1. **Generating Function**

   Let

$$g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] \tag{1}$$

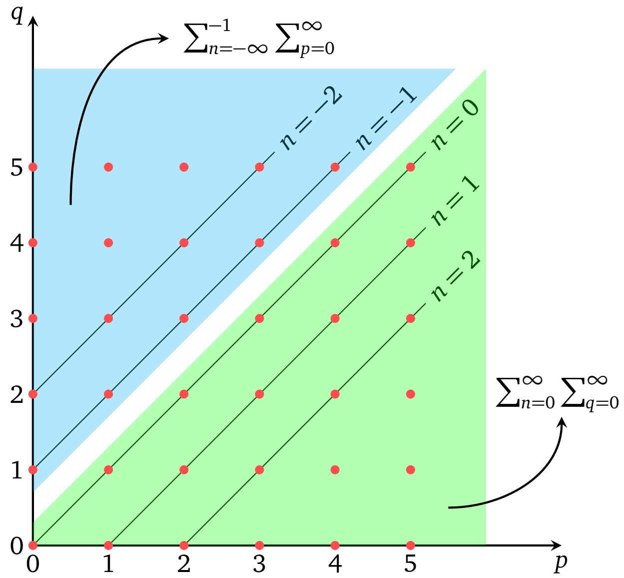
then we define, for each order n, the *Bessel function of the first kind* $J_n(x)$ in terms of the expansion

$$g(x,t) = \sum_{n=-\infty}^{\infty} J_n(x) t^n \tag{2}$$

We can match $J_n(x)$ to each of the t^n power terms in the expansion of (1)

$$\begin{aligned} g(x,t) &= e^{xt/2} \cdot e^{-x/2t} \\ &= \left[\sum_{p=0}^{\infty} \frac{\left(\frac{x}{2}\right)^p t^p}{p!}\right] \cdot \left[\sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^q t^{-q}}{q!}\right] \\ &= \sum_{p,q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{p+q} t^{p-q}}{p!q!} \end{aligned} \tag{3}$$

Now the double sum of p and q runs through all the grid points of the p-q plane's first quadrant. If we define $n = p - q$, we can convert the double sum $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}$ into a double sum whose outer sum runs through all the diagonal rays representing different n (from $-\infty$ to ∞), and whose inner sum runs through all different grid points along the diagonal ray for a given n. But depending on whether $n \geq 0$ or $n < 0$, the inner sum's index will be chosen differently (see figure below).



The sum in (3) is equivalent to

$$g(x,t) = \sum_{n=-\infty}^{-1}\sum_{p=0}^{\infty} \frac{(-1)^{p-n}\left(\frac{x}{2}\right)^{2p-n} t^n}{p!(p-n)!} + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^q\left(\frac{x}{2}\right)^{2q+n} t^n}{q!(q+n)!} \tag{4}$$

Compare (4) with (2), we can easily see that for $n \geq 0$,

$$J_n(x) = \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+n}}{q!(q+n)!} \tag{5}$$

$$J_{-n}(x) = (-1)^n J_n(x) \tag{6}$$

2. **Non-integer Order**

   Now we consider to generalize $J_n(x)$ to non-integer orders, denoted $J_\nu(x)$, where ν is real but not necessarily an integer. Our starting point is (5) where $(x/2)^{2q+\nu}$ is already well defined, so we only need to generalize $(q+\nu)!$. The natural generalization of factorial is the Γ-function (keep in mind that $\Gamma(n+1) = n!$ for positive integer n). Therefore we define, for arbitrary real order ν, the Bessel function of the first kind $J_\nu(x)$ as

$$J_\nu(x) = \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} \tag{7}$$

   Since Γ function is singular at non-positive integers, this means when ν is a negative integer $-n$, for (7) to be well defined, we have to restrict q to avoid those singularities. I.e., when $\nu = -n$, the sum has to be $\sum_{q=n}^{\infty}$ instead of $\sum_{q=0}^{\infty}$.

   The new definition (7) is compatible with (6), since when $\nu = -n$, by (7) we have (note we have shifted q to start from n instead of 0 given the comments above)

$$\begin{aligned}
J_{-n}(x) &= \sum_{q=n}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q-n}}{q!\Gamma(q-n+1)} && \text{(relabel } q' = q-n\text{)} \\
&= \sum_{q'=0}^{\infty} \frac{(-1)^{q'+n} \left(\frac{x}{2}\right)^{2q'+n}}{(q'+n)!\Gamma(q'+1)} && \text{(note } (q'+n)! = \Gamma(q'+n+1), \Gamma(q'+1) = q'!\text{)} \\
&= (-1)^n J_n(x) && (8)
\end{aligned}$$

3. **Recurrence Relations**

   There are many recurrence relations for $J_\nu(x)$, here we prove one (which can be used to give the exact form of spherical Bessel functions later).

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x) \tag{9}$$

   Proof. By definition (7), we have

$$\begin{aligned}
\left(\frac{x}{2}\right) J_{\nu+1}(x) &= \left(\frac{x}{2}\right) \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu+1}}{q!\Gamma(q+\nu+2)} \\
&= \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2(q+1)+\nu}}{q!\Gamma(q+\nu+2)} && \text{(relabel } q' = q+1\text{)} \\
&= \sum_{q'=1}^{\infty} \frac{-(-1)^{q'} q' \left(\frac{x}{2}\right)^{2q'+\nu}}{q'!\Gamma(q'+\nu+1)} && \text{(add } q' = 0 \text{ term)} \\
&= \sum_{q'=0}^{\infty} \frac{-(-1)^{q'} q' \left(\frac{x}{2}\right)^{2q'+\nu}}{q'!\Gamma(q'+\nu+1)} && (10) \\
\left(\frac{x}{2}\right) J_{\nu-1}(x) &= \left(\frac{x}{2}\right) \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu-1}}{q!\Gamma(q+\nu)} \\
&= \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu)} && (11)
\end{aligned}$$

Adding (10) and (11) gives

$$\begin{aligned}
\left(\frac{x}{2}\right)[J_{\nu+1}(x)+J_{\nu-1}(x)] &= \sum_{q=0}^{\infty}\left[\frac{1}{q!\Gamma(q+\nu)}-\frac{q}{q!\Gamma(q+\nu+1)}\right](-1)^q\left(\frac{x}{2}\right)^{2q+\nu} && (\text{note } \Gamma(x+1)=x\Gamma(x)) \\
&= \sum_{q=0}^{\infty}\left[\frac{q+\nu}{q!\Gamma(q+\nu+1)}-\frac{q}{q!\Gamma(q+\nu+1)}\right](-1)^q\left(\frac{x}{2}\right)^{2q+\nu} \\
&= \nu\cdot\sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} \\
&= \nu J_\nu(x) && (12)
\end{aligned}$$

□

4. **Bessel Equation**

We claim $J_\nu(x)$ satisfies the *Bessel equation*

$$x^2F''(x)+xF'(x)+(x^2-\nu^2)F(x)=0 \tag{13}$$

Proof. By taking the derivative of $J_\nu(x)$, we have

$$\begin{aligned}
xJ_\nu'(x) &= x\cdot\sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{2q+\nu}{2}\right)\left(\frac{x}{2}\right)^{2q+\nu-1}}{q!\Gamma(q+\nu+1)} \\
&= \sum_{q=0}^{\infty}\frac{(-1)^q(2q+\nu)\left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} && (14) \\
x^2J_\nu''(x) &= x^2\cdot\sum_{q=0}^{\infty}\frac{(-1)^q\left[\frac{(2q+\nu)(2q+\nu-1)}{2\cdot 2}\right]\left(\frac{x}{2}\right)^{2q+\nu-2}}{q!\Gamma(q+\nu+1)} \\
&= \sum_{q=0}^{\infty}\frac{(-1)^q(2q+\nu)(2q+\nu-1)\left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} && (15)
\end{aligned}$$

Adding (14) and (15) gives

$$xJ_\nu'(x)+x^2J_\nu''(x)=\sum_{q=0}^{\infty}\frac{(-1)^q(2q+\nu)^2\left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} \tag{16}$$

On the other hand

$$\begin{aligned}
x^2J_\nu(x) &= x^2\sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)} \\
&= \sum_{q=0}^{\infty}\frac{(-1)^q 4\left(\frac{x}{2}\right)^{2(q+1)+\nu}}{q!\Gamma(q+\nu+1)} && (\text{relabel } q'=q+1) \\
&= \sum_{q'=0}^{\infty}\frac{-(-1)^{q'}4q'(q'+\nu)\left(\frac{x}{2}\right)^{2q'+\nu}}{q'!\Gamma(q'+\nu+1)} && (17)
\end{aligned}$$

Finally (13) is proved by adding (16) and (17) since $(2q+\nu)^2-4q(q+\nu)=\nu^2$. □

5. **Orthonormality**

The notion of orthonormality between Bessel functions is different than usual. Let $J_\nu(x)$ be a Bessel function of real order ν, and let $\alpha_{\nu l}$ and $\alpha_{\nu k}$ be its l-th and k-th root. Then the orthonomality condition claims

$$\int_0^L J_\nu\left(\alpha_{\nu l}\frac{x}{L}\right)J_\nu\left(\alpha_{\nu k}\frac{x}{L}\right)x\,dx = \frac{1}{2}L^2[J'_\nu(\alpha_{\nu k})]^2\delta_{lk} \tag{18}$$

Note that it applies to Bessel functions of the same order ν, whose arguments are scaled by different roots.

Proof. Since the order ν is fixed, it will be omitted from the subscripts in this proof. The Bessel equation (13) can be rewritten as

$$\begin{aligned} xJ''(x)+J'(x)+\left(x-\frac{\nu^2}{x}\right)J(x)=0 \qquad &\Longrightarrow \\ \frac{d}{dx}\left[x\frac{dJ(x)}{dx}\right]+\left(x-\frac{\nu^2}{x}\right)J(x)=0 \qquad & \end{aligned} \tag{19}$$

Now for a given real number β, define $x=\beta y$, then (19) can be rewritten as

$$\begin{aligned} \frac{1}{\beta}\frac{d}{dy}\left[\beta y\frac{dJ(\beta y)}{dy}\frac{1}{\beta}\right]+\left(\beta y-\frac{\nu^2}{\beta y}\right)J(\beta y)=0 \qquad &\Longrightarrow \\ \frac{d}{dy}\left[y\frac{dJ(\beta y)}{dy}\right]+\left(\beta^2 y-\frac{\nu^2}{y}\right)J(\beta y)=0 \qquad & \end{aligned} \tag{20}$$

Denote $J(\beta y)=J_\beta(y)$, and similarly define $J_\gamma(y)$ for another factor γ, together we have

$$\frac{d}{dy}\left[y\frac{dJ_\beta(y)}{dy}\right]+\left(\beta^2 y-\frac{\nu^2}{y}\right)J_\beta(y)=0 \tag{21}$$

$$\frac{d}{dy}\left[y\frac{dJ_\gamma(y)}{dy}\right]+\left(\gamma^2 y-\frac{\nu^2}{y}\right)J_\gamma(y)=0 \tag{22}$$

Multiply (21) by J_γ and multiply (22) by J_β and subtract

$$J_\gamma\frac{d}{dy}\left(y\frac{dJ_\beta}{dy}\right)-J_\beta\frac{d}{dy}\left(y\frac{dJ_\gamma}{dy}\right)=\left(\gamma^2-\beta^2\right)yJ_\beta J_\gamma \tag{23}$$

Integrate (23) from 0 to L, we have

$$\int_0^L\left[J_\gamma\frac{d}{dy}\left(y\frac{dJ_\beta}{dy}\right)-J_\beta\frac{d}{dy}\left(y\frac{dJ_\gamma}{dy}\right)\right]dy=\left(\gamma^2-\beta^2\right)\int_0^L J_\beta J_\gamma y\,dy \tag{24}$$

whose LHS gives

$$\begin{aligned} \text{LHS} &= \left[J_\gamma y\frac{dJ_\beta}{dy}\Big|_0^L-\int_0^L y\frac{dJ_\beta}{dy}\frac{dJ_\gamma}{dy}dy\right]-\left[J_\beta y\frac{dJ_\gamma}{dy}\Big|_0^L-\int_0^L y\frac{dJ_\gamma}{dy}\frac{dJ_\beta}{dy}dy\right] \\ &= J_\gamma(L)L\frac{dJ_\beta}{dy}\Big|_L-J_\beta(L)L\frac{dJ_\gamma}{dy}\Big|_L \end{aligned} \tag{25}$$

Then (24) becomes

$$J_\gamma(L)L\frac{dJ_\beta}{dy}\Big|_L-J_\beta(L)L\frac{dJ_\gamma}{dy}\Big|_L=\left(\gamma^2-\beta^2\right)\int_0^L J_\beta J_\gamma y\,dy \tag{26}$$

When we take $\beta=\alpha_l/L$ and $\gamma=\alpha_k/L$, where $l\neq k$ and α_l,α_k are roots of $J(x)$, LHS of (24) vanishes, which implies the integral on the RHS must vanish, proving orthogonality.

Now choose $\gamma=\alpha/L$ and $\beta=\alpha_k/L$, where α is not necessarily a root, (26) becomes

$$\begin{aligned} J\left(\alpha\frac{L}{L}\right)L\frac{dJ(x)}{dx/\beta}\Big|_{x=\alpha_k}=\frac{\alpha^2-\alpha_k^2}{L^2}\int_0^L J\left(\alpha\frac{y}{L}\right)J\left(\alpha_k\frac{y}{L}\right)y\,dy \qquad &\Longrightarrow \\ \int_0^L J\left(\alpha\frac{y}{L}\right)J\left(\alpha_k\frac{y}{L}\right)y\,dy=J(\alpha)\alpha_k J'(\alpha_k)\frac{L^2}{\alpha^2-\alpha_k^2}=\underbrace{\left[\frac{J(\alpha)-\overbrace{J(\alpha_k)}^{0}}{\alpha-\alpha_k}\right]}_{(\Delta J/\Delta\alpha)|_{\alpha_k}}\frac{\alpha_k L^2}{\alpha+\alpha_k}J'(\alpha_k) & \end{aligned} \tag{27}$$

(18) is proved by taking the limit $\alpha\to\alpha_k$. □

6. **Bessel Function of the Second Kind**

The Bessel equation (13) is a second order ODE, so it should have two linearly independent solutions. Since (13) is dependent on ν^2, it's clear that both $J_\nu(x)$ and $J_{-\nu}$ are solutions. It is also easy to see that for non-integer ν, the two solutions J_ν and $J_{-\nu}$ are linearly independent. This is because by definition

$$J_\nu(x) = \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+\nu}}{q!\Gamma(q+\nu+1)}$$
$$J_{-\nu}(x) = \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q-\nu}}{q!\Gamma(q-\nu+1)}$$

they cannot be related by a multiplicative factor unless they have the same set of x power terms, i.e., there must be q, q' to satisfy

$$2q + \nu = 2q' - \nu \qquad \Longrightarrow \qquad \nu = q' - q$$

in other words, ν must be integer for $J_{\pm\nu}$ to be linearly dependent. In fact, due to (6), this condition is also sufficient.

Thus for ν a non-integer, we have two linearly independent solutions $J_\nu(x)$ and $J_{-\nu}(x)$. But the Bessel function of the second kind is defined as

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \qquad (\nu \text{ non-integer}) \tag{28}$$

Note $Y_\nu(x)$, being an linear combination of $J_{\pm\nu}(x)$, is indeed a solution of (13).

Such a definition will yield convenient relations when the order is half integer $\nu = n + 1/2$:

$$J_{-(n+1/2)}(x) = (-1)^{n+1} Y_{n+1/2}(x) \tag{29}$$
$$Y_{-(n+1/2)}(x) = (-1)^n J_{n+1/2}(x) \tag{30}$$

which follow trivially from (28).

In these notes, we will omit the discussion of Y_ν where ν is integer. Subsequent discussion will focus on spherical Bessel functions which are defined using half-integer order Bessel functions of both kinds.

7. **Spherical Bessel Functions**

The radial part of the wave function of a free particle is the solution to the equation (see Sakurai eq (3.281)):

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[1 - \frac{l(l+1)}{\rho^2}\right]R = 0 \tag{31}$$

If we write $R(\rho) = \rho^{-1/2}u(\rho)$, we will have

$$R' = -\frac{1}{2}\rho^{-3/2}u + \rho^{-1/2}u' \qquad \Longrightarrow$$
$$\frac{2}{\rho}R' = -\rho^{-5/2}u + 2\rho^{-3/2}u' \tag{32}$$
$$R'' = \frac{3}{4}\rho^{-5/2}u - 2\cdot\frac{1}{2}\rho^{-3/2}u' + \rho^{-1/2}u'' \tag{33}$$
$$\left[1 - \frac{l(l+1)}{\rho^2}\right]R = \rho^{-1/2}u - l(l+1)\rho^{-5/2}u \tag{34}$$

Add (32)-(34):

$$\rho^{-1/2}u'' + \rho^{-3/2}u' + \left\{\rho^{-1/2} - \left[l(l+1) + \frac{1}{4}\right]\rho^{-5/2}\right\}u = 0 \tag{35}$$

Multiply $\rho^{5/2}$ to both sides

$$\rho^2 u'' + \rho u' + \left[\rho^2 - \left(l + \frac{1}{2}\right)^2\right]u = 0 \tag{36}$$

which is exactly (13) with $\nu = l + 1/2$. This means

$$R(\rho) = \frac{1}{\sqrt{\rho}} J_{\pm(l+1/2)}(\rho) \tag{37}$$

will be the solutions of (31).

We call

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) \tag{38}$$

$$y_l(x) = \sqrt{\frac{\pi}{2x}} Y_{l+1/2}(x) = (-1)^{l+1} \sqrt{\frac{\pi}{2x}} J_{-(l+1/2)}(x) \tag{39}$$

the *Spherical Bessel Functions* of order l. They are the two linearly independent solutions to equation (31).

8. **Closed Form of Spherical Bessel Functions**

Unlike ordinary Bessel functions, spherical Bessel functions can be written in closed forms. First, let's calculate $j_0(x)$.

$$\begin{aligned}
j_0(x) &= \sqrt{\frac{\pi}{2x}} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q+1/2}}{q!\Gamma(q+3/2)} \\
&= \sqrt{\frac{\pi}{4}} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q}}{q!\Gamma(q+3/2)} \\
&= \sqrt{\frac{\pi}{4}} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x^{2q}}{4^q}\right)}{q!\frac{(2q+2)!}{4^{q+1}(q+1)!}\sqrt{\pi}} \\
&= \frac{1}{2} \sum_{q=0}^{\infty} \frac{(-1)^q x^{2q} 4(q+1)}{(2q+2)!} \\
&= \frac{1}{x} \sum_{q=0}^{\infty} \frac{(-1)^q x^{2q+1}}{(2q+1)!} = \frac{\sin x}{x}
\end{aligned} \tag{40}$$

where in the third line, we have used $\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}(2n)!/(4^n n!)$. Similarly, for $y_0(x)$:

$$\begin{aligned}
y_0(x) &= -\sqrt{\frac{\pi}{2x}} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q-1/2}}{q!\Gamma(q+1/2)} \\
&= -\frac{\sqrt{\pi}}{x} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x}{2}\right)^{2q}}{q!\Gamma(q+1/2)} \\
&= -\frac{\sqrt{\pi}}{x} \sum_{q=0}^{\infty} \frac{(-1)^q \left(\frac{x^{2q}}{4^q}\right)}{q!\frac{(2q)!}{4^q q!}\sqrt{\pi}} \\
&= -\frac{1}{x} \sum_{q=0}^{\infty} \frac{(-1)^q x^{2q}}{(2q)!} = -\frac{\cos x}{x}
\end{aligned} \tag{41}$$

Here we see that the $\sqrt{\pi/2}$ factor in the definition of spherical Bessel functions is to give j_0 and y_0 a "normalized" form.

We can use (9) to continue to obtain $j_1(x), y_1(x)$:

$$j_1(x) = \sqrt{\frac{\pi}{2x}} J_{3/2}(x) = \sqrt{\frac{\pi}{2x}} \left[\frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)\right] = \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2x}{\pi}} \left[\frac{1}{x} j_0(x) + y_0(x)\right] = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{42}$$

$$y_1(x) = \sqrt{\frac{\pi}{2x}} J_{-3/2}(x) = \sqrt{\frac{\pi}{2x}} \left[-\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)\right] = \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2x}{\pi}} \left[\frac{1}{x} y_0(x) - j_0(x)\right] = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \tag{43}$$

The general recurrence relation for j_l, y_l can be obtained from (9), (38) and (39):

$$J_{l+3/2} = \frac{2l+1}{x} J_{l+1/2} - J_{l-1/2} \qquad \Longrightarrow \qquad j_{l+1} = \frac{2l+1}{x} j_l - j_{l-1} \tag{44}$$

$$J_{-(l+3/2)} = -\frac{2l+1}{x} J_{-(l+1/2)} - J_{-(l-1/2)} \qquad \Longrightarrow \qquad y_{l+1} = \frac{2l+1}{x} y_l - y_{l-1} \tag{45}$$

We can then calculate the closed form formula for any spherical Bessel functions j_l, y_l using (40)-(45).

9. **Rayleigh's Formulae**

There is neat differential relation of spherical Bessel functions, called Rayleigh's Formulae

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l j_0(x) \tag{46}$$

$$y_l(x) = (-x)^{l+1} \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x} = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l y_0(x) \tag{47}$$

Proof. Since they are of the same form, we are going to use f_l to denote either j_l or y_l. Now define

$$g_l(x) = \frac{f_l(x)}{(-x)^l} \tag{48}$$

We are going to use induction for the proof, for which the $l = 0, 1$ cases are trivially true (for both j and y). Now the goal is to inductively prove

$$g_{l+1}(x) = \frac{f_{l+1}(x)}{(-x)^{l+1}} = \frac{1}{x}\frac{d}{dx}\left[\left(\frac{1}{x}\frac{d}{dx}\right)^l f_0(x)\right] = \frac{1}{x} g_l'(x) \tag{49}$$

To prove (49), we first establish the differential equation $g_l(x)$ must satisfy. Take the first and second derivative of (48), we have

$$f_l' = -l(-x)^{l-1} g_l + (-x)^l g_l' \tag{50}$$

$$f_l'' = l(l-1)(-x)^{l-2} g_l - 2l(-x)^{l-1} g_l' + (-x)^l g_l'' \tag{51}$$

Then (31) becomes

$$\begin{aligned}
0 &= x^2 f_l'' + 2x f_l' + [x^2 - l(l+1)] f_l \\
&= x^2 \left[l(l-1)(-x)^{l-2} g_l - 2l(-x)^{l-1} g_l' + (-x)^l g_l''\right] + 2x\left[-l(-x)^{l-1} g_l + (-x)^l g_l'\right] + [x^2 - l(l+1)](-x)^l g_l \\
&= (-x)^{l+2} g_l'' + (-x)^{l+1}(-2l-2) g_l' + (-x)^l [l(l-1) + 2l + x^2 - l(l+1)] g_l \\
&= (-x)^{l+2} g_l'' - 2(l+1)(-x)^{l+1} g_l' + (-x)^{l+2} g_l
\end{aligned} \tag{52}$$

After canceling the common factor $(-x)^{l+1}$, we get the differential equation satisfied by all g_ls:

$$x g_l'' + 2(l+1) g_l' + x g_l = 0 \tag{53}$$

To prove (49), by (44) or (45), it's equivalent to prove

$$\begin{aligned}
\frac{2l+1}{x}(-x)^l g_l - (-x)^{l-1} g_{l-1} &= (-x)^{l+1} \frac{1}{x} g_l' && \text{(by canceling } (-x)^{l-1}\text{)} && \Longleftrightarrow \\
-(2l+1) g_l - g_{l-1} &= x g_l' && \text{(by induction assumption } g_l = \tfrac{1}{x} g_{l-1}'\text{)} && \Longleftrightarrow \\
-\frac{2l+1}{x} g_{l-1}' - g_{l-1} &= x\left(\frac{1}{x} g_{l-1}'' - \frac{1}{x^2} g_{l-1}'\right) && && \Longleftrightarrow \\
x g_{l-1}'' + 2l g_{l-1}' + x g_{l-1} &= 0
\end{aligned} \tag{54}$$

which is exactly what we have shown in (53) with $l \to l-1$. □