The Hamiltonian of the system is

$$H = \eta \boldsymbol{S} \cdot \boldsymbol{B} = \frac{\eta B \hbar}{2} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{B}}$$

where $\hat{\boldsymbol{B}}$ is the unit vector along $\boldsymbol{B}$'s direction, which is represented by the spherical angles (θ, ϕ).
Up to a global phase factor, the "upper" eigenket for $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{B}}$ is (see eq 3.70)

$$|\hat{\boldsymbol{B}}_+\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \\ e^{i\phi} \sin\frac{\theta}{2} \end{bmatrix} \tag{1}$$

then by definition of Berry's potential

$$\boldsymbol{A}_{\hat{\boldsymbol{B}}_+}(r, \theta, \phi) = i \langle \hat{\boldsymbol{B}}_+ | \boldsymbol{\nabla} | \hat{\boldsymbol{B}}_+ \rangle \tag{2}$$

Recall the gradient in spherical coordinate has representation

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r} \hat{\boldsymbol{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \hat{\boldsymbol{\phi}} \tag{3}$$

Then we have

$$\boldsymbol{\nabla}|\hat{\boldsymbol{B}}_+\rangle = \begin{bmatrix} -\frac{1}{2r} \sin\frac{\theta}{2} \hat{\boldsymbol{\theta}} \\ \\ \frac{1}{2r} e^{i\phi} \cos\frac{\theta}{2} \hat{\boldsymbol{\theta}} + \frac{1}{2r\cos\frac{\theta}{2}} i e^{i\phi} \hat{\boldsymbol{\phi}} \end{bmatrix} \qquad \Longrightarrow$$

$$\langle \hat{\boldsymbol{B}}_+ | \boldsymbol{\nabla} | \hat{\boldsymbol{B}}_+ \rangle = \begin{bmatrix} \cos\frac{\theta}{2} & e^{-i\phi} \sin\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2r} \sin\frac{\theta}{2} \hat{\boldsymbol{\theta}} \\ \\ \frac{1}{2r} e^{i\phi} \cos\frac{\theta}{2} \hat{\boldsymbol{\theta}} + \frac{1}{2r\cos\frac{\theta}{2}} i e^{i\phi} \hat{\boldsymbol{\phi}} \end{bmatrix} = \frac{i}{2r} \tan\frac{\theta}{2} \hat{\boldsymbol{\phi}} \tag{4}$$

which gives Berry's phase

$$\begin{aligned} \gamma_{\hat{\boldsymbol{B}}_+}(C) &= \oint_C \boldsymbol{A}_{\hat{\boldsymbol{B}}_+} \cdot d\boldsymbol{R} \\ &= \oint_C \boldsymbol{A}_{\hat{\boldsymbol{B}}_+} \cdot r \sin\theta \, d\hat{\boldsymbol{\phi}} \\ &= -\frac{1}{2} \tan\frac{\theta}{2} \sin\theta \cdot 2\pi = -2\pi \sin^2\frac{\theta}{2} \end{aligned} \tag{5}$$

which is exactly $-1/2$ times the solid angle given by the circle C, since the solid angle

$$\Omega = \frac{1}{r^2} \int_0^\theta 2\pi r \sin\theta' \cdot r \, d\theta' = 2\pi(1 - \cos\theta) = 4\pi \sin^2\frac{\theta}{2} \tag{6}$$

agreeing with eq 5.253.