The $n=2$ orbital states are $|nlm\rangle = |211\rangle, |210\rangle, |21-1\rangle, |200\rangle$. Considering spin, each of the above state will be tensored with $\left|\frac{1}{2}\right\rangle$ or $\left|-\frac{1}{2}\right\rangle$, so $n=2$ has degeneracy of 8.

In the following, we will omit the 2 in $|nlm\rangle$, which simplifies the eight degenerate states as $|lm\rangle \otimes \left|\pm\frac{1}{2}\right\rangle$.

1.

$$\begin{aligned} V &= \frac{A}{2\hbar^2}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2\right) + \frac{B}{\hbar}(J_z + S_z) \\ &= \underbrace{\left[\frac{A}{2\hbar^2}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2\right) + \frac{B}{\hbar}J_z\right]}_{V_1} + \underbrace{\frac{B}{\hbar}S_z}_{V_2} \end{aligned} \tag{1}$$

where V_1 is diagonalizable in the $\boldsymbol{L}^2, \boldsymbol{S}^2, \boldsymbol{J}^2, J_z$ basis, and V_2 is diagonalizable in the $\boldsymbol{S}^2, S_z$ basis.

2. Using C-G coefficients, we can express the 8 (spherical) basis in $\boldsymbol{L}^2, \boldsymbol{S}^2, \boldsymbol{J}^2, J_z$ as linear combination of the (Cartesian) basis $|lm\rangle \otimes \left|\pm\frac{1}{2}\right\rangle$

$$\begin{array}{lll} & |ls; jm\rangle & |lm\rangle \otimes |s_z\rangle \\ |a\rangle & \left|1\,\frac{1}{2}; \frac{3}{2}\,\frac{3}{2}\right\rangle & = |11\rangle \otimes \left|\frac{1}{2}\right\rangle \end{array} \tag{2}$$

$$\begin{array}{lll} |b\rangle & \left|1\,\frac{1}{2}; \frac{3}{2}\,\frac{1}{2}\right\rangle & = \sqrt{\frac{1}{3}}|11\rangle \otimes \left|-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}|10\rangle \otimes \left|\frac{1}{2}\right\rangle \end{array} \tag{3}$$

$$\begin{array}{lll} |c\rangle & \left|1\,\frac{1}{2}; \frac{3}{2}\,\frac{-1}{2}\right\rangle & = \sqrt{\frac{1}{3}}|1-1\rangle \otimes \left|\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}|10\rangle \otimes \left|-\frac{1}{2}\right\rangle \end{array} \tag{4}$$

$$\begin{array}{lll} |d\rangle & \left|1\,\frac{1}{2}; \frac{3}{2}\,\frac{-3}{2}\right\rangle & = |1-1\rangle \otimes \left|-\frac{1}{2}\right\rangle \end{array} \tag{5}$$

$$\begin{array}{lll} |e\rangle & \left|1\,\frac{1}{2}; \frac{1}{2}\,\frac{1}{2}\right\rangle & = \sqrt{\frac{2}{3}}|11\rangle \otimes \left|-\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}|10\rangle \otimes \left|\frac{1}{2}\right\rangle \end{array} \tag{6}$$

$$\begin{array}{lll} |f\rangle & \left|1\,\frac{1}{2}; \frac{1}{2}\,\frac{-1}{2}\right\rangle & = -\sqrt{\frac{2}{3}}|1-1\rangle \otimes \left|\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}|10\rangle \otimes \left|-\frac{1}{2}\right\rangle \end{array} \tag{7}$$

$$\begin{array}{lll} |g\rangle & \left|0\,\frac{1}{2}; \frac{1}{2}\,\frac{1}{2}\right\rangle & = |00\rangle \otimes \left|\frac{1}{2}\right\rangle \end{array} \tag{8}$$

$$\begin{array}{lll} |h\rangle & \left|0\,\frac{1}{2}; \frac{1}{2}\,\frac{-1}{2}\right\rangle & = |00\rangle \otimes \left|-\frac{1}{2}\right\rangle \end{array} \tag{9}$$

Apply V to all

$$V|a\rangle = \overbrace{\left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2}-1\cdot 2-\frac{1}{2}\frac{3}{2}\right)+\frac{3B}{2}\right]|a\rangle}^{V_1|a\rangle} + \overbrace{\frac{B}{2}|a\rangle}^{V_2|a\rangle} = \left(\frac{A}{2}+2B\right)|a\rangle \tag{10}$$

$$\begin{aligned} V|b\rangle &= \overbrace{\left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2}-1\cdot 2-\frac{1}{2}\frac{3}{2}\right)+\frac{B}{2}\right]|b\rangle}^{V_1|b\rangle} \overbrace{-\frac{B}{2}\sqrt{\frac{1}{3}}|11\rangle\otimes\left|-\frac{1}{2}\right\rangle + \frac{B}{2}\sqrt{\frac{2}{3}}|10\rangle\otimes\left|\frac{1}{2}\right\rangle}^{V_2|b\rangle} \\ &= \frac{A}{2}\sqrt{\frac{1}{3}}|11\rangle\otimes\left|-\frac{1}{2}\right\rangle + \left(\frac{A}{2}+B\right)\sqrt{\frac{2}{3}}|10\rangle\otimes\left|\frac{1}{2}\right\rangle \end{aligned} \tag{11}$$

$$\begin{aligned} V|c\rangle &= \left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2}-1\cdot 2-\frac{1}{2}\frac{3}{2}\right)-\frac{B}{2}\right]|c\rangle + \frac{B}{2}\sqrt{\frac{1}{3}}|1-1\rangle\otimes\left|\frac{1}{2}\right\rangle - \frac{B}{2}\sqrt{\frac{2}{3}}|10\rangle\otimes\left|-\frac{1}{2}\right\rangle \\ &= \frac{A}{2}\sqrt{\frac{1}{3}}|1-1\rangle\otimes\left|\frac{1}{2}\right\rangle + \left(\frac{A}{2}-B\right)\sqrt{\frac{2}{3}}|10\rangle\otimes\left|-\frac{1}{2}\right\rangle \end{aligned} \tag{12}$$

$$V|d\rangle = \left[\frac{A}{2}\left(\frac{3}{2}\frac{5}{2}-1\cdot 2-\frac{1}{2}\frac{3}{2}\right)-\frac{3B}{2}\right]|d\rangle - \frac{B}{2}|d\rangle = \left(\frac{A}{2}-2B\right)|d\rangle \tag{13}$$

$$\begin{aligned} V|e\rangle &= \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2}-1\cdot 2-\frac{1}{2}\frac{3}{2}\right)+\frac{B}{2}\right]|e\rangle - \frac{B}{2}\sqrt{\frac{2}{3}}|11\rangle\otimes\left|-\frac{1}{2}\right\rangle - \frac{B}{2}\sqrt{\frac{1}{3}}|10\rangle\otimes\left|\frac{1}{2}\right\rangle \\ &= -A\sqrt{\frac{2}{3}}|11\rangle\otimes\left|-\frac{1}{2}\right\rangle + (A-B)\sqrt{\frac{1}{3}}|10\rangle\otimes\left|\frac{1}{2}\right\rangle \end{aligned} \tag{14}$$

$$\begin{aligned} V|f\rangle &= \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2}-1\cdot 2-\frac{1}{2}\frac{3}{2}\right)-\frac{B}{2}\right]|f\rangle - \frac{B}{2}\sqrt{\frac{2}{3}}|1-1\rangle\otimes\left|\frac{1}{2}\right\rangle - \frac{B}{2}\sqrt{\frac{1}{3}}|10\rangle\otimes\left|-\frac{1}{2}\right\rangle \\ &= A\sqrt{\frac{2}{3}}|1-1\rangle\otimes\left|\frac{1}{2}\right\rangle - (A+B)\sqrt{\frac{1}{3}}|10\rangle\otimes\left|-\frac{1}{2}\right\rangle \end{aligned} \tag{15}$$

$$V|g\rangle = \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2}-0\cdot 1-\frac{1}{2}\frac{3}{2}\right)+\frac{B}{2}\right]|g\rangle + \frac{B}{2}|g\rangle = B|g\rangle \tag{16}$$

$$V|h\rangle = \left[\frac{A}{2}\left(\frac{1}{2}\frac{3}{2}-0\cdot 1-\frac{1}{2}\frac{3}{2}\right)-\frac{B}{2}\right]|h\rangle - \frac{B}{2}|g\rangle = -B|h\rangle \tag{17}$$

With (2)-(17), we can construct the full 8×8 matrix of V in the spherical basis.

$$\begin{array}{c c}
\begin{array}{cccccccc} |a\rangle & |d\rangle & |g\rangle & |h\rangle & |b\rangle & |e\rangle & |c\rangle & |f\rangle \end{array} & \\
\left[\begin{array}{cccccccc}
\frac{A}{2}+2B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{A}{2}-2B & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{A}{2}+\frac{2B}{3} & -\frac{\sqrt{2}B}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\sqrt{2}B}{3} & -A+\frac{B}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{A}{2}-\frac{2B}{3} & -\frac{\sqrt{2}B}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}B}{3} & -A-\frac{B}{3}
\end{array}\right] & \begin{array}{c} |a\rangle \\ |d\rangle \\ |g\rangle \\ |h\rangle \\ |b\rangle \\ |e\rangle \\ |c\rangle \\ |f\rangle \end{array}
\end{array}$$

3. The diagonal elements for the a, d, g, h block are the energy eigenvalues for these states:

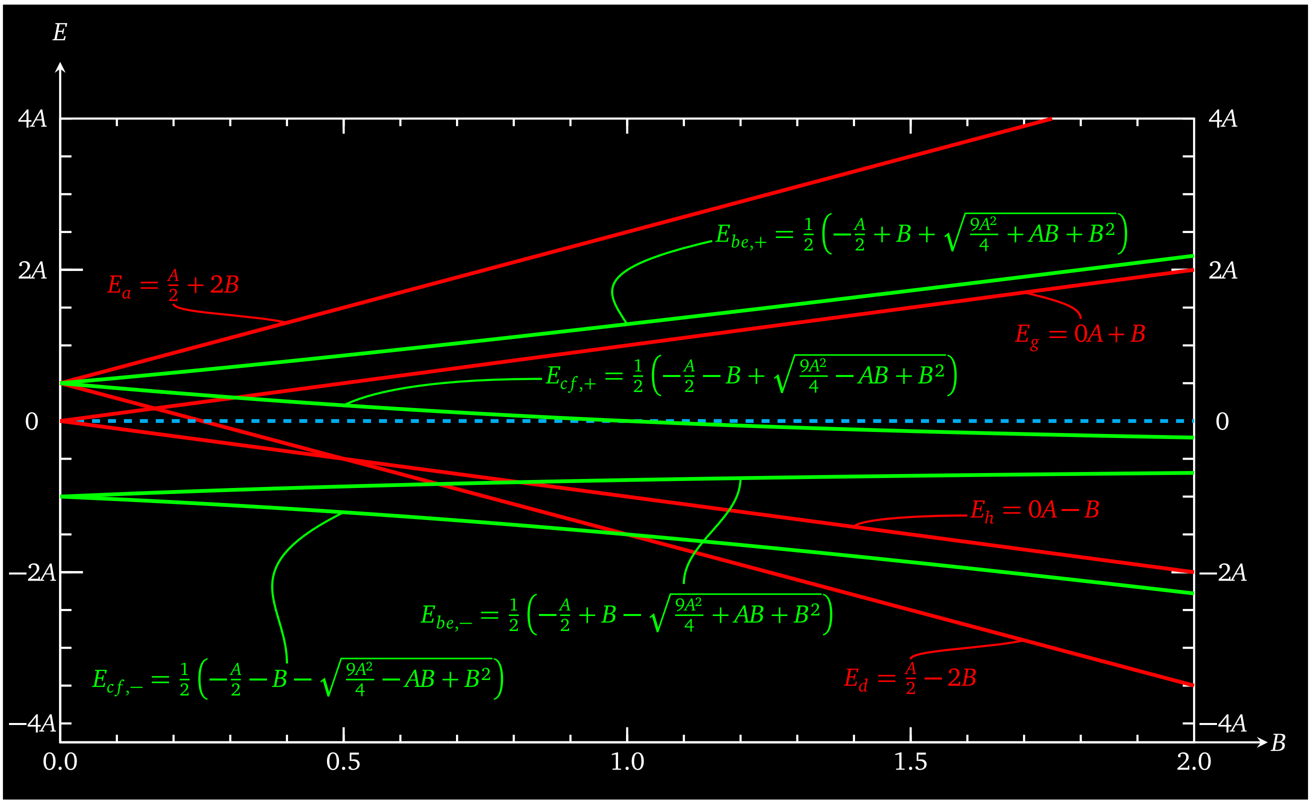
$$E_a = \frac{A}{2}+2B \tag{18}$$

$$E_d = \frac{A}{2}-2B \tag{19}$$

$$E_g = 0A+B \tag{20}$$

$$E_h = 0A-B \tag{21}$$

These are depicted in the $E\sim B$ diagram below as red **straight** lines.



For the b, e block, the energy eigenvalues can be found via

$$\left[E_{be} - \left(\frac{A}{2} + \frac{2B}{3}\right)\right]\left[E_{be} - \left(-A + \frac{B}{3}\right)\right] - \frac{2B^2}{9} = 0 \qquad \Longrightarrow$$

$$E_{be}^2 + \left(\frac{A}{2} - B\right)E_{be} - \left(\frac{A^2}{2} + \frac{AB}{2}\right) = 0 \qquad \Longrightarrow$$

$$E_{be,\pm} = \frac{1}{2}\left(-\frac{A}{2} + B \pm \sqrt{\frac{9A^2}{4} + AB + B^2}\right) \tag{22}$$

Similarly for the c, f block:

$$\left[E_{cf} - \left(\frac{A}{2} - \frac{2B}{3}\right)\right]\left[E_{cf} - \left(-A - \frac{B}{3}\right)\right] - \frac{2B^2}{9} = 0 \qquad \Longrightarrow$$

$$E_{cf}^2 + \left(\frac{A}{2} + B\right)E_{cf} - \left(\frac{A^2}{2} - \frac{AB}{2}\right) = 0 \qquad \Longrightarrow$$

$$E_{cf,\pm} = \frac{1}{2}\left(-\frac{A}{2} - B \pm \sqrt{\frac{9A^2}{4} - AB + B^2}\right) \tag{23}$$

The B-dependency of $E_{be,\pm}, E_{cf,\pm}$ are plotted in the diagram with green **curved** lines.

The asymptotic behaviors of $E_{be,\pm}, E_{cf,\pm}$ as $B/A \to 0$ and $B/A \to \infty$ are

$$\begin{array}{ccccc}
\frac{A}{2} + \frac{2B}{3} = \frac{1}{2}\left[-\frac{A}{2} + B + \frac{3A}{2}\left(1 + \frac{2B}{9A}\right)\right] & \xleftarrow[B/A\to 0]{} & E_{be,+} & \xrightarrow[B/A\to\infty]{} & \frac{1}{2}\left[-\frac{A}{2} + B + B\left(1 + \frac{A}{2B}\right)\right] = B \\
-A + \frac{B}{3} = \frac{1}{2}\left[-\frac{A}{2} + B - \frac{3A}{2}\left(1 + \frac{2B}{9A}\right)\right] & \xleftarrow[B/A\to 0]{} & E_{be,-} & \xrightarrow[B/A\to\infty]{} & \frac{1}{2}\left[-\frac{A}{2} + B - B\left(1 + \frac{A}{2B}\right)\right] = -\frac{A}{2} \\
\frac{A}{2} - \frac{2B}{3} = \frac{1}{2}\left[-\frac{A}{2} - B + \frac{3A}{2}\left(1 - \frac{2B}{9A}\right)\right] & \xleftarrow[B/A\to 0]{} & E_{cf,+} & \xrightarrow[B/A\to\infty]{} & \frac{1}{2}\left[-\frac{A}{2} - B + B\left(1 - \frac{A}{2B}\right)\right] = -\frac{A}{2} \\
-A - \frac{B}{3} = \frac{1}{2}\left[-\frac{A}{2} - B - \frac{3A}{2}\left(1 - \frac{2B}{9A}\right)\right] & \xleftarrow[B/A\to 0]{} & E_{cf,-} & \xrightarrow[B/A\to\infty]{} & \frac{1}{2}\left[-\frac{A}{2} - B - B\left(1 - \frac{A}{2B}\right)\right] = -B
\end{array}$$

In particular, as $B/A \to \infty$, $E_{be,+}$ will approach E_g, both $E_{cf,+}$ and $E_{be,-}$ will converge to $-A/2$, and $E_{cf,-}$ will approach E_h. These are the "five states" of high field as described in Fig 5.3's caption.