1. Using

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \qquad p = i\sqrt{\frac{m\hbar\omega}{2}}\left(-a + a^\dagger\right)$$

we get

$$\begin{aligned} L_1 = x_2 p_3 - x_3 p_2 &= \frac{i\hbar}{2}\left[\left(a_2 + a_2^\dagger\right)\left(-a_3 + a_3^\dagger\right) - \left(a_3 + a_3^\dagger\right)\left(-a_2 + a_2^\dagger\right)\right] \\ &= i\hbar\left(a_2 a_3^\dagger - a_3 a_2^\dagger\right) \end{aligned}$$

then it follows

$$L_i = i\hbar\epsilon_{ijk} a_j a_k^\dagger$$

To get $\mathbf{L}^2$, note that

$$\begin{aligned} L_1^2 &= -\hbar^2\left(a_2 a_3^\dagger - a_3 a_2^\dagger\right)^2 \\ &= -\hbar^2\left(a_2 a_3^\dagger a_2 a_3^\dagger + a_3 a_2^\dagger a_3 a_2^\dagger - a_2 a_3^\dagger a_3 a_2^\dagger - a_3 a_2^\dagger a_2 a_3^\dagger\right) \qquad (\text{use } a^\dagger a = N, [a, a^\dagger] = 1) \\ &= -\hbar^2\left(a_2 a_3^\dagger a_2 a_3^\dagger + a_3 a_2^\dagger a_3 a_2^\dagger - N_3(N_2+1) - N_3(N_2+1)\right) \\ &= \hbar^2\left(2N_2N_3 + N_2 + N_3 - a_2 a_3^\dagger a_2 a_3^\dagger - a_3 a_2^\dagger a_3 a_2^\dagger\right) \end{aligned}$$

Then adding similar terms for L_2^2, L_3^2, we have

$$\mathbf{L}^2 = \hbar^2\left[2(N_2N_3 + N_3N_1 + N_1N_2) + 2N - a_k^\dagger a_k^\dagger a_j a_j + a_j^\dagger a_j^\dagger a_j a_j\right]$$

Since

$$a_j^\dagger a_j^\dagger a_j a_j = a_j^\dagger(a_j a_j^\dagger - [a_j, a_j^\dagger])a_j = N_j^2 - N_j$$

it follows that

$$\mathbf{L}^2 = \hbar^2\left[N(N+1) - a_k^\dagger a_k^\dagger a_j a_j\right]$$

2. For $|qlm\rangle = |01m\rangle$, we have $N = 2q + l = 1$. Then in the Cartesian basis, the 3 degenerated states for $N = 1$ are

$$|n_x n_y n_z\rangle \in \{|100\rangle, |010\rangle, |001\rangle\}$$

where the "1" indicates the eigenstate for the harmonic oscillator corresponding to $n = 1$.
Let

$$|01m\rangle = \alpha_1|100\rangle + \alpha_2|010\rangle + \alpha_3|001\rangle$$

Apply the eigenequation $L_z|01m\rangle = m\hbar|01m\rangle$, we get

$$\begin{aligned} L_z|01m\rangle &= i\hbar\left(a_1 a_2^\dagger - a_2 a_1^\dagger\right)(\alpha_1|100\rangle + \alpha_2|010\rangle + \alpha_3|001\rangle) \\ &= i\hbar(\alpha_1|010\rangle - \alpha_2|100\rangle) \\ &= m\hbar\left(\alpha_1|100\rangle + \alpha_2|010\rangle + \alpha_3|001\rangle\right) \qquad \Longrightarrow \\ i\alpha_1 = m\alpha_2 &\qquad -i\alpha_2 = m\alpha_1 \qquad m\alpha_3 = 0 \end{aligned}$$

So

$$\begin{aligned} |01m_{=1}\rangle &= C_1\left(|100\rangle + i|010\rangle\right) \\ |01m_{=0}\rangle &= C_0|001\rangle \\ |01m_{m=-1}\rangle &= C_{-1}\left(|100\rangle - i|010\rangle\right) \end{aligned}$$

For $q = 0$, by equation (3.295), $f(\rho) = a_0$, thus $u(\rho) \propto \rho^{l+1}e^{-\rho^2/2}$, with $\rho = r/\sqrt{h/m\omega} \equiv r/a$, and the radial wavefunction is

$$R(r) = \frac{u(r)}{r} \propto re^{-\frac{r^2}{2a^2}}$$

Of course the directional wavefunctions are the spherical harmonics

$$Y_1^{-1}(\theta,\phi) \propto e^{-i\phi}\sin\theta = \frac{x-iy}{r}$$
$$Y_1^{0}(\theta,\phi) \propto \cos\theta = \frac{z}{r}$$
$$Y_1^{1}(\theta,\phi) \propto e^{i\phi}\sin\theta = \frac{x+iy}{r}$$

Now coming back to the $|n_x n_y n_z\rangle$ basis, we shall recover the radial function and spherical harmonics. Indeed, if we recall equation (2.150-2.152), we know for each dimension $\tau \in \{x, y, z\}$,

$$\langle\tau|0\rangle \propto e^{-\frac{\tau^2}{2a^2}}$$
$$\langle\tau|1\rangle \propto \left(\tau - a^2\frac{d}{d\tau}\right)e^{-\frac{\tau^2}{2a^2}} \propto \tau e^{-\frac{\tau^2}{2a^2}}$$

Multiplying all 3 dimensions together, we have

$$\langle\mathbf{x}|100\rangle \propto xe^{-\frac{x^2+y^2+z^2}{2a^2}} = xe^{-\frac{r^2}{2a^2}}$$
$$\langle\mathbf{x}|010\rangle \propto ye^{-\frac{r^2}{2a^2}}$$
$$\langle\mathbf{x}|001\rangle \propto ze^{-\frac{r^2}{2a^2}}$$

Thus we have

$$\langle\mathbf{x}|01m_{=1}\rangle \propto \langle\mathbf{x}|100\rangle + i\langle\mathbf{x}|010\rangle = re^{-\frac{r^2}{2a^2}}\cdot\frac{x+iy}{r}$$
$$\langle\mathbf{x}|01m_{=0}\rangle \propto \langle\mathbf{x}|001\rangle = re^{-\frac{r^2}{2a^2}}\cdot\frac{z}{r}$$
$$\langle\mathbf{x}|01m_{m=-1}\rangle \propto \langle\mathbf{x}|100\rangle - i\langle\mathbf{x}|010\rangle = re^{-\frac{r^2}{2a^2}}\cdot\frac{x-iy}{r}$$

which is exactly the product form of radial wavefunction and spherical harmonics.

3. In this case, $q=1, l=m=0$, which gives $N=2$, which has 6-fold degeneracy:

$$|n_x n_y n_z\rangle \in \{|200\rangle, |020\rangle, |002\rangle, |011\rangle, |101\rangle, |110\rangle\}$$

Let

$$|qlm\rangle = \alpha_1|200\rangle + \alpha_2|020\rangle + \alpha_3|002\rangle + \alpha_4|011\rangle + \alpha_5|101\rangle + \alpha_6|110\rangle$$

then the L_z eigenequation gives

$$\begin{aligned}
L_z|qlm\rangle &= i\hbar\left(a_1 a_2^\dagger - a_2 a_1^\dagger\right)\left(\alpha_1|200\rangle + \alpha_2|020\rangle + \alpha_3|002\rangle + \alpha_4|011\rangle + \alpha_5|101\rangle + \alpha_6|110\rangle\right)\\
&= i\hbar\left[(\alpha_1-\alpha_2)|110\rangle - \alpha_4|101\rangle + \alpha_5|011\rangle + \alpha_6|020\rangle - \alpha_6|200\rangle\right]\\
&= 0\hbar\left(\alpha_1|200\rangle + \alpha_2|020\rangle + \alpha_3|002\rangle + \alpha_4|011\rangle + \alpha_5|101\rangle + \alpha_6|110\rangle\right) \qquad \Longrightarrow\\
\alpha_4 = \alpha_5 = \alpha_6 = 0 &\qquad \alpha_1 = \alpha_2 \qquad \alpha_3 = \text{arbitrary value}
\end{aligned}$$

For the $\mathbf{L}^2$ eigenequation, notice

$$\begin{aligned}
&N(N+1)|200\rangle = 2\cdot 3|200\rangle\\
&a_k^\dagger a_k^\dagger a_j a_j|200\rangle = a_k^\dagger a_k^\dagger a_1 a_1|200\rangle = a_k^\dagger a_k^\dagger\sqrt{2}|000\rangle = \sqrt{2}\cdot\sqrt{2}(|200\rangle + |020\rangle + |002\rangle)
\end{aligned}$$

and similarly for $|020\rangle, |002\rangle$ states, we have

$$\begin{aligned}
\mathbf{L}^2|qlm\rangle &= \hbar^2\left[N(N+1) - a_k^\dagger a_k^\dagger a_j a_j\right]\left[\alpha_1(|200\rangle + |020\rangle) + \alpha_3|002\rangle\right]\\
&= \hbar^2\left[6\alpha_1(|200\rangle + |020\rangle) + 6\alpha_3|002\rangle\right] - \hbar^2\left[(4\alpha_1 + 2\alpha_3)(|200\rangle + |020\rangle + |002\rangle)\right]\\
&= \hbar^2\left[(2\alpha_1 - 2\alpha_3)(|200\rangle + |020\rangle) + (4\alpha_3 - 4\alpha_1)|002\rangle\right]\\
&= 0\cdot(0+1)|qlm\rangle \qquad \Longrightarrow\\
\alpha_1 = \alpha_2 &= \alpha_3 \qquad \Longrightarrow\\
|qlm\rangle &\propto |200\rangle + |020\rangle + |002\rangle
\end{aligned}$$

In this case, $\lambda = 4q + 2l + 3 = 7$, and we expect $f(\rho)$ to have form $f(\rho) = a_2\rho^2 + a_0$, where by equation (3.298)

$$a_2 = -\frac{2}{3}a_0$$

Then the radial wavefunction becomes

$$R(r) \propto r^0 e^{-\frac{r^2}{2a^2}} \left(-\frac{2}{3}\frac{r^2}{a^2} + 1\right)$$

and the directional wavefunction is just Y_0^0, i.e., a constant.

On the other hand, from equation (2.152)

$$\langle\tau|2\rangle \propto \left(\tau - a^2\frac{d}{d\tau}\right)^2 e^{-\frac{\tau^2}{2a^2}} \propto \left(\tau - a^2\frac{d}{d\tau}\right)\left(\tau e^{-\frac{\tau^2}{2a^2}}\right) = \left(2\tau^2 - a^2\right)e^{-\frac{r^2}{2a^2}}$$

Then

$$\begin{aligned}\langle\mathbf{x}|qlm\rangle &\propto \langle\mathbf{x}|200\rangle + \langle\mathbf{x}|020\rangle + \langle\mathbf{x}|002\rangle \\ &\propto \left(2r^2 - 3a^2\right)e^{-\frac{r^2}{2a^2}}\end{aligned}$$

which agrees with above.

4. omitted, no additional insights to be gained.