The outside radial wavefunction is given by eq (6.138):

$$\begin{aligned} A_0(r) &= e^{i\delta_0}\left[\cos\delta_0 j_0(kr) - \sin\delta_0 n_0(kr)\right] \\ &= e^{i\delta_0}\left(\cos\delta_0 \frac{\sin kr}{kr} + \sin\delta_0 \frac{\cos kr}{kr}\right) \qquad \Longrightarrow \\ u_0(r) = rA_0(r) &= \frac{e^{i\delta_0}}{k}\left(\cos\delta_0 \sin kr + \sin\delta_0 \cos kr\right) \end{aligned} \tag{1}$$

On the other hand, inside the sphere $r < R$, we have $V = 0$, but it includes the origin, so only $j_0(kr)$ contributes,

$$A_0(r) = C j_0(kr) = C\frac{\sin kr}{kr} \qquad \Longrightarrow \qquad u_0(r) = \frac{C}{k}\sin kr \tag{2}$$

Wavefunction is continuous at $r = R$, this implies

$$\begin{aligned} \frac{C}{k}\sin kR &= \frac{e^{i\delta_0}}{k}\left(\cos\delta_0 \sin kR + \sin\delta_0 \cos kR\right) \qquad \Longrightarrow \\ C &= e^{i\delta_0}\left(\cos\delta_0 + \sin\delta_0 \cot kR\right) \end{aligned} \tag{3}$$

The first-order derivative $u_0'(r)$ is not continuous at R due to the δ-function in V. The standard treatment is to integrate the radial Schrödinger equation eq (6.141) in the range $[R-\epsilon, R+\epsilon]$, and then take the limit $\epsilon \to 0$:

$$\begin{aligned} \int_{R-\epsilon}^{R+\epsilon} dr \left\{\frac{d^2u_0}{dr^2} + \left[k^2 - \gamma\delta(r-R) + \frac{l(l+1)}{r^2}\right]u_0\right\} &= 0 \qquad \Longrightarrow \\ \left[u_0'(R+\epsilon) - u_0'(R-\epsilon)\right] - \gamma u_0(R) &= 0 \end{aligned} \tag{4}$$

$u_0'(R+\epsilon)$ and $u_0'(R-\epsilon)$ can be computed from (1) and (2), which turns (4) into

$$\begin{aligned} \left[\frac{e^{i\delta_0}}{k}\left(\cos\delta_0 k\cos kR - \sin\delta_0 k \sin kR\right) - \frac{C}{k}k\cos kR\right] - \frac{\gamma C}{k}\sin kR &= 0 \qquad \Longrightarrow \\ e^{i\delta_0}\left(\cos\delta_0\cos kR - \sin\delta_0 \sin kR\right) - C\left(\cos kR + \frac{\gamma}{k}\sin kR\right) &= 0 \qquad \Longrightarrow \\ e^{i\delta_0}\left(\cos\delta_0\cos kR - \sin\delta_0 \sin kR\right) - e^{i\delta_0}\left(\cos\delta_0 + \sin\delta_0\cot kR\right)\left(\cos kR + \frac{\gamma}{k}\sin kR\right) &= 0 \qquad \Longrightarrow \\ -\cos\delta_0\left(\frac{\gamma}{k}\right)\sin kR - \sin\delta_0\left[\frac{1 + \left(\frac{\gamma}{k}\right)\sin kR\cos kR}{\sin kR}\right] &= 0 \end{aligned} \tag{5}$$

which gives the equation for the phase shift δ_0:

$$\tan\delta_0 = \frac{-\left(\frac{\gamma}{k}\right)\sin^2 kR}{1 + \left(\frac{\gamma}{k}\right)\sin kR\cos kR} \qquad \text{or} \qquad \cot\delta_0 = \frac{-\left[1 + \left(\frac{\gamma}{k}\right)\sin kR\cos kR\right]}{\left(\frac{\gamma}{k}\right)\sin^2 kR} \tag{6}$$

Now we examine the nodes of $\cot\delta_0$ under the assumption that $\gamma R \gg 1$. The numerator of $\cot\delta_0$ is

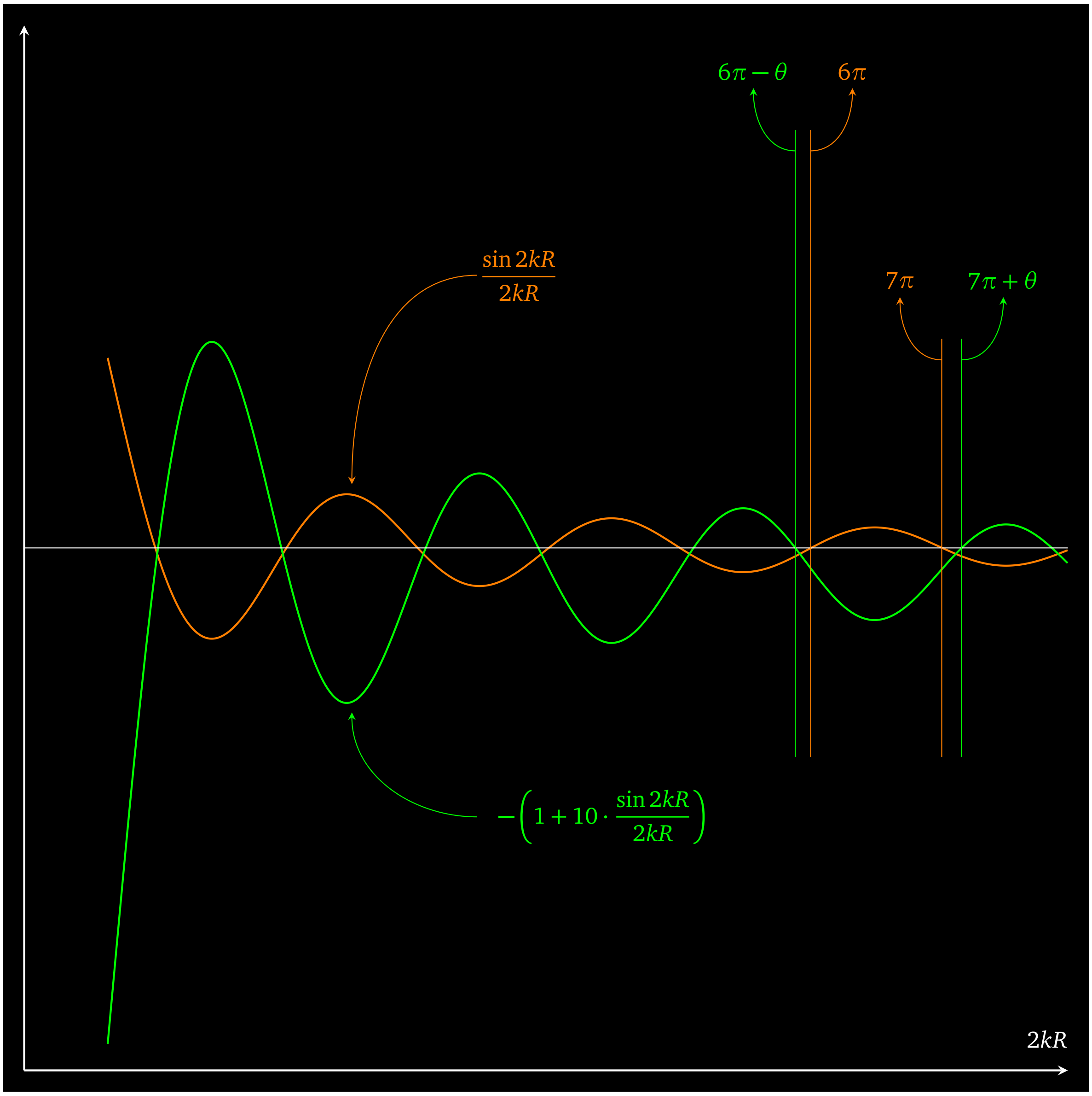
$$\text{numerator}(\cot\delta_0) = -\left(1 + \gamma R\frac{\sin 2kR}{2kR}\right) \tag{7}$$

The diagram below plots (7) (with $\gamma R = 10$) together with $\sin 2kR/2kR$ which has nodes at $2kR = n\pi$. It is understood that when $\gamma R \gg 1$, the nodes of (7) should be close to the nodes of $\sin 2kR/2kR$. In more details, when (7) vanishes, we have

$$\frac{\sin 2kR}{2kR} = -\frac{1}{\gamma R} \tag{8}$$

which can be satisfied by

$$2kR = \begin{cases} n\pi + \theta & n \text{ odd} \\ n\pi - \theta & n \text{ even} \end{cases} \tag{9}$$



Therefore

when $n=2p+1$:

$$\frac{1}{\gamma R}=\frac{\sin\theta}{n\pi+\theta}\approx\frac{\theta}{n\pi+\theta} \qquad \Longrightarrow$$

$$\theta\approx\frac{n\pi}{\gamma R-1} \qquad \text{or}$$

$$k=\frac{n\pi}{2}\frac{\gamma}{\gamma R-1}\approx\frac{n\pi}{2R}\left(1+\frac{1}{\gamma R}\right)=\frac{(2p+1)\pi}{2R}\left(1+\frac{1}{\gamma R}\right) \tag{10}$$

when $n=2p$:

$$\frac{1}{\gamma R}=\frac{\sin\theta}{n\pi-\theta}\approx\frac{\theta}{n\pi-\theta} \qquad \Longrightarrow$$

$$\theta\approx\frac{n\pi}{\gamma R+1} \qquad \text{or}$$

$$k=\frac{n\pi}{2}\frac{\gamma}{\gamma R+1}\approx\frac{n\pi}{2R}\left(1-\frac{1}{\gamma R}\right)=\frac{p\pi}{R}\left(1-\frac{1}{\gamma R}\right) \tag{11}$$

Both cases would correspond to a sharp increase in the s-wave scattering cross section, but only the one that corresponds to the rising δ_0, in which case $\cot\delta_0$ will cross zero from the positive side, i.e., the $n=2p$ case, can be physically attributed to the *quasi-bound state*.

Recall the text's *Editor's Note* on page 413: "Such a sharp rise in the phase shift is, in the time-independent Schrödinger equation, associated with a delay of the emergence of the trapped particles, rather than an unphysical advance, as would be the case for a sharp decrease through $\pi/2$".

To make the text's point clear "Let's call such a state a quasi-bound state because it would be an honest bound state if the barrier were infinitely high": if $V = \infty$ at $r = R$ and zero elsewhere, the $l = 0$ partial wave has been solved, with bound energies given by equation 3.287, which is exactly the same as what (11) will produce if $\gamma = \infty$.

For the last part, by (6),

$$\begin{aligned}
\cot\delta_0 &= -\frac{1}{\gamma}\frac{k}{\sin^2 kR} - \cot kR & \Longrightarrow \\
\frac{d\cot\delta_0}{dk} &= -\frac{1}{\gamma}\left[\frac{1}{\sin^2 kR} + k\cdot\frac{-2R\cos kR}{\sin^3 kR}\right] + \frac{R}{\sin^2 kR} & \qquad (12)
\end{aligned}$$

Recall that at resonance (11)

$$\sin kR \approx (-1)^{p+1}\frac{p\pi}{\gamma R} \qquad \cos kR \approx (-1)^p \tag{13}$$

which turns (12) into

$$\begin{aligned}
\frac{d\cot\delta_0}{dk} &= -\frac{1}{\gamma}\left[\left(\frac{\gamma R}{p\pi}\right)^2 - 2kR\frac{(-1)^p}{(-1)^{3p+3}}\left(\frac{\gamma R}{p\pi}\right)^3\right] + R\left(\frac{\gamma R}{p\pi}\right)^2 \\
&= -\frac{1}{\gamma}\left[\left(\frac{\gamma R}{p\pi}\right)^2 + 2kR\left(\frac{\gamma R}{p\pi}\right)^3\right] + R\left(\frac{\gamma R}{p\pi}\right)^2 \\
&\approx \frac{-\gamma^2R^3 - \gamma R^2}{(p\pi)^2} \approx -\frac{\gamma^2 R^3}{(p\pi)^2} & \Longrightarrow \qquad (14) \\
\frac{d\cot\delta_0}{dE} &= \frac{d\cot\delta_0}{dk}\frac{dk}{dE} = -\frac{\gamma^2R^3}{(p\pi)^2}\frac{m}{\hbar^2 k} \approx -\frac{\gamma^2 R^4 m}{\hbar^2 (p\pi)^3} & \qquad (15)
\end{aligned}$$

Therefore

$$\Gamma = \frac{-2}{d\cot\delta_0/dE\Big|_{E=E_r}} \propto \frac{1}{\gamma^2}$$