- First we look at the WKB method from a different angle than the text. Let $Q(x) = 2m[E - V(x)]$, so the wave function PDE

$$\frac{-\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)u(x) = Eu(x)$$

becomes

$$\hbar^2 u'' + Q(x)u = 0 \tag{1}$$

where $Q(x)$ can be of either sign.

Now let

$$u(x) = e^{iW(x)/\hbar} \tag{2}$$

we know

$$u'(x) = \frac{i}{\hbar}W' e^{iW/\hbar} \tag{3}$$

$$u''(x) = \left(\frac{i}{\hbar}W'' - \frac{W'^2}{\hbar^2}\right)e^{iW/\hbar} \tag{4}$$

Then (1) becomes an equivalent PDE for $W(x)$:

$$i\hbar W'' - W'^2 + Q(x) = 0 \tag{5}$$

Now assume $W(x)$ takes the form

$$W(x) = W_0(x) + \hbar W_1(x) + \hbar^2 W_2(x) + \cdots \tag{6}$$

Then (5) becomes

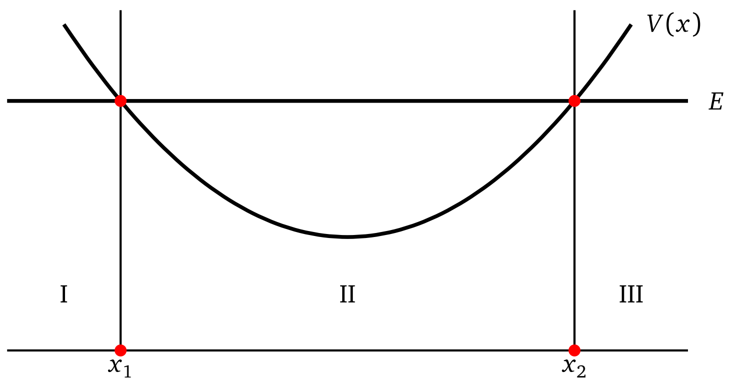
$$\begin{aligned} i\hbar[W_0'' + \hbar W_1'' + O(\hbar^2)] - [W_0' + \hbar W_1' + O(\hbar^2)]^2 + Q(x) = 0 &\quad \Longrightarrow \\ i\hbar W_0'' + i\hbar^2 W_1'' - (W_0'^2 + \hbar^2 W_1'^2 + 2\hbar W_0' W_1') + O(\hbar^3) + Q(x) = 0 &\quad \Longrightarrow \\ [-W_0'^2 + Q(x)] + \hbar(iW_0'' - 2W_0'W_1') + \hbar^2(iW_1'' - W_1'^2) + O(\hbar^3) = 0 & \end{aligned} \tag{7}$$

WKB method is basically solving (7) with ever-increasing power of $\hbar$. In the text, as well as in this document, we solve up to order of $\hbar^1$, i.e.,

$$W_0'^2 = Q(x) \tag{8}$$

$$W_1' = \frac{iW_0''}{2W_0'} \tag{9}$$

We see clearly that when $Q(x) = 0$, the approximation breaks down since (9) has zero in the denominator.



In region II where $E > V(x)$, hence $Q(x) > 0$, (8) and (9) become

$$W_0' = \pm\sqrt{Q(x)} \tag{10}$$

$$W_1' = \frac{i}{2}\frac{\left(\pm\frac{1}{2}\frac{Q'}{\sqrt{Q}}\right)}{\pm\sqrt{Q}} = \frac{i}{4}\frac{Q'}{Q} \tag{11}$$

Therefore

$$\begin{aligned} u_{II}(x) &= e^{iW_0/\hbar}e^{i\hbar W_1/\hbar} \\ &= \exp\left(\pm\frac{i}{\hbar}\int^x \sqrt{Q(t)}dt\right)\cdot\exp\left(\int^x -\frac{1}{4}\frac{Q'(t)}{Q(t)}dt\right) \\ &= \exp\left(-\frac{1}{4}\ln Q(x)\right)\cdot\exp\left(\pm\frac{i}{\hbar}\int^x \sqrt{Q(t)}dt\right) \\ &= \frac{1}{Q(x)^{1/4}}\cdot\exp\left(\pm\frac{i}{\hbar}\int^x \sqrt{Q(t)}dt\right) \end{aligned} \tag{12}$$

i.e., the general solution $u(x)$ in region II is a combination

$$u_{II}(x) = \frac{1}{Q(x)^{1/4}}\left[A\exp\left(\frac{i}{\hbar}\int_0^x \sqrt{Q(t)}dt\right) + B\exp\left(-\frac{i}{\hbar}\int_0^x \sqrt{Q(t)}dt\right)\right] \tag{13}$$

In region I and III where $E < V(x)$, hence $Q(x) < 0$, let $q(x) = |Q(x)|$, (8) and (9) become

$$W_0' = \pm i\sqrt{q(x)} \tag{14}$$

$$W_1' = \frac{i}{2}\frac{\left(\pm\frac{i}{2}\frac{q'}{\sqrt{q}}\right)}{\pm i\sqrt{q}} = \frac{i}{4}\frac{q'}{q} \tag{15}$$

Compare these with (10) and (11), we can see the general solution in region I and III is

$$u_{I,III}(x) = \frac{1}{q(x)^{1/4}}\left[C\exp\left(-\frac{1}{\hbar}\int_0^x \sqrt{q(t)}dt\right) + D\exp\left(\frac{1}{\hbar}\int_0^x \sqrt{q(t)}dt\right)\right] \tag{16}$$

The boundary condition at $\pm\infty$ should require $u \to 0$, and clearly in region III, the D term will blow up when $x \to +\infty$, and in region I, the C term will blow up when $x \to -\infty$, therefore the general *physical* solution will be

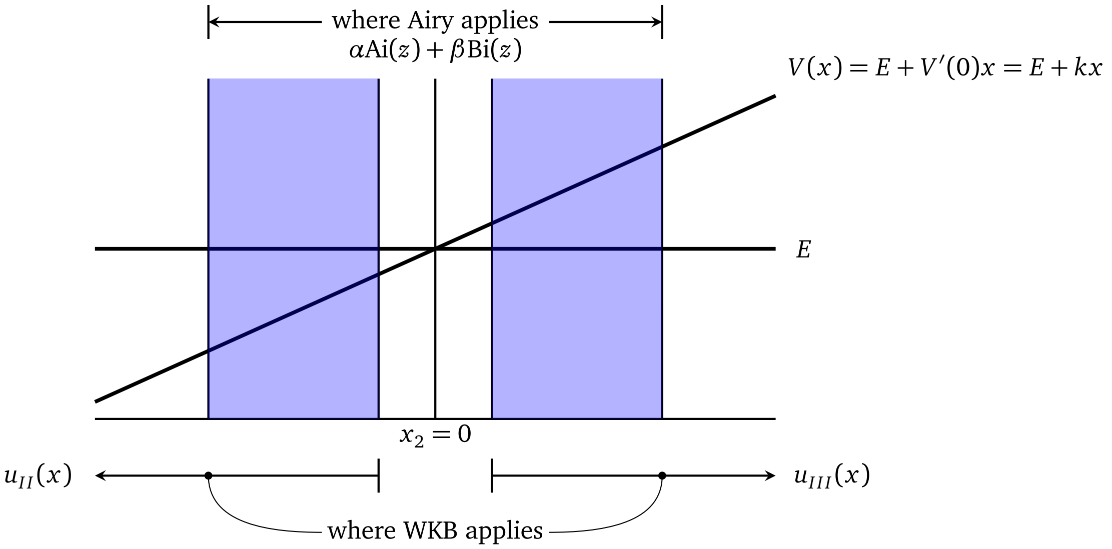
$$u_I(x) = \frac{1}{q(x)^{1/4}}\cdot D\exp\left(\frac{1}{\hbar}\int_0^x \sqrt{q(t)}dt\right) \tag{17}$$

$$u_{III}(x) = \frac{1}{q(x)^{1/4}}\cdot C\exp\left(-\frac{1}{\hbar}\int_0^x \sqrt{q(t)}dt\right) \tag{18}$$

- Here is our strategy: first we focus on the right turning point x_2, where we now consider as the origin. In the vicinity of $x_2 = 0$, the potential $V(x)$ can be taken as linear with positive slope k, so we consider a region where this linear approximation is applicable, where the wave function solution is known to have general form

  $$u(z) = \alpha\mathrm{Ai}(z) + \beta\mathrm{Bi}(z)$$

  Since the WKB approximation breaks down near the origin (where $Q(x) = 0$), we now look at the region where both WKB approximation and linear potential approximation are valid (shaded area in the figure). We will find out that the asymptotic form of Airy function agrees with u_{II} to the left and with u_{III} to the right, so we can establish the relation between $(\alpha,\beta) \sim (A,B)$ on the left, and $(\alpha,\beta) \sim C$ on the right, which gives the restriction between $(A,B) \sim C$.



Recall for a linear potential $V(x)=E+kx$, the Schrödinger equation is equivalent to (see eq 2.236, where $\epsilon=0$)

$$u''(z)+zu(z)=0 \qquad \text{where} \tag{19}$$

$$z=2^{1/3}\left(\frac{\hbar^2}{mk}\right)^{-1/3}x=\left(\frac{2mk}{\hbar^2}\right)^{1/3}x \tag{20}$$

Then we see in the left shaded region where u_{II} is applicable, the integral in (13) can be approximated as

$$\begin{aligned}\int_0^x\sqrt{Q(t)}dt&=\int_0^x\sqrt{2m(E-V(t))}dt=\int_0^x\sqrt{2mk}\sqrt{-t}dt\\&=\sqrt{2mk}\int_0^{-x}\sqrt{\tau}d(-\tau)=-\sqrt{2mk}\,\frac{2}{3}\tau^{3/2}\Big|_0^{-x}\\&=-\frac{2}{3}\sqrt{2mk}(-x)^{3/2}\end{aligned} \tag{21}$$

Then (13) becomes

$$\begin{aligned}u_{II}(x)&=\frac{1}{[2mk(-x)]^{1/4}}\left[A\exp\left(-i\frac{2}{3}\sqrt{\frac{2mk}{\hbar^2}}(-x)^{3/2}\right)+B\exp\left(i\frac{2}{3}\sqrt{\frac{2mk}{\hbar^2}}(-x)^{3/2}\right)\right]\\&=[2mk(-x)]^{-1/4}\left[A\exp\left(-i\frac{2}{3}(-z)^{3/2}\right)+B\exp\left(i\frac{2}{3}(-z)^{3/2}\right)\right]\\&=\left[(2mk)^{2/3}\hbar^{2/3}\left(\frac{2mk}{\hbar^2}\right)^{1/3}(-x)\right]^{-1/4}\left[A\exp\left(-i\frac{2}{3}(-z)^{3/2}\right)+B\exp\left(i\frac{2}{3}(-z)^{3/2}\right)\right]\\&=(2mk\hbar)^{-1/6}(-z)^{-1/4}\left[A\exp\left(-i\frac{2}{3}(-z)^{3/2}\right)+B\exp\left(i\frac{2}{3}(-z)^{3/2}\right)\right]\qquad\left(\theta\equiv\frac{2}{3}(-z)^{3/2}\right)\\&=(2mk\hbar)^{-1/6}(-z)^{-1/4}(A\cos\theta-iA\sin\theta+B\cos\theta+iB\sin\theta)\equiv u_{II}(z)\end{aligned} \tag{22}$$

For $z\ll 0$, the asymptotic form of Airy functions are known to be

$$\mathrm{Ai}(z)=\frac{1}{\sqrt{\pi}(-z)^{1/4}}\sin\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)=\frac{1}{\sqrt{\pi}(-z)^{1/4}}\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta) \tag{23}$$

$$\mathrm{Bi}(z)=\frac{1}{\sqrt{\pi}(-z)^{1/4}}\cos\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)=\frac{1}{\sqrt{\pi}(-z)^{1/4}}\frac{1}{\sqrt{2}}(\cos\theta-\sin\theta) \tag{24}$$

If we identify the general Airy function solution with $u_{II}(z)$, we should have

$$\begin{aligned}\alpha\mathrm{Ai}(z)+\beta\mathrm{Bi}(z)=u_{II}(z)\qquad&\Longrightarrow\\ \frac{\alpha(\cos\theta+\sin\theta)+\beta(\cos\theta-\sin\theta)}{\sqrt{2}\sqrt{\pi}}=(2mk\hbar)^{-1/6}&(A\cos\theta+B\cos\theta+iB\sin\theta-iA\sin\theta)\end{aligned} \tag{25}$$

By identifying cos and sin terms respectively, we have

$$\begin{cases} \frac{1}{\sqrt{2\pi}}(\alpha+\beta) = (2mk\hbar)^{-1/6}(A+B) \\ \frac{1}{\sqrt{2\pi}}(\alpha-\beta) = (2mk\hbar)^{-1/6}i(B-A) \end{cases} \implies$$

$$\begin{cases} \alpha = \frac{\sqrt{2\pi}(2mk\hbar)^{-1/6}}{2}(A-iA+B+iB) \\ \beta = \frac{\sqrt{2\pi}(2mk\hbar)^{-1/6}}{2}(A+iA+B-iB) \end{cases} \tag{26}$$

We now look at the right shaded region, where for (18) $q(x) = \sqrt{2m(V(x)-E)} = \sqrt{2mkx}$, and

$$\begin{aligned} u_{III}(x) &= \frac{1}{(2mkx)^{1/4}} \cdot C\exp\left(-\frac{1}{\hbar}\int_0^x \sqrt{2mkt}dt\right) \\ &= \frac{1}{(2mkx)^{1/4}} \cdot C\exp\left(-\int_0^x \sqrt{\frac{2mk}{\hbar^2}}\sqrt{t}dt\right) \\ &= (2mk\hbar)^{-1/6}z^{-1/4} \cdot C\exp\left(-\frac{2}{3}z^{3/2}\right) \equiv u_{III}(z) \end{aligned} \tag{27}$$

for $z \gg 0$, the asymptotic form of Airy functions are known to be

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}z^{1/4}}\exp\left(-\frac{2}{3}z^{3/2}\right) \tag{28}$$

$$\mathrm{Bi}(z) = \frac{1}{\sqrt{\pi}z^{1/4}}\exp\left(\frac{2}{3}z^{3/2}\right) \tag{29}$$

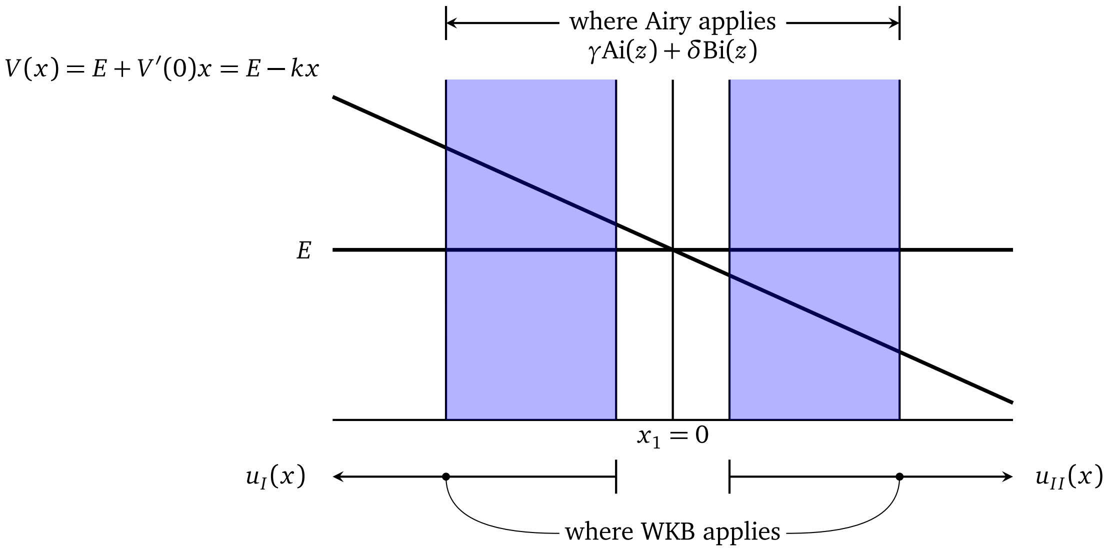
Equating $\alpha\mathrm{Ai}(z)+\beta\mathrm{Bi}(z)$ with $u_{III}(z)$, we have

$$\begin{cases} \alpha = 2\sqrt{\pi}(2mk\hbar)^{-1/6}C \\ \beta = 0 \end{cases} \tag{30}$$

Now compare (30) with (26), we must have

$$A = iB \tag{31}$$

Now coming to the left turning point x_1.



In the right shaded area of the figure above, the integral in (13) becomes

$$\int_0^x \sqrt{Q(t)}dt = \int_0^x \sqrt{2m(E-V(t))}dt = \int_0^x \sqrt{2mkt}dt = \frac{2}{3}\sqrt{2mk}x^{3/2} \tag{32}$$

Thus (13) becomes

$$\begin{aligned}
u_{II}(x) &= \frac{1}{(2mkx)^{1/4}}\left[A'\exp\left(i\frac{2}{3}\sqrt{\frac{2mk}{\hbar^2}}x^{3/2}\right)+B'\exp\left(-i\frac{2}{3}\sqrt{\frac{2mk}{\hbar^2}}x^{3/2}\right)\right] \\
&= (2mk\hbar)^{-1/6}z^{-1/4}\left[A'\exp\left(i\frac{2}{3}z^{3/2}\right)+B'\exp\left(-i\frac{2}{3}z^{3/2}\right)\right] \qquad (\theta\equiv\frac{2}{3}z^{3/2}) \\
&= (2mk\hbar)^{-1/6}z^{-1/4}(A'\cos\theta+iA'\sin\theta+B'\cos\theta-iB'\sin\theta)\equiv u_{II}(z) \qquad (33)
\end{aligned}$$

Compare (33) with (22) we see the correspondence $A' \longleftrightarrow B$ and $B' \longleftrightarrow A$.

To see how the Airy asymptotics (23-24) become for $z \gg 0$ near the left turning point, notice the figure for left turning point is the x−reflection of the figure for right turning point. So if (23-24) are the asymptotics form for the right turning point with $z \ll 0$, replacing $z \to -z$ will give the asymptotic form of them around the left turning point with $z \gg 0$, i.e.,

$$\mathrm{Ai}(z) = \frac{1}{\sqrt{\pi}z^{1/4}}\sin\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right) = \frac{1}{\sqrt{\pi}z^{1/4}}\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta) \qquad (34)$$

$$\mathrm{Bi}(z) = \frac{1}{\sqrt{\pi}z^{1/4}}\cos\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right) = \frac{1}{\sqrt{\pi}z^{1/4}}\frac{1}{\sqrt{2}}(\cos\theta-\sin\theta) \qquad (35)$$

Then (26) is converted via $A' \longleftrightarrow B, B' \longleftrightarrow A, \alpha \longleftrightarrow \gamma, \beta \longleftrightarrow \delta$:

$$\begin{cases} \gamma = \frac{\sqrt{2\pi}(2mk\hbar)^{-1/6}}{2}(A'+iA'+B'-iB') \\ \delta = \frac{\sqrt{2\pi}(2mk\hbar)^{-1/6}}{2}(A'-iA'+B'+iB') \end{cases} \qquad (36)$$

Similarly (30) and (31) become

$$\begin{cases} \gamma = 2\sqrt{\pi}(2mk\hbar)^{-1/6}D \\ \delta = 0 \end{cases} \qquad (37)$$

and

$$B' = iA' \qquad \text{or} \qquad A' = -iB' \qquad (38)$$

- As the last step, we'll see how the same wave function $u_{II}(x)$ (13) can be consistently expressed in both of the forms below.

$$u_{II}(x) \propto B\left[i\exp\left(\frac{i}{\hbar}\int_{x_2}^{x}\sqrt{Q(t)}dt\right)+\exp\left(-\frac{i}{\hbar}\int_{x_2}^{x}\sqrt{Q(t)}dt\right)\right] \qquad (39)$$

$$u_{II}(x) \propto B'\left[-i\exp\left(\frac{i}{\hbar}\int_{x_1}^{x}\sqrt{Q(t)}dt\right)+\exp\left(-\frac{i}{\hbar}\int_{x_1}^{x}\sqrt{Q(t)}dt\right)\right] \qquad (40)$$

Note here we have replaced the lower integration limit of (13) with the proper turning point, matching $\int_{x_2}^{x} \longleftrightarrow B$ and $\int_{x_1}^{x} \longleftrightarrow B'$.

Pick (39) and subtract the integration ranges, we have

$$\begin{aligned}
u_{II}(x) &\propto B\left[i\exp\left(\frac{i}{\hbar}\int_{x_1}^{x}-\frac{i}{\hbar}\int_{x_1}^{x_2}\right)+\exp\left(-\frac{i}{\hbar}\int_{x_1}^{x}+\frac{i}{\hbar}\int_{x_1}^{x_2}\right)\right] \\
&\propto B\left[i\exp\left(\frac{i}{\hbar}\int_{x_1}^{x}\right)\exp\left(-\frac{i}{\hbar}\int_{x_1}^{x_2}\right)+\exp\left(-\frac{i}{\hbar}\int_{x_1}^{x}\right)\exp\left(\frac{i}{\hbar}\int_{x_1}^{x_2}\right)\right] \qquad (41)
\end{aligned}$$

which needs to be consistent with the other form

$$u_{II}(x) \propto B'\left[-i\exp\left(\frac{i}{\hbar}\int_{x_1}^{x}\right)+\exp\left(-\frac{i}{\hbar}\int_{x_1}^{x}\right)\right] \qquad (42)$$

For (41) and (42) to be consistent, they must differ by an overall factor, i.e.,

$$
\begin{aligned}
-\exp\left(-\frac{i}{\hbar}\int_{x_1}^{x_2}\right) &= \exp\left(\frac{i}{\hbar}\int_{x_1}^{x_2}\right) \qquad \text{or} \\
\exp\left(\frac{2i}{\hbar}\int_{x_1}^{x_2}\sqrt{Q(t)}dt\right) &= e^{i(2n+1)\pi} \qquad \text{or} \\
\frac{1}{\hbar}\int_{x_1}^{x_2}\sqrt{Q(t)}dt = \frac{1}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(E-V(t))}dt &= \left(n+\frac{1}{2}\right)\pi
\end{aligned}
\tag{43}
$$