Here we will supply the proof for Sakurai equation (7.200) from which we can conclude the minimum uncertainty of the squeezed light. Recall the electric field can be written as (eq 7.194),

$$E(\chi) = \frac{1}{2} a e^{-i\chi} + \frac{1}{2} a^\dagger e^{i\chi} \tag{1}$$

The squeezed light state $|\zeta\rangle$ is given by the result of applying the operator

$$Z = \exp\left(\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta a^{\dagger 2}\right) \qquad \text{where } \zeta = se^{i\theta}\, \textbf{(note the typo from the text!)} \tag{2}$$

to the vacuum state $|\mathbf{0}\rangle$, i.e.

$$|\zeta\rangle = Z|\mathbf{0}\rangle = \exp\left(\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta a^{\dagger 2}\right)|\mathbf{0}\rangle \tag{3}$$

Recall

$$[\Delta E(\chi)]^2 = \left\langle E(\chi)^2 \right\rangle - \langle E(\chi)\rangle^2 \tag{4}$$

For state $|\zeta\rangle$, it's easy to see that $\langle E(\chi)\rangle = 0$ because if we expand Z, we end up with terms that create or annihilate two photons at a time, so $|\zeta\rangle$ is necessarily composed of only even-numbered states. Then $E|\zeta\rangle$ changes these into odd-numbered states, whose inner product with $\langle\zeta|$ will be zero.

Then we just need to calculate $\left\langle E(\chi)^2 \right\rangle$, where, by (1)

$$E(\chi)^2 = \frac{1}{4}\left(a^2 e^{-2i\chi} + a^{\dagger 2} e^{2i\chi} + aa^\dagger + a^\dagger a\right) \tag{5}$$

Recall the Baker-Campbell-Hausdorff formula

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots + \frac{1}{n!}\overbrace{[A,[A,\cdots,[A}^{n},B]\cdots]] + \cdots \tag{6}$$

Now let

$$A = \frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2 \tag{7}$$

i.e., the negative of the exponent of Z (see (2)), let's calculate $e^A a e^{-A}$ and $e^A a^\dagger e^{-A}$ respectively.

- $e^A a e^{-A}$
  - $k = 0$:

$$B = a \tag{8}$$

  - $k = 1$:

$$\begin{aligned} [A,B] &= \left[\frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2, a\right] \\ &= \frac{1}{2}\zeta\left[a^{\dagger 2}, a\right] \\ &= \frac{1}{2}\zeta\left\{a^\dagger[a^\dagger, a] + [a^\dagger, a]a^\dagger\right\} \\ &= \frac{1}{2}\zeta(-2a^\dagger) = -\zeta a^\dagger \end{aligned} \tag{9}$$

  - $k = 2$:

$$\begin{aligned} [A,[A,B]] &= \left[\frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2, -\zeta a^\dagger\right] \\ &= \frac{1}{2}\zeta^*\zeta\left[a^2, a^\dagger\right] \\ &= \frac{1}{2}\zeta^*\zeta\left\{a[a, a^\dagger] + [a, a^\dagger]a\right\} \\ &= \frac{1}{2}\zeta^*\zeta(2a) = s^2 a \end{aligned} \tag{10}$$

Now the pattern should be clear, the BCH formula should give

$$\begin{aligned}
e^{A}ae^{-A} &= \sum_{l=0}^{\infty}\frac{s^{2l}a}{(2l)!}+\sum_{l=0}^{\infty}\frac{(-\zeta)s^{2l}a^{\dagger}}{(2l+1)!} \\
&= \sum_{l=0}^{\infty}\frac{s^{2l}a}{(2l)!}-e^{i\theta}\sum_{l=0}^{\infty}\frac{s^{2l+1}a^{\dagger}}{(2l+1)!} \\
&= a\cosh s-e^{i\theta}a^{\dagger}\sinh s
\end{aligned} \tag{11}$$

- $e^{A}a^{\dagger}e^{-A}$
  - $k=0$:

$$B=a^{\dagger} \tag{12}$$

  - $k=1$:

$$\begin{aligned}
[A,B] &= \left[\frac{1}{2}\zeta a^{\dagger 2}-\frac{1}{2}\zeta^{*}a^{2},a^{\dagger}\right] \\
&= -\frac{1}{2}\zeta^{*}\left[a^{2},a^{\dagger}\right] \\
&= -\frac{1}{2}\zeta^{*}\left\{a[a,a^{\dagger}]+[a,a^{\dagger}]a\right\} \\
&= -\frac{1}{2}\zeta^{*}(2a)=-\zeta^{*}a
\end{aligned} \tag{13}$$

  - $k=2$:

$$\begin{aligned}
[A,[A,B]] &= \left[\frac{1}{2}\zeta a^{\dagger 2}-\frac{1}{2}\zeta^{*}a^{2},-\zeta^{*}a\right] \\
&= -\frac{1}{2}\zeta\zeta^{*}\left[a^{\dagger 2},a\right] \\
&= -\frac{1}{2}\zeta\zeta^{*}\left\{a^{\dagger}[a^{\dagger},a]+[a^{\dagger},a]a^{\dagger}\right\} \\
&= -\frac{1}{2}\zeta\zeta^{*}(-2a^{\dagger})=s^{2}a^{\dagger}
\end{aligned} \tag{14}$$

And similarly by BCH formula

$$\begin{aligned}
e^{A}a^{\dagger}e^{-A} &= \sum_{l=0}^{\infty}\frac{s^{2l}a^{\dagger}}{(2l)!}+\sum_{l=0}^{\infty}\frac{(-\zeta^{*})s^{2l}a}{(2l+1)!} \\
&= \sum_{l=0}^{\infty}\frac{s^{2l}a^{\dagger}}{(2l)!}-e^{-i\theta}\sum_{l=0}^{\infty}\frac{s^{2l+1}a}{(2l+1)!} \\
&= a^{\dagger}\cosh s-e^{-i\theta}a\sinh s
\end{aligned} \tag{15}$$

With these, we can finally calculate $[\Delta E(\chi)]^2$:

$$\begin{aligned}
[\Delta E(\chi)]^{2} &= \langle E(\chi)^{2}\rangle=\left\langle \mathbf{0}\middle|Z^{\dagger}E(\chi)^{2}Z\middle|\mathbf{0}\right\rangle \\
&= \frac{1}{4}\left\langle \mathbf{0}\middle|e^{A}\left(a^{2}e^{-2i\chi}+a^{\dagger 2}e^{2i\chi}+aa^{\dagger}+a^{\dagger}a\right)e^{-A}\middle|\mathbf{0}\right\rangle \\
&\equiv \frac{1}{4}\left(X_{1}+X_{2}+X_{3}+X_{4}\right)
\end{aligned} \tag{16}$$

where

$$X_{1}=\left\langle \mathbf{0}\middle|e^{A}a^{2}e^{-2i\chi}e^{-A}\middle|\mathbf{0}\right\rangle \tag{17}$$
$$X_{2}=\left\langle \mathbf{0}\middle|e^{A}a^{\dagger 2}e^{2i\chi}e^{-A}\middle|\mathbf{0}\right\rangle \tag{18}$$
$$X_{3}=\left\langle \mathbf{0}\middle|e^{A}aa^{\dagger}e^{-A}\middle|\mathbf{0}\right\rangle \tag{19}$$
$$X_{4}=\left\langle \mathbf{0}\middle|e^{A}a^{\dagger}ae^{-A}\middle|\mathbf{0}\right\rangle \tag{20}$$

Let's take a note that

$$\left\langle \mathbf{0} \middle| a^{\dagger 2} \middle| \mathbf{0} \right\rangle = \left\langle \mathbf{0} \middle| a^{\dagger} a \middle| \mathbf{0} \right\rangle = \left\langle \mathbf{0} \middle| a^{2} \middle| \mathbf{0} \right\rangle = 0 \qquad \left\langle \mathbf{0} \middle| a a^{\dagger} \middle| \mathbf{0} \right\rangle = 1 \tag{21}$$

This leaves only a few surviving terms when we expand X_1-X_4 using (11) and (15).

$$\begin{aligned}
X_1 &= \left\langle \mathbf{0} \middle| e^{A} a^2 e^{-2i\chi} e^{-A} \middle| \mathbf{0} \right\rangle \\
&= e^{-2i\chi} \left\langle \mathbf{0} \middle| \left(a \cosh s - e^{i\theta} a^{\dagger} \sinh s\right)\left(a \cosh s - e^{i\theta} a^{\dagger} \sinh s\right) \middle| \mathbf{0} \right\rangle \\
&= e^{-2i\chi} (-e^{i\theta}) \sinh s \cosh s
\end{aligned} \tag{22}$$

$$\begin{aligned}
X_2 &= \left\langle \mathbf{0} \middle| e^{A} a^{\dagger 2} e^{2i\chi} e^{-A} \middle| \mathbf{0} \right\rangle \\
&= e^{2i\chi} \left\langle \mathbf{0} \middle| \left(a^{\dagger} \cosh s - e^{-i\theta} a \sinh s\right)\left(a^{\dagger} \cosh s - e^{-i\theta} a \sinh s\right) \middle| \mathbf{0} \right\rangle \\
&= e^{2i\chi} (-e^{-i\theta}) \sinh s \cosh s
\end{aligned} \tag{23}$$

$$\begin{aligned}
X_3 &= \left\langle \mathbf{0} \middle| e^{A} a a^{\dagger} e^{-A} \middle| \mathbf{0} \right\rangle \\
&= \left\langle \mathbf{0} \middle| \left(a \cosh s - e^{i\theta} a^{\dagger} \sinh s\right)\left(a^{\dagger} \cosh s - e^{-i\theta} a \sinh s\right) \middle| \mathbf{0} \right\rangle \\
&= \cosh^2 s
\end{aligned} \tag{24}$$

$$\begin{aligned}
X_4 &= \left\langle \mathbf{0} \middle| e^{A} a^{\dagger} a e^{-A} \middle| \mathbf{0} \right\rangle \\
&= \left\langle \mathbf{0} \middle| \left(a^{\dagger} \cosh s - e^{-i\theta} a \sinh s\right)\left(a \cosh s - e^{i\theta} a^{\dagger} \sinh s\right) \middle| \mathbf{0} \right\rangle \\
&= \sinh^2 s
\end{aligned} \tag{25}$$

which produces

$$\begin{aligned}
4[\Delta E(\chi)]^2 &= X_1 + X_2 + X_3 + X_4 \\
&= -\sinh s \cosh s \left(e^{i\theta} e^{-2i\chi} + e^{-i\theta} e^{2i\chi}\right) + \cosh^2 s + \sinh^2 s \\
&= -\left(\frac{e^{s} - e^{-s}}{2}\right)\left(\frac{e^{s} + e^{-s}}{2}\right)\left[e^{i(2\chi-\theta)} + e^{-i(2\chi-\theta)}\right] + \left(\frac{e^{s} + e^{-s}}{2}\right)^2 + \left(\frac{e^{s} - e^{-s}}{2}\right)^2 \\
&= -\frac{e^{2s} - e^{-2s}}{2} \cos(2\chi - \theta) + \frac{e^{2s} + e^{-2s}}{2} \\
&= e^{2s} \frac{1 - \cos(2\chi - \theta)}{2} + e^{-2s} \frac{1 + \cos(2\chi - \theta)}{2} \\
&= e^{2s} \sin^2\left(\chi - \frac{\theta}{2}\right) + e^{-2s} \cos^2\left(\chi - \frac{\theta}{2}\right)
\end{aligned} \tag{26}$$

which achieves minimum electric field variance

$$\Delta E_{\text{min}} = \frac{1}{2} e^{-s} \qquad \text{for } \chi = \frac{\theta}{2} + m\pi \tag{27}$$

and maximum variance

$$\Delta E_{\text{max}} = \frac{1}{2} e^{s} \qquad \text{for } \chi = \frac{\theta}{2} + \left(m + \frac{1}{2}\right)\pi \tag{28}$$