The energy level given by the Dirac equation is (eq (8.150))

$$E = \frac{mc^2}{\left[1 + \dfrac{(Z\alpha)^2}{\left[\sqrt{\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2} + n'\right]^2}\right]^{1/2}} \tag{1}$$

which can be rewritten as

$$E = mc^2 \cdot A^{-1/2} \tag{2}$$

$$A = 1 + \frac{(Z\alpha)^2}{B^2} \tag{3}$$

$$B = \left[\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2\right]^{1/2} + n' \tag{4}$$

Expand $A^{-1/2}$ into power series

$$A^{-1/2} = 1 - \frac{1}{2}\frac{(Z\alpha)^2}{B^2} + \frac{1}{2!}\frac{3}{4}\frac{(Z\alpha)^4}{B^4} + O(Z^6\alpha^6) \tag{5}$$

Similar for B:

$$\begin{aligned} B &= \left(j+\frac{1}{2}\right)\left[1 - \frac{(Z\alpha)^2}{\left(j+\frac{1}{2}\right)^2}\right]^{1/2} + n' \\ &= j + \frac{1}{2} + n' - \frac{(Z\alpha)^2}{2\left(j+\frac{1}{2}\right)} + O(Z^4\alpha^4) \\ &= n - \frac{(Z\alpha)^2}{2\left(j+\frac{1}{2}\right)} + O(Z^4\alpha^4) \end{aligned} \tag{6}$$

Therefore

$$\begin{aligned} B^{-2} &= n^{-2}\left[1 + \frac{(Z\alpha)^2}{n\left(j+\frac{1}{2}\right)}\right] + O(Z^4\alpha^4) \\ &= \frac{1}{n^2} + \frac{(Z\alpha)^2}{n^3\left(j+\frac{1}{2}\right)} + O(Z^4\alpha^4) \end{aligned} \tag{7}$$

These can approximate $A^{-1/2}$ up to $(Z\alpha)^4$, by (5)

$$A^{-1/2} = 1 - \frac{1}{2}(Z\alpha)^2\left[\frac{1}{n^2} + \frac{(Z\alpha)^2}{n^3\left(j+\frac{1}{2}\right)}\right] + \frac{3}{8}\frac{(Z\alpha)^4}{n^4} + O(Z^6\alpha^6) \tag{8}$$

Then the energy from (2) is given by

$$E = mc^2 - \frac{1}{2}\frac{mc^2(Z\alpha)^2}{n^2} - \frac{1}{2}\frac{mc^2(Z\alpha)^4}{n^3\left(j+\frac{1}{2}\right)} + \frac{3}{8}\frac{mc^2(Z\alpha)^4}{n^4} + O(Z^6\alpha^6) \tag{9}$$

where the first term is rest energy, and the second term is the unperturbed energy $E_n^{(0)}$ (a negative quantity). We are now going to prove that the third and fourth term here together account for the combined calculation from section 5.3.1 (*Relativistic Correction to the Kinetic Energy*) and section 5.3.2 (*Spin-Orbit Interaction and Fine Structure*).

The energy shift from section 5.3.1 is given by equation (5.104b):

$$\Delta_{nl}^{(1)} = -\frac{1}{2}mc^2(Z\alpha)^4\left[-\frac{3}{4n^4}+\frac{1}{n^3\left(l+\frac{1}{2}\right)}\right] \tag{10}$$

and that from 5.3.2 is given by equation (5.125):

$$\begin{aligned}
\Delta_{nlj} &= -\frac{(Z\alpha)^2}{2nl(l+1)\left(l+\frac{1}{2}\right)}E_n^{(0)}\cdot\begin{cases} l & j=l+\frac{1}{2} \\ -(l+1) & j=l-\frac{1}{2}\end{cases} \\
&= \frac{mc^2(Z\alpha)^4}{4n^3l(l+1)\left(l+\frac{1}{2}\right)}\cdot\begin{cases} l & j=l+\frac{1}{2} \\ -(l+1) & j=l-\frac{1}{2}\end{cases} \\
&= \begin{cases} \dfrac{mc^2(Z\alpha)^4}{4n^3(l+1)\left(l+\frac{1}{2}\right)} & j=l+\frac{1}{2} \\ -\dfrac{mc^2(Z\alpha)^4}{4n^3l\left(l+\frac{1}{2}\right)} & j=l-\frac{1}{2}\end{cases}
\end{aligned} \tag{11}$$

Note that

- when $j=l+1/2$:

$$\begin{aligned}
-\frac{mc^2(Z\alpha)^4}{2n^3\left(l+\frac{1}{2}\right)}+\frac{mc^2(Z\alpha)^4}{4n^3(l+1)\left(l+\frac{1}{2}\right)} &= -\frac{mc^2(Z\alpha)^4}{n^3}\left[\frac{1}{2\left(l+\frac{1}{2}\right)}-\frac{1}{4(l+1)\left(l+\frac{1}{2}\right)}\right] \\
&= -\frac{mc^2(Z\alpha)^4}{n^3}\left[\frac{2(l+1)-1}{4(l+1)\left(l+\frac{1}{2}\right)}\right] \\
&= -\frac{mc^2(Z\alpha)^4}{2n^3(l+1)} \\
&= -\frac{mc^2(Z\alpha)^4}{2n^3\left(j+\frac{1}{2}\right)}
\end{aligned} \tag{12}$$

- when $j=l-1/2$:

$$\begin{aligned}
-\frac{mc^2(Z\alpha)^4}{2n^3\left(l+\frac{1}{2}\right)}-\frac{mc^2(Z\alpha)^4}{4n^3l\left(l+\frac{1}{2}\right)} &= -\frac{mc^2(Z\alpha)^4}{n^3}\left[\frac{1}{2\left(l+\frac{1}{2}\right)}+\frac{1}{4l\left(l+\frac{1}{2}\right)}\right] \\
&= -\frac{mc^2(Z\alpha)^4}{n^3}\left[\frac{2l+1}{4l\left(l+\frac{1}{2}\right)}\right] \\
&= -\frac{mc^2(Z\alpha)^4}{2n^3l} \\
&= -\frac{mc^2(Z\alpha)^4}{2n^3\left(j+\frac{1}{2}\right)}
\end{aligned} \tag{13}$$

This shows that $\Delta_{nl}^{(1)}+\Delta_{nlj}$ exactly matches the third and fourth term in (9).