

We shall derive equation (3.220), referring to the diagram above.

Let $x = (r = 1, \theta, \phi)$ be a point in spherical coordinates. Consider an infinitesimal rotation around x-axis by angle $d\xi$. We would like to find the expression of the corresponding change in $d\theta$ and $d\phi$ by such an infinitesimal rotation.

In order to do this, we shall consider the infinitesimal arc sweeped by $d\xi$, and project this arc onto the $\hat{\theta}$ and $\hat{\phi}$ direction. Referring to the diagram, we know the infinitesimal sweeped arc vector is

$$d\mathbf{s} = \sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi} d\xi \hat{\xi} \tag{1}$$

where the unit vector

$$\hat{\boldsymbol{\xi}} = 0\hat{\mathbf{x}} - \sin\xi\hat{\mathbf{y}} + \cos\xi\hat{\mathbf{z}} \tag{2}$$

We also know that

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \tag{3}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \tag{4}$$

From which we obtain the projection coefficients

$$\hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\theta}} = -\sin \xi \cos \theta \sin \phi - \sin \theta \cos \xi \tag{5}$$

$$\hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\phi}} = -\sin \boldsymbol{\xi} \cos \boldsymbol{\phi} \tag{6}$$

But by geometry, we know

$$\sin \xi = \frac{\cos \theta}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} \tag{7}$$

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$$\cos \xi = \frac{\sin \theta \sin \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}}$$
(8)

Thus (5) and (6) become

$$\hat{\xi} \cdot \hat{\theta} = -\frac{\cos^2 \theta \sin \phi + \sin^2 \theta \sin \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} = -\frac{\sin \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}}$$

$$\hat{\xi} \cdot \hat{\phi} = -\frac{\cos \theta \cos \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}}$$

$$(10)$$

$$\hat{\xi} \cdot \hat{\phi} = -\frac{\cos \theta \cos \phi}{\sqrt{\cos^2 \theta + \sin^2 \theta \sin^2 \phi}} \tag{10}$$

It is straightforward to verify that the projection is complete in the sense that $(\hat{\xi} \cdot \hat{\theta})^2 + (\hat{\xi} \cdot \hat{\phi})^2 = 1$, as it should since $\hat{\xi}$ does not have a radial component.

Using (1), (9) and (10) to project the arc onto $\hat{\theta}$ and $\hat{\phi}$ direction, the projected arc lengths are

$$d\mathbf{s} \cdot \hat{\boldsymbol{\theta}} = -\sin\phi \, d\xi \tag{11}$$

$$d\mathbf{s} \cdot \hat{\boldsymbol{\phi}} = -\cos\theta\cos\phi \,d\xi\tag{12}$$

Then $d\theta$ and $d\phi$ can be obtained by dividing (11) and (12) by the corresponding radius 1 and sin θ , which gives the relation

$$d\theta = -\sin\phi \, d\xi \tag{13}$$

$$d\phi = -\cot\theta\cos\phi d\xi\tag{14}$$

To summarize, when an infinitesimal rotation $d\xi$ around x-axis is applied to position eigenket $|r, \theta, \phi\rangle$, the state changes to $|r, \theta + d\theta, \phi + d\phi\rangle$ according to (13) and (14), i.e.,

$$\left(1 - \frac{id\xi L_x}{\hbar}\right)|r,\theta,\phi\rangle = |r,\theta+d\theta,\phi+d\phi\rangle \tag{15}$$

Equivalently

$$\langle r, \theta, \phi | \left(1 - \frac{id\xi L_x}{\hbar} \right) = \left[\left(1 + \frac{id\xi L_x}{\hbar} \right) | r, \theta, \phi \rangle \right]^{\dagger}$$

$$= \left[\left(1 - \frac{i(-d\xi)L_x}{\hbar} \right) | r, \theta, \phi \rangle \right]^{\dagger}$$

$$= \left(| r, \theta - d\theta, \phi - d\phi \rangle \right)^{\dagger} = \langle r, \theta - d\theta, \phi - d\phi |$$

$$(16)$$

Therefore

$$\begin{split} \left\langle r,\theta,\phi \left| 1 - \frac{id\xi L_x}{\hbar} \right| \alpha \right\rangle &= \left\langle r,\theta - d\theta,\phi - d\phi \right| \alpha \right\rangle \\ &= \left\langle r,\theta,\phi \right| \alpha \right\rangle - \left(\frac{\partial}{\partial \theta} d\theta + \frac{\partial}{\partial \phi} d\phi \right) \left\langle r,\theta,\phi \right| \alpha \right\rangle \end{split}$$

From which we identify

$$\langle r, \theta, \phi | L_x | \alpha \rangle = \frac{\hbar}{id\xi} \left(\frac{\partial}{\partial \theta} d\theta + \frac{\partial}{\partial \phi} \phi \right) \langle r, \theta, \phi | \alpha \rangle$$

$$= -i\hbar \left(-\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \langle r, \theta, \phi | \alpha \rangle$$
(17)