1. Expanding $t/(1-t)$ into Taylor series, we have

$$\frac{\partial g(x,t)}{\partial x} = \frac{-t}{1-t} g(x,t) = \left(\sum_{q=1}^{\infty} -t^q\right)\left(\sum_{p=0}^{\infty} L_p(x)\frac{t^p}{p!}\right) = \sum_{p=0}^{\infty} L_p'(x)\frac{t^p}{p!}$$

We require the t^p term of both sides to be equal, i.e.,

$$\begin{aligned}
\sum_{s=0}^{p-1} L_s(x)\frac{t^s}{s!}\left(-t^{p-s}\right) = L_p'(x)\frac{t^p}{p!} \qquad &\Longrightarrow \\
\frac{L_p'(x)}{p!} &= -\sum_{s=0}^{p-1}\frac{L_s(x)}{s!} \\
&= -\left[\sum_{s=0}^{p-2}\frac{L_s(x)}{s!} + \frac{L_{p-1}(x)}{(p-1)!}\right] \\
&= \frac{L_{p-1}'(x)}{(p-1)!} - \frac{L_{p-1}(x)}{(p-1)!} \qquad \Longrightarrow \\
L_p'(x) - pL_{p-1}'(x) = -pL_{p-1}(x) &
\end{aligned} \tag{1}$$

Next, we look at the derivative with respect to t:

$$\begin{aligned}
\frac{\partial g(x,t)}{\partial t} &= \left[\frac{-x}{(1-t)^2} + \frac{1}{1-t}\right] g(x,t) = \left[\frac{-x}{(1-t)^2} + \frac{1}{1-t}\right]\sum_{p=0}^{\infty} L_p(x)\frac{t^p}{p!} \\
&= \sum_{p=0}^{\infty} L_p(x)\frac{\partial}{\partial t}\frac{t^p}{p!} \\
&= \sum_{p=0}^{\infty} L_{p+1}(x)\frac{t^p}{p!}
\end{aligned} \tag{2}$$

But

$$\left[\frac{-x}{(1-t)^2} + \frac{1}{1-t}\right]\sum_{p=0}^{\infty} L_p(x)\frac{t^p}{p!} = -x\left[\sum_{q=0}^{\infty}(q+1)t^q\right]\left[\sum_{p=0}^{\infty} L_p(x)\frac{t^p}{p!}\right] + \left(\sum_{q=0}^{\infty} t^q\right)\left[\sum_{p=0}^{\infty} L_p(x)\frac{t^p}{p!}\right]$$

On the RHS, the term of t^p for a given p is

$$-x\left[\sum_{s=0}^{p} L_s(x)\frac{t^s}{s!}(p-s+1)t^{p-s}\right] + \sum_{s=0}^{p} L_s(x)\frac{t^s}{s!}t^{p-s}$$

Now compare with (2), we get

$$\frac{L_{p+1}(x)}{p!} = -x\left[\sum_{s=0}^{p}\frac{L_s(x)(p-s+1)}{s!}\right] + \sum_{s=0}^{p}\frac{L_s(x)}{s!} \tag{3}$$

$$\Longrightarrow \qquad \frac{L_p(x)}{(p-1)!} = -x\left[\sum_{s=0}^{p-1}\frac{L_s(x)(p-s)}{s!}\right] + \sum_{s=0}^{p-1}\frac{L_s(x)}{s!} \tag{4}$$

Subtracting (4) from (3), we get

$$\frac{L_{p+1}(x)}{p!} - \frac{L_p(x)}{(p-1)!} = -x\left[\sum_{s=0}^{p-1}\frac{L_s(x)}{s!} + \frac{L_p(x)}{p!}\right] + \frac{L_p(x)}{p!} \tag{5}$$

$$\Longrightarrow \qquad \frac{L_p(x)}{(p-1)!} - \frac{L_{p-1}(x)}{(p-2)!} = -x\left[\sum_{s=0}^{p-2}\frac{L_s(x)}{s!} + \frac{L_{p-1}(x)}{(p-1)!}\right] + \frac{L_{p-1}(x)}{(p-1)!} \tag{6}$$

Once again, subtracting (6) from (5):

$$\frac{L_{p+1}(x)}{p!} - \frac{2L_p(x)}{(p-1)!} + \frac{L_{p-1}(x)}{(p-2)!} = -x\frac{L_p(x)}{p!} + \frac{L_p(x)}{p!} - \frac{L_{p-1}(x)}{(p-1)!} \qquad \Longrightarrow$$
$$L_{p+1}(x) - 2pL_p(x) + p(p-1)L_{p-1}(x) = (-x+1)L_p(x) - pL_{p-1}(x)$$
$$L_{p+1}(x) - (2p+1-x)L_p(x) + p^2L_{p-1}(x) = 0 \tag{7}$$

Applying $p \to p+1$ to (1), we have

$$L'_{p+1} - (p+1)L'_p = -(p+1)L_p \tag{8}$$

Differentiating (7) we have

$$L'_{p+1} - (2p+1)L'_p + L_p + xL'_p + p^2L'_{p-1} = 0 \tag{9}$$

Subtracting (9) from (8):

$$(p-x)L'_p + pL_p - p^2L'_{p-1} = 0 \tag{10}$$
$$\Longrightarrow \quad p(L'_p - pL'_{p-1}) - xL'_p + pL_p = 0 \tag{11}$$
$$\text{(by (1))} \quad \Longrightarrow \quad p(L_p - pL_{p-1}) = xL'_p \tag{12}$$
$$\text{(differentiating)} \quad \Longrightarrow \quad p(L'_p - pL'_{p-1}) = xL''_p + L'_p \tag{13}$$
$$\text{(by (11))} \quad \Longrightarrow \quad xL''_p + (1-x)L'_p + pL_p = 0 \tag{14}$$

2. Now apply $p \to p+q$ to (14), we hve

$$xL''_{p+q} + (1-x)L'_{p+q} + (p+q)L_{p+q} = 0 \tag{15}$$

Take the q-th derivative of (15), and use the relation

$$\frac{d^q}{x^q}(fg) = \sum_{l=0}^{q}\binom{q}{l}\frac{d^lf}{dx^l}\frac{d^{q-l}g}{dx^{q-l}}$$

then multiply by $(-1)^q$, we obtain the desired identity

$$xL_p^{q\prime\prime} + (q+1-x)L_p^{q\prime} + pL_p^q = 0 \tag{16}$$

Now recall in the generating function

$$g(x,t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_{p=0}^{\infty} L_p(x)\frac{t^p}{p!}$$

$L_p(x)$ is by definition an order-p polynomial, whose existence is guaranteed by Talor expansion of e^y with $y = -xt/(1-t)$.

Now apply q-th derivative with respect to x,

$$\begin{aligned}
\frac{\partial^q g(x,t)}{\partial x^q} &= (-1)^q\frac{t^q}{(1-t)^{q+1}}e^{-xt/(1-t)} \\
&= \sum_{p=0}^{\infty}\frac{d^qL_p(x)}{dx^q}\frac{t^p}{p!} \qquad (\text{since } d^qL_p/dx^q = 0 \text{ for } p<q) \\
&= \sum_{p=0}^{\infty}\frac{d^qL_{p+q}(x)}{dx^q}\frac{t^{p+q}}{(p+q)!} \\
&= (-1)^q t^q \sum_{p=0}^{\infty} L_p^q(x)\frac{t^p}{(p+q)!}
\end{aligned}$$

which gives

$$h(x,t) = \frac{e^{-xt/(1-t)}}{(1-t)^{q+1}} = \sum_{p=0}^{\infty} L_p^q(x)\frac{t^p}{(p+q)!} \tag{17}$$

Note here I think the book has an extra factor of $(-1)^q$ in either the definition of L_p^q or the definition of $h(x,t)$.

At $x=0$, expand $h(x,t)$ in Taylor series, we have

$$h(0,t)=(1-t)^{-(q+1)}=\sum_{p=0}^{\infty}\frac{d^p(1-t)^{-(q+1)}/dt^p|_{t=0}}{p!}t^p=\sum_{p=0}^{\infty}\frac{(p+q)!}{p!q!}t^p$$

Compare with (17), we have

$$L_p^q(0)=\frac{[(p+q)!]^2}{p!q!}$$

Also note the disagreement of factor $(-1)^q$ with the book. If we want $L_p^q(0)$ to retain its $(-1)^q$, we must change the definition of L_p^q to drop it, which does not affect the derivation of (16).

3. When using the exponential form of $h(x,t)$

$$\begin{aligned}\int_0^{\infty}x^{q+1}e^{-x}h(x,t)h(x,s)dx&=\frac{1}{(1-t)^{q+1}}\frac{1}{(1-s)^{q+1}}\int_0^{\infty}x^{q+1}e^{-x}e^{-xt/(1-t)}e^{-xs/(1-s)}\\&=\frac{1}{[(1-t)(1-s)]^{q+1}}\int_0^{\infty}x^{q+1}e^{-Ax}dx\qquad\left(A\equiv 1+\frac{t}{1-t}+\frac{s}{1-s}\right)\end{aligned}\tag{18}$$

But

$$\begin{aligned}\int_0^{\infty}x^{q+1}e^{-Ax}dx&=-\frac{1}{A}\int_0^{\infty}x^{q+1}d\left(e^{-Ax}\right)=-\frac{1}{A}\left[x^{q+1}e^{-Ax}\Big|_0^{\infty}-\int_0^{\infty}(q+1)x^qe^{-Ax}dx\right]\\&=\frac{q+1}{A}\int_0^{\infty}x^qe^{-Ax}dx=\cdots=\frac{(q+1)!}{A^{q+1}}\int_0^{\infty}e^{-Ax}dx\\&=\frac{(q+1)!}{A^{q+2}}\end{aligned}\tag{19}$$

Insert (19) into (18) and note $A=1+\frac{t}{1-t}+\frac{s}{1-s}=\frac{1-ts}{(1-t)(1-s)}$, we have

$$\begin{aligned}\int_0^{\infty}x^{q+1}e^{-x}h(x,t)h(x,s)dx&=(q+1)!\frac{(1-t)(1-s)}{(1-ts)^{q+2}}\\&=(q+1)!(1-t-s+ts)\sum_{l=0}^{\infty}\binom{q+l+1}{l}t^ls^l\end{aligned}\tag{20}$$

On the other hand, when we use the summation form of $h(x,t)$, we get

$$\int_0^{\infty}x^{q+1}e^{-x}h(x,t)h(x,s)dx=\sum_{p,p'=0}^{\infty}\frac{t^p}{(p+q)!}\frac{s^{p'}}{(p'+q)!}\overbrace{\int_0^{\infty}x^{q+1}e^{-x}L_p^q(x)L_{p'}^q(x)dx}^{I_{pp'}^q}\tag{21}$$

Equating the t^ps^p term of (20) and (21), we have

$$\begin{aligned}I_{pp}^q&=[(p+q)!]^2(q+1)!\left[\binom{q+p+1}{p}+\binom{q+p}{p-1}\right]\\&=[(p+q)!]^2(q+1)!\left[\frac{(q+p+1)!}{p!(q+1)!}+\frac{(q+p)!}{(p-1)!(q+1)!}\right]\\&=[(p+q)!]^2\left[\frac{(q+p+1)!+p(q+p)!}{p!}\right]\\&=\frac{[(p+q)!]^3(q+2p+1)}{p!}\end{aligned}\tag{22}$$

As a side note, when $|p-p'|=1$, we can compare (20) and (21) and see that $I_{pp'}^q\neq 0$. This seems to imply the radial wavefunction in $L_p^q(x)$ form may not be orthogonal between, say p and $p+1$, but it is actually not the case (see comments at the end).

Also as a side note, the orthogonality condition for $L_p^q(x)$ is obtained by modifying the weight in (18) from $x^{q+1}e^{-x}$ to x^qe^{-x}, in which case (19) will become $q!A^{-(q+1)}$ and (20) will become $q!\sum_l^{\infty}\binom{q+l}{l}t^ls^l$, which only has the same power for both t and s. Despite the nice orthogonality given by the weight $x^q e^{-x}$, it doesn't seem to be particularly useful in the calculation of radial wavefunction of hydrogen atom.

Now let's see how (22) can give the overall normalization constant for the radial wavefunction.

Recall the following relevant equations from the text

$$R_{El}(r) = \frac{u_{El}(r)}{r} \tag{3.270}$$

$$1 = \int dr u_{El}^*(r) u_{El}(r) \tag{3.272}$$

$$u_{El}(\rho) = \rho^{l+1} e^{-\rho} w(\rho) \qquad \text{where } \rho = \frac{Zr}{na_0} \tag{3.278, 3.316}$$

$$w(\rho) = F(l+1-\frac{\rho_0}{2}; 2(l+1); 2\rho) \qquad \text{where } \rho_0 = 2n \tag{3.311, 3.314}$$

$$0 = x\frac{d^2F}{dx^2} + (2l+2-x)\frac{dF}{dx} + (n-l-1)F \qquad \text{where } x = 2\rho = \frac{2Zr}{na_0} \tag{3.308}$$

$$F(a;c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!}\cdots \tag{3.310}$$

Comparing (16) and (3.308), if we let

$$q = 2l+1 \tag{23}$$

$$p = n-l-1 \tag{24}$$

we see that $L_p^q(x)$ is identical to $F(a;c;x)$ up to a constant factor, which can be determined by $L_p^q(0)$ and $F(a;c;0) = 1$, i.e.,

$$L_p^q(x) = \frac{[(p+q)!]^2}{p!q!} F(a;c;x) = \frac{[(n+l)!]^2}{(2l+1)!(n-l-1)!} F(a;c;x) \equiv \Lambda F(a;c;x) \tag{25}$$

Let the overall radial normalization constant be C, the normalization condition requires

$$\begin{aligned}
\frac{1}{C^2} &= \int_{r=0}^{\infty} \rho^{2l+2} e^{-2\rho} [F(a;c;x)]^2 dr \\
&= \int_{r=0}^{\infty} \left(\frac{Zr}{na_0}\right)^{2l+2} e^{-2Zr/na_0} \frac{1}{\Lambda^2} \left[L_p^q\left(\frac{2Zr}{na_0}\right)\right]^2 dr \\
&= \frac{na_0}{2Z}\frac{1}{\Lambda^2} \int_{x=0}^{\infty} \left(\frac{x}{2}\right)^{q+1} e^{-x} L_p^q(x) L_p^q(x) dx \\
&= \frac{na_0}{2Z}\frac{1}{\Lambda^2} \left(\frac{1}{2}\right)^{2l+2} \cdot I_{pp}^q
\end{aligned} \tag{26}$$

Plugging in everything we have so far, we obtain

$$\begin{aligned}
C &= 2^{l+1}\Lambda \sqrt{\frac{2Z}{na_0}} \frac{1}{\sqrt{I_{pp}^q}} \\
&= 2^{l+1} \frac{[(n+l)!]^2}{(2l+1)!(n-l-1)!} \sqrt{\frac{p!}{[(p+q)!]^3(q+2p+1)}} \\
&= 2^{l+1} \sqrt{\frac{2Z}{na_0}} \frac{[(n+l)!]^2}{(2l+1)!(n-l-1)!} \sqrt{\frac{(n-l-1)!}{[(n+l)!]^3 \cdot 2n}} \\
&= 2^{l+1} \sqrt{\frac{2Z}{na_0}} \frac{1}{(2l+1)!} \sqrt{\frac{(n+l)!}{2n(n-l-1)!}}
\end{aligned} \tag{27}$$

Thus we obtain the full normalized radial wavefunction, in terms of $F(a;c;x)$

$$
\begin{aligned}
R_{nl}(r) = \frac{Cu_{nl}(r)}{r} &= \frac{1}{r}\sqrt{\frac{2Z}{na_0}}\frac{1}{(2l+1)!}\sqrt{\frac{(n+l)!}{2n(n-l-1)!}}(2\rho)^{l+1}e^{-\rho}w(\rho) \\
&= \frac{1}{r}\sqrt{\frac{2Z}{na_0}}\frac{1}{(2l+1)!}\sqrt{\frac{(n+l)!}{2n(n-l-1)!}}\left(\frac{2Zr}{na_0}\right)^{l+1}e^{-Zr/na_0}F\left(l+1-n;2l+2;\frac{2Zr}{na_0}\right) \\
&= \frac{1}{(2l+1)!}\left(\frac{2Zr}{na_0}\right)^{l}e^{-Zr/na_0}\sqrt{\left(\frac{2Z}{na_0}\right)^3\frac{(n+l)!}{2n(n-l-1)!}}F\left(l+1-n;2l+2;\frac{2Zr}{na_0}\right)
\end{aligned}
\tag{28}
$$

Or, in terms of $L_p^q(x)$:

$$
R_{nl}(r) = \left(\frac{2Zr}{na_0}\right)^{l}\sqrt{\left(\frac{2Z}{na_0}\right)^3\frac{(n-l-1)!}{2n[(n+l)!]^3}}e^{-Zr/na_0}L_{n-l-1}^{2l+1}\left(\frac{2Zr}{na_0}\right) \tag{29}
$$

Coming back to the earlier comments about non-vanishing weighted inner product $I_{pp'}^q$ where $|p-p'|=1$, does it imply non-vanishing inner product of two wavefunctions u_{nl} and $u_{n'l}$ (recall by (23), (24), q depends only on l, but p depends on both n and l)?

Actually, despite the appearance of identical variable x in the definition of $I_{pp'}^q$ in (21), when we we calculate the inner product of $u_{nl}(r)$ and $u_{n'l}(r)$, the x variable now depends on n, i.e., $x = 2Zr/na_0$ (refer to (26)). Therefore the inner product between $u_{nl}(r)$ and $u_{n'l}(r)$ with r as free variable does not have direct relation with the quantity $I_{pp'}^q$, whose non-zeroness thus is not applicable towards the orthogonality of the wavefunctions.