We will provide a brute-force proof of equation (8.125), which was shown in the text via symmetry arguments.
Start with

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{r}} = \begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix} \tag{1}$$

and the definition of spin-angular functions (see eq (3.384))

$$\mathscr{Y}_l^{j=l\pm1/2,m} = \frac{1}{\sqrt{2l+1}} \begin{bmatrix} \pm\sqrt{l\pm m+\frac{1}{2}}\,Y_l^{m-1/2}(\theta,\phi) \\ \sqrt{l\mp m+\frac{1}{2}}\,Y_l^{m+1/2}(\theta,\phi) \end{bmatrix} \tag{2}$$

When j, instead of l, is fixed, we have

$$\mathscr{Y}_{l=j+1/2}^{j,m} = \frac{1}{\sqrt{2j+2}} \begin{bmatrix} -\sqrt{j-m+1}\,Y_{j+1/2}^{m-1/2}(\theta,\phi) \\ \sqrt{j+m+1}\,Y_{j+1/2}^{m+1/2}(\theta,\phi) \end{bmatrix} \qquad \mathscr{Y}_{l=j-1/2}^{j,m} = \frac{1}{\sqrt{2j}} \begin{bmatrix} \sqrt{j+m}\,Y_{j-1/2}^{m-1/2}(\theta,\phi) \\ \sqrt{j-m}\,Y_{j-1/2}^{m+1/2}(\theta,\phi) \end{bmatrix} \tag{3}$$

We wish to prove equation (8.125)

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,\mathscr{Y}_{l=j\mp1/2}^{j,m}(\theta,\phi) = -\mathscr{Y}_{l=j\pm1/2}^{j,m}(\theta,\phi) \tag{4}$$

But only one sign needs to be proved, since $(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})^2=1$ implies the other. Now we pick to prove

$$\begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \sqrt{\frac{j+m}{2j}}\,Y_{j-1/2}^{m-1/2}(\theta,\phi) \\ \sqrt{\frac{j-m}{2j}}\,Y_{j-1/2}^{m+1/2}(\theta,\phi) \end{bmatrix} = - \begin{bmatrix} -\sqrt{\frac{j-m+1}{2j+2}}\,Y_{j+1/2}^{m-1/2}Y(\theta,\phi) \\ \sqrt{\frac{j+m+1}{2j+2}}\,Y_{j+1/2}^{m+1/2}Y(\theta,\phi) \end{bmatrix} \tag{5}$$

Our strategy is to keep writing (5) as equivalent identities, until the end where they are recognized as well known facts.
Expand (5) into component identities:

$$\cos\theta\sqrt{\frac{j+m}{2j}}\,Y_{j-1/2}^{m-1/2}(\theta,\phi) + e^{-i\phi}\sin\theta\sqrt{\frac{j-m}{2j}}\,Y_{j-1/2}^{m+1/2}(\theta,\phi) = \sqrt{\frac{j-m+1}{2j+2}}\,Y_{j+1/2}^{m-1/2}Y(\theta,\phi) \tag{6}$$

$$e^{i\phi}\sin\theta\sqrt{\frac{j+m}{2j}}\,Y_{j-1/2}^{m-1/2}(\theta,\phi) - \cos\theta\sqrt{\frac{j-m}{2j}}\,Y_{j-1/2}^{m+1/2}(\theta,\phi) = -\sqrt{\frac{j+m+1}{2j+2}}\,Y_{j+1/2}^{m+1/2}Y(\theta,\phi) \tag{7}$$

With the general spherical harmonics formula

$$Y_l^m(\theta,\phi) = C_l^m e^{im\phi} P_l^m(\cos\theta) \qquad \text{where} \qquad C_l^m = (-1)^m\sqrt{\frac{(l-m)!}{(l+m)!}}\sqrt{\frac{2l+1}{4}} \tag{8}$$

we can see in (6) and (7) the ϕ dependency is clearly equal on both sides. So now we need to prove

$$\cos\theta\sqrt{\frac{j+m}{2j}}\,C_{j-1/2}^{m-1/2}P_{j-1/2}^{m-1/2} + \sin\theta\sqrt{\frac{j-m}{2j}}\,C_{j-1/2}^{m+1/2}P_{j-1/2}^{m+1/2} = \sqrt{\frac{j-m+1}{2j+2}}\,C_{j+1/2}^{m-1/2}P_{j+1/2}^{m-1/2} \tag{9}$$

$$\sin\theta\sqrt{\frac{j+m}{2j}}\,C_{j-1/2}^{m-1/2}P_{j-1/2}^{m-1/2} - \cos\theta\sqrt{\frac{j-m}{2j}}\,C_{j-1/2}^{m+1/2}P_{j-1/2}^{m+1/2} = -\sqrt{\frac{j+m+1}{2j+2}}\,C_{j+1/2}^{m+1/2}P_{j+1/2}^{m+1/2} \tag{10}$$

Divide (9) and (10) by $C_{j-1/2}^{m-1/2}$, while noticing (8) gives

$$\frac{C_{j-1/2}^{m+1/2}}{C_{j-1/2}^{m-1/2}} = -\sqrt{\frac{(j-m-1)!}{(j+m)!}}\sqrt{\frac{(j+m-1)!}{(j-m)!}} = -\frac{1}{\sqrt{(j+m)(j-m)}} \tag{11}$$

$$\frac{C_{j+1/2}^{m-1/2}}{C_{j-1/2}^{m-1/2}} = \sqrt{\frac{(j-m+1)!}{(j+m)!}}\sqrt{2j+2}\sqrt{\frac{(j+m-1)!}{(j-m)!}}\frac{1}{\sqrt{2j}} = \sqrt{\frac{2j+2}{2j}}\sqrt{\frac{j-m+1}{j+m}} \tag{12}$$

$$\frac{C_{j+1/2}^{m+1/2}}{C_{j-1/2}^{m-1/2}} = -\sqrt{\frac{(j-m)!}{(j+m+1)!}}\sqrt{2j+2}\sqrt{\frac{(j+m-1)!}{(j-m)!}}\frac{1}{\sqrt{2j}} = -\sqrt{\frac{2j+2}{2j}}\frac{1}{\sqrt{(j+m+1)(j+m)}} \tag{13}$$

we end up with the equivalent claims

$$\cos\theta\sqrt{\frac{j+m}{2j}}P_{j-1/2}^{m-1/2}-\sin\theta\sqrt{\frac{j-m}{2j}}\frac{1}{\sqrt{(j+m)(j-m)}}P_{j-1/2}^{m+1/2}=\sqrt{\frac{j-m+1}{2j+2}}\sqrt{\frac{2j+2}{2j}}\sqrt{\frac{j-m+1}{j+m}}P_{j+1/2}^{m-1/2} \tag{14}$$

$$\sin\theta\sqrt{\frac{j+m}{2j}}P_{j-1/2}^{m-1/2}+\cos\theta\sqrt{\frac{j-m}{2j}}\frac{1}{\sqrt{(j+m)(j-m)}}P_{j-1/2}^{m+1/2}=\sqrt{\frac{j+m+1}{2j+2}}\sqrt{\frac{2j+2}{2j}}\frac{1}{\sqrt{(j+m+1)(j+m)}}P_{j+1/2}^{m+1/2} \tag{15}$$

which are simplified into

$$\cos\theta\sqrt{\frac{j+m}{2j}}P_{j-1/2}^{m-1/2}-\frac{\sin\theta}{\sqrt{2j(j+m)}}P_{j-1/2}^{m+1/2}=\frac{j-m+1}{\sqrt{2j(j+m)}}P_{j+1/2}^{m-1/2} \tag{16}$$

$$\sin\theta\sqrt{\frac{j+m}{2j}}P_{j-1/2}^{m-1/2}+\frac{\cos\theta}{\sqrt{2j(j+m)}}P_{j-1/2}^{m+1/2}=\frac{1}{\sqrt{2j(j+m)}}P_{j+1/2}^{m+1/2} \tag{17}$$

or,

$$\cos\theta(j+m)P_{j-1/2}^{m-1/2}-\sin\theta P_{j-1/2}^{m+1/2}=(j-m+1)P_{j+1/2}^{m-1/2} \tag{18}$$

$$\sin\theta(j+m)P_{j-1/2}^{m-1/2}+\cos\theta P_{j-1/2}^{m+1/2}=P_{j+1/2}^{m+1/2} \tag{19}$$

If we denote $l'=j-1/2, m'=m-1/2, x=\cos\theta$, these are equivalent to

$$x(l'+m'+1)P_{l'}^{m'}-\sqrt{1-x^2}P_{l'}^{m'+1}=(l'-m'+1)P_{l'+1}^{m'} \tag{20}$$

$$\sqrt{1-x^2}(l'+m'+1)P_{l'}^{m'}+xP_{l'}^{m'+1}=P_{l'+1}^{m'+1} \tag{21}$$

But these are well known recurrence relations of the associated Legendre functions (for reference, see equation (2.5.23) and (2.5.22) in *A. R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press, 2nd edition (1960)*).