We wish to establish the triangle inequality $|j_1 - j_2| \leq j \leq |j_1 + j_2|$ where $\hbar^2 j(j+1)$ is the eigenvalue of the total angular momentum operator $\boldsymbol{J}^2 = (\boldsymbol{J_1} + \boldsymbol{J_2}) \cdot (\boldsymbol{J_1} + \boldsymbol{J_2}) = \boldsymbol{J_1}^2 + \boldsymbol{J_2}^2 + 2\boldsymbol{J_1} \cdot \boldsymbol{J_2}$.

Let $|\psi\rangle$ be a simultaneous eigenket of $\boldsymbol{J}^2, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2$ and J_z. Then $|\psi\rangle$ must be an eigenket of $\boldsymbol{J}_1 \cdot \boldsymbol{J}_2$. I.e., if we let $(\boldsymbol{J_1} \cdot \boldsymbol{J_2})|\psi\rangle = \hbar^2 M |\psi\rangle$, the eigenequation for $\boldsymbol{J}^2$ becomes

$$j(j+1) = j_1(j_1+1) + j_2(j_2+1) + 2M \tag{1}$$

For real λ, define $\boldsymbol{J}_\lambda = \boldsymbol{J_1} + \lambda \boldsymbol{J_2}$, which is a group of 3 Hermitian operators. Let $\boldsymbol{J}_\lambda^2 = \boldsymbol{J}_\lambda \cdot \boldsymbol{J}_\lambda = \boldsymbol{J_1}^2 + \lambda^2 \boldsymbol{J_2}^2 + 2\lambda \boldsymbol{J_1} \cdot \boldsymbol{J_2}$. Then $|\psi\rangle$ is an eigenket of $\boldsymbol{J}_\lambda^2$ of eigenvalue

$$j_1(j_1+1) + \lambda^2 j_2(j_2+1) + 2\lambda M \tag{2}$$

which is non-negative for any real λ due to the definition of $\boldsymbol{J}_\lambda^2$.

For this to be true, if we view (2) as a quadratic polynomial in λ, its determinant must satisfy

$$\begin{aligned} 4M^2 - 4j_1 j_2(j_1+1)(j_2+1) \leq 0 \qquad \text{or} \\ -\sqrt{j_1 j_2(j_1+1)(j_2+1)} \leq M \leq \sqrt{j_1 j_2(j_1+1)(j_2+1)} \end{aligned} \tag{3}$$

According to (1), this means the total-j must satisfy

$$j(j+1) \geq j_1(j_1+1) + j_2(j_2+1) - 2\sqrt{j_1 j_2(j_1+1)(j_2+1)} \tag{4}$$

$$j(j+1) \leq j_1(j_1+1) + j_2(j_2+1) + 2\sqrt{j_1 j_2(j_1+1)(j_2+1)} \tag{5}$$

Without loss of generality, assume $j_1 \geq j_2$.

1. To show $j \geq j_1 - j_2$, assume the opposite, i.e., $j < j_1 - j_2$. Since j is a total angular momentum number, which increments in units of $\frac{1}{2}$, we must have $j \leq j_1 - j_2 - \frac{1}{2}$. But this conflicts with (4):

$$\begin{aligned} j(j+1) &\leq \left(j_1 - j_2 - \frac{1}{2}\right)\left(j_1 - j_2 + \frac{1}{2}\right) \\ &= j_1^2 - 2j_1 j_2 + j_2^2 - \frac{1}{4} \\ &< j_1^2 - 2j_1 j_2 + j_2^2 \\ &= j_1(j_1+1) + j_2(j_2+1) - [j_1(j_2+1) + j_2(j_1+1)] \\ &\leq j_1(j_1+1) + j_2(j_2+1) - 2\sqrt{j_1 j_2(j_1+1)(j_2+1)} \end{aligned}$$

2. Similarly, to show $j \leq j_1 + j_2$, assume the opposite that $j \geq j_1 + j_2 + \frac{1}{2}$. But this conflicts with (5):

$$\begin{aligned} j(j+1) &\geq \left(j_1 + j_2 + \frac{1}{2}\right)\left(j_1 + j_2 + \frac{3}{2}\right) \\ &= j_1^2 + j_2^2 + 2j_1 j_2 + 2j_1 + 2j_2 + \frac{3}{4} \\ &> j_1(j_1+1) + j_2(j_2+1) + [j_1(j_2+1) + j_2(j_1+1)] \\ &\geq j_1(j_1+1) + j_2(j_2+1) + 2\sqrt{j_1 j_2(j_1+1)(j_2+1)} \end{aligned}$$