1} - \left[l(l+1) - (Z\alpha)^2 \right] c_{i+2} = 0 \tag{15}$$

which gives the recurrence relation

$$\frac{c_{i+2}}{c_{i+1}} = \frac{(k+i+1) - 2Z\alpha E/\gamma}{(k+i+2)(k+i+1) - l(l+1) + (Z\alpha)^2} \quad \to \frac{1}{i} \quad \text{as } i \to \infty \tag{16}$$

With this, the polynomial $\sum_i c_i \rho^i$ will approach e^ρ, which renders the whole $u_l(\rho)$ divergent for large ρ unless it terminates at some point, e.g., $c_{N+1} = 0$:

$$k + N = \frac{2Z\alpha E}{\gamma} \tag{17}$$

4. Combine (17) with (11), we have the restriction on E:

$$\begin{aligned}
k &= \frac{2Z\alpha E}{\gamma} - N = \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2} && \Longrightarrow \\
2Z\alpha E &= \left[N + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2} \right] \gamma = \left[N + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2} \right] 2\sqrt{m^2 - E^2} && \Longrightarrow \\
E &= \frac{m}{\sqrt{1 + \dfrac{(Z\alpha)^2}{\left[N + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2} \right]^2}}}
\end{aligned} \tag{18}$$

By following the convention of equation (3.313), where $n = N + l + 1$ is called principal quantum number, E is rewritten as

$$E = \frac{m}{\sqrt{1 + \dfrac{(Z\alpha)^2}{\left[n - l - \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - (Z\alpha)^2} \right]^2}}} \tag{19}$$

5. When $Z\alpha \ll 1$, (19) is reduced to

$$E \approx \frac{m}{\sqrt{1+\frac{(Z\alpha)^2}{\left[n-l-\frac{1}{2}+\left(l+\frac{1}{2}\right)\right]^2}}} \approx m\left[1-\frac{(Z\alpha)^2}{2n^2}\right] \tag{20}$$

which is manifestly the rest energy plus $E_n^{(0)}$.
Let's try to expand E to one more order of $Z\alpha$. Write

$$E = mA^{-1/2} \tag{21}$$

$$A = 1+\frac{(Z\alpha)^2}{B^2} \tag{22}$$

$$\begin{aligned} B &= n-l-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-(Z\alpha)^2} \\ &\approx n-l-\frac{1}{2}+\left(l+\frac{1}{2}\right)-\frac{(Z\alpha)^2}{2\left(l+\frac{1}{2}\right)} \\ &= n-\frac{(Z\alpha)^2}{2\left(l+\frac{1}{2}\right)} \end{aligned} \tag{23}$$

Then

$$\begin{aligned} E &\approx m\left[1-\frac{1}{2}\frac{(Z\alpha)^2}{B^2}+\frac{1}{2!}\frac{3}{4}\frac{(Z\alpha)^4}{B^4}\right] \\ &\approx m\left\{1-\frac{1}{2}(Z\alpha)^2n^{-2}\left[1+\frac{(Z\alpha)^2}{n\left(l+\frac{1}{2}\right)}\right]+\frac{3}{8}\frac{(Z\alpha)^4}{n^4}\right\} \\ &= m-\frac{m(Z\alpha)^2}{2n^2}-\frac{m(Z\alpha)^4}{2n^3\left(l+\frac{1}{2}\right)}+\frac{3}{8}\frac{(Z\alpha)^4}{n^4} \end{aligned} \tag{24}$$

whose third and fourth term match section 5.3.1 (*The Relativistic Correction to the Kinetic Energy*, in particular eq (5.104b)) perfectly.