(a) First let's prove $D\Theta = \Theta D$, where $D = e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi/\hbar}$.
Indeed

$$\begin{aligned}
\Theta D \Theta^{-1} &= \Theta\left[\sum_k \frac{1}{k!}\left(\frac{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi}{\hbar}\right)^k\right]\Theta^{-1} \\
&= \sum_k \frac{1}{k!}\left[\frac{\Theta(-i\boldsymbol{J}\cdot\boldsymbol{n}\phi)\Theta^{-1}}{\hbar}\right]^k \\
&= \sum_k \frac{1}{k!}\left[\frac{-(i\boldsymbol{J}\cdot\boldsymbol{n}\phi)\Theta\Theta^{-1}}{\hbar}\right]^k \\
&= \sum_k \frac{1}{k!}\left(\frac{-i\boldsymbol{J}\cdot\boldsymbol{n}\phi}{\hbar}\right)^k = D
\end{aligned}$$

where in the third equal sign we used $\Theta i|\rangle = -i\Theta|\rangle$, and $\Theta\boldsymbol{J} = -\boldsymbol{J}\Theta$.
Now from the eigenequation

$$J_z\Theta|j,m\rangle = -\Theta J_z|j,m\rangle = -m\hbar\Theta|j,m\rangle$$

we see that $\Theta|j,m\rangle$ is the same as $|j,m\rangle$ up to a phase factor, i.e.,

$$\Theta|j,m\rangle = \eta_j^m|j,-m\rangle$$

Also observe that

$$\Theta J_+ = \Theta(J_x + iJ_y) = (-J_x + iJ_y)\Theta = -J_-\Theta$$

then

$$\begin{aligned}
\Theta J_+|j,m\rangle &= \Theta\sqrt{(j-m)(j+m+1)}\hbar|j,m+1\rangle \\
&= \sqrt{(j-m)(j+m+1)}\hbar\Theta|j,m+1\rangle \\
&= \sqrt{(j-m)(j+m+1)}\hbar\eta_j^{m+1}|j,-m-1\rangle \qquad (1)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\Theta J_+|j,m\rangle &= -J_-\Theta|j,m\rangle \\
&= -\eta_j^m J_-|j,-m\rangle \\
&= -\eta_j^m\sqrt{[j+(-m)][(j-(-m)+1]}\hbar|j,-m-1\rangle \qquad (2)
\end{aligned}$$

Comparing (1) and (2), we know

$$\eta_j^{m+1} = -\eta_j^m$$

or equivalently

$$\eta_j^m = e^{i\delta}(-1)^m$$

for some global phase factor $e^{i\delta}$ independent of m (which could be dependent of j).

(b)

$$\Theta D|j,m\rangle = D\Theta|j,m\rangle = e^{i\delta}(-1)^m D|j,-m\rangle$$

(c) First observe that the proof of $\Theta D\Theta^{-1} = D$ applies similarly to the relation $\Theta D^\dagger\Theta^{-1} = D^\dagger$. Then

$$\begin{aligned}
D_{m'm}^{j*} &= \langle j,m'|D|j,m\rangle^* \\
&= \langle j,m|D^\dagger|j,m'\rangle \\
&= \langle j,m|\Theta D^\dagger\Theta^{-1}|j,m'\rangle \qquad (3)
\end{aligned}$$

But Since

$$\Theta|j,-m'\rangle = e^{i\delta}(-1)^{-m'}|j,m'\rangle \qquad \Longrightarrow$$
$$e^{-i\delta}(-1)^{m'}\Theta|j,-m'\rangle = \Theta e^{i\delta}(-1)^{m'}|j,-m'\rangle = |j,m'\rangle \qquad \Longrightarrow$$
$$\Theta^{-1}|j,m'\rangle = e^{i\delta}(-1)^{m'}|j,-m'\rangle$$

thus (3) becomes

$$\begin{aligned}
& & D^{j*}_{m'm} &= \langle j,m|\Theta D^\dagger e^{i\delta}(-1)^{m'}|j,-m'\rangle \\
&(\Theta \text{ antiunitary}) & &= e^{-i\delta}(-1)^{m'}\langle j,m|\Theta D^\dagger|j,-m'\rangle \\
& & &= e^{-i\delta}(-1)^{m'}\langle j,m|\Theta \sum_{m''}|j,m''\rangle\langle j,m''|D^\dagger|j,-m'\rangle \\
&(\Theta \text{ antiunitary}) & &= e^{-i\delta}(-1)^{m'}\sum_{m''}\langle j,m''|D^\dagger|j,-m'\rangle^*\langle j,m|\Theta|j,m''\rangle \\
& & &= e^{-i\delta}(-1)^{m'}\sum_{m''}\langle j,-m'|D|j,m''\rangle\langle j,m|e^{i\delta}(-1)^{m''}|j,-m''\rangle \\
&(\text{only } m''=-m \text{ survives the sum}) & &= e^{-i\delta}(-1)^{m'}\langle j,-m'|D|j,-m\rangle e^{i\delta}(-1)^{-m} \\
& & &= (-1)^{m'-m}D^j_{-m',-m} \\
&(m-m' \text{ is integer}) & &= (-1)^{m-m'}D^j_{-m',-m}
\end{aligned}$$

(d) Obvious, given (a), (c).