In these notes, we give the full treatment of Runge-Lenz operator

$$\mathbf{M} = \frac{1}{2m}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) - \frac{Ze^2}{r}\mathbf{x} \tag{1}$$

in a central Coulomb potential $V(r) = -Ze^2/r$, i.e., with Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} - \frac{Ze^2}{r} \tag{2}$$

We shall show the following relationships

$$[M_i, L_j] = i\hbar\epsilon_{ijk}M_k \tag{3}$$
$$[\mathbf{M}, H] = 0 \tag{4}$$
$$\mathbf{L}\cdot\mathbf{M} = 0 \tag{5}$$
$$\mathbf{M}\cdot\mathbf{L} = 0 \tag{6}$$
$$\mathbf{M}^2 = \frac{2}{m}H\left(\mathbf{L}^2 + \hbar^2\right) + Z^2e^4 \tag{7}$$
$$[M_i, M_j] = -i\hbar\epsilon_{ijk}\frac{2}{m}HL_k \tag{8}$$

First we recall a few results that are used throughout the proofs.

1. From excercise 1.31,

$$[p_k, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_k} \tag{9}$$

2. Both $\mathbf{p}$ and $\mathbf{x}$ are vector operators:

$$[p_i, L_j] = \epsilon_{lkj}[p_i, x_l p_k] = \epsilon_{lkj}[p_i, x_l]p_k = -i\hbar\epsilon_{lkj}\delta_{il}p_k = i\hbar\epsilon_{ijk}p_k \tag{10}$$
$$[x_i, L_j] = \epsilon_{lkj}[x_i, x_l p_k] = \epsilon_{lkj}x_l[x_i, p_k] = i\hbar\epsilon_{lkj}x_l\delta_{ik} = i\hbar\epsilon_{ijl}x_l \tag{11}$$

3. For two vector operators $\mathbf{u}, \mathbf{v}$, $\mathbf{u}\cdot\mathbf{v}$ is a scalar operator, i.e., it commutes with L_j. Since

$$\begin{aligned}\left[\mathbf{u}\cdot\mathbf{v}, L_j\right] &= \left[u_i v_i, L_j\right] = \left[u_i, L_j\right]v_i + u_i\left[v_i, L_j\right] \\ &= i\hbar\epsilon_{ijk}u_k v_i + i\hbar\epsilon_{ijk}u_i v_k = u_k\left[L_j, v_k\right] + \left[L_j, u_k\right]v_k = \left[L_j, u_k v_k\right] = \left[L_j, \mathbf{u}\cdot\mathbf{v}\right]\end{aligned} \tag{12}$$

4. Any spherically symmetric function of position $F(\mathbf{x}) = f(r)$ is a scalar operator, i.e., it commutes with the generator of rotation, i.e.,

$$[L_k, f(r)] = 0 \tag{13}$$

Indeed, with the antisymmetry of ϵ_{ijk} symbol,

$$[L_k, f(r)] = \epsilon_{ijk}[x_i p_j, f(r)] = \epsilon_{ijk}x_i[p_j, f(r)] = -i\hbar\epsilon_{ijk}x_i\frac{\partial f(r)}{\partial x_j} = -i\hbar\epsilon_{ijk}x_i f'(r)\frac{x_j}{r} = 0$$

5. For a scalar operator s and a vector operator $\mathbf{v}$, both $s\mathbf{v}$ and $\mathbf{v}s$ are vector operators:

$$\left[sv_i, L_j\right] = s\left[v_i, L_j\right] = i\hbar\epsilon_{ijk}sv_k \tag{14}$$
$$\left[v_i s, L_j\right] = \left[v_i, L_j\right]s = i\hbar\epsilon_{ijk}v_k s \tag{15}$$

6.

$$\mathbf{L}\cdot\mathbf{p} = L_k p_k = \epsilon_{ijk}x_i p_j p_k = 0 \tag{16}$$
$$\mathbf{p}\cdot\mathbf{L} = p_k L_k = \epsilon_{ijk}p_k x_i p_j = \epsilon_{ijk}p_k\left(p_j x_i + \delta_{ij}i\hbar\right) = \epsilon_{ijk}p_k p_j x_i + \epsilon_{ijk}\delta_{ij}i\hbar = 0 \tag{17}$$

7. For any given $i, j, l, m \in \{1,2,3\}$,

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{18}$$

   As usual, k is to be summed over.

   The proof is straightforward by observing for a fixed k, say $k = 3$, and enumerating possible combinations of i, j, l and m.

8. For vector operators $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ is also a vector operator.

   Indeed

$$\begin{aligned}
\left[(\mathbf{u}\times\mathbf{v})_i, L_j\right] &= \epsilon_{lmi}\left[u_l v_m, L_j\right] = \epsilon_{lmi}\left(\left[u_l, L_j\right] v_m + u_l\left[v_m, L_j\right]\right) \\
&= \epsilon_{lmi}\left(i\hbar\epsilon_{ljk}u_k v_m + i\hbar\epsilon_{mjk}u_l v_k\right) \\
&= i\hbar\left(\epsilon_{lmi}\epsilon_{ljk}u_k v_m + \epsilon_{lmi}\epsilon_{mjk}u_l v_k\right) \\
&= i\hbar\left[\left(\delta_{jm}\delta_{ik} - \delta_{mk}\delta_{ij}\right)u_k v_m - \left(\delta_{ik}\delta_{jl} - \delta_{lk}\delta_{ij}\right)u_l v_k\right] \\
&= i\hbar\left(u_i v_j - \delta_{ij}u_k v_k - u_j v_i + \delta_{ij}u_k v_k\right) \\
&= i\hbar\left(u_i v_j - u_j v_i\right) \\
&= i\hbar\epsilon_{ijk}(\mathbf{u}\times\mathbf{v})_k
\end{aligned} \tag{19}$$

---

1. **Proof of (3).**

   This is obvious since due to (19), both $\mathbf{p}\times\mathbf{L}$ and $\mathbf{L}\times\mathbf{p}$ are vector operators. Furthermore, $\mathbf{x}/r$ is the product of a vector operator and a scalar (i.e., setting $f(r) = 1/r$ in (13)), so $\mathbf{M}$ is indeed a vector operator.

2. **Proof of (4).**

   By definition of $\mathbf{M}$, we have

$$[M_k, H] = \overbrace{\frac{1}{2m}\epsilon_{ijk}\left[p_i L_j - L_i p_j, H\right]}^{A} - \overbrace{\left[Ze^2\frac{x_k}{r}, H\right]}^{B}$$

   where

$$\begin{aligned}
B &= Ze^2\left[\frac{x_k}{r}, \frac{p_i p_i}{2m}\right] = \frac{Ze^2}{2m}\left(p_i\left[\frac{x_k}{r}, p_i\right] + \left[\frac{x_k}{r}, p_i\right]p_i\right) \\
&= \frac{i\hbar Ze^2}{2m}\left[p_i\left(\frac{\delta_{ik}}{r} - \frac{x_i x_k}{r^3}\right) + \left(\frac{\delta_{ik}}{r} - \frac{x_i x_k}{r^3}\right)p_i\right] \\
&= \frac{i\hbar Ze^2}{2m}\underbrace{\left[\left(p_k\frac{1}{r} - p_i\frac{x_i x_k}{r^3}\right) + \left(\frac{1}{r}p_k - \frac{x_i x_k}{r^3}p_i\right)\right]}_{B'}
\end{aligned} \tag{20}$$

   On the other hand, recall that both $\mathbf{p}^2$ and $1/r$ in H are scalars, so $[L_i, H] = 0$ for all i, which gives

$$\begin{aligned}
A &= \frac{1}{2m}\epsilon_{ijk}\left([p_i, H]L_j - L_i[p_j, H]\right) \\
&= -\frac{Ze^2}{2m}\epsilon_{ijk}\left(\left[p_i, \frac{1}{r}\right]L_j - L_i\left[p_j, \frac{1}{r}\right]\right) \\
&= -\frac{i\hbar Ze^2}{2m}\epsilon_{ijk}\left(\frac{x_i}{r^3}L_j - L_i\frac{x_j}{r^3}\right) \\
&= \frac{i\hbar Ze^2}{2m}\underbrace{\epsilon_{ijk}\left(L_i\frac{x_j}{r^3} - \frac{x_i}{r^3}L_j\right)}_{A'}
\end{aligned} \tag{21}$$

Now compare (20) with (21), indeed they are equal because

$$
\begin{aligned}
A' &= \epsilon_{ijk}\left(L_i \frac{x_j}{r^3} - \frac{x_i}{r^3} L_j\right) \\
&= \epsilon_{ijk}\left(\epsilon_{lmi} x_l p_m \frac{x_j}{r^3} - \epsilon_{lmj} \frac{x_i}{r^3} x_l p_m\right) \\
&= \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) x_l p_m \frac{x_j}{r^3} + \left(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}\right) \frac{x_i}{r^3} x_l p_m \\
&= x_j p_k \frac{x_j}{r^3} - x_k p_j \frac{x_j}{r^3} + \frac{x_i x_i}{r^3} p_k - \frac{x_i x_k}{r^3} p_i \\
&= \left(p_k x_j + \left[x_j, p_k\right]\right) \frac{x_j}{r^3} - \left(p_j x_k + \left[x_k, p_j\right]\right) \frac{x_j}{r^3} + \frac{x_i x_i}{r^3} p_k - \frac{x_i x_k}{r^3} p_i \\
&= p_k \frac{x_j x_j}{r^3} + i\hbar \delta_{jk} \frac{x_j}{r^3} - p_j \frac{x_j x_k}{r^3} - i\hbar \delta_{jk} \frac{x_j}{r^3} + \frac{x_i x_i}{r^3} p_k - \frac{x_i x_k}{r^3} p_i \\
&= B'
\end{aligned}
$$

3. **Proof of (5)**.

$$
\mathbf{L} \cdot \mathbf{M} = L_k M_k = \frac{1}{2m} \overbrace{\epsilon_{ijk}\left[L_k\left(p_i L_j - L_i p_j\right)\right]}^{A} - Z e^2 \overbrace{L_k \frac{x_k}{r}}^{B} \tag{22}
$$

where

$$
\begin{aligned}
A &= \epsilon_{ijk}\left(L_k p_i L_j - L_k L_i p_j\right) = \epsilon_{ijk} L_k p_i L_j - \epsilon_{ijk} L_k L_i p_j \\
&= \epsilon_{ijk}\left(p_i L_k - \left[p_i, L_k\right]\right) L_j - \epsilon_{ijk} L_k L_i p_j \\
&= \epsilon_{ijk} p_i L_k L_j - \epsilon_{ijk}\left(i\hbar \epsilon_{ikj} p_j\right) L_j - \epsilon_{ijk} L_k L_i p_j \\
\text{(use } \epsilon_{ijk} L_i L_j = (\mathbf{L} \times \mathbf{L})_k = i\hbar L_k \text{ etc.)} \quad &= -i\hbar p_i L_i + i\hbar \epsilon_{ijk}^2 p_j L_j - i\hbar L_j p_j \\
\text{(use (16), (17))} \quad &= 0 \\
B &= L_k \frac{x_k}{r} = \epsilon_{ijk} x_i p_j \frac{x_k}{r} = \epsilon_{ijk} x_i \left(\frac{x_k}{r} p_j + \left[p_j, \frac{x_k}{r}\right]\right) \\
&= \epsilon_{ijk} x_i \frac{x_k}{r} p_j - i\hbar \epsilon_{ijk} x_i \left(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3}\right) \\
&= 0
\end{aligned}
$$

4. **Proof of (6)**.

   This trivially follows from (3) (by setting $i = j$) and (5).

5. **Proof of (7)**.

   Expanding $\mathbf{M}^2$ on the LHS and H on the RHS, we get

$$
\begin{aligned}
\text{LHS} &= M_k M_k = \left[\frac{1}{2m}\epsilon_{ijk}\left(p_i L_j - L_i p_j\right) - Z e^2 \frac{x_k}{r}\right] \cdot \left[\frac{1}{2m}\epsilon_{lmk}\left(p_l L_m - L_l p_m\right) - Z e^2 \frac{x_k}{r}\right] \\
&= \overbrace{\frac{1}{4m^2}\epsilon_{ijk}\epsilon_{lmk}\left(p_i L_j - L_i p_j\right)\left(p_l L_m - L_l p_m\right)}^{A} - \overbrace{\frac{Z e^2}{2m}\epsilon_{ijk}\left[\left(p_i L_j - L_i p_j\right)\frac{x_k}{r} + \frac{x_k}{r}\left(p_i L_j - L_i p_j\right)\right]}^{B} + \overbrace{Z^2 e^4 \frac{x_k x_k}{r^2}}^{C} \\
\text{RHS} &= \frac{2}{m}\left(\frac{\mathbf{p}^2}{2m} - Z e^2 \frac{1}{r}\right)\left(\mathbf{L}^2 + \hbar^2\right) + Z^2 e^4 \\
&= \overbrace{\frac{\mathbf{p}^2}{m^2}\left(\mathbf{L}^2 + \hbar^2\right)}^{A'} - \overbrace{\frac{2 Z e^2}{m}\frac{1}{r}\left(\mathbf{L}^2 + \hbar^2\right)}^{B'} + \overbrace{Z^2 e^4}^{C'}
\end{aligned}
$$

   For dimensions to match, we must have $A = A', B = B', C = C'$, where $C = C'$ is already trivial to see.

   To prove $B = B'$, it's equivalent to prove

$$
\epsilon_{ijk}\left[\left(p_i L_j - L_i p_j\right)\frac{x_k}{r} + \frac{x_k}{r}\left(p_i L_j - L_i p_j\right)\right] = \frac{4}{r}\left(\mathbf{L}^2 + \hbar^2\right) \tag{23}
$$

Let's break the LHS of (23) into

$$\overbrace{\epsilon_{ijk}p_iL_j\frac{x_k}{r}}^{B_1}-\overbrace{\epsilon_{ijk}L_ip_j\frac{x_k}{r}}^{B_2}+\overbrace{\epsilon_{ijk}\frac{x_k}{r}p_iL_j}^{B_3}-\overbrace{\epsilon_{ijk}\frac{x_k}{r}L_ip_j}^{B_4}$$

where we immediately see that

$$B_3=\frac{1}{r}L_jL_j=\frac{1}{r}\mathbf{L}^2$$

Furthermore

$$B_2=\epsilon_{ijk}L_i\left(\frac{x_k}{r}p_j+\left[p_j,\frac{x_k}{r}\right]\right)=\epsilon_{ijk}L_i\left[\frac{x_k}{r}p_j-i\hbar\left(\frac{\delta_{jk}}{r}-\frac{x_jx_k}{r^3}\right)\right]=L_i\left(-\frac{1}{r}L_i\right)=-\frac{1}{r}\mathbf{L}^2$$

where in the last step, we have used (13).

Lastly

$$\begin{aligned}
B_1&=\epsilon_{ijk}p_i\left(\frac{x_k}{r}L_j-\left[\frac{x_k}{r},L_j\right]\right)\\
&=\epsilon_{ijk}\left(\frac{x_k}{r}p_i+\left[p_i,\frac{x_k}{r}\right]\right)L_j-\epsilon_{ijk}p_i\left(i\hbar\epsilon_{kji}\frac{x_i}{r}\right)\\
&=\frac{1}{r}L_jL_j-i\hbar\epsilon_{ijk}\left(\frac{\delta_{ik}}{r}-\frac{x_ix_k}{r^3}\right)L_j+i\hbar\epsilon_{ijk}^2p_i\frac{x_i}{r}\\
&=\frac{1}{r}\mathbf{L}^2+i\hbar\epsilon_{ijk}^2p_i\frac{x_i}{r}\\
B_4&=\epsilon_{ijk}\left(L_i\frac{x_k}{r}+\left[\frac{x_k}{r},L_i\right]\right)p_j\\
&=-L_i\frac{1}{r}L_i+\epsilon_{ijk}\left(i\hbar\epsilon_{kij}\frac{x_j}{r}\right)p_j\\
&=-\frac{1}{r}\mathbf{L}^2+i\hbar\epsilon_{ijk}^2\frac{x_j}{r}p_j
\end{aligned}$$

Thus

$$B_1-B_2+B_3-B_4=\frac{4}{r}\mathbf{L}^2+i\hbar\epsilon_{ijk}^2\left[p_i,\frac{x_i}{r}\right]=\frac{4}{r}\mathbf{L}^2+\hbar^2\epsilon_{ijk}^2\left(\frac{1}{r}-\frac{x_ix_i}{r^3}\right)=\frac{4}{r}\left(\mathbf{L}^2+\hbar^2\right)\tag{24}$$

It still remains to show $A=A'$, which is equivalent to

$$\epsilon_{ijk}\epsilon_{lmk}\left(p_iL_j-L_ip_j\right)\left(p_lL_m-L_lp_m\right)=4\mathbf{p}^2\left(\mathbf{L}^2+\hbar^2\right)\tag{25}$$

Using (18), the LHS of (25) can be broken into

$$\begin{aligned}
\text{LHS}&=\left(p_iL_j-L_ip_j\right)\left(p_iL_j-L_ip_j\right)-\left(p_iL_j-L_ip_j\right)\left(p_jL_i-L_jp_i\right)\\
&=\overbrace{p_iL_jp_iL_j}^{A_1}-\overbrace{p_iL_jL_ip_j}^{A_2}-\overbrace{L_ip_jp_iL_j}^{A_3}+\overbrace{L_ip_jL_ip_j}^{A_4}-\overbrace{p_iL_jp_jL_i}^{A_5}+\overbrace{p_iL_jL_jp_i}^{A_6}+\overbrace{L_ip_jp_jL_i}^{A_7}-\overbrace{L_ip_jL_jp_i}^{A_8}
\end{aligned}$$

Attacking them one by one,

$$\begin{aligned}
A_1&=p_iL_jp_iL_j=p_i\left(p_iL_j-\left[p_i,L_j\right]\right)L_j=p_i^2L_j^2-i\hbar\epsilon_{ijk}p_ip_kL_j=\mathbf{p}^2\mathbf{L}^2\\
A_2&=p_iL_jL_ip_j=p_i\left(L_iL_j-\left[L_i,L_j\right]\right)p_j=(\mathbf{p}\cdot\mathbf{L})\cdot(\mathbf{L}\cdot\mathbf{p})-i\hbar\epsilon_{ijk}p_iL_kp_j=-i\hbar\epsilon_{ijk}p_iL_kp_j\\
&=-i\hbar\epsilon_{ijk}\left(L_kp_i+[p_i,L_k]\right)p_j=-i\hbar\epsilon_{ijk}L_kp_ip_j-i\hbar\epsilon_{ijk}(i\hbar\epsilon_{ikj}p_j)p_j\\
&=-\hbar^2\epsilon_{ijk}^2p_j^2=-2\hbar^2\mathbf{p}^2\\
A_3&=L_ip_jp_iL_j=L_ip_ip_jL_j=(\mathbf{L}\cdot\mathbf{p})\cdot(\mathbf{p}\cdot\mathbf{L})=0\\
A_4&=L_ip_jL_ip_j=L_i\left(L_ip_j+[p_j,L_i]\right)p_j=L_i^2p_j^2-i\hbar\epsilon_{ijk}L_ip_kp_j=\mathbf{L}^2\mathbf{p}^2\\
A_5&=p_iL_jp_jL_i=p_i\left(\mathbf{L}\cdot\mathbf{p}\right)L_i=0\\
A_6&=p_iL_jL_jp_i=\left(L_jp_i+\left[p_i,L_j\right]\right)\left(p_iL_j-\left[p_i,L_j\right]\right)=\left(L_jp_i+i\hbar\epsilon_{ijk}p_k\right)\left(p_iL_j-i\hbar\epsilon_{ijk}p_k\right)\\
&=L_j\mathbf{p}^2L_j+i\hbar\epsilon_{ijk}\left(p_kp_iL_j-L_jp_ip_k\right)+\hbar^2\epsilon_{ijk}^2p_k^2\qquad(\mathbf{p}^2\text{ is scalar})\\
&=\mathbf{p}^2\mathbf{L}^2+2\hbar^2\mathbf{p}^2\\
A_7&=L_ip_jp_jL_i=L_i\left(\mathbf{p}^2\right)L_i=\mathbf{p}^2\mathbf{L}^2\\
A_8&=L_ip_jL_jp_i=L_i\left(\mathbf{p}\cdot\mathbf{L}\right)L_i=0
\end{aligned}$$

Noting that $\mathbf{L}^2\mathbf{p}^2=\mathbf{p}^2\mathbf{L}^2$ since $\mathbf{p}^2$ is a scalar, thus we have proved (25), hence (7).

6. **Proof of (8)**.

   First, before going into details, we should note that for $\epsilon_{ijk}=1$, $[M_i,M_j]$ is the k-component of the vector operator $\mathbf{M}\times\mathbf{M}$. Let

$$\mathbf{W}=\mathbf{p}\times\mathbf{L}-\mathbf{L}\times\mathbf{p}$$

   be a vector operator, then

$$M_i=\frac{1}{2m}W_i-Ze^2\frac{x_i}{r}$$
$$M_j=\frac{1}{2m}W_j-Ze^2\frac{x_j}{r}$$

   Thus

$$[M_i,M_j]=\frac{1}{4m^2}[W_i,W_j]-\frac{Ze^2}{2m}\left(\left[W_i,\frac{x_j}{r}\right]+\left[\frac{x_i}{r},W_j\right]\right)\tag{26}$$

   But the RHS of (8) is

$$-i\hbar\epsilon_{ijk}\frac{2}{m}\left(\frac{\mathbf{p}^2}{2m}-\frac{Ze^2}{r}\right)L_k=-i\hbar\epsilon_{ijk}\frac{\mathbf{p}^2}{m^2}L_k+i\hbar\epsilon_{ijk}\frac{2Ze^2}{m}\frac{1}{r}L_k\tag{27}$$

   Comparing (26) with (27), it's clear that we should eventually prove

$$[W_i,W_j]=-4i\hbar\epsilon_{ijk}\mathbf{p}^2L_k\tag{28}$$
$$\left[W_i,\frac{x_j}{r}\right]+\left[\frac{x_i}{r},W_j\right]=-4i\hbar\epsilon_{ijk}\frac{1}{r}L_k\tag{29}$$

   (a) **Proof of (28)**.

   By virtue of being a vector operator, $\mathbf{W}$ satisfies

$$[W_i,L_j]=i\hbar\epsilon_{ijk}W_k\tag{30}$$

   Also

$$\begin{aligned}[W_i,p_j]&=\epsilon_{lmi}\left[p_lL_m-L_lp_m,p_j\right]=\epsilon_{lmi}\left(p_l[L_m,p_j]-[L_l,p_j]p_m\right)\\&=i\hbar\epsilon_{lmi}\left(\epsilon_{mjk}p_lp_k-\epsilon_{ljk}p_kp_m\right)\\&=i\hbar\epsilon_{lmi}\epsilon_{mjk}p_lp_k-i\hbar\epsilon_{lmi}\epsilon_{ljk}p_kp_m\qquad(l\longleftrightarrow m\text{ in 2nd term})\\&=i\hbar\epsilon_{lmi}\epsilon_{mjk}p_lp_k-i\hbar\epsilon_{mli}\epsilon_{mjk}p_kp_l\\&=i\hbar\epsilon_{lmi}\epsilon_{mjk}p_lp_k+i\hbar\epsilon_{lmi}\epsilon_{mjk}p_kp_l\\&=2i\hbar\left(\delta_{lk}\delta_{ij}-\delta_{lj}\delta_{ik}\right)p_lp_k\\&=2i\hbar\left(\delta_{ij}\mathbf{p}^2-p_ip_j\right)\end{aligned}\tag{31}$$

   It follows that

$$\begin{aligned}[W_i,W_j]&=\epsilon_{lmj}[W_i,p_lL_m-L_lp_m]\\&=\epsilon_{lmj}([W_i,p_l]L_m+p_l[W_i,L_m]-[W_i,L_l]p_m-L_l[W_i,p_m])\\&=\overbrace{\epsilon_{lmj}[W_i,p_l]L_m}^{A}+\overbrace{\epsilon_{lmj}p_l[W_i,L_m]}^{B}-\overbrace{\epsilon_{lmj}[W_i,L_l]p_m}^{C}-\overbrace{\epsilon_{lmj}L_l[W_i,p_m]}^{D}\end{aligned}$$

   Using (30) and (31), we have

$$\begin{aligned}A&=2i\hbar\epsilon_{lmj}\left(\delta_{il}\mathbf{p}^2-p_ip_l\right)L_m=2i\hbar\epsilon_{imj}\mathbf{p}^2L_m-2i\hbar\epsilon_{lmj}p_ip_lL_m\\B&=i\hbar\epsilon_{lmj}p_l\epsilon_{imk}W_k=i\hbar\left(\delta_{il}\delta_{jk}-\delta_{ij}\delta_{lk}\right)p_lW_k=i\hbar p_iW_j-i\hbar\delta_{ij}(\mathbf{p}\cdot\mathbf{W})\\C&=i\hbar\epsilon_{lmj}\epsilon_{ilk}W_kp_m=i\hbar\left(\delta_{ij}\delta_{mk}-\delta_{im}\delta_{jk}\right)W_kp_m=i\hbar\delta_{ij}(\mathbf{W}\cdot\mathbf{p})-i\hbar W_jp_i\\D&=2i\hbar\epsilon_{lmj}L_l\left(\delta_{im}\mathbf{p}^2-p_ip_m\right)=2i\hbar\epsilon_{lij}L_l\mathbf{p}^2-2i\hbar\epsilon_{lmj}L_lp_ip_m\end{aligned}$$

   Compare $A+B-C-D$ with RHS of (28), all that remains to show is

$$-2\epsilon_{lmj}p_ip_lL_m+p_iW_j-\delta_{ij}\left(\mathbf{p}\cdot\mathbf{W}+\mathbf{W}\cdot\mathbf{p}\right)+W_jp_i+2\epsilon_{lmj}L_lp_ip_m=0$$

which is easily proved considering the following

$$
\begin{aligned}
\mathbf{p}\cdot\mathbf{W}+\mathbf{W}\cdot\mathbf{p} &= \epsilon_{lmk}\left[p_k\left(p_l L_m - L_l p_m\right)+\left(p_l L_m - L_l p_m\right)p_k\right] \\
&= \epsilon_{lmk}p_k p_l L_m - \epsilon_{lmk}p_k L_l p_m + \epsilon_{lmk}p_l L_m p_k - \epsilon_{lmk}L_l p_m p_k \qquad (lmk \to mkl \text{ in 2nd term}) \\
&= 0 \\
p_i W_j + W_j p_i &= \epsilon_{lmj}\left[p_i\left(p_l L_m - L_l p_m\right)+\left(p_l L_m - L_l p_m\right)p_i\right] \\
&= \epsilon_{lmj}p_i p_l L_m - \epsilon_{lmj}p_i L_l p_m + \epsilon_{lmj}p_l L_m p_i - \epsilon_{lmj}L_l p_m p_i \\
&= \epsilon_{lmj}p_i p_l L_m - \epsilon_{lmj}p_i\left(p_m L_l + [L_l, p_m]\right) + \epsilon_{lmj}\left(L_m p_l + [p_l, L_m]\right)p_i - \epsilon_{lmj}L_l p_m p_i \\
&= \underbrace{\epsilon_{lmj}p_i p_l L_m - \epsilon_{lmj}p_i p_m L_l}_{2\epsilon_{lmj}p_i p_l L_m} - \overbrace{\epsilon_{lmj}p_i[L_l, p_m]}^{i\hbar\epsilon_{lmj}^2 p_i p_j} + \overbrace{\epsilon_{lmj}[p_l, L_m]p_i}^{i\hbar\epsilon_{lmj}^2 p_j p_i} + \underbrace{\epsilon_{lmj}L_m p_l p_i - \epsilon_{lmj}L_l p_m p_i}_{-2\epsilon_{lmj}L_l p_m p_i} \\
&= 2\epsilon_{lmj}p_i p_l L_m - 2\epsilon_{lmj}L_l p_m p_i
\end{aligned}
$$

(b) **Proof of (29)**.

Rewriting LHS of (29), we have

$$
\begin{aligned}
\text{LHS} &= \epsilon_{lmi}\left[p_l L_m - L_l p_m, \frac{x_j}{r}\right] + \epsilon_{stj}\left[\frac{x_i}{r}, p_s L_t - L_s p_t\right] \\
&= \overbrace{\epsilon_{lmi}\left[p_l L_m, \frac{x_j}{r}\right]}^{A} - \overbrace{\epsilon_{stj}\left[p_s L_t, \frac{x_i}{r}\right]}^{B} + \overbrace{\epsilon_{stj}\left[L_s p_t, \frac{x_i}{r}\right]}^{C} - \overbrace{\epsilon_{lmi}\left[L_l p_m, \frac{x_j}{r}\right]}^{D}
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \epsilon_{lmi}\left(p_l\left[L_m, \frac{x_j}{r}\right] + \left[p_l, \frac{x_j}{r}\right]L_m\right) \\
&= \epsilon_{lmi}\left[i\hbar\epsilon_{mjk}p_l\frac{x_k}{r} - i\hbar\left(\frac{\delta_{lj}}{r} - \frac{x_l x_j}{r^3}\right)L_m\right] \\
&= i\hbar\epsilon_{lmi}\epsilon_{mjk}p_l\frac{x_k}{r} - i\hbar\epsilon_{jmi}\frac{1}{r}L_m + i\hbar\epsilon_{lmi}\frac{x_l x_j}{r^3}L_m \\
&= i\hbar\left(\delta_{ij}\delta_{kl} - \delta_{jl}\delta_{ik}\right)p_l\frac{x_k}{r} - i\hbar\epsilon_{jmi}\frac{1}{r}L_m + i\hbar\epsilon_{lmi}\frac{x_l x_j}{r^3}L_m \\
&= i\hbar\delta_{ij}p_k\frac{x_k}{r} - i\hbar p_j\frac{x_i}{r} - i\hbar\epsilon_{ijm}\frac{1}{r}L_m + i\hbar\epsilon_{lmi}\frac{x_l x_j}{r^3}L_m \\
C &= \epsilon_{stj}\left(L_s\left[p_t, \frac{x_i}{r}\right] + \left[L_s, \frac{x_i}{r}\right]p_t\right) \\
&= \epsilon_{stj}\left[-i\hbar L_s\left(\frac{\delta_{ti}}{r} - \frac{x_t x_i}{r^3}\right) + i\hbar\epsilon_{sik}\frac{x_k}{r}p_t\right] \\
&= -i\hbar\epsilon_{sij}L_s\frac{1}{r} + i\hbar\epsilon_{stj}L_s\frac{x_t x_i}{r^3} + i\hbar\epsilon_{stj}\epsilon_{sik}\frac{x_k}{r}p_t \\
&= -i\hbar\epsilon_{sij}L_s\frac{1}{r} + i\hbar\epsilon_{stj}L_s\frac{x_t x_i}{r^3} + i\hbar\left(\delta_{it}\delta_{jk} - \delta_{ij}\delta_{tk}\right)\frac{x_k}{r}p_t \\
&= -i\hbar\epsilon_{ijs}L_s\frac{1}{r} + i\hbar\epsilon_{stj}L_s\frac{x_t x_i}{r^3} + i\hbar\frac{x_j}{r}p_i - i\hbar\delta_{ij}\frac{x_k}{r}p_k
\end{aligned}
$$

With mapping of indices $i \leftrightarrow j, l \leftrightarrow s, m \leftrightarrow t$, we obtain the form of B, D from A, C:

$$
\begin{aligned}
B &= i\hbar\delta_{ij}p_k\frac{x_k}{r} - i\hbar p_i\frac{x_j}{r} - i\hbar\epsilon_{jit}\frac{1}{r}L_t + i\hbar\epsilon_{stj}\frac{x_s x_i}{r^3}L_t \\
D &= -i\hbar\epsilon_{jil}L_l\frac{1}{r} + i\hbar\epsilon_{lmi}L_l\frac{x_m x_j}{r^3} + i\hbar\frac{x_i}{r}p_j - i\hbar\delta_{ij}\frac{x_k}{r}p_k
\end{aligned}
$$

Now combine $A-B+C-D$, notice all terms with δ_{ij} cancel out. Also since $1/r$ commutes with L_i, we have also accounted for the $-4i\hbar\epsilon_{ijk}L_k/r$ term on the RHS of (29), all it remains to prove is

$$
-p_j\frac{x_i}{r} + \overbrace{\epsilon_{lmi}\frac{x_l x_j}{r^3}L_m}^{A_1} + p_i\frac{x_j}{r} - \overbrace{\epsilon_{stj}\frac{x_s x_i}{r^3}L_t}^{B_1} + \overbrace{\epsilon_{stj}L_s\frac{x_t x_i}{r^3}}^{C_1} + \frac{x_j}{r}p_i - \overbrace{\epsilon_{lmi}L_l\frac{x_m x_j}{r^3}}^{D_1} - \frac{x_i}{r}p_j = 0 \tag{32}
$$

Next notice

$$\begin{aligned}
A_1 &= \epsilon_{lmi}\frac{x_l x_j}{r^3}\epsilon_{uvm}x_u p_v = -(\delta_{lu}\delta_{iv} - \delta_{lv}\delta_{iu})\frac{x_l x_j}{r^3}x_u p_v \\
&= -\frac{x_l x_j x_l}{r^3}p_i + \frac{x_l x_j x_i}{r^3}p_l = -\frac{x_j}{r}p_i + \frac{x_i x_j}{r^3}(\mathbf{x}\cdot\mathbf{p}) \\
C_1 &= \epsilon_{stj}\epsilon_{uvs}x_u p_v \frac{x_t x_i}{r^3} = \left(\delta_{tu}\delta_{jv} - \delta_{tv}\delta_{ju}\right)x_u p_v \frac{x_t x_i}{r^3} \\
&= \left(x_t p_j - x_j p_t\right)\frac{x_t x_i}{r^3} = \left(p_j x_t - p_t x_j\right)\frac{x_t x_i}{r^3} \\
&= p_j \frac{x_t x_t x_i}{r^3} - p_t \frac{x_j x_t x_i}{r^3} = p_j \frac{x_i}{r} - (\mathbf{p}\cdot\mathbf{x})\frac{x_i x_j}{r^3}
\end{aligned}$$

Lastly, with the usual index mapping $i \longleftrightarrow j$, we obtain B_1, D_1 as

$$\begin{aligned}
B_1 &= -\frac{x_i}{r}p_j + \frac{x_i x_j}{r^3}(\mathbf{x}\cdot\mathbf{p}) \\
D_1 &= p_i \frac{x_j}{r} - (\mathbf{p}\cdot\mathbf{x})\frac{x_i x_j}{r^3}
\end{aligned}$$

We see indeed all terms on the LHS of (32) cancel out.