

We prove the properties of coherent state listed in problem 2.21.

- $|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger} |0\rangle$  is the normalized eigenstate of  $a$ .

Proof: Expanding  $e^{\lambda a^\dagger}$ , we have

$$\begin{aligned} a e^{\lambda a^\dagger} |0\rangle &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} a (a^\dagger)^k |0\rangle \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} a \sqrt{k!} |k\rangle \\ &= \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sqrt{k} \sqrt{k!} |k-1\rangle \\ &= \sum_{k=0}^{\infty} \lambda \cdot \frac{\lambda^k}{\sqrt{k!}} |k\rangle \\ &= \lambda \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (a^\dagger)^k |0\rangle = \lambda e^{\lambda a^\dagger} |0\rangle \end{aligned}$$

which means  $e^{\lambda a^\dagger} |0\rangle$  is an unnormalized eigenstate of  $a$ .

The overall normalization constant can be obtained by noting

$$\langle 0 | (e^{\lambda a^\dagger})^\dagger \cdot e^{\lambda a^\dagger} | 0 \rangle = \sum_{k,l=0}^{\infty} \left\langle k \left| \frac{\lambda^{*k}}{\sqrt{k!}} \cdot \frac{\lambda^l}{\sqrt{l!}} \right| l \right\rangle = \sum_{k=0}^{\infty} \frac{(|\lambda|^2)^k}{k!} = e^{|\lambda|^2}$$

- To see  $|\lambda\rangle$  satisfies the minimal uncertainty, we should calculate both  $\langle \lambda | x | \lambda \rangle$  and  $\langle \lambda | x^2 | \lambda \rangle$  (similar for  $p$ ).

First note

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) \\ x^2 &= \frac{\hbar}{2m\omega} (a^{\dagger 2} + a^2 + a^\dagger a + a a^\dagger) \\ p &= i \sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a) \\ p^2 &= -\frac{m\hbar\omega}{2} (a^{\dagger 2} + a^2 - a^\dagger a - a a^\dagger) \end{aligned}$$

Therefore

$$\begin{aligned} \langle \lambda | x | \lambda \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle \lambda | a^\dagger + a | \lambda \rangle \quad \text{using } a|\lambda\rangle = \lambda|\lambda\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\lambda^* + \lambda) = 2\text{Re}\lambda \cdot \sqrt{\frac{\hbar}{2m\omega}} \\ \langle \lambda | x^2 | \lambda \rangle &= \frac{\hbar}{2m\omega} \langle \lambda | a^{\dagger 2} + a^2 + a^\dagger a + a a^\dagger | \lambda \rangle \\ &= \frac{\hbar}{2m\omega} \langle \lambda | a^{\dagger 2} + a^2 + 2a^\dagger a + [a, a^\dagger] | \lambda \rangle \\ &= \frac{\hbar}{2m\omega} (\lambda^{*2} + \lambda^2 + 2\lambda^* \lambda + 1) = \frac{\hbar}{2m\omega} [4(\text{Re}\lambda)^2 + 1] \implies \\ \langle (\Delta x)^2 \rangle &= \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{2m\omega} \end{aligned}$$

and

$$\begin{aligned}
\langle \lambda | p | \lambda \rangle &= i \sqrt{\frac{m\hbar\omega}{2}} \langle \lambda | a^\dagger - a | \lambda \rangle \\
&= i \sqrt{\frac{m\hbar\omega}{2}} (\lambda^* - \lambda) = 2\text{Im}\lambda \cdot \sqrt{\frac{m\hbar\omega}{2}} \\
\langle \lambda | p^2 | \lambda \rangle &= -\frac{m\hbar\omega}{2} \langle \lambda | a^{\dagger 2} + a^2 - a^\dagger a - a a^\dagger | \lambda \rangle \\
&= -\frac{m\hbar\omega}{2} \langle \lambda | a^{\dagger 2} + a^2 - 2a^\dagger a - [a, a^\dagger] | \lambda \rangle \\
&= -\frac{m\hbar\omega}{2} (\lambda^{*2} + \lambda^2 - 2\lambda^* \lambda - 1) = \frac{m\hbar\omega}{2} [4(\text{Im}\lambda)^2 + 1] \quad \Rightarrow \\
\langle (\Delta p)^2 \rangle &= \langle p^2 \rangle - \langle p \rangle^2 = \frac{m\hbar\omega}{2}
\end{aligned}$$

So  $\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = \hbar^2/4$  is the minimal uncertainty.

- We have already seen that

$$|\lambda\rangle = e^{-|\lambda|^2/2} \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}} |k\rangle$$

So by letting  $\mu = |\lambda|^2$ , we have

$$|f(k)|^2 = e^{-|\lambda|^2} \frac{|\lambda|^{2k}}{k!} = e^{-\mu} \frac{\mu^k}{k!}$$

where the most probable  $n$  is near the mean value  $\mu$ .

- To see how  $|\lambda\rangle$  can be achieved by applying  $e^{-ip/\hbar}$  to  $|0\rangle$ , we first note the claim that when  $[A, B]$  commutes with both  $A$  and  $B$ ,

$$e^{A+B} = e^A e^B e^{-[A,B]/2}$$

Also note that  $e^{-ip/\hbar} = e^{\sqrt{\frac{m\omega}{2\hbar}} l (a^\dagger - a)}$ , and  $[a^\dagger, a] = 1$ , by letting  $\lambda = \sqrt{\frac{m\omega}{2\hbar}} l$ , we have

$$\begin{aligned}
e^{-ip/\hbar} |0\rangle &= e^{\lambda a^\dagger - \lambda a} |0\rangle = e^{\lambda a^\dagger} e^{-\lambda a} e^{-\lambda^2 [a^\dagger, -a]/2} |0\rangle \\
&= e^{-\lambda^2/2} e^{\lambda a^\dagger} e^{-\lambda a} |0\rangle \\
&= e^{-\lambda^2/2} e^{\lambda a^\dagger} |0\rangle = |\lambda\rangle
\end{aligned}$$