We prove the properties of coherent state listed in problem 2.21.

- $|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle$ is the normalized eigenstate of a.

  Proof: Expanding $e^{\lambda a^\dagger}$, we have

$$
\begin{aligned}
ae^{\lambda a^\dagger}|0\rangle &= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}a(a^\dagger)^k|0\rangle \\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}a\sqrt{k!}|k\rangle \\
&= \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\sqrt{k}\sqrt{k!}|k-1\rangle \\
&= \sum_{k=0}^{\infty}\lambda\cdot\frac{\lambda^k}{\sqrt{k!}}|k\rangle \\
&= \lambda\cdot\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}(a^\dagger)^k|0\rangle = \lambda e^{\lambda a^\dagger}|0\rangle
\end{aligned}
$$

  which means $e^{\lambda a^\dagger}|0\rangle$ is an unnormalized eigenstate of a.

  The overall normalization constant can be obtained by noting

$$
\langle 0|(e^{\lambda a^\dagger})^\dagger\cdot e^{\lambda a^\dagger}|0\rangle = \sum_{k,l=0}^{\infty}\left\langle k\left|\frac{\lambda^{*k}}{\sqrt{k!}}\cdot\frac{\lambda^l}{\sqrt{l!}}\right|l\right\rangle = \sum_{k=0}^{\infty}\frac{(|\lambda|^2)^k}{k!} = e^{|\lambda|^2}
$$

- To see $|\lambda\rangle$ satisfies the minimal uncertainty, we should calculate both $\langle\lambda|x|\lambda\rangle$ and $\langle\lambda|x^2|\lambda\rangle$ (similar for p).

  First note

$$
\begin{aligned}
x &= \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \\
x^2 &= \frac{\hbar}{2m\omega}(a^{\dagger 2} + a^2 + a^\dagger a + aa^\dagger) \\
p &= i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a) \\
p^2 &= -\frac{m\hbar\omega}{2}(a^{\dagger 2} + a^2 - a^\dagger a - aa^\dagger)
\end{aligned}
$$

  Therefore

$$
\begin{aligned}
\langle\lambda|x|\lambda\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|a^\dagger + a|\lambda\rangle \qquad \text{using } a|\lambda\rangle = \lambda|\lambda\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\lambda^* + \lambda) = 2\mathrm{Re}\lambda\cdot\sqrt{\frac{\hbar}{2m\omega}} \\
\langle\lambda|x^2|\lambda\rangle &= \frac{\hbar}{2m\omega}\langle\lambda|a^{\dagger 2} + a^2 + a^\dagger a + aa^\dagger|\lambda\rangle \\
&= \frac{\hbar}{2m\omega}\langle\lambda|a^{\dagger 2} + a^2 + 2a^\dagger a + [a, a^\dagger]|\lambda\rangle \\
&= \frac{\hbar}{2m\omega}(\lambda^{*2} + \lambda^2 + 2\lambda^*\lambda + 1) = \frac{\hbar}{2m\omega}[4(\mathrm{Re}\lambda)^2 + 1] \qquad \Longrightarrow \\
\langle(\Delta x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2 = \frac{h}{2m\omega}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\lambda|p|\lambda\rangle &= i\sqrt{\frac{m\hbar\omega}{2}}\langle\lambda|a^\dagger - a|\lambda\rangle \\
&= i\sqrt{\frac{m\hbar\omega}{2}}(\lambda^* - \lambda) = 2\mathrm{Im}\lambda\cdot\sqrt{\frac{m\hbar\omega}{2}} \\
\langle\lambda|p^2|\lambda\rangle &= -\frac{m\hbar\omega}{2}\langle\lambda|a^{\dagger 2} + a^2 - a^\dagger a - aa^\dagger|\lambda\rangle \\
&= -\frac{m\hbar\omega}{2}\langle\lambda|a^{\dagger 2} + a^2 - 2a^\dagger a - [a, a^\dagger]|\lambda\rangle \\
&= -\frac{m\hbar\omega}{2}(\lambda^{*2} + \lambda^2 - 2\lambda^*\lambda - 1) = \frac{m\hbar\omega}{2}[4(\mathrm{Im}\lambda)^2 + 1] \qquad \Longrightarrow \\
\langle(\Delta p)^2\rangle &= \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2}
\end{aligned}
$$

So $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$ is the minimal uncertainty.

- We have already seen that

  $$|\lambda\rangle = e^{-|\lambda|^2/2}\sum_{k=0}\frac{\lambda^k}{\sqrt{k!}}|k\rangle$$

  So by letting $\mu = |\lambda|^2$, we have

  $$|f(k)|^2 = e^{-|\lambda|^2}\frac{|\lambda|^2 k}{k!} = e^{-\mu}\frac{\mu^k}{k!}$$

  where the most probable n is near the mean value μ.

- To see how $|\lambda\rangle$ can be achieved by applying $e^{-ipl/\hbar}$ to $|0\rangle$, we first note the claim that when $[A, B]$ commutes with both A and B,

  $$e^{A+B} = e^A e^B e^{-[A,B]/2}$$

  Also note that $e^{-ipl/\hbar} = e^{\sqrt{\frac{m\omega}{2\hbar}}l(a^\dagger - a)}$, and $[a, a^\dagger] = 1$, by letting $\lambda = \sqrt{\frac{m\omega}{2\hbar}}l$, we have

  $$
  \begin{aligned}
  e^{-ipl/\hbar}|0\rangle &= e^{\lambda a^\dagger - \lambda a}|0\rangle = e^{\lambda a^\dagger}e^{-\lambda a}e^{-\lambda^2[a^\dagger, -a]/2}|0\rangle \\
  &= e^{-\lambda^2/2}e^{\lambda a^\dagger}e^{-\lambda a}|0\rangle \\
  &= e^{-\lambda^2/2}e^{\lambda a^\dagger}|0\rangle = |\lambda\rangle
  \end{aligned}
  $$