These notes fill the missing steps leading from eq (5.338) to eq (5.342). The calculation is straightforward but fairly tedious.

First eq (5.338) can be simplified to

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{4\pi^2\alpha\hbar}{m_e^2\omega}\left|\overbrace{\left\langle \boldsymbol{k}_f \middle| e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{p}\middle| i\right\rangle}^{A}\right|^2 \frac{m_e k_f L^3}{\hbar^2(2\pi)^3} \\
&= \frac{\alpha k_f L^3}{2\pi m_e\omega\hbar}|A|^2
\end{aligned} \tag{1}
$$

With $|i\rangle$ being the ground state, we can calculate A as

$$
\begin{aligned}
A &= \hat{\boldsymbol{\epsilon}}\cdot\int d^3x \frac{e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}}{L^{3/2}} e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}(-i\hbar\boldsymbol{\nabla})\left[\frac{1}{\sqrt{\pi}}e^{-Zr/a_0}\left(\frac{Z}{a_0}\right)^{3/2}\right] \\
&= \frac{-i\hbar}{\sqrt{\pi}L^{3/2}}\left(\frac{Z}{a_0}\right)^{3/2}\underbrace{\hat{\boldsymbol{\epsilon}}\cdot\int d^3x e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}} e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\boldsymbol{\nabla}\left(e^{-Zr/a_0}\right)}_{B}
\end{aligned} \tag{2}
$$

Combining with (1), we have

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\alpha k_f L^3}{2\pi m_e\omega\hbar}\left|\frac{-i\hbar}{\sqrt{\pi}L^{3/2}}\left(\frac{Z}{a_0}\right)^{3/2}\right|^2 |B|^2 \\
&= \frac{\alpha k_f\hbar}{2\pi^2 m_e\omega}\left(\frac{Z}{a_0}\right)^3 |B|^2 \qquad \text{(recall } \alpha\hbar = e^2/c\text{)} \\
&= \frac{e^2 k_f}{2\pi^2 m_e c\omega}\left(\frac{Z}{a_0}\right)^3 |B|^2
\end{aligned} \tag{3}
$$

Now we calculate B using integration-by-parts:

$$
\begin{aligned}
B &= \hat{\boldsymbol{\epsilon}}\cdot\int d^3x e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}} e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\boldsymbol{\nabla}\left(e^{-Zr/a_0}\right) \\
&= \hat{\boldsymbol{\epsilon}}\cdot\left[\underbrace{e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}} e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})} e^{-Zr/a_0}\Big|_{-\infty}^{+\infty}}_{=0} - \int d^3x \boldsymbol{\nabla}\left(e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}} e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\right)e^{-Zr/a_0}\right] \\
&= -\hat{\boldsymbol{\epsilon}}\cdot\int d^3x\left[\left(\boldsymbol{\nabla}e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}\right)e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}e^{-Zr/a_0} + e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}\left(\boldsymbol{\nabla}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}\right)e^{-Zr/a_0}\right]
\end{aligned} \tag{4}
$$

We can drop the second term in the integral since the gradient $\boldsymbol{\nabla}e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}$ is along the $\hat{\boldsymbol{n}}$ direction, and $\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{n}} = 0$.

Now (4) becomes

$$
\begin{aligned}
B &= -\hat{\boldsymbol{\epsilon}}\cdot\int d^3x\left(\boldsymbol{\nabla}e^{-i\boldsymbol{k}_f\cdot\boldsymbol{x}}\right)e^{i(\omega/c)(\hat{\boldsymbol{n}}\cdot\boldsymbol{x})}e^{-Zr/a_0} \\
&= i\left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{k}_f\right)\int d^3x e^{-i(\boldsymbol{k}_f-\omega\hat{\boldsymbol{n}}/c)\cdot\boldsymbol{x}}e^{-Zr/a_0} \qquad \left(\text{define } \boldsymbol{q}\equiv\boldsymbol{k}_f-\left(\frac{\omega}{c}\right)\hat{\boldsymbol{n}}\right) \\
&= i\left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{k}_f\right)\underbrace{\int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}e^{-Zr/a_0}}_{C}
\end{aligned} \tag{5}
$$

Lastly, we evaluate the Fourier transform integral C in spherical coordinates where $\boldsymbol{q}$ aligns with the z-axis.

$$
\begin{aligned}
C &= \int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} e^{-Zr/a_0} \\
&= \int_0^{2\pi} d\phi \int_0^{\infty} r^2 e^{-Zr/a_0} dr \int_0^{\pi} \sin\theta d\theta e^{-iqr\cos\theta} \qquad (\text{let } y \equiv -\cos\theta) \\
&= 2\pi \int r^2 e^{-Zr/a_0} dr \int_{y=-1}^{1} dy e^{iqry} \\
&= \frac{2\pi}{iq} \int r e^{-Zr/a_0} dr \left(e^{iqr} - e^{-iqr}\right)
\end{aligned} \tag{6}
$$

Since for complex γ,

$$
\int_0^{\infty} e^{\gamma r} r dr = \frac{1}{\gamma} e^{\gamma r} r \Big|_0^{\infty} - \frac{1}{\gamma} \int_0^{\infty} e^{\gamma r} dr = \frac{1}{\gamma^2} \tag{7}
$$

We can evaluate (6) with $\gamma = iq - Z/a_0$ and $\gamma = -iq - Z/a_0$ respectively, which gives

$$
\begin{aligned}
C &= \frac{2\pi}{iq} \left[\frac{1}{(iq - Z/a_0)^2} - \frac{1}{(-iq - Z/a_0)^2} \right] \\
&= \frac{2\pi}{iq} \frac{4iqZ/a_0}{[q^2 + (Z/a_0)^2]^2} \\
&= \frac{8\pi Z/a_0}{[q^2 + (Z/a_0)^2]^2}
\end{aligned} \tag{8}
$$

Finally, combining (3), (5) and (8), we obtain eq (5.342)

$$
\frac{d\sigma}{d\Omega} = \frac{32e^2 k_f (\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{k}_f)^2}{m_e c\omega} \left(\frac{Z}{a_0}\right)^5 \frac{1}{[q^2 + (Z/a_0)^2]^4} \tag{9}
$$