Update: There seems to be a much simpler proof if we take either $\hat{\boldsymbol{k}}$ or $\hat{\boldsymbol{r}}$ to be the $\hat{\boldsymbol{z}}$, as outlined in Sakurai (6.111)-(6.115).

We will prove the referenced *addition theorem of spherical harmonics*

$$\sum_m Y_l^m(\hat{\boldsymbol{r}})Y_l^{m*}(\hat{\boldsymbol{k}}) = \frac{2l+1}{4\pi}P_l(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{k}}) \tag{1}$$

By Sakurai equation (3.260)

$$Y_l^m(\hat{\boldsymbol{r}}) = \sqrt{\frac{2l+1}{4\pi}}\mathscr{D}_{m0}^{(l)*}(\phi_{\hat{r}},\theta_{\hat{r}},0) \tag{2}$$

$$Y_l^{m*}(\hat{\boldsymbol{k}}) = \sqrt{\frac{2l+1}{4\pi}}\mathscr{D}_{m0}^{(l)}(\phi_{\hat{k}},\theta_{\hat{k}},0) \tag{3}$$

Then we have

$$\begin{aligned}\sum_m Y_l^m(\hat{\boldsymbol{r}})Y_l^{m*}(\hat{\boldsymbol{k}}) &= \frac{2l+1}{4\pi}\sum_m \mathscr{D}_{m0}^{(l)*}(\phi_{\hat{r}},\theta_{\hat{r}},0)\mathscr{D}_{m0}^{(l)}(\phi_{\hat{k}},\theta_{\hat{k}},0)\\ &= \frac{2l+1}{4\pi}\sum_m \left[\mathscr{D}^{(l)\dagger}(\phi_{\hat{r}},\theta_{\hat{r}},0)\right]^T{}_{m0}\mathscr{D}_{m0}^{(l)}(\phi_{\hat{k}},\theta_{\hat{k}},0)\\ &= \frac{2l+1}{4\pi}\sum_m \left[\mathscr{D}^{(l)\dagger}(\phi_{\hat{r}},\theta_{\hat{r}},0)\right]_{0m}\mathscr{D}_{m0}^{(l)}(\phi_{\hat{k}},\theta_{\hat{k}},0)\\ &= \frac{2l+1}{4\pi}\left[\mathscr{D}^{(l)\dagger}(\phi_{\hat{r}},\theta_{\hat{r}},0)\cdot\mathscr{D}^{(l)}(\phi_{\hat{k}},\theta_{\hat{k}},0)\right]_{00}\end{aligned} \tag{4}$$

Recall that $\mathscr{D}^{(l)}(\alpha,\beta,0)$ is just the l-dimensional representation of the rotation operator $R_z(\alpha)R_y(\beta)$ (Sakurai Fig 3.3), so

$$\mathscr{D}^{(l)\dagger}(\phi_{\hat{r}},\theta_{\hat{r}},0)\cdot\mathscr{D}^{(l)}(\phi_{\hat{k}},\theta_{\hat{k}},0)$$

is just the l-dimensional representation of the rotation operator

$$R_y^\dagger(\theta_{\hat{r}})R_z^\dagger(\phi_{\hat{r}})R_z(\phi_{\hat{k}})R_y(\theta_{\hat{k}}) = R_y(-\theta_{\hat{r}})R_z(\phi_{\hat{k}}-\phi_{\hat{r}})R_y(\theta_{\hat{k}}) \tag{5}$$

Also recall Sakurai equation (3.262)

$$\mathscr{D}_{00}^{(l)}(\phi,\theta,0) = P_l(\cos\theta) \tag{6}$$

Compare (4)-(6) with (1), it remains to prove that

$$R_y(-\theta_{\hat{r}})R_z(\phi_{\hat{k}}-\phi_{\hat{r}})R_y(\theta_{\hat{k}}) = R_z(\phi)R_y(\angle_{\hat{k},\hat{r}}) \tag{7}$$

which is a rotation with polar angle equal to the angle between $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{r}}$, and some azimuthal angle ϕ for which we don't care about.

To see the LHS of (7) has an effective polar angle of $\angle_{\hat{k},\hat{r}}$, it's sufficient to prove that it rotates the $\hat{\boldsymbol{z}}$ vector into a vector whose z component equals to $\cos\angle_{\hat{k},\hat{r}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}$.

Indeed, on the one hand

$$\begin{aligned}R_y(-\theta_{\hat{r}})R_z(\phi_{\hat{k}}-\phi_{\hat{r}})R_y(\theta_{\hat{k}})\begin{bmatrix}0\\0\\1\end{bmatrix} &= R_y(-\theta_{\hat{r}})R_z(\phi_{\hat{k}}-\phi_{\hat{r}})\begin{bmatrix}\cos\theta_{\hat{k}} & 0 & \sin\theta_{\hat{k}}\\ 0 & 1 & 0\\ -\sin\theta_{\hat{k}} & 0 & \cos\theta_{\hat{k}}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\\ &= R_y(-\theta_{\hat{r}})\begin{bmatrix}\cos(\phi_{\hat{k}}-\phi_{\hat{r}}) & -\sin(\phi_{\hat{k}}-\phi_{\hat{r}}) & 0\\ \sin(\phi_{\hat{k}}-\phi_{\hat{r}}) & \cos(\phi_{\hat{k}}-\phi_{\hat{r}}) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sin\theta_{\hat{k}}\\0\\\cos\theta_{\hat{k}}\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta_{\hat{r}} & 0 & -\sin\theta_{\hat{r}}\\ 0 & 1 & 0\\ \sin\theta_{\hat{r}} & 0 & \cos\theta_{\hat{r}}\end{bmatrix}\begin{bmatrix}\sin\theta_{\hat{k}}\cos(\phi_{\hat{k}}-\phi_{\hat{r}})\\ \sin\theta_{\hat{k}}\sin(\phi_{\hat{k}}-\phi_{\hat{r}})\\ \cos\theta_{\hat{k}}\end{bmatrix}\\ &= \begin{bmatrix}\cdots\\ \cdots\\ \sin\theta_{\hat{r}}\sin\theta_{\hat{k}}\cos(\phi_{\hat{k}}-\phi_{\hat{r}})+\cos\theta_{\hat{r}}\cos\theta_{\hat{k}}\end{bmatrix}\end{aligned} \tag{8}$$

and on the other hand

$$
\begin{aligned}
\cos \angle_{\hat{k},\hat{r}} &= \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}} \\
&= \begin{bmatrix} \sin\theta_{\hat{k}} \cos\phi_{\hat{k}} \\ \sin\theta_{\hat{k}} \sin\phi_{\hat{k}} \\ \cos\theta_{\hat{k}} \end{bmatrix} \cdot \begin{bmatrix} \sin\theta_{\hat{r}} \cos\phi_{\hat{r}} \\ \sin\theta_{\hat{r}} \sin\phi_{\hat{r}} \\ \cos\theta_{\hat{r}} \end{bmatrix} \\
&= \sin\theta_{\hat{k}} \sin\theta_{\hat{r}} \left(\cos\phi_{\hat{k}} \cos\phi_{\hat{r}} + \sin\phi_{\hat{k}} \sin\phi_{\hat{r}} \right) + \cos\theta_{\hat{k}} \cos\theta_{\hat{r}} \\
&= \sin\theta_{\hat{k}} \sin\theta_{\hat{r}} \cos(\phi_{\hat{k}} - \phi_{\hat{r}}) + \cos\theta_{\hat{k}} \cos\theta_{\hat{r}}
\end{aligned} \tag{9}
$$