

The $n = 2$ orbital states are $|nlm\rangle = |211\rangle, |210\rangle, |21-1\rangle, |200\rangle$. Considering spin, each of the above state will be tensored with $|\frac{1}{2}\rangle$ or $|\frac{-1}{2}\rangle$, so $n = 2$ has degeneracy of 8.

In the following, we will omit the 2 in $|nlm\rangle$, which simplifies the eight degenerate states as $|lm\rangle \otimes |\pm\frac{1}{2}\rangle$.

1.

$$\begin{aligned} V &= \frac{A}{2\hbar^2} (\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2) + \frac{B}{\hbar} (J_z + S_z) \\ &= \underbrace{\left[\frac{A}{2\hbar^2} (\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2) + \frac{B}{\hbar} J_z \right]}_{V_1} + \underbrace{\frac{B}{\hbar} S_z}_{V_2} \end{aligned} \quad (1)$$

where V_1 is diagonalizable in the $\mathbf{L}^2, \mathbf{S}^2, \mathbf{J}^2, J_z$ basis, and V_2 is diagonalizable in the \mathbf{S}^2, S_z basis.

2. Using C-G coefficients, we can express the 8 (spherical) basis in $\mathbf{L}^2, \mathbf{S}^2, \mathbf{J}^2, J_z$ as linear combination of the (Cartesian) basis $|lm\rangle \otimes |\pm\frac{1}{2}\rangle$

$$\begin{array}{ll} |ls; jm\rangle & |lm\rangle \otimes |s_z\rangle \\ |a\rangle & \left| 1 \frac{1}{2}; \frac{3}{2} \frac{3}{2} \right\rangle = |11\rangle \otimes \left| \frac{1}{2} \right\rangle \end{array} \quad (2)$$

$$|b\rangle \quad \left| 1 \frac{1}{2}; \frac{3}{2} \frac{1}{2} \right\rangle = \sqrt{\frac{1}{3}} |11\rangle \otimes \left| -\frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}} |10\rangle \otimes \left| \frac{1}{2} \right\rangle \quad (3)$$

$$|c\rangle \quad \left| 1 \frac{1}{2}; \frac{3}{2} \frac{-1}{2} \right\rangle = \sqrt{\frac{1}{3}} |1-1\rangle \otimes \left| \frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}} |10\rangle \otimes \left| -\frac{1}{2} \right\rangle \quad (4)$$

$$|d\rangle \quad \left| 1 \frac{1}{2}; \frac{3}{2} \frac{-3}{2} \right\rangle = |1-1\rangle \otimes \left| -\frac{1}{2} \right\rangle \quad (5)$$

$$|e\rangle \quad \left| 1 \frac{1}{2}; \frac{1}{2} \frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} |11\rangle \otimes \left| -\frac{1}{2} \right\rangle - \sqrt{\frac{1}{3}} |10\rangle \otimes \left| \frac{1}{2} \right\rangle \quad (6)$$

$$|f\rangle \quad \left| 1 \frac{1}{2}; \frac{1}{2} \frac{-1}{2} \right\rangle = -\sqrt{\frac{2}{3}} |1-1\rangle \otimes \left| \frac{1}{2} \right\rangle + \sqrt{\frac{1}{3}} |10\rangle \otimes \left| -\frac{1}{2} \right\rangle \quad (7)$$

$$|g\rangle \quad \left| 0 \frac{1}{2}; \frac{1}{2} \frac{1}{2} \right\rangle = |00\rangle \otimes \left| \frac{1}{2} \right\rangle \quad (8)$$

$$|h\rangle \quad \left| 0 \frac{1}{2}; \frac{1}{2} \frac{-1}{2} \right\rangle = |00\rangle \otimes \left| -\frac{1}{2} \right\rangle \quad (9)$$

Apply V to all

$$V|a\rangle = \overbrace{\left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2} - 1 \cdot 2 - \frac{1}{2}\frac{3}{2}\right) + \frac{3B}{2}\right]}^{V_1|a\rangle}|a\rangle + \overbrace{\frac{B}{2}|a\rangle}^{V_2|a\rangle} = \left(\frac{A}{2} + 2B\right)|a\rangle \quad (10)$$

$$\begin{aligned} V|b\rangle &= \overbrace{\left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2} - 1 \cdot 2 - \frac{1}{2}\frac{3}{2}\right) + \frac{B}{2}\right]}^{V_1|b\rangle}|b\rangle - \overbrace{\frac{B}{2}\sqrt{\frac{1}{3}}|11\rangle \otimes \left|-\frac{1}{2}\right\rangle + \frac{B}{2}\sqrt{\frac{2}{3}}|10\rangle \otimes \left|\frac{1}{2}\right\rangle}^{V_2|b\rangle} \\ &= \frac{A}{2}\sqrt{\frac{1}{3}}|11\rangle \otimes \left|-\frac{1}{2}\right\rangle + \left(\frac{A}{2} + B\right)\sqrt{\frac{2}{3}}|10\rangle \otimes \left|\frac{1}{2}\right\rangle \end{aligned} \quad (11)$$

$$\begin{aligned} V|c\rangle &= \left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2} - 1 \cdot 2 - \frac{1}{2}\frac{3}{2}\right) - \frac{B}{2}\right]|c\rangle + \frac{B}{2}\sqrt{\frac{1}{3}}|1-1\rangle \otimes \left|\frac{1}{2}\right\rangle - \frac{B}{2}\sqrt{\frac{2}{3}}|10\rangle \otimes \left|-\frac{1}{2}\right\rangle \\ &= \frac{A}{2}\sqrt{\frac{1}{3}}|1-1\rangle \otimes \left|\frac{1}{2}\right\rangle + \left(\frac{A}{2} - B\right)\sqrt{\frac{2}{3}}|10\rangle \otimes \left|-\frac{1}{2}\right\rangle \end{aligned} \quad (12)$$

$$V|d\rangle = \left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2} - 1 \cdot 2 - \frac{1}{2}\frac{3}{2}\right) - \frac{3B}{2}\right]|d\rangle - \frac{B}{2}|d\rangle = \left(\frac{A}{2} - 2B\right)|d\rangle \quad (13)$$

$$\begin{aligned} V|e\rangle &= \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2} - 1 \cdot 2 - \frac{1}{2}\frac{3}{2}\right) + \frac{B}{2}\right]|e\rangle - \frac{B}{2}\sqrt{\frac{2}{3}}|11\rangle \otimes \left|-\frac{1}{2}\right\rangle - \frac{B}{2}\sqrt{\frac{1}{3}}|10\rangle \otimes \left|\frac{1}{2}\right\rangle \\ &= -A\sqrt{\frac{2}{3}}|11\rangle \otimes \left|-\frac{1}{2}\right\rangle + (A - B)\sqrt{\frac{1}{3}}|10\rangle \otimes \left|\frac{1}{2}\right\rangle \end{aligned} \quad (14)$$

$$\begin{aligned} V|f\rangle &= \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2} - 1 \cdot 2 - \frac{1}{2}\frac{3}{2}\right) - \frac{B}{2}\right]|f\rangle - \frac{B}{2}\sqrt{\frac{2}{3}}|1-1\rangle \otimes \left|\frac{1}{2}\right\rangle - \frac{B}{2}\sqrt{\frac{1}{3}}|10\rangle \otimes \left|-\frac{1}{2}\right\rangle \\ &= A\sqrt{\frac{2}{3}}|1-1\rangle \otimes \left|\frac{1}{2}\right\rangle - (A + B)\sqrt{\frac{1}{3}}|10\rangle \otimes \left|-\frac{1}{2}\right\rangle \end{aligned} \quad (15)$$

$$V|g\rangle = \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2} - 0 \cdot 1 - \frac{1}{2}\frac{3}{2}\right) + \frac{B}{2}\right]|g\rangle + \frac{B}{2}|g\rangle = B|g\rangle \quad (16)$$

$$V|h\rangle = \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2} - 0 \cdot 1 - \frac{1}{2}\frac{3}{2}\right) - \frac{B}{2}\right]|h\rangle - \frac{B}{2}|g\rangle = -B|h\rangle \quad (17)$$

With (2)-(17), we can construct the full 8×8 matrix of V in the spherical basis.

$$\begin{bmatrix} \frac{A}{2} + 2B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A}{2} - 2B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{A}{2} + \frac{2B}{3} & -\frac{\sqrt{2}B}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}B}{3} & -A + \frac{B}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{A}{2} - \frac{2B}{3} & -\frac{\sqrt{2}B}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}B}{3} & -A - \frac{B}{3} \end{bmatrix} \begin{matrix} |a\rangle \\ |d\rangle \\ |g\rangle \\ |h\rangle \\ |b\rangle \\ |e\rangle \\ |c\rangle \\ |f\rangle \end{matrix}$$

3. The diagonal elements for the a, d, g, h block are the energy eigenvalues for these states:

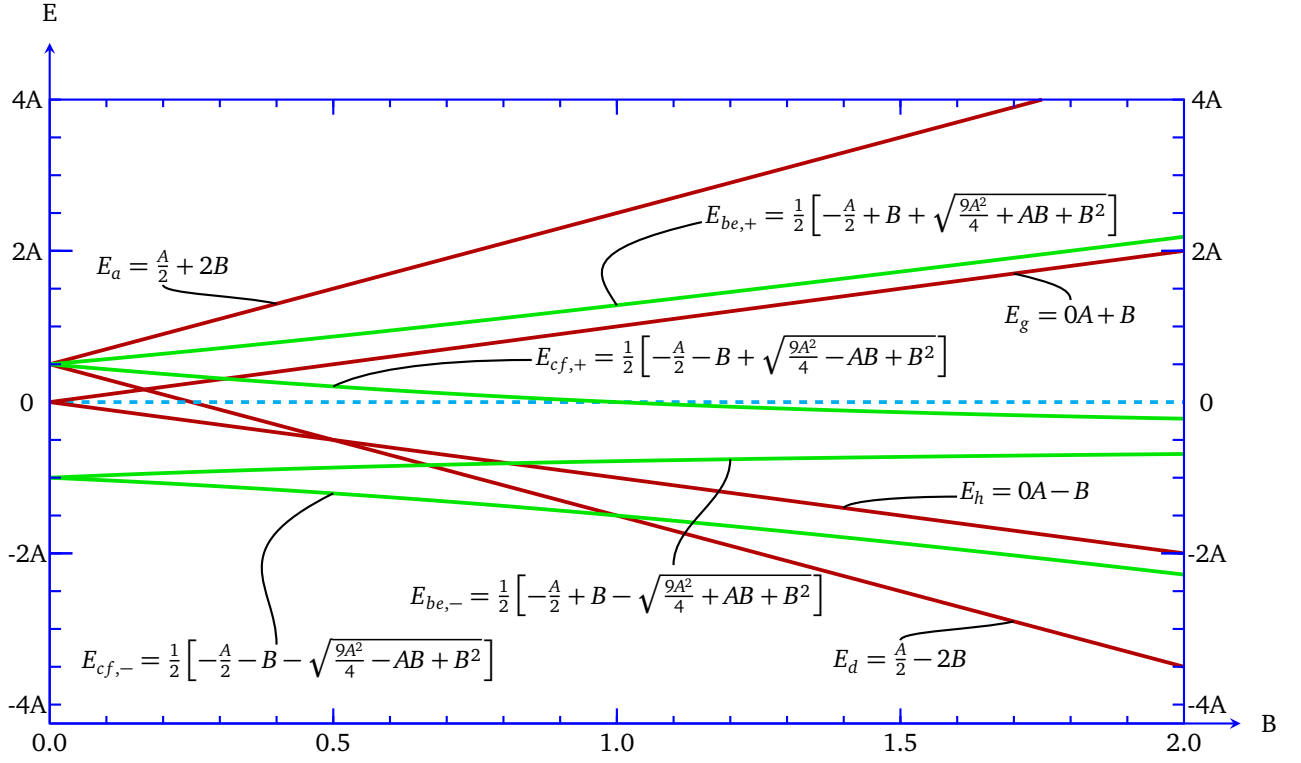
$$E_a = \frac{A}{2} + 2B \quad (18)$$

$$E_d = \frac{A}{2} - 2B \quad (19)$$

$$E_g = 0A + B \quad (20)$$

$$E_h = 0A - B \quad (21)$$

These are depicted in the $E \sim B$ diagram below as red **straight** lines.



For the b, e block, the energy eigenvalues can be found via

$$\begin{aligned}
 \left[E_{be} - \left(\frac{A}{2} + \frac{2B}{3} \right) \right] \left[E_{be} - \left(-A + \frac{B}{3} \right) \right] - \frac{2B^2}{9} &= 0 & \Rightarrow \\
 E_{be}^2 + \left(\frac{A}{2} - B \right) E_{be} - \left(\frac{A^2}{2} + \frac{AB}{2} \right) &= 0 & \Rightarrow \\
 E_{be,\pm} = \frac{1}{2} \left[-\frac{A}{2} + B \pm \sqrt{\frac{9A^2}{4} + AB + B^2} \right] & & (22)
 \end{aligned}$$

Similarly for the c, f block:

$$\begin{aligned}
 \left[E_{cf} - \left(\frac{A}{2} - \frac{2B}{3} \right) \right] \left[E_{cf} - \left(-A - \frac{B}{3} \right) \right] - \frac{2B^2}{9} &= 0 & \Rightarrow \\
 E_{cf}^2 + \left(\frac{A}{2} + B \right) E_{cf} - \left(\frac{A^2}{2} - \frac{AB}{2} \right) &= 0 & \Rightarrow \\
 E_{cf,\pm} = \frac{1}{2} \left[-\frac{A}{2} - B \pm \sqrt{\frac{9A^2}{4} - AB + B^2} \right] & & (23)
 \end{aligned}$$

The B -dependency of $E_{be,\pm}, E_{cf,\pm}$ are plotted in the diagram with green **curved** lines.

The asymptotic behaviors of $E_{be,\pm}, E_{cf,\pm}$ as $B/A \rightarrow 0$ and $B/A \rightarrow \infty$ are

$$\begin{array}{llll}
 \frac{A}{2} + \frac{2B}{3} = \frac{1}{2} \left[-\frac{A}{2} + B + \frac{3A}{2} \left(1 + \frac{2B}{9A} \right) \right] & \xleftarrow{B/A \rightarrow 0} & E_{be,+} & \xrightarrow{B/A \rightarrow \infty} & \frac{1}{2} \left[-\frac{A}{2} + B + B \left(1 + \frac{A}{2B} \right) \right] = B \\
 -A + \frac{B}{3} = \frac{1}{2} \left[-\frac{A}{2} + B - \frac{3A}{2} \left(1 + \frac{2B}{9A} \right) \right] & \xleftarrow{B/A \rightarrow 0} & E_{be,-} & \xrightarrow{B/A \rightarrow \infty} & \frac{1}{2} \left[-\frac{A}{2} + B - B \left(1 + \frac{A}{2B} \right) \right] = -\frac{A}{2} \\
 \frac{A}{2} - \frac{2B}{3} = \frac{1}{2} \left[-\frac{A}{2} - B + \frac{3A}{2} \left(1 - \frac{2B}{9A} \right) \right] & \xleftarrow{B/A \rightarrow 0} & E_{cf,+} & \xrightarrow{B/A \rightarrow \infty} & \frac{1}{2} \left[-\frac{A}{2} - B + B \left(1 - \frac{A}{2B} \right) \right] = -\frac{A}{2} \\
 -A - \frac{B}{3} = \frac{1}{2} \left[-\frac{A}{2} - B - \frac{3A}{2} \left(1 - \frac{2B}{9A} \right) \right] & \xleftarrow{B/A \rightarrow 0} & E_{cf,-} & \xrightarrow{B/A \rightarrow \infty} & \frac{1}{2} \left[-\frac{A}{2} - B - B \left(1 - \frac{A}{2B} \right) \right] = -B
 \end{array}$$

In particular, as $B/A \rightarrow \infty$, $E_{be,+}$ will approach E_g , both $E_{cf,+}$ and $E_{be,-}$ will converge to $-A/2$, and $E_{cf,-}$ will approach E_h . These are the "five states" of high field as described in Fig 5.3's caption.