

In these notes, we give the full treatment of Runge-Lenz operator

$$\mathbf{M} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{Ze^2}{r}\mathbf{x} \quad (1)$$

in a central Coulomb potential  $V(r) = -Ze^2/r$ , i.e., with Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} - \frac{Ze^2}{r} \quad (2)$$

We shall show the following relationships

$$[M_i, L_j] = i\hbar\epsilon_{ijk}M_k \quad (3)$$

$$[\mathbf{M}, H] = 0 \quad (4)$$

$$\mathbf{L} \cdot \mathbf{M} = 0 \quad (5)$$

$$\mathbf{M} \cdot \mathbf{L} = 0 \quad (6)$$

$$\mathbf{M}^2 = \frac{2}{m}H(\mathbf{L}^2 + \hbar^2) + Z^2e^4 \quad (7)$$

$$[M_i, M_j] = -i\hbar\epsilon_{ijk}\frac{2}{m}HL_k \quad (8)$$

First we recall a few results that are used throughout the proofs.

1. From exercise 1.31,

$$[p_k, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_k} \quad (9)$$

2. Both  $\mathbf{p}$  and  $\mathbf{x}$  are vector operators:

$$[p_i, L_j] = \epsilon_{lkj}[p_i, x_l p_k] = \epsilon_{lkj}[p_i, x_l]p_k = -i\hbar\epsilon_{lkj}\delta_{il}p_k = i\hbar\epsilon_{ijk}p_k \quad (10)$$

$$[x_i, L_j] = \epsilon_{lkj}[x_i, x_l p_k] = \epsilon_{lkj}x_l[x_i, p_k] = i\hbar\epsilon_{lkj}x_l\delta_{ik} = i\hbar\epsilon_{ijl}x_l \quad (11)$$

3. For two vector operators  $\mathbf{u}, \mathbf{v}$ ,  $\mathbf{u} \cdot \mathbf{v}$  is a scalar operator, i.e., it commutes with  $L_j$ . Since

$$\begin{aligned} [\mathbf{u} \cdot \mathbf{v}, L_j] &= [u_i v_i, L_j] = [u_i, L_j]v_i + u_i[v_i, L_j] \\ &= i\hbar\epsilon_{ijk}u_k v_i + i\hbar\epsilon_{ijk}u_i v_k = u_k[L_j, v_k] + [L_j, u_k]v_k = [L_j, u_k v_k] = [L_j, \mathbf{u} \cdot \mathbf{v}] \end{aligned} \quad (12)$$

4. Any spherically symmetric function of position  $F(\mathbf{x}) = f(r)$  is a scalar operator, i.e., it commutes with the generator of rotation, i.e.,

$$[L_k, f(r)] = 0 \quad (13)$$

Indeed, with the antisymmetry of  $\epsilon_{ijk}$  symbol,

$$[L_k, f(r)] = \epsilon_{ijk}[x_i p_j, f(r)] = \epsilon_{ijk}x_i[p_j, f(r)] = -i\hbar\epsilon_{ijk}x_i\frac{\partial f(r)}{\partial x_j} = -i\hbar\epsilon_{ijk}x_i f'(r)\frac{x_j}{r} = 0$$

5. For a scalar operator  $s$  and a vector operator  $\mathbf{v}$ , both  $s\mathbf{v}$  and  $\mathbf{v}s$  are vector operators:

$$[sv_i, L_j] = s[v_i, L_j] = i\hbar\epsilon_{ijk}sv_k \quad (14)$$

$$[v_i s, L_j] = [v_i, L_j]s = i\hbar\epsilon_{ijk}v_k s \quad (15)$$

6.

$$\mathbf{L} \cdot \mathbf{p} = L_k p_k = \epsilon_{ijk}x_i p_j p_k = 0 \quad (16)$$

$$\mathbf{p} \cdot \mathbf{L} = p_k L_k = \epsilon_{ijk}p_k x_i p_j = \epsilon_{ijk}p_k (p_j x_i + \delta_{ij}i\hbar) = \epsilon_{ijk}p_k p_j x_i + \epsilon_{ijk}\delta_{ij}i\hbar = 0 \quad (17)$$

7. For any given  $i, j, l, m, k \in \{1, 2, 3\}$ ,

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \quad (18)$$

Note here we temporarily suspend the repeated-index summation convention, i.e., the LHS is to be treated as a single term.

The proof is straightforward by fixing, say  $k = 3$ , and enumerating possible combinations of  $i, j, l$  and  $m$ .

8. For vector operators  $\mathbf{u}$  and  $\mathbf{v}$ ,  $\mathbf{u} \times \mathbf{v}$  is also a vector operator.

Indeed

$$\begin{aligned} [(\mathbf{u} \times \mathbf{v})_i, L_j] &= \epsilon_{lmi} [u_l v_m, L_j] = \epsilon_{lmi} ([u_l, L_j] v_m + u_l [v_m, L_j]) \\ &= \epsilon_{lmi} (i\hbar \epsilon_{ljk} u_k v_m + i\hbar \epsilon_{mjk} u_l v_k) \\ &= i\hbar (\epsilon_{lmi} \epsilon_{ljk} u_k v_m + \epsilon_{lmi} \epsilon_{mjk} u_l v_k) \\ &= i\hbar (\delta_{jm} \delta_{ik} - \delta_{mk} \delta_{ij}) u_k v_m - (\delta_{ik} \delta_{jl} - \delta_{lk} \delta_{ij}) u_l v_k \\ &= i\hbar (u_i v_j - \delta_{ij} u_k v_k - u_j v_i + \delta_{ij} u_k v_k) \\ &= i\hbar (u_i v_j - u_j v_i) \\ &= i\hbar \epsilon_{ijk} (\mathbf{u} \times \mathbf{v})_k \end{aligned} \quad (19)$$

### 1. Proof of (3).

This is obvious since due to (19), both  $\mathbf{p} \times \mathbf{L}$  and  $\mathbf{L} \times \mathbf{p}$  are vector operators. Furthermore,  $\mathbf{x}/r$  is the product of a vector operator and a scalar (i.e., setting  $f(r) = 1/r$  in (13)), so  $\mathbf{M}$  is indeed a vector operator.

### 2. Proof of (4).

By definition of  $\mathbf{M}$ , we have

$$[M_k, H] = \overbrace{\frac{1}{2m} \epsilon_{ijk} [p_i L_j - L_i p_j, H]}^A - \overbrace{\left[ Z e^2 \frac{x_k}{r}, H \right]}^B$$

where

$$\begin{aligned} B &= Z e^2 \left[ \frac{x_k}{r}, \frac{p_i p_i}{2m} \right] = \frac{Z e^2}{2m} \left( p_i \left[ \frac{x_k}{r}, p_i \right] + \left[ \frac{x_k}{r}, p_i \right] p_i \right) \\ &= \frac{i\hbar Z e^2}{2m} \left[ p_i \left( \frac{\delta_{ik}}{r} - \frac{x_i x_k}{r^3} \right) + \left( \frac{\delta_{ik}}{r} - \frac{x_i x_k}{r^3} \right) p_i \right] \\ &= \frac{i\hbar Z e^2}{2m} \underbrace{\left[ \left( p_k \frac{1}{r} - p_i \frac{x_i x_k}{r^3} \right) + \left( \frac{1}{r} p_k - \frac{x_i x_k}{r^3} p_i \right) \right]}_{B'} \end{aligned} \quad (20)$$

On the other hand, recall that both  $\mathbf{p}^2$  and  $1/r$  in  $H$  are scalars, so  $[L_i, H] = 0$  for all  $i$ , which gives

$$\begin{aligned} A &= \frac{1}{2m} \epsilon_{ijk} ([p_i, H] L_j - L_i [p_j, H]) \\ &= -\frac{Z e^2}{2m} \epsilon_{ijk} \left( \left[ p_i, \frac{1}{r} \right] L_j - L_i \left[ p_j, \frac{1}{r} \right] \right) \\ &= -\frac{i\hbar Z e^2}{2m} \epsilon_{ijk} \left( \frac{x_i}{r^3} L_j - L_i \frac{x_j}{r^3} \right) \\ &= \frac{i\hbar Z e^2}{2m} \underbrace{\epsilon_{ijk} \left( L_i \frac{x_j}{r^3} - \frac{x_i}{r^3} L_j \right)}_{A'} \end{aligned} \quad (21)$$

Now compare (20) with (21), indeed they are equal because

$$\begin{aligned}
A' &= \epsilon_{ijk} \left( L_i \frac{x_j}{r^3} - \frac{x_i}{r^3} L_j \right) \\
&= \epsilon_{ijk} \left( \epsilon_{lmi} x_l p_m \frac{x_j}{r^3} - \epsilon_{lmj} \frac{x_i}{r^3} x_l p_m \right) \\
&= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) x_l p_m \frac{x_j}{r^3} + (\delta_{il} \delta_{km} - \delta_{im} \delta_{kl}) \frac{x_i}{r^3} x_l p_m \\
&= x_j p_k \frac{x_j}{r^3} - x_k p_j \frac{x_j}{r^3} + \frac{x_i x_i}{r^3} p_k - \frac{x_i x_k}{r^3} p_i \\
&= (p_k x_j + [x_j, p_k]) \frac{x_j}{r^3} - (p_j x_k + [x_k, p_j]) \frac{x_j}{r^3} + \frac{x_i x_i}{r^3} p_k - \frac{x_i x_k}{r^3} p_i \\
&= p_k \frac{x_j x_j}{r^3} + i\hbar \delta_{jk} \frac{x_j}{r^3} - p_j \frac{x_j x_k}{r^3} - i\hbar \delta_{jk} \frac{x_j}{r^3} + \frac{x_i x_i}{r^3} p_k - \frac{x_i x_k}{r^3} p_i \\
&= B'
\end{aligned}$$

### 3. Proof of (5).

$$L \cdot M = L_k M_k = \frac{1}{2m} \overbrace{\epsilon_{ijk} [L_k (p_i L_j - L_i p_j)]}^A - Z e^2 \overbrace{L_k \frac{x_k}{r}}^B \quad (22)$$

where

$$\begin{aligned}
A &= \epsilon_{ijk} (L_k p_i L_j - L_k L_i p_j) = \epsilon_{ijk} L_k p_i L_j - \epsilon_{ijk} L_k L_i p_j \\
&= \epsilon_{ijk} (p_i L_k - [p_i, L_k]) L_j - \epsilon_{ijk} L_k L_i p_j \\
&= \epsilon_{ijk} p_i L_k L_j - \epsilon_{ijk} (i\hbar \epsilon_{ikj} p_j) L_j - \epsilon_{ijk} L_k L_i p_j \\
&= -i\hbar p_i L_i + i\hbar \epsilon_{ijk}^2 p_j L_j - i\hbar L_j p_j \\
&= 0 \\
B &= L_k \frac{x_k}{r} = \epsilon_{ijk} x_i p_j \frac{x_k}{r} = \epsilon_{ijk} x_i \left( \frac{x_k}{r} p_j + \left[ p_j, \frac{x_k}{r} \right] \right) \\
&= \epsilon_{ijk} x_i \frac{x_k}{r} p_j - i\hbar \epsilon_{ijk} x_i \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \\
&= 0
\end{aligned}$$

(use  $\epsilon_{ijk} L_i L_j = (\mathbf{L} \times \mathbf{L})_k = i\hbar L_k$  etc.)  
(use (16), (17))

### 4. Proof of (6).

This trivially follows from (3) (by setting  $i = j$ ) and (5).

### 5. Proof of (7).

Expanding  $M^2$  on the LHS and  $H$  on the RHS, we get

$$\begin{aligned}
\text{LHS} &= M_k M_k = \left[ \frac{1}{2m} \epsilon_{ijk} (p_i L_j - L_i p_j) - Z e^2 \frac{x_k}{r} \right] \cdot \left[ \frac{1}{2m} \epsilon_{lmk} (p_l L_m - L_l p_m) - Z e^2 \frac{x_k}{r} \right] \\
&= \overbrace{\frac{1}{4m^2} \epsilon_{ijk} \epsilon_{lmk} (p_i L_j - L_i p_j) (p_l L_m - L_l p_m)}^A - \overbrace{\frac{Z e^2}{2m} \epsilon_{ijk} \left[ (p_i L_j - L_i p_j) \frac{x_k}{r} + \frac{x_k}{r} (p_i L_j - L_i p_j) \right]}^B + \overbrace{Z^2 e^4 \frac{x_k x_k}{r^2}}^C \\
\text{RHS} &= \frac{2}{m} \left( \frac{\mathbf{p}^2}{2m} - Z e^2 \frac{1}{r} \right) (\mathbf{L}^2 + \hbar^2) + Z^2 e^4 \\
&= \overbrace{\frac{\mathbf{p}^2}{m^2} (\mathbf{L}^2 + \hbar^2)}^{A'} - \overbrace{\frac{2Z e^2}{m} \frac{1}{r} (\mathbf{L}^2 + \hbar^2)}^{B'} + \overbrace{Z^2 e^4}^{C'}
\end{aligned}$$

For dimensions to match, we must have  $A = A', B = B', C = C'$ , where  $C = C'$  is already trivial to see.

To prove  $B = B'$ , it's equivalent to prove

$$\epsilon_{ijk} \left[ (p_i L_j - L_i p_j) \frac{x_k}{r} + \frac{x_k}{r} (p_i L_j - L_i p_j) \right] = \frac{4}{r} (\mathbf{L}^2 + \hbar^2) \quad (23)$$

Let's break the LHS of (23) into

$$\overbrace{\epsilon_{ijk} p_i L_j \frac{x_k}{r}}^{B_1} - \overbrace{\epsilon_{ijk} L_i p_j \frac{x_k}{r}}^{B_2} + \overbrace{\epsilon_{ijk} \frac{x_k}{r} p_i L_j}^{B_3} - \overbrace{\epsilon_{ijk} \frac{x_k}{r} L_i p_j}^{B_4}$$

where we immediately see that

$$B_3 = \frac{1}{r} L_j L_j = \frac{1}{r} \mathbf{L}^2$$

Furthermore

$$B_2 = \epsilon_{ijk} L_i \left( \frac{x_k}{r} p_j + \left[ p_j, \frac{x_k}{r} \right] \right) = \epsilon_{ijk} L_i \left[ \frac{x_k}{r} p_j - i\hbar \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \right] = L_i \left( -\frac{1}{r} L_i \right) = -\frac{1}{r} \mathbf{L}^2$$

where in the last step, we have used (13).

Lastly

$$\begin{aligned} B_1 &= \epsilon_{ijk} p_i \left( \frac{x_k}{r} L_j - \left[ \frac{x_k}{r}, L_j \right] \right) \\ &= \epsilon_{ijk} \left( \frac{x_k}{r} p_i + \left[ p_i, \frac{x_k}{r} \right] \right) L_j - \epsilon_{ijk} p_i \left( i\hbar \epsilon_{kji} \frac{x_i}{r} \right) \\ &= \frac{1}{r} L_j L_j - i\hbar \epsilon_{ijk} \left( \frac{\delta_{ik}}{r} - \frac{x_i x_k}{r^3} \right) L_j + i\hbar \epsilon_{ijk}^2 p_i \frac{x_i}{r} \\ &= \frac{1}{r} \mathbf{L}^2 + i\hbar \epsilon_{ijk}^2 p_i \frac{x_i}{r} \\ B_4 &= \epsilon_{ijk} \left( L_i \frac{x_k}{r} + \left[ \frac{x_k}{r}, L_i \right] \right) p_j \\ &= -L_i \frac{1}{r} L_i + \epsilon_{ijk} \left( i\hbar \epsilon_{kij} \frac{x_j}{r} \right) p_j \\ &= -\frac{1}{r} \mathbf{L}^2 + i\hbar \epsilon_{ijk}^2 \frac{x_j}{r} p_j \end{aligned}$$

Thus

$$B_1 - B_2 + B_3 - B_4 = \frac{4}{r} \mathbf{L}^2 + i\hbar \epsilon_{ijk}^2 \left[ p_i, \frac{x_i}{r} \right] = \frac{4}{r} \mathbf{L}^2 + \hbar^2 \epsilon_{ijk}^2 \left( \frac{1}{r} - \frac{x_i x_i}{r^3} \right) = \frac{4}{r} (\mathbf{L}^2 + \hbar^2) \quad (24)$$

It still remains to show  $A = A'$ , which is equivalent to

$$\epsilon_{ijk} \epsilon_{lmk} (p_i L_j - L_i p_j) (p_l L_m - L_l p_m) = 4\mathbf{p}^2 (\mathbf{L}^2 + \hbar^2) \quad (25)$$

Using (18), the LHS of (25) can be broken into

$$\begin{aligned} \text{LHS} &= (p_i L_j - L_i p_j) (p_i L_j - L_i p_j) - (p_i L_j - L_i p_j) (p_j L_i - L_j p_i) \\ &= \overbrace{p_i L_j p_i L_j}^{A_1} - \overbrace{p_i L_j L_i p_j}^{A_2} - \overbrace{L_i p_j p_i L_j}^{A_3} + \overbrace{L_i p_j L_i p_j}^{A_4} - \overbrace{p_i L_j p_j L_i}^{A_5} + \overbrace{p_i L_j L_j p_i}^{A_6} + \overbrace{L_i p_j p_j L_i}^{A_7} - \overbrace{L_i p_j L_j p_i}^{A_8} \end{aligned}$$

Attacking them one by one,

$$\begin{aligned} A_1 &= p_i L_j p_i L_j = p_i (p_i L_j - [p_i, L_j]) L_j = p_i^2 L_j^2 - i\hbar \epsilon_{ijk} p_i p_k L_j = \mathbf{p}^2 \mathbf{L}^2 \\ A_2 &= p_i L_j L_i p_j = p_i (L_i L_j - [L_i, L_j]) p_j = (\mathbf{p} \cdot \mathbf{L}) \cdot (\mathbf{L} \cdot \mathbf{p}) - i\hbar \epsilon_{ijk} p_i L_k p_j = -i\hbar \epsilon_{ijk} p_i L_k p_j \\ &= -i\hbar \epsilon_{ijk} (L_k p_i + [p_i, L_k]) p_j = -i\hbar \epsilon_{ijk} L_k p_i p_j - i\hbar \epsilon_{ijk} (i\hbar \epsilon_{ikj} p_j) p_j \\ &= -\hbar^2 \epsilon_{ijk}^2 p_j^2 = -2\hbar^2 \mathbf{p}^2 \\ A_3 &= L_i p_j p_i L_j = L_i p_i p_j L_j = (\mathbf{L} \cdot \mathbf{p}) \cdot (\mathbf{p} \cdot \mathbf{L}) = 0 \\ A_4 &= L_i p_j L_i p_j = L_i (L_i p_j + [p_j, L_i]) p_j = L_i^2 p_j^2 - i\hbar \epsilon_{ijk} L_i p_k p_j = \mathbf{L}^2 \mathbf{p}^2 \\ A_5 &= p_i L_j p_j L_i = p_i (\mathbf{L} \cdot \mathbf{p}) L_i = 0 \\ A_6 &= p_i L_j L_j p_i = (L_j p_i + [p_i, L_j]) (p_i L_j - [p_i, L_j]) = (L_j p_i + i\hbar \epsilon_{ijk} p_k) (p_i L_j - i\hbar \epsilon_{ijk} p_k) \\ &= L_j p_i^2 L_j + i\hbar \epsilon_{ijk} (p_k p_i L_j - L_j p_i p_k) + \hbar^2 \epsilon_{ijk}^2 p_k^2 \quad (\mathbf{p}^2 \text{ is scalar}) \\ &= \mathbf{p}^2 \mathbf{L}^2 + 2\hbar^2 \mathbf{p}^2 \\ A_7 &= L_i p_j p_j L_i = L_i (\mathbf{p}^2) L_i = \mathbf{p}^2 \mathbf{L}^2 \\ A_8 &= L_i p_j L_j p_i = L_i (\mathbf{p} \cdot \mathbf{L}) L_i = 0 \end{aligned}$$

Noting that  $\mathbf{L}^2 \mathbf{p}^2 = \mathbf{p}^2 \mathbf{L}^2$  since  $\mathbf{p}^2$  is a scalar, thus we have proved (25), hence (7).

## 6. Proof of (8).

First, before going into details, we should note that for  $\epsilon_{ijk} = 1$ ,  $[M_i, M_j]$  is the  $k$ -component of the vector operator  $\mathbf{M} \times \mathbf{M}$ . Let

$$\mathbf{W} = \mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}$$

be a vector operator, then

$$\begin{aligned} M_i &= \frac{1}{2m} W_i - Z e^2 \frac{x_i}{r} \\ M_j &= \frac{1}{2m} W_j - Z e^2 \frac{x_j}{r} \end{aligned}$$

Thus

$$[M_i, M_j] = \frac{1}{4m^2} [W_i, W_j] - \frac{Z e^2}{2m} \left( \left[ W_i, \frac{x_j}{r} \right] + \left[ \frac{x_i}{r}, W_j \right] \right) \quad (26)$$

But the RHS of (8) is

$$-i\hbar\epsilon_{ijk} \frac{2}{m} \left( \frac{\mathbf{p}^2}{2m} - \frac{Z e^2}{r} \right) L_k = -i\hbar\epsilon_{ijk} \frac{\mathbf{p}^2}{m^2} L_k + i\hbar\epsilon_{ijk} \frac{2Z e^2}{m} \frac{1}{r} L_k \quad (27)$$

Comparing (26) with (27), it's clear that we should eventually prove

$$[W_i, W_j] = -4i\hbar\epsilon_{ijk} \mathbf{p}^2 L_k \quad (28)$$

$$\left[ W_i, \frac{x_j}{r} \right] + \left[ \frac{x_i}{r}, W_j \right] = -4i\hbar\epsilon_{ijk} \frac{1}{r} L_k \quad (29)$$

(a) **Proof of (28).** By virtue of being a vector operator,  $\mathbf{W}$  satisfies

$$[W_i, L_j] = i\hbar\epsilon_{ijk} W_k \quad (30)$$

Also

$$\begin{aligned} [W_i, p_j] &= \epsilon_{lmi} [p_l L_m - L_l p_m, p_j] = \epsilon_{lmi} (p_l [L_m, p_j] - [L_l, p_j] p_m) \\ &= i\hbar\epsilon_{lmi} (\epsilon_{mjk} p_l p_k - \epsilon_{ljk} p_k p_m) \\ &= i\hbar\epsilon_{lmi} \epsilon_{mjk} p_l p_k - i\hbar\epsilon_{lmi} \epsilon_{ljk} p_k p_m \quad (l \leftrightarrow m \text{ in 2nd term}) \\ &= i\hbar\epsilon_{lmi} \epsilon_{mjk} p_l p_k - i\hbar\epsilon_{mli} \epsilon_{mjk} p_k p_l \\ &= i\hbar\epsilon_{lmi} \epsilon_{mjk} p_l p_k + i\hbar\epsilon_{lmi} \epsilon_{mjk} p_k p_l \\ &= 2i\hbar (\delta_{lk} \delta_{ij} - \delta_{lj} \delta_{ik}) p_l p_k \\ &= 2i\hbar (\delta_{ij} \mathbf{p}^2 - p_i p_j) \end{aligned} \quad (31)$$

It follows that

$$\begin{aligned} [W_i, W_j] &= \epsilon_{lmj} [W_i, p_l L_m - L_l p_m] \\ &= \epsilon_{lmj} ([W_i, p_l] L_m + p_l [W_i, L_m] - [W_i, L_l] p_m - L_l [W_i, p_m]) \\ &= \underbrace{\epsilon_{lmj} [W_i, p_l] L_m}_A + \underbrace{\epsilon_{lmj} p_l [W_i, L_m]}_B - \underbrace{\epsilon_{lmj} [W_i, L_l] p_m}_C - \underbrace{\epsilon_{lmj} L_l [W_i, p_m]}_D \end{aligned}$$

Using (30) and (31), we have

$$\begin{aligned} A &= 2i\hbar\epsilon_{lmj} (\delta_{il} \mathbf{p}^2 - p_i p_l) L_m = 2i\hbar\epsilon_{lmj} \mathbf{p}^2 L_m - 2i\hbar\epsilon_{lmj} p_i p_l L_m \\ B &= i\hbar\epsilon_{lmj} p_l \epsilon_{imk} W_k = i\hbar (\delta_{il} \delta_{jk} - \delta_{ij} \delta_{lk}) p_l W_k = i\hbar p_i W_j - i\hbar \delta_{ij} (\mathbf{p} \cdot \mathbf{W}) \\ C &= i\hbar\epsilon_{lmj} \epsilon_{ilk} W_k p_m = i\hbar (\delta_{ij} \delta_{mk} - \delta_{im} \delta_{jk}) W_k p_m = i\hbar \delta_{ij} (\mathbf{W} \cdot \mathbf{p}) - i\hbar W_j p_i \\ D &= 2i\hbar\epsilon_{lmj} L_l (\delta_{im} \mathbf{p}^2 - p_i p_m) = 2i\hbar\epsilon_{lij} L_l \mathbf{p}^2 - 2i\hbar\epsilon_{lmj} L_l p_i p_m \end{aligned}$$

Compare  $A + B - C - D$  with RHS of (28), all that remains to show is

$$-2\epsilon_{lmj} p_i p_l L_m + p_i W_j - \delta_{ij} (\mathbf{p} \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{p}) + W_j p_i + 2\epsilon_{lmj} L_l p_i p_m = 0$$

which is easily proved considering the following

$$\begin{aligned}
\mathbf{p} \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{p} &= \epsilon_{lmk} [p_k (p_l L_m - L_l p_m) + (p_l L_m - L_l p_m) p_k] \\
&= \epsilon_{lmk} p_k p_l L_m - \epsilon_{lmk} p_k L_l p_m + \epsilon_{lmk} p_l L_m p_k - \epsilon_{lmk} L_l p_m p_k \quad (lmk \rightarrow mkl \text{ in 2nd term}) \\
&= 0 \\
p_i W_j + W_j p_i &= \epsilon_{lmj} [p_i (p_l L_m - L_l p_m) + (p_l L_m - L_l p_m) p_i] \\
&= \epsilon_{lmj} p_i p_l L_m - \epsilon_{lmj} p_i L_l p_m + \epsilon_{lmj} p_l L_m p_i - \epsilon_{lmj} L_l p_m p_i \\
&= \epsilon_{lmj} p_i p_l L_m - \epsilon_{lmj} p_i (p_m L_l + [L_l, p_m]) + \epsilon_{lmj} (L_m p_l + [p_l, L_m]) p_i - \epsilon_{lmj} L_l p_m p_i \\
&= \underbrace{\epsilon_{lmj} p_i p_l L_m - \epsilon_{lmj} p_i p_m L_l}_{2\epsilon_{lmj} p_i p_l L_m} - \overbrace{\epsilon_{lmj} p_i [L_l, p_m]}^{i\hbar p_i p_j} + \overbrace{\epsilon_{lmj} [p_l, L_m] p_i}^{i\hbar p_j p_i} + \underbrace{\epsilon_{lmj} L_m p_l p_i - \epsilon_{lmj} L_l p_m p_i}_{-2\epsilon_{lmj} L_l p_m p_i} \\
&= 2\epsilon_{lmj} p_i p_l L_m - 2\epsilon_{lmj} L_l p_m p_i
\end{aligned}$$

(b) **Proof of (29).** Rewriting LHS of (29), we have

$$\begin{aligned}
\text{LHS} &= \epsilon_{lmi} \left[ p_l L_m - L_l p_m, \frac{x_j}{r} \right] + \epsilon_{stj} \left[ \frac{x_i}{r}, p_s L_t - L_s p_t \right] \\
&= \overbrace{\epsilon_{lmi} \left[ p_l L_m, \frac{x_j}{r} \right]}^A - \overbrace{\epsilon_{stj} \left[ p_s L_t, \frac{x_i}{r} \right]}^B + \overbrace{\epsilon_{stj} \left[ L_s p_t, \frac{x_i}{r} \right]}^C - \overbrace{\epsilon_{lmi} \left[ L_l p_m, \frac{x_j}{r} \right]}^D
\end{aligned}$$

where

$$\begin{aligned}
A &= \epsilon_{lmi} \left( p_l \left[ L_m, \frac{x_j}{r} \right] + \left[ p_l, \frac{x_j}{r} \right] L_m \right) \\
&= \epsilon_{lmi} \left[ i\hbar \epsilon_{mjk} p_l \frac{x_k}{r} - i\hbar \left( \frac{\delta_{lj}}{r} - \frac{x_l x_j}{r^3} \right) L_m \right] \\
&= i\hbar \epsilon_{lmi} \epsilon_{mjk} p_l \frac{x_k}{r} - i\hbar \epsilon_{jmi} \frac{1}{r} L_m + i\hbar \epsilon_{lmi} \frac{x_l x_j}{r^3} L_m \\
&= i\hbar (\delta_{ij} \delta_{kl} - \delta_{jl} \delta_{ik}) p_l \frac{x_k}{r} - i\hbar \epsilon_{jmi} \frac{1}{r} L_m + i\hbar \epsilon_{lmi} \frac{x_l x_j}{r^3} L_m \\
&= i\hbar \delta_{ij} p_k \frac{x_k}{r} - i\hbar p_j \frac{x_i}{r} - i\hbar \epsilon_{ijm} \frac{1}{r} L_m + i\hbar \epsilon_{lmi} \frac{x_l x_j}{r^3} L_m \\
C &= \epsilon_{stj} \left( L_s \left[ p_t, \frac{x_i}{r} \right] + \left[ L_s, \frac{x_i}{r} \right] p_t \right) \\
&= \epsilon_{stj} \left[ -i\hbar L_s \left( \frac{\delta_{ti}}{r} - \frac{x_t x_i}{r^3} \right) + i\hbar \epsilon_{sik} \frac{x_k}{r} p_t \right] \\
&= -i\hbar \epsilon_{sij} L_s \frac{1}{r} + i\hbar \epsilon_{stj} L_s \frac{x_t x_i}{r^3} + i\hbar \epsilon_{stj} \epsilon_{sik} \frac{x_k}{r} p_t \\
&= -i\hbar \epsilon_{sij} L_s \frac{1}{r} + i\hbar \epsilon_{stj} L_s \frac{x_t x_i}{r^3} + i\hbar (\delta_{it} \delta_{jk} - \delta_{ij} \delta_{tk}) \frac{x_k}{r} p_t \\
&= -i\hbar \epsilon_{ijs} L_s \frac{1}{r} + i\hbar \epsilon_{stj} L_s \frac{x_t x_i}{r^3} + i\hbar \frac{x_j}{r} p_i - i\hbar \delta_{ij} \frac{x_k}{r} p_k
\end{aligned}$$

With mapping of indices  $i \leftrightarrow j, l \leftrightarrow s, m \leftrightarrow t$ , we obtain the form of  $B, D$  from  $A, C$ :

$$\begin{aligned}
B &= i\hbar \delta_{ij} p_k \frac{x_k}{r} - i\hbar p_i \frac{x_j}{r} - i\hbar \epsilon_{jit} \frac{1}{r} L_t + i\hbar \epsilon_{stj} \frac{x_s x_i}{r^3} L_t \\
D &= -i\hbar \epsilon_{jil} L_l \frac{1}{r} + i\hbar \epsilon_{lmi} L_l \frac{x_m x_j}{r^3} + i\hbar \frac{x_i}{r} p_j - i\hbar \delta_{ij} \frac{x_k}{r} p_k
\end{aligned}$$

Now combine  $A - B + C - D$ , notice all terms with  $\delta_{ij}$  cancel out. Also since  $1/r$  commutes with  $L_i$ , we have also accounted for the  $-4i\hbar \epsilon_{ijk} L_k / r$  term on the RHS of (29), all it remains to prove is

$$-p_j \frac{x_i}{r} + \overbrace{\epsilon_{lmi} \frac{x_l x_j}{r^3} L_m}^{A_1} + p_i \frac{x_j}{r} - \overbrace{\epsilon_{stj} \frac{x_s x_i}{r^3} L_t}^{B_1} + \overbrace{\epsilon_{stj} L_s \frac{x_t x_i}{r^3}}^{C_1} + \frac{x_j}{r} p_i - \overbrace{\epsilon_{lmi} L_l \frac{x_m x_j}{r^3}}^{D_1} - \frac{x_i}{r} p_j = 0 \quad (32)$$

Next notice

$$\begin{aligned}
A_1 &= \epsilon_{lmi} \frac{x_l x_j}{r^3} \epsilon_{uvm} x_u p_v = -(\delta_{lu} \delta_{iv} - \delta_{lv} \delta_{iu}) \frac{x_l x_j}{r^3} x_u p_v \\
&= -\frac{x_l x_j x_l}{r^3} p_i + \frac{x_l x_j x_i}{r^3} p_l = -\frac{x_j}{r} p_i + \frac{x_i x_j}{r^3} (\mathbf{x} \cdot \mathbf{p}) \\
C_1 &= \epsilon_{stj} \epsilon_{uvs} x_u p_v \frac{x_t x_i}{r^3} = (\delta_{tu} \delta_{jv} - \delta_{tv} \delta_{ju}) x_u p_v \frac{x_t x_i}{r^3} \\
&= (x_t p_j - x_j p_t) \frac{x_t x_i}{r^3} = (p_j x_t - p_t x_j) \frac{x_t x_i}{r^3} \\
&= p_j \frac{x_t x_t x_i}{r^3} - p_t \frac{x_j x_t x_i}{r^3} = p_j \frac{x_i}{r} - (\mathbf{p} \cdot \mathbf{x}) \frac{x_i x_j}{r^3}
\end{aligned}$$

Lastly, with the usual index mapping  $i \leftrightarrow j$ , we obtain  $B_1, D_1$  as

$$\begin{aligned}
B_1 &= -\frac{x_i}{r} p_j + \frac{x_i x_j}{r^3} (\mathbf{x} \cdot \mathbf{p}) \\
D_1 &= p_i \frac{x_j}{r} - (\mathbf{p} \cdot \mathbf{x}) \frac{x_i x_j}{r^3}
\end{aligned}$$

We see indeed all terms on the LHS of (32) cancel out.