1. **Legendre Polynomials**

   (a) **Generating Function**
   Let
   $$g(x,t) = (1-2xt+t^2)^{-1/2}$$
   be expanded into Taylor series around $t=0$
   $$g(x,t) = \sum_{l=0}^{\infty} P_l(x)t^l \tag{1}$$
   then
   $$P_l(x) = \frac{1}{l!}\left.\frac{\partial^l g(x,t)}{\partial t^l}\right|_{t=0}$$
   is called the Legendre Polynomials (of degree $l$). The fact that $P_l(x)$ is a polynomial of degree $l$ will be clear after showing the recurrence relation below.

   (b) **Recurrence Relations**
   Take the derivative of (1) with respect to $t$
   $$\begin{aligned}\frac{\partial g(x,t)}{\partial t} &= \sum_{l=1}^{\infty} lP_l(x)t^{l-1} = \sum_{l=0}^{\infty}(l+1)P_{l+1}(x)t^l\\ &= -\frac{1}{2}(1-2xt+t^2)^{-3/2}(2t-2x)\\ &= \frac{x-t}{1-2xt+t^2}g(x,t) = \frac{x-t}{1-2xt+t^2}\sum_{l=0}^{\infty}P_l(x)t^l \qquad \Longrightarrow\\ (x-t)\sum_{l=0}^{\infty}P_l(x)t^l &= (1-2xt+t^2)\sum_{l=0}(l+1)P_{l+1}(x)t^l \qquad \Longrightarrow\\ xP_l - P_{l-1} &= (l+1)P_{l+1} - 2xlP_l + (l-1)P_{l-1} \qquad \Longrightarrow\\ (l+1)P_{l+1} &= (2l+1)xP_l - lP_{l-1} \end{aligned} \tag{2}$$
   With (2) and together with the fact that
   $$P_0(x) = 1 \qquad P_1(x) = x$$
   We know $P_l(x)$ is a polynomial of degree $l$.
   Rewrite (2) with $l \to l-1$,
   $$lP_l - (2l-1)xP_{l-1} + (l-1)P_{l-2} = 0 \tag{3}$$

   (c) **Differential Equation**
   Now take the derivative of (1) with respect to $x$
   $$\begin{aligned}\frac{\partial g(x,t)}{\partial x} &= \sum_{l=0}^{\infty}P_l'(x)t^l\\ &= -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t)\\ &= \frac{t}{1-2xt+t^2}g(x,t) = \frac{t}{1-2xt+t^2}\sum_{l=0}^{\infty}P_l(x)t^l \qquad \Longrightarrow\\ t\sum_{l=0}^{\infty}P_l(x)t^l &= (1-2xt+t^2)\sum_{l=0}^{\infty}P_l'(x)t^l \qquad \Longrightarrow\\ P_{l-1} &= P_l' - 2xP_{l-1}' + P_{l-2}'\end{aligned} \tag{4}$$

Take the derivative of (3):

$$lP_l' - (2l-1)xP_{l-1}' - (2l-1)P_{l-1} + (l-1)P_{l-2}' = 0 \tag{5}$$

Multiply $l-1$ on (4):

$$-(l-1)P_{l-1} + (l-1)P_l' - 2x(l-1)P_{l-1}' + (l-1)P_{l-2}' = 0 \tag{6}$$

Subtract (6) from (5):

$$P_l' - lP_{l-1} - xP_{l-1}' = 0 \tag{7}$$

Multiply $l$ on (4):

$$lP_l' - 2lxP_{l-1}' + lP_{l-2}' - lP_{l-1} = 0 \tag{8}$$

Subtract (8) from (5):

$$\begin{aligned} xP_{l-1}' - (l-1)P_{l-1} - P_{l-2}' &= 0 \qquad \text{(substitute } l \to l+1\text{)} \\ xP_l' - lP_l - P_{l-1}' &= 0 \end{aligned} \tag{9}$$

Insert (9) into (7):

$$P_l' - lP_{l-1} - x(xP_l' - lP_l) = (1-x^2)P_l' - lP_{l-1} + lxP_l = 0 \tag{10}$$

Take the derivative of (10)

$$(1-x^2)P_l'' - 2xP_l' - lP_{l-1}' + lxP_l' + lP_l = 0 \tag{11}$$

Insert (9) into (11)

$$\begin{aligned} (1-x^2)P_l'' - 2xP_l' - l(xP_l' - lP_l) + lxP_l' + lP_l = 0 \qquad &\Longrightarrow \\ (1-x^2)P_l'' - 2xP_l' + l(l+1)P_l = 0 & \end{aligned} \tag{12}$$

(12) is the differential equation the Legendre Polynomials satisfy, which can also be equivalently expressed as

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] + l(l+1)P_l = 0 \tag{13}$$

(d) **Orthonormality**

Multiply $P_k$ to (13), we get

$$P_k\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right] + l(l+1)P_kP_l = 0 \tag{14}$$

By symmetry between $k$ and $l$

$$P_l\frac{d}{dx}\left[(1-x^2)\frac{dP_k}{dx}\right] + k(k+1)P_kP_l = 0 \tag{15}$$

Subtract, then integrate over $[-1,1]$, we have

$$\int_{-1}^{1}\left\{P_k\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right] - P_l\frac{d}{dx}\left[(1-x^2)\frac{dP_k}{dx}\right]\right\}dx = [k(k+1) - l(l+1)]\int_{-1}^{1}P_kP_l\,dx \tag{16}$$

The LHS is easily shown to vanish using integration by parts, which means

$$\int_{-1}^{1}P_kP_l\,dx = 0 \qquad \text{if } k \neq l \tag{17}$$

Multiply (3) by $P_l$:

$$\begin{aligned} lP_l^2 - (2l-1)xP_{l-1}P_l + (l-2)P_{l-2}P_l = 0 \qquad &\Longrightarrow \\ l\int_{-1}^{1}P_l^2\,dx = (2l-1)\int_{-1}^{1}xP_{l-1}P_l\,dx & \end{aligned} \tag{18}$$

Multiply (2) by $P_{l-1}$:

$$\begin{aligned}(l+1)P_{l+1}P_{l-1} &= (2l+1)xP_lP_{l-1} - lP_{l-1}^2 \qquad \Longrightarrow \\ l\int_{-1}^{1} P_{l-1}^2 dx &= (2l+1)\int_{-1}^{1} xP_lP_{l-1}dx \end{aligned} \tag{19}$$

Compare (18) and (19) we know

$$\int_{-1}^{1} P_l^2 dx = \frac{2l-1}{2l+1}\int_{-1}^{1} P_{l-1}^2 dx = \frac{1}{2l+1}\int_{-1}^{1} P_0 dx = \frac{2}{2l+1} \tag{20}$$

2. **Associated Legendre Functions**

   (a) **Definition**
   Now for $0 \le m \le l$, define the Associated Legendre Functions

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m P_l(x)}{dx^m} \tag{21}$$

   (b) **Differential Equation**
   First we claim that $P_l^m(x)$ satisfies the differential equation

$$(1-x^2)\frac{d^2P_l^m}{dx^2} - 2x\frac{dP_l^m}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0 \tag{22}$$

   To see this, first define

$$U_l^m(x) = \frac{d^m P_l(x)}{dx^m}$$

   Then

$$\begin{aligned}\frac{dP_l^m}{dx} &= (1-x^2)^{m/2}\frac{dU_l^m}{dx} + \frac{m}{2}(1-x^2)^{m/2-1}(-2x)U_l^m \\ &= (1-x^2)^{m/2}\frac{dU_l^m}{dx} - mx(1-x^2)^{m/2-1}U_l^m \end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d^2P_l^m}{dx^2} &= (1-x^2)^{m/2}\frac{d^2U_l^m}{dx^2} - 2mx(1-x^2)^{m/2-1}\frac{dU_l^m}{dx} - U_l^m\cdot\frac{d}{dx}\left[mx(1-x^2)^{m/2-1}\right] \\ &= (1-x^2)^{m/2}\left[\frac{d^2U_l^m}{dx^2} - \frac{2mx}{1-x^2}\frac{dU_l^m}{dx}\right] - U_l^m\cdot\underbrace{\left[mx\left(\frac{m}{2}-1\right)(1-x^2)^{m/2-2}(-2x) + m(1-x^2)^{m/2-1}\right]}_{(1-x^2)^{m/2-1}\left[m-\frac{mx^2(m-2)}{1-x^2}\right]=(1-x^2)^{m/2-1}\left[\frac{m(1-mx^2+x^2)}{1-x^2}\right]} \end{aligned} \tag{24}$$

   Now the three terms of (22) are

$$(1-x^2)\frac{d^2P_l^m}{dx^2} = (1-x^2)^{m/2}\left[(1-x^2)\frac{d^2U_l^m}{dx^2} - 2mx\frac{dU_l^m}{dx} - \frac{m(1-mx^2+x^2)}{1-x^2}U_l^m\right] \tag{25}$$

$$-2x\frac{dP_l^m}{dx} = (1-x^2)^{m/2}\left[-2x\frac{dU_l^m}{dx} + \frac{2mx^2}{1-x^2}U_l^m\right] \tag{26}$$

$$\left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = (1-x^2)^{m/2}\left[l(l+1) - \frac{m^2}{1-x^2}\right]U_l^m \tag{27}$$

   Adding (25)-(27), we see that (22) is equivalent to

$$(1-x^2)^{m/2}\left\{(1-x^2)\frac{d^2U_l^m}{dx^2} - 2(m+1)x\frac{dU_l^m}{dx} + \left[l(l+1) - \frac{m(1-mx^2+x^2) - 2mx^2 + m^2}{1-x^2}\right]U_l^m\right\} = 0 \tag{28}$$

   or

$$(1-x^2)\frac{d^2U_l^m}{dx^2} - 2(m+1)x\frac{dU_l^m}{dx} + [l(l+1) - m(m+1)]\,U_l^m = 0 \tag{29}$$

Noting

$$\frac{d^m(fg)}{dx^m} = \sum_{k=0}^{m} \binom{m}{k} \frac{d^k f}{dx^k} \frac{d^{m-k} g}{dx^{m-k}}$$

(29) can be shown by taking the $m$−th derivative of (12), where

$$\begin{aligned}
0 &= \frac{d^m}{dx^m}\left[(1-x^2)P_l''\right] - \frac{d^m}{dx^m}(2xP_l') + l(l+1)\frac{d^m}{dx^m}P_l \\
&= (1-x^2)\frac{d^2U_l^m}{dx^2} + m(-2x)\frac{dU_l^m}{dx} + \frac{m(m-1)}{2}(-2)U_l^m - 2x\frac{dU_l^m}{dx} - m\cdot 2U_l^m + l(l+1)U_l^m \\
&= (1-x^2)\frac{d^2U_l^m}{dx^2} - 2(m+1)x\frac{dU_l^m}{dx} + [l(l+1) - m(m+1)]U_l^m
\end{aligned} \tag{30}$$

(c) **Recurrence Relations**

Equation (29) can also be written as

$$(1-x^2)U_l^{m+2} - 2(m+1)xU_l^{m+1} + (l+m+1)(l-m)U_l^m = 0 \tag{31}$$

Then using the relation $U_l^m = (1-x^2)^{-m/2}P_l^m$, we get the recurrence relation for $P_l^m$s:

$$(1-x^2)(1-x^2)^{-m/2-1}P_l^{m+2} - 2(m+1)x(1-x^2)^{-m/2-1/2}P_l^{m+1} + (l+m+1)(l-m)(1-x^2)^{-m/2}P_l^m$$

or,

$$P_l^{m+2} = \frac{2(m+1)x}{\sqrt{1-x^2}}P_l^{m+1} - (l+m+1)(l-m)P_l^m \qquad \text{(substitute } m \to m-1\text{)} \tag{32}$$

$$P_l^{m+1} = \frac{2mx}{\sqrt{1-x^2}}P_l^m - (l+m)(l-m+1)P_l^{m-1} \tag{33}$$

(d) **Orthonormality**

First, rewrite (22) as equivalent form

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m = 0 \tag{34}$$

Similar to (14)-(16), we have

$$\int_{-1}^{1}\left\{P_k^m\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m}{dx}\right] - P_l^m\frac{d}{dx}\left[(1-x^2)\frac{dP_k^m}{dx}\right]\right\}dx = [k(k+1) - l(l+1)]\int_{-1}^{1}P_k^m P_l^m dx \tag{35}$$

which establishes the orthogonality

$$\int_{-1}^{1}P_k^m P_l^m dx = 0 \qquad \text{if } k \neq l \tag{36}$$

Next, multiply $(1-x^2)^m$ to (31), we have

$$\begin{aligned}
(1-x^2)^{m+1}U_l^{m+2} - 2x(m+1)(1-x^2)^m U_l^{m+1} + (l+m+1)(l-m)(1-x^2)^m U_l^m = 0 \qquad &\Longrightarrow \\
\frac{d}{dx}\left[(1-x^2)^{m+1}U_l^{m+1}\right] + (l+m+1)(l-m)(1-x^2)^m U_l^m = 0 &
\end{aligned} \tag{37}$$

Multiply $U_l^m$ to (37) and integrate,

$$\begin{aligned}
(l+m+1)(l-m)\int_{-1}^{1}(1-x^2)^m U_l^m U_l^m dx &= -\int_{-1}^{1}U_l^m\frac{d}{dx}\left[(1-x^2)^{m+1}U_l^{m+1}\right]dx \\
&= \int_{-1}^{1}\frac{dU_l^m}{dx}(1-x^2)^{m+1}U_l^{m+1}dx \\
&= \int_{-1}^{1}(1-x^2)^{m+1}U_l^{m+1}U_l^{m+1}dx \qquad \Longrightarrow
\end{aligned}$$

$$\int_{-1}^{1}P_l^{m+1}P_l^{m+1}dx = (l+m+1)(l-m)\int_{-1}^{1}P_l^m P_l^m dx \tag{38}$$

Considering $P_l^0 = P_l$ and (20), finally we have

$$\begin{aligned}
\int_{-1}^{1} P_l^m P_l^m dx &= (l+m)(l-m+1)\int_{-1}^{1} P_l^{m-1} P_l^{m-1} dx \\
&= [(l+m)(l+m-1)][(l-m+1)(l-m+2)]\int_{-1}^{1} P_l^{m-2} P_l^{m-2} dx \\
&= [(l+m)(l+m-1)\cdots(l+1)][(l-m+1)(l-m+2)\cdots l]\int_{-1}^{1} P_l^{0} P_l^{0} dx \\
&= \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1} \qquad (39)
\end{aligned}$$

(e) **Explicit Form of $P_l^l$**

Since $U_l^l = d^l P_l / dx^l$ is a constant, $P_l^l(x) = A(1-x^2)^{l/2}$, then the normalization condition gives

$$\frac{2(2l)!}{2l+1} = \int_{-1}^{1} P_l^l P_l^l dx = \int_{-1}^{1} A^2 (1-x^2)^l dx \qquad (40)$$

Define $M_l \equiv \int (1-x^2)^l dx$, then

$$\begin{aligned}
M_l &= \int_{-1}^{1} (1-x^2)^l dx = x(1-x^2)^l\Big|_{-1}^{1} - \int_{-1}^{1} xl(1-x^2)^{l-1}(-2x)dx \\
&= \int_{-1}^{1} 2lx^2(1-x^2)^{l-1} dx \\
&= 2l\int_{-1}^{1}(1-x^2)^{l-1}dx - 2l\int_{-1}^{1}(1-x^2)^l dx = 2lM_{l-1} - 2lM_l \qquad \Longrightarrow \\
M_l &= \frac{2l}{2l+1}M_{l-1} = \frac{2l}{2l+1}\frac{2l-2}{2l-1}M_{l-2} = \cdots = \frac{(2^l l!)^2}{(2l+1)!}M_0 = \frac{2\cdot 2^{2l}(l!)^2}{(2l+1)!} \qquad (41)
\end{aligned}$$

Combine (40) and (41), we have

$$A^2\frac{2\cdot 2^{2l}(l!)^2}{(2l+1)!} = \frac{2(2l)!}{2l+1} \qquad \Longrightarrow \qquad A = \frac{(2l)!}{2^l l!} \qquad \text{i.e.,}$$

$$P_l^l(x) = \frac{(2l)!}{2^l l!}(1-x^2)^{l/2} \qquad (42)$$

3. **Spherical Harmonics**

(a) **Change of Variable**

Consider the change of variable $x = \cos\theta$ with $\theta \in [0, \pi]$. Then

$$\begin{aligned}
dx &= -\sin\theta \, d\theta \\
\frac{d}{dx} &= -\frac{1}{\sin\theta}\frac{d}{d\theta} \\
\frac{d^2}{dx^2} &= \frac{d}{dx}\left(\frac{d}{dx}\right) = -\frac{1}{\sin\theta}\frac{d}{d\theta}\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\right) \\
&= \frac{1}{\sin\theta}\left(-\frac{1}{\sin^2\theta}\cos\theta\frac{d}{d\theta} + \frac{1}{\sin\theta}\frac{d^2}{d\theta^2}\right) \\
&= -\frac{\cot\theta}{\sin^2\theta}\frac{d}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2}{d\theta^2}
\end{aligned}$$

Now (22) becomes

$$\sin^2\theta\left(-\frac{\cot\theta}{\sin^2\theta}\frac{dP_l^m}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2P_l^m}{d\theta^2}\right) + 2\cot\theta\frac{dP_l^m}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P_l^m = 0 \qquad \Longrightarrow$$

$$\frac{d^2P_l^m}{d\theta^2} + \cot\theta\frac{dP_l^m}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P_l^m = 0 \qquad (43)$$

Or, equivalently, (34) becomes

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_l^m}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P_l^m=0 \tag{44}$$

(b) **Spherical Harmonics**
In spherical coordinates,

$$L_z \longleftrightarrow -i\hbar\frac{\partial}{\partial\phi} \tag{3.218}$$

$$\boldsymbol{L}^2 \longleftrightarrow -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] \tag{3.224}$$

Now consider the simultaneous eigenket $|l,m\rangle$ of $L_z$ and $\boldsymbol{L}^2$, and let $Y_l^m(\theta,\phi)$, called spherical harmonics, be its representation in spherical coordinate basis $\langle\hat{\boldsymbol{n}}|l,m\rangle$. The eigenequations are now:

$$-i\hbar\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi)=m\hbar Y_l^m(\theta,\phi) \tag{45}$$

$$-\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]Y_l^m(\theta,\phi)=l(l+1)\hbar^2Y_l^m(\theta,\phi) \tag{46}$$

From (45), we can see that

$$Y_l^m(\theta,\phi)=e^{im\phi}W_l^m(\theta) \tag{47}$$

Then (46) becomes

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dW_l^m}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]W_l^m=0 \tag{48}$$

Compare (48) with (44), we can see that $W_l^m \propto P_l^m$, hence

$$Y_l^m(\theta,\phi)=C_l^m e^{im\phi}P_l^m(\cos\theta) \tag{49}$$

(c) **Orthonormality**
To see orthonormality of the spherical harmonics, note that

$$\begin{aligned}\int_{d\Omega}Y_{l'}^{m'*}(\theta,\phi)Y_l^m(\theta,\phi)d\Omega &= C_l^mC_{l'}^{m'*}\int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta)e^{i(m-m')\phi}P_l^m(\cos\theta)P_{l'}^{m'}(\cos\theta)\\ &= C_l^mC_{l'}^{m'*}\underbrace{\left(\int_0^{2\pi}e^{i(m-m')\phi}d\phi\right)}_{X}\underbrace{\left(\int_{-1}^{1}d(\cos\theta)P_l^m(\cos\theta)P_{l'}^{m'}(\cos\theta)\right)}_{Y}\end{aligned}$$

Obviously, when $m\neq m'$, $X$ vanishes, and when $m=m'$, $X=2\pi$, in which case, according to (36),(39), $Y=\frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{ll'}$. In summary

$$\int_{d\Omega}Y_{l'}^{m'*}(\theta,\phi)Y_l^m(\theta,\phi)d\Omega=C_l^mC_{l'}^{m'*}\frac{(l+m)!}{(l-m)!}\frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'} \tag{50}$$

To make $Y_l^m(\theta,\phi)$ orthonormal, we must choose $C_l^m$ such that

$$|C_l^m|=\sqrt{\frac{(l-m)!}{(l+m)!}\frac{2l+1}{4\pi}} \tag{51}$$

(d) **Phase Convention**
By applying the lowering operator $L_-$ to the $|l,m\rangle$ state, we have (Sakurai 3.245)

$$\langle\hat{\boldsymbol{n}}|l,m-1\rangle=\frac{1}{\sqrt{(l+m)(l-m+1)}}e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)\langle\hat{\boldsymbol{n}}|l,m\rangle \implies$$

$$C_l^{m-1}e^{i(m-1)\phi}P_l^{m-1}(\cos\theta)=C_l^m\frac{1}{\sqrt{(l+m)(l-m+1)}}e^{-i\phi}\left[-e^{im\phi}\frac{dP_l^m(\cos\theta)}{d\theta}+i\cot\theta(im)e^{im\phi}P_l^m(\cos\theta)\right] \implies$$

$$P_l^{m-1}=\frac{C_l^m}{C_l^{m-1}}\frac{-1}{\sqrt{(l+m)(l-m+1)}}\left(\frac{dP_l^m}{d\theta}+m\cot\theta P_l^m\right) \tag{52}$$

Recall that

$$
\begin{aligned}
\frac{dP_l^m}{dx} &= \frac{d}{dx}\left[(1-x^2)^{m/2}U_l^m\right] \\
&= (1-x^2)^{m/2}U_l^{m+1} - \frac{mx}{1-x^2}(1-x^2)^{m/2}U_l^m \\
&= \frac{(1-x^2)^{(m+1)/2}U_l^{m+1}}{\sqrt{1-x^2}} - \frac{mx}{1-x^2}P_l^m \\
&= \frac{P_l^{m+1}}{\sqrt{1-x^2}} - \frac{mx}{1-x^2}P_l^m \qquad \Longrightarrow \\
P_l^{m+1} &= \sqrt{1-x^2}\frac{dP_l^m}{dx} + \frac{mx}{\sqrt{1-x^2}}P_l^m
\end{aligned}
\tag{53}
$$

Now plug (50) into (33),

$$
\begin{aligned}
\sqrt{1-x^2}\frac{dP_l^m}{dx} + \frac{mx}{\sqrt{1-x^2}}P_l^m &= \frac{2mx}{\sqrt{1-x^2}}P_l^m - (l+m)(l-m+1)P_l^{m-1} && \Longrightarrow \\
(l+m)(l-m+1)P_l^{m-1} &= -\sqrt{1-x^2}\frac{dP_l^m}{dx} + \frac{mx}{\sqrt{1-x^2}}P_l^m && \Longrightarrow \\
P_l^{m-1} &= \frac{1}{(l+m)(l-m+1)}\left[-\sin\theta\left(-\frac{1}{\sin\theta}\frac{dP_l^m}{d\theta}\right) + m\cot\theta P_l^m\right] \\
&= \frac{1}{(l+m)(l-m+1)}\left[\frac{dP_l^m}{d\theta} + m\cot\theta P_l^m\right]
\end{aligned}
\tag{54}
$$

Compare (54) with (52), we have

$$
\frac{C_l^m}{C_l^{m-1}} = \frac{-1}{\sqrt{(l+m)(l-m+1)}} \tag{55}
$$

but by (51),

$$
\frac{|C_l^m|}{|C_l^{m-1}|} = \sqrt{\frac{(l-m)!}{(l+m)!}}\sqrt{\frac{(l+m-1)!}{(l-m+1)!}} = \frac{1}{\sqrt{(l+m)(l-m+1)}} \tag{56}
$$

This shows there is a relative phase factor of $-1$ between $C_l^m$ and $C_l^{m-1}$.
If we (arbitrarily) define $C_l^0$ to have the $+1$ sign, we have fixed all $C_l^m$ to be

$$
C_l^m = (-1)^m\sqrt{\frac{(l-m)!}{(l+m)!}\frac{2l+1}{4\pi}} \tag{57}
$$

In summary, this convention of phase factor of the spherical harmonics is determined by two choices: 1) the arbitrarily chosen $+1$ sign of $Y_l^0$, and 2) the usual phase convention of the $L_-$ operator.

(e) **Sakurai (3.246)**

Next, let's derive Sakurai (3.246):

$$
Y_l^m(\theta,\phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}}e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{l-m}}{d(\cos\theta)^{l-m}}\sin^{2l}\theta \tag{3.246}
$$

Define

$$
Q_l^m(x) = (1-x^2)^m U_l^m(x) \tag{58}
$$

Then by definition,

$$
P_l^m(x) = (1-x^2)^{-m/2}Q_l^m(x) \tag{59}
$$

$$
Y_l^m(\theta,\phi) = C_l^m e^{im\phi}P_l^m(\cos\theta) = C_l^m e^{im\phi}\frac{1}{\sin^m\theta}Q_l^m(\cos\theta) \tag{60}
$$

And (37) becomes

$$
\begin{aligned}
Q_l^m &= -\frac{1}{(l+m+1)(l-m)}\frac{dQ_l^{m+1}}{dx} \\
&= (-1)^2\frac{1}{(l+m+1)(l-m)}\frac{1}{(l+m+2)(l-m-1)}\frac{d^2Q_l^{m+2}}{dx^2} \\
&= (-1)^{l-m}\frac{1}{(l+m+1)(l-m)}\frac{1}{(l+m+2)(l-m-1)}\cdots\frac{1}{2l\cdot 1}\frac{d^{l-m}Q_l^l}{dx^{l-m}} \\
&= (-1)^{l-m}\frac{(l+m)!}{(2l)!(l-m)!}\frac{d^{l-m}Q_l^l}{dx^{l-m}} \qquad (61)
\end{aligned}
$$

Finally (60) becomes

$$
\begin{aligned}
Y_l^m(\theta,\phi) &= C_l^m e^{im\phi}\frac{1}{\sin^m\theta}Q_l^m(\cos\theta) \\
\text{(by (57),(61))} \quad &= (-1)^m\sqrt{\frac{(l-m)!}{(l+m)!}\frac{2l+1}{4\pi}}(-1)^{l-m}\frac{(l+m)!}{(2l)!(l-m)!}e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{l-m}Q_l^l}{dx^{l-m}} \\
&= (-1)^l\sqrt{\frac{(l+m)!}{(l-m)!}\frac{2l+1}{4\pi}}\frac{1}{(2l)!}e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{l-m}Q_l^l}{dx^{l-m}} \\
\text{(by (59),(42))} \quad &= (-1)^l\sqrt{\frac{(l+m)!}{(l-m)!}\frac{2l+1}{4\pi}}\frac{1}{(2l)!}e^{im\phi}\frac{1}{\sin^m\theta}\frac{(2l)!}{2^l l!}\frac{d^{l-m}\sin^{2l}\theta}{d(\cos\theta)^{l-m}} \\
&= \frac{(-1)^l}{2^l l!}\sqrt{\frac{(l+m)!}{(l-m)!}\frac{2l+1}{4\pi}}e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{l-m}\sin^{2l}\theta}{d(\cos\theta)^{l-m}} \qquad (62)
\end{aligned}
$$

which proves (3.246).

(f) **Extension of $m$ into Negative Range**

At last, let's justify the convention used to extend $P_l^m$ and $Y_l^m$ to negative range $-l \le m < 0$. Recall Saukurai (3.222) for the spherical coordinate representation of $L_\pm$:

$$
L_+ \longleftrightarrow \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad (63)
$$

$$
L_- \longleftrightarrow \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad (64)
$$

For $m>0$, the effect of $L_- : m \to m-1$ is (see 3.245):

$$
L_- : \qquad Y_l^{m-1}(\theta,\phi) = \frac{1}{\sqrt{(l+m)(l-m+1)}}e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)Y_l^m(\theta,\phi) \qquad (65)
$$

For $-m<0$, we would like the effect of $L_+ : -m \to -(m-1)$ to be:

$$
L_+ : \qquad Y_l^{-(m-1)}(\theta,\phi) = \frac{1}{\sqrt{[l-(-m)][l+(-m)+1]}}e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)Y_l^{-m}(\theta,\phi) \qquad (66)
$$

But conjugate (65) and multiply by $(-1)^{m-1}$, we get

$$
(-1)^{m-1}Y_l^{m-1*}(\theta,\phi) = (-1)^{m-1}\frac{1}{\sqrt{(l+m)(l-m+1)}}e^{i\phi}\left(-\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\phi}\right)Y_l^{m*}(\theta,\phi) \qquad (67)
$$

which shows that if we define

$$
Y_l^{-m}(\theta,\phi) = (-1)^m Y_l^{m*}(\theta,\phi) \qquad (68)
$$

we get the desired raising operator relation of (66) in the negative $m$ range.
Furthermore, if we want to keep the form parity between $Y_l^m$ and $Y_l^{-m}$, so that

$$
Y_l^m(\theta,\phi) = C_l^m e^{im\phi}P_l^m(\cos\theta) \qquad (69)
$$

$$
Y_l^{-m}(\theta,\phi) = C_l^{-m} e^{-im\phi}P_l^{-m}(\cos\theta) \qquad (70)
$$

We would require

$$
\begin{aligned}
C_l^{-m} e^{-im\phi} P_l^{-m}(\cos\theta) = (-1)^m Y_l^{m*}(\theta,\phi) = (-1)^m C_l^m e^{-im\phi} P_l^m(\cos\theta) \quad \Longrightarrow \\
P_l^{-m} = (-1)^m \frac{C_l^m}{C_l^{-m}} P_l^m = (-1)^m \frac{(-1)^m \sqrt{(l-m)!/(l+m)!}}{(-1)^{-m}\sqrt{(l+m)!/(l-m)!}} P_l^m \\
= (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m
\end{aligned}
\tag{71}
$$

which ends up to be the conventional way to extend associated Legendre functions into the negative $m$ range.