Here we prove the integral representation of $j_l(x)$, i.e., Sakurai equation (6.105).

$$j_l(x) = \frac{1}{2i^l}\int_0^\pi e^{ix\cos\theta} P_l(\cos\theta)\sin\theta d\theta \tag{1}$$

Proof. Let $y = \cos\theta$, and define

$$u_l(x) = \frac{1}{2i^l}\int_{-1}^1 e^{ixy} P_l(y) dy \tag{2}$$

It's straightforward to see that $j_l(x) = u_l(x)$ for $l = 0, 1$:

$$u_0(x) = \frac{1}{2}\int_{-1}^1 e^{ixy} dy = \frac{1}{2ix}(e^{ix} - e^{-ix}) = \frac{\sin x}{x} = j_0(x) \tag{3}$$

$$\begin{aligned} u_1(x) &= \frac{1}{2i}\int_{-1}^1 e^{ixy} y dy = \frac{1}{2i}\frac{1}{ix}\int_{-1}^1 \frac{de^{ixy}}{dy} y dy \\ &= -\frac{1}{2x}\left[e^{ixy} y\Big|_{-1}^1 - \int_{-1}^1 e^{ixy} dy \right] \\ &= -\frac{\cos x}{x} + \frac{\sin x}{x^2} = j_1(x) \end{aligned} \tag{4}$$

To prove $u_l(x) = j_l(x)$ for any $l >= 2$, we will use the recurrence relation of $j_l(x)$ (which will be proved shortly)

$$(l+1) j_{l+1}(x) = l j_{l-1}(x) - (2l+1) j_l'(x) \tag{5}$$

and show that $u_l(x)$ satisfies the same relation

$$(l+1) u_{l+1}(x) = l u_{l-1}(x) - (2l+1) u_l'(x) \tag{6}$$

Once both (5) and (6) are proved, induction argument can be used to conclude that $u_l(x) = j_l(x)$ for any l.

First let's prove (5). Recall from the last notes, we have proved (equation (49))

$$g_l(x) \equiv \frac{j_l(x)}{(-x)^l} = \frac{1}{x} g_{l-1}'(x) \tag{7}$$

for any l.

Then (5) is equivalent to

$$\begin{aligned} (l+1)(-x)^{l+1} g_{l+1} &= l(-x)^{l-1} g_{l-1} - (2l+1)\left[(-x)^l g_l\right]' && \iff \\ (l+1)(-x)^{l+1}\left(\frac{1}{x} g_l'\right) &= l(-x)^{l-1} g_{l-1} - (2l+1)\left[(-x)^l g_l' - l(-x)^{l-1} g_l\right] && \text{(drop factor } (-x)^{l-1}) \quad \iff \\ (l+1) x g_l' &= l g_{l-1} - (2l+1)\left[-x g_l' - l g_l\right] && \iff \\ 0 &= l g_{l-1} + l x g_l' + l(2l+1) g_l && \text{(drop factor } l) \quad \iff \\ 0 &= g_{l-1} + \underbrace{(x g_l' + g_l)}_{(x g_l)' = g_{l-1}''} + 2l g_l && \iff \\ 0 &= g_{l-1} + g_{l-1}'' + 2l\left(\frac{1}{x} g_{l-1}'\right) \end{aligned} \tag{8}$$

which is exactly what we have proved in the last notes (equation (53)), so (5) is proved.

On the other hand (6) is equivalent to

$$\begin{aligned} \int_{-1}^1 e^{ixy}\left[\frac{l+1}{2i^{l+1}} P_{l+1}(y) - \frac{l}{2i^{l-1}} P_{l-1}(y) + \frac{2l+1}{2i^l}(iy) P_l(y)\right] dy = 0 \qquad \iff \\ \int_{-1}^1 e^{ixy}\frac{1}{2i^{l-1}}\left[-(l+1) P_{l+1}(y) - l P_{l-1}(y) + (2l+1) y P_l(y)\right] dy = 0 \end{aligned} \tag{9}$$

which is obviously true given the recurrence relation of Legendre polynomials (proved in earlier notes)

$$(l+1) P_{l+1}(y) = (2l+1) y P_l(y) - l P_{l-1}(y) \tag{10}$$

□

Moreover, since we know that $P_l(y)$'s form an orthogonal set of functions over $[-1,1]$, with normalizing constant

$$\int_{-1}^{1} P_l(y)P_l(y)dy = \frac{2}{2l+1} \tag{11}$$

then any function $f(y)$ on $[-1,1]$ should be decomposed into the sum

$$f(y) = \sum_l \left\langle \sqrt{\frac{2l+1}{2}}P_l(y), f(y) \right\rangle \cdot \sqrt{\frac{2l+1}{2}}P_l(y) \tag{12}$$

where the inner product $\langle u,v\rangle$ is defined by the integral

$$\langle u,v\rangle = \int_{-1}^{1} u^*(t)v(t)dt \tag{13}$$

Take $f(y) = e^{ixy}$, then we have

$$\begin{aligned} e^{ixy} &= \sum_l \frac{2l+1}{2}\left[\int_{-1}^{1} e^{ixt}\cdot P_l(t)dt\right]P_l(y) \\ &= \sum_l \frac{2l+1}{2}\cdot 2i^l j_l(x)P_l(y) \\ &= \sum_l (2l+1)i^l j_l(x)P_l(y) \end{aligned} \tag{14}$$

Or, equivalently, if we reinterpret y as $\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}$ and x as $|\boldsymbol{k}||\boldsymbol{r}| = kr$, we have just proved Sakurai (6.104)

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_l (2l+1)i^l j_l(kr)P_l(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}) \tag{15}$$