Here we elaborate the derivation of the Kohn-Sham ground energy (equation 7.91) with slightly more rigorous language of tensor product.

The totally symmetric or anti-symmetric multi-particle state $|\Psi\rangle$ should be the totally symmetric or anti-symmetric combination of the tensor product states of each particle, i.e.,

$$|\Psi\rangle = \frac{1}{\sqrt{N!}} \sum_{p\in S_N} \eta_p \left| \phi_{p(1)} \right\rangle \otimes \left| \phi_{p(2)} \right\rangle \otimes \cdots \otimes \left| \phi_{p(N)} \right\rangle \tag{1}$$

where $\eta_p = 1$ for symmetric states, and $\eta_p = \epsilon_{p(1)p(2)\cdots p(N)}$ for anti-symmetric states.

That $|\Psi\rangle$ is normalized is guaranteed by requiring the set of $\{|\phi_j\rangle\}$ to be orthonormal, in which case

$$\begin{aligned}\langle\Psi|\Psi\rangle &= \frac{1}{N!} \sum_{p,q\in S_N} (\eta_p\eta_q) \overbrace{\left(\langle\phi_{p(1)}|\phi_{q(1)}\rangle\right)}^{\delta_{p(1)q(1)}} \underbrace{\left(\langle\phi_{p(2)}|\phi_{q(2)}\rangle\right)}_{\delta_{p(2)q(2)}} \cdots \overbrace{\left(\langle\phi_{p(N)}|\phi_{q(N)}\rangle\right)}^{\delta_{p(N)q(N)}} \\ &= \frac{1}{N!} \sum_{p\in S_N} \eta_p^2 = 1 \end{aligned}\tag{2}$$

Thus, the wave function given by eq (7.88) should actually be written as

$$\begin{aligned}\langle \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N|\Psi\rangle &= \Psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N) \\ &= \frac{1}{\sqrt{N!}} \sum_{p\in S_N} \eta_p \phi_{p(1)}(\boldsymbol{x}_1)\phi_{p(2)}(\boldsymbol{x}_2)\cdots\phi_{p(N)}(\boldsymbol{x}_N) \end{aligned}\tag{3}$$

We also assume that each $|\phi_j\rangle$ is an energy eigenstate satisfying eq (7.90), i.e.,

$$H_{KS}|\phi_j\rangle = \epsilon_j|\phi_j\rangle \tag{4}$$

Care must be taken to construct the multi-particle Hamiltonian H_{KS}^N from single particle Hamiltonian. H_{KS}^N should be written as

$$\begin{aligned}H_{KS}^N =& H_{KS}\otimes I\otimes\cdots\otimes I + \\ & I\otimes H_{KS}\otimes I\otimes\cdots\otimes I + \\ & \cdots + \\ & I\otimes\cdots\otimes I\otimes H_{KS}\end{aligned}\tag{5}$$

In this spirit, we should have

$$\begin{aligned}\left\langle \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N \left| H_{KS}^N \right| \Psi \right\rangle &= \left(\langle \boldsymbol{x}_1| \otimes \langle \boldsymbol{x}_2| \otimes \cdots \otimes \langle \boldsymbol{x}_N|\right) H_{KS}^N \left(\frac{1}{\sqrt{N!}} \sum_{p\in S_N} \eta_p \left| \phi_{p(1)} \right\rangle \otimes \left| \phi_{p(2)} \right\rangle \otimes \cdots \otimes \left| \phi_{p(N)} \right\rangle \right) \\ &= \frac{1}{\sqrt{N!}} \sum_{p\in S_N} \eta_p \cdot \left[\sum_{j=1}^N \phi_{p(1)}(\boldsymbol{x}_1)\cdots\epsilon_j\phi_{p(j)}(\boldsymbol{x}_j)\cdots\phi_{p(N)}(\boldsymbol{x}_N) \right] \\ &= \frac{1}{\sqrt{N!}} \sum_{p\in S_N} \eta_p \left(\sum_j \epsilon_j \right) \left[\phi_{p(1)}(\boldsymbol{x}_1)\phi_{p(2)}(\boldsymbol{x}_2)\cdots\phi_{p(N)}(\boldsymbol{x}_N) \right] \\ &= \left(\sum_j \epsilon_j \right) \left[\frac{1}{\sqrt{N!}} \sum_{p\in S_N} \eta_p \phi_{p(1)}(\boldsymbol{x}_1)\phi_{p(2)}(\boldsymbol{x}_2)\cdots\phi_{p(N)}(\boldsymbol{x}_N) \right] \\ &= \left(\sum_j \epsilon_j \right) \langle \boldsymbol{x}_1, \boldsymbol{x}_2 \cdots, \boldsymbol{x}_N|\Psi\rangle \end{aligned}\tag{6}$$

which shows that $|\Psi\rangle$ is an energy eigenstate of H_{KS}^N with eigenvalue $\sum_j \epsilon_j$, which gives eq (7.91).

Note that in this derivation, we assume all ϕ_js are distinct, but in fact they are allowed to be the same for i, j (even all N are allowed to be the same). When some of them are the same, the totally symmetric/anti-symmetric construction of $|\Psi\rangle$ will be different than (1), see eq (7.63),(7.64) for example, but the remaining of the derivation still holds (with properly constructed $|\Psi\rangle$).