



We shall derive equation (3.220), referring to the diagram above.

Let $\mathbf{x} = (r = 1, \theta, \phi)$ be a point in spherical coordinates. Consider an infinitesimal rotation around x -axis by angle $d\xi$. We would like to find the expression of the corresponding change in $d\theta$ and $d\phi$ by such an infinitesimal rotation.

In order to do this, we shall consider the infinitesimal arc swept by $d\xi$, and project this arc onto the $\hat{\theta}$ and $\hat{\phi}$ direction. Referring to the diagram, we know the infinitesimal swept arc vector is

$$d\mathbf{s} = \sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi} d\xi \hat{\xi} \quad (1)$$

where the unit vector

$$\hat{\xi} = 0\hat{\mathbf{x}} - \sin \xi \hat{\mathbf{y}} + \cos \xi \hat{\mathbf{z}} \quad (2)$$

We also know that

$$\hat{\theta} = \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \quad (3)$$

$$\hat{\phi} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} + 0\hat{\mathbf{z}} \quad (4)$$

From which we obtain the projection coefficients

$$\hat{\xi} \cdot \hat{\theta} = -\sin \xi \cos \theta \sin \phi - \sin \theta \cos \xi \quad (5)$$

$$\hat{\xi} \cdot \hat{\phi} = -\sin \xi \cos \phi \quad (6)$$

But by geometry, we know

$$\sin \xi = \frac{\cos \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} \quad (7)$$

$$\cos \xi = \frac{\sin \theta \sin \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} \quad (8)$$

Thus (5) and (6) become

$$\hat{\xi} \cdot \hat{\theta} = -\frac{\cos^2 \theta \sin \phi + \sin^2 \theta \sin \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} = -\frac{\sin \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} \quad (9)$$

$$\hat{\xi} \cdot \hat{\phi} = -\frac{\cos \theta \cos \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} \quad (10)$$

It is straightforward to verify that the projection is complete in the sense that $(\hat{\xi} \cdot \hat{\theta})^2 + (\hat{\xi} \cdot \hat{\phi})^2 = 1$, as it should since $\hat{\xi}$ does not have a radial component.

Using (1), (9) and (10) to project the arc onto $\hat{\theta}$ and $\hat{\phi}$ direction, the projected arc lengths are

$$ds \cdot \hat{\theta} = -\sin \phi d\xi \quad (11)$$

$$ds \cdot \hat{\phi} = -\cos \theta \cos \phi d\xi \quad (12)$$

Then $d\theta$ and $d\phi$ can be obtained by dividing (11) and (12) by the corresponding radius 1 and $\sin \theta$, which gives the relation

$$d\theta = -\sin \phi d\xi \quad (13)$$

$$d\phi = -\cot \theta \cos \phi d\xi \quad (14)$$

To summarize, when an infinitesimal rotation $d\xi$ around x -axis is applied to position eigenket $|r, \theta, \phi\rangle$, the state changes to $|r, \theta + d\theta, \phi + d\phi\rangle$ according to (13) and (14), i.e.,

$$\left(1 - \frac{id\xi L_x}{\hbar}\right)|r, \theta, \phi\rangle = |r, \theta + d\theta, \phi + d\phi\rangle \quad (15)$$

Conjugating (15) yields

$$\begin{aligned} \left[\left(1 - \frac{id\xi L_x}{\hbar}\right)|r, \theta, \phi\rangle\right]^\dagger &= \left[\left(1 - \frac{i(-d\xi)L_x}{\hbar}\right)|r, \theta, \phi\rangle\right]^\dagger \\ (|r, \theta - d\theta, \phi - d\phi\rangle)^\dagger &= \langle r, \theta - d\theta, \phi - d\phi| \end{aligned} \quad (16)$$

Therefore

$$\begin{aligned} \left\langle r, \theta, \phi \left| 1 - \frac{id\xi L_x}{\hbar} \right| \alpha \right\rangle &= \langle r, \theta - d\theta, \phi - d\phi | \alpha \rangle \\ &= \langle r, \theta, \phi | \alpha \rangle - \left(\frac{\partial}{\partial \theta} d\theta + \frac{\partial}{\partial \phi} d\phi \right) \langle r, \theta, \phi | \alpha \rangle \end{aligned}$$

From which we identify

$$\begin{aligned} \langle r, \theta, \phi | L_x | \alpha \rangle &= \frac{\hbar}{id\xi} \left(\frac{\partial}{\partial \theta} d\theta + \frac{\partial}{\partial \phi} d\phi \right) \langle r, \theta, \phi | \alpha \rangle \\ &= -i\hbar \left(-\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \langle r, \theta, \phi | \alpha \rangle \end{aligned} \quad (17)$$