We shall prove the optical theorem

$$\mathrm{Im}[f(\boldsymbol{k},\boldsymbol{k})] = \frac{k\sigma_{\mathrm{tot}}}{4\pi} \tag{1}$$

Recall the scattering amplitude

$$f(\boldsymbol{k},\boldsymbol{k}_s) \equiv -\frac{mL^3}{2\pi\hbar^2}\left\langle \boldsymbol{k}_s\,|V|\,\psi^{(+)}\right\rangle \tag{2}$$

and the Lippmann-Schwinger equation

$$|\psi^{(+)}\rangle = |i\rangle + \frac{1}{E_i - H_0 + i\hbar\epsilon}V|\psi^{(+)}\rangle \tag{3}$$

Left multiply $\langle\psi^{(+)}|V$ to (3), we have

$$\langle\psi^{(+)}|V|\psi^{(+)}\rangle = \langle\psi^{(+)}|V|\boldsymbol{k}\rangle + \left\langle \psi^{(+)}\left| V\frac{1}{E_i - H_0 + i\epsilon}V\right|\psi^{(+)}\right\rangle \tag{4}$$

We then recognize the LHS of (4) as real since V is Hermitian. Then the imaginary part of the RHS gives

$$\begin{aligned}
-\mathrm{Im}\langle\psi^{(+)}|V|\boldsymbol{k}\rangle &= \mathrm{Im}\left\langle \psi^{(+)}\left| V\frac{1}{E_i - H_0 + i\epsilon}V\right|\psi^{(+)}\right\rangle && \Longrightarrow\\
\mathrm{Im}\langle\boldsymbol{k}|V|\psi^{(+)}\rangle &= \mathrm{Im}\left\langle \psi^{(+)}\left| V\frac{1}{E_i - H_0 + i\epsilon}V\right|\psi^{(+)}\right\rangle && \Longrightarrow\\
\mathrm{Im}[f(\boldsymbol{k},\boldsymbol{k})] &= -\frac{mL^3}{2\pi\hbar^2}\cdot\mathrm{Im}\underbrace{\left\langle \psi^{(+)}\left| V\frac{1}{E_i - H_0 + i\epsilon}V\right|\psi^{(+)}\right\rangle}_{A} && (5)
\end{aligned}$$

Before proceeding further, let's first prove the following useful claim:
Claim: For real x

$$\lim_{\epsilon\to 0}\mathrm{Im}\left(\frac{1}{x+i\epsilon}\right) = -\pi\delta(x) \tag{6}$$

Proof. Indeed, if we let $g(x) \equiv 1/(x+i\epsilon)$, then

$$\begin{aligned}
\int_{-\infty}^{\infty}\mathrm{Im}[g(x)]dx &= \frac{1}{2i}\int_{-\infty}^{\infty}[g(x)-g^*(x)]\,dx = \frac{1}{2i}\int_{-\infty}^{\infty}\left(\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon}\right)dx\\
&= \int_{-\infty}^{\infty}\frac{-\epsilon}{x^2+\epsilon^2}dx && (\text{let } x = \epsilon\tan\theta)\\
&= -\int_{-\pi/2}^{\pi/2}\frac{\epsilon^2 d(\tan\theta)}{\epsilon^2/\cos^2\theta}\\
&= -\int_{-\pi/2}^{\pi/2}d\theta\\
&= -\pi && (7)
\end{aligned}$$

Now (7), together with the fact that $\lim_{\epsilon\to 0}\mathrm{Im}[g(x)] = 0$ for all $x \neq 0$, gives (7) by definition of the δ-function. □

Look back at (5), we now insert two complete energy eigenkets F, F' into A:

$$\begin{aligned}
\mathrm{Im}(A) &= \mathrm{Im}\left[\sum_{F,F'}\langle\psi^{(+)}|V|F\rangle\left\langle F\left|\frac{1}{E_i - H_0 + i\epsilon}\right|F'\right\rangle\langle F'|V|\psi^{(+)}\rangle\right]\\
&= \mathrm{Im}\left[\sum_{F}\langle\psi^{(+)}|V|F\rangle\left(\frac{1}{E_i - F + i\epsilon}\right)\langle F|V|\psi^{(+)}\rangle\right]\\
&= \mathrm{Im}\left[\sum_{F}\left(\frac{1}{E_i - F + i\epsilon}\right)\left|\langle F|V|\psi^{(+)}\rangle\right|^2\right] && (\text{by (6)})\\
&\xrightarrow[\epsilon\to 0]{} -\pi\sum_F\delta(E_i - F)\left|\langle F|V|\psi^{(+)}\rangle\right|^2 && (8)
\end{aligned}$$

Then (5) becomes

$$\begin{aligned}\mathrm{Im}[f(\boldsymbol{k},\boldsymbol{k})] &= -\frac{mL^3}{2\pi\hbar^2}\cdot\left[-\pi\sum_F \delta(E_i - F)\left|\langle F|V|\psi^{(+)}\rangle\right|^2\right] \\ &= \frac{mL^3}{2\hbar^2}\cdot\left[\sum_F \delta(E_i - F)\left|\langle F|V|\psi^{(+)}\rangle\right|^2\right] \\ &= \frac{mL^3}{2\hbar^2}\cdot\left[\sum_{\boldsymbol{k}_s}\delta\left(E_i - \frac{\hbar^2 k_s^2}{2m}\right)\left|\langle \boldsymbol{k}_s|V|\psi^{(+)}\rangle\right|^2\right]\end{aligned} \tag{9}$$

The sum in (9) can be seen to be done over all the grid points (i.e., $\Delta n_x = \Delta n_y = \Delta n_z = 1$), thus with large L, the sum can be replaced by $\boldsymbol{k}_s$-space integral

$$\sum_{\boldsymbol{k}} = \sum_{n_x,n_y,n_z}\Delta n_x \Delta n_y \Delta n_z = \sum_{k_x,k_y,k_z}\Delta k_x \Delta k_y \Delta k_z \frac{L^3}{(2\pi)^3} = \int_{\boldsymbol{k}} d^3\boldsymbol{k}\frac{L^3}{(2\pi)^3}$$

which makes

$$\begin{aligned}\mathrm{Im}[f(\boldsymbol{k},\boldsymbol{k})] &= \frac{mL^3}{2\hbar^2}\frac{L^3}{8\pi^3}\int_{\boldsymbol{k}_s} d^3\boldsymbol{k}_s \delta\left(E_i - \frac{\hbar^2 k_s^2}{2m}\right)\left|\langle \boldsymbol{k}_s|V|\psi^{(+)}\rangle\right|^2 \\ &= \frac{\hbar^2}{4\pi m}\int_{\boldsymbol{k}_s} d^3\boldsymbol{k}_s \delta\left(E_i - \frac{\hbar^2 k_s^2}{2m}\right)\underbrace{\left(\frac{m^2L^6}{4\pi^2\hbar^4}\left|\langle \boldsymbol{k}_s|V|\psi^{(+)}\rangle\right|^2\right)}_{d\sigma/d\Omega} \qquad \text{(see (6.58), (6.59))} \\ &= \frac{\hbar^2}{4\pi m}\int d\Omega \int_{k_s=0}^{\infty} k_s^2 dk_s \delta\left(E_i - \frac{\hbar^2 k_s^2}{2m}\right)\frac{d\sigma}{d\Omega}\end{aligned} \tag{10}$$

We have to be careful integrating with δ-function. Let $\kappa = \hbar^2 k_s^2/2m$ (hence $k_s = \sqrt{2m/\hbar^2}\kappa^{1/2}$),

$$\begin{aligned}\int_{k_s=0}^{\infty} k_s^2 dk_s \delta\left(E_i - \frac{\hbar^2 k_s^2}{2m}\right)h(k_s) &= \int_{\kappa=0}^{\infty}\frac{2m}{\hbar^2}\kappa\sqrt{\frac{2m}{\hbar^2}}\frac{d\kappa}{2\sqrt{\kappa}}\delta(E_i-\kappa)h(\kappa) \\ &= \int_{\kappa=0}^{\infty}\frac{m}{\hbar^2}\sqrt{\frac{2m}{\hbar^2}}\sqrt{\kappa}d\kappa\delta(E_i-\kappa)h(\kappa) \\ &= \frac{m}{\hbar^2}\sqrt{\frac{2m}{\hbar^2}}\sqrt{\kappa}h(\kappa)\Big|_{\kappa=E_i}\end{aligned} \tag{11}$$

which finally turns (10) into

$$\begin{aligned}\mathrm{Im}[f(\boldsymbol{k},\boldsymbol{k})] &= \frac{\hbar^2}{4\pi m}\int d\Omega\frac{m}{\hbar^2}\sqrt{\frac{2m}{\hbar^2}}\sqrt{\frac{\hbar^2k^2}{2m}}\frac{d\sigma}{d\Omega} \\ &= \frac{k\sigma_{\mathrm{tot}}}{4\pi}\end{aligned} \tag{12}$$