In these notes, we fill the gaps between equation (7.41) and equation (7.46).
We start with the 1st-order perturbed energy shift (eq 7.41)

$$\Delta_{(1s)^2} = \left\langle \frac{e^2}{r_{12}} \right\rangle_{(1s)^2} = \iint d^3x_1 d^3x_2 \frac{Z^6}{\pi^2 a_0^6} e^{-2Z(r_1+r_2)/a_0} \frac{e^2}{r_{12}} \tag{1}$$

Let $\angle_{r_1,r_2} = \gamma$, then

$$\begin{aligned} \frac{1}{r_{12}} &= \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\gamma}} \\ &= \frac{1}{r_> \sqrt{1 - 2\cos\gamma \left(\frac{r_<}{r_>}\right) + \left(\frac{r_<}{r_>}\right)^2}} \end{aligned} \tag{2}$$

But recall $g(x,t) = (1-2tx+t^2)^{-1/2}$, for $x,t \in [-1,1]$ is the generating function for the Legendre polynomials

$$(1-2tx+t^2)^{-1/2} = \sum_l P_l(x) t^l \tag{3}$$

thus (2) becomes equation (7.42)

$$\frac{1}{r_{12}} = \frac{1}{r_>} \sum_l P_l(\cos\gamma) \left(\frac{r_<}{r_>}\right)^l = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{4}$$

Next, we can use the Addition Theorem of spherical harmonics (proved in earlier notes)

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_m Y_l^{m*}(\theta_1,\phi_1) Y_l^m(\theta_2,\phi_2) \tag{5}$$

Insert (4),(5) back into (1) and do the double integral in spherical coordinates,

$$\Delta_{(1s)^2} = \frac{Z^6 e^2}{\pi^2 a_0^6} \int r_1^2 dr_1 \int r_2^2 dr_2 e^{-2Z(r_1+r_2)/a_0} \sum_{l,m} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} \int d\Omega_1 \int d\Omega_2 Y_l^{m*}(\theta_1,\phi_1) Y_l^m(\theta_2,\phi_2) \tag{6}$$

Next, notice

$$\int d\Omega Y_l^m(\theta,\phi) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta C_l^m e^{im\phi} P_l^m(\cos\theta) \tag{7}$$

where the ϕ integral will vanish unless $m=0$, in which case the θ integral in (7) becomes (let $y = \cos\theta$)

$$\int_{-1}^{1} dy P_l(y) = \int_{-1}^{1} dy P_l(y) \overbrace{P_0(y)}^{=1} = 2\delta_{l0} \tag{8}$$

Recall $C_{l=0}^{m=0} = 1/\sqrt{4\pi}$, so we have

$$\int d\Omega Y_l^m(\theta,\phi) = \frac{4\pi}{\sqrt{4\pi}} \delta_{l0}\delta_{m0} = \sqrt{4\pi}\delta_{l0}\delta_{m0} \tag{9}$$

Now plugging (9) back into (6) produces

$$\Delta_{(1s)^2} = \frac{Z^6 e^2}{\pi^2 a_0^6} (4\pi)^2 \overbrace{\int r_1^2 dr_1 \int r_2^2 dr_2 e^{-2Z(r_1+r_2)/a_0} \frac{1}{r_>}}^{R} \tag{10}$$

where the radial integral is equivalent to

$$\begin{aligned} R &= \int_0^\infty r_1^2 dr_1 \left\{ \left[\int_0^{r_1} r_2^2 dr_2 e^{-2Z(r_1+r_2)/a_0} \frac{1}{r_1} \right] + \left[\int_{r_1}^\infty r_2^2 dr_2 e^{-2Z(r_1+r_2)/a_0} \frac{1}{r_2} \right] \right\} \qquad \left(\text{define } \beta \equiv \frac{2Z}{a_0}\right) \\ &= \int_0^\infty r_1 dr_1 e^{-\beta r_1} \underbrace{\int_0^{r_1} r_2^2 dr_2 e^{-\beta r_2}}_{A} + \int_0^\infty r_1^2 dr_1 e^{-\beta r_1} \underbrace{\int_{r_1}^\infty r_2 dr_2 e^{-\beta r_2}}_{B} \end{aligned} \tag{11}$$

For general integration range $[u,v]$,

$$
\begin{aligned}
\int_u^v re^{-\lambda r}dr &= -\frac{1}{\lambda}e^{-\lambda r}r\Big|_u^v + \frac{1}{\lambda}\int_u^v e^{-\lambda r}dr \\
&= -\frac{1}{\lambda}e^{-\lambda r}r\Big|_u^v - \frac{1}{\lambda^2}e^{-\lambda r}\Big|_u^v \\
&= -\left(\frac{1}{\lambda}e^{-\lambda r}r + \frac{1}{\lambda^2}e^{-\lambda r}\right)\Big|_u^v \qquad (12) \\
\int_u^v r^2e^{-\lambda r}dr &= -\frac{1}{\lambda}e^{-\lambda r}r^2\Big|_u^v + \frac{1}{\lambda}\int_u^v e^{-\lambda r}2rdr \qquad \text{(by (12))} \\
&= -\left(\frac{1}{\lambda}e^{-\lambda r}r^2 + \frac{2}{\lambda^2}e^{-\lambda r}r + \frac{2}{\lambda^3}e^{-\lambda r}\right)\Big|_u^v \qquad (13)
\end{aligned}
$$

Then the A, B in (11) can be evaluated as

$$
\begin{aligned}
A &= -\left(\frac{1}{\beta}e^{-\beta r_1}r_1^2 + \frac{2}{\beta^2}e^{-\beta r_1}r_1 + \frac{2}{\beta^3}e^{-\beta r_1} - \frac{2}{\beta^3}\right) \qquad (14) \\
B &= \frac{1}{\beta}e^{-\beta r_1}r_1 + \frac{1}{\beta^2}e^{-\beta r_1} \qquad (15)
\end{aligned}
$$

Now the radial integral (11) becomes

$$
\begin{aligned}
R &= \int_0^\infty -dr_1\left(\frac{1}{\beta}e^{-2\beta r_1}r_1^3 + \frac{2}{\beta^2}e^{-2\beta r_1}r_1^2 + \frac{2}{\beta^3}e^{-2\beta r_1}r_1 - \frac{2}{\beta^3}e^{-\beta r_1}r_1\right) + \int_0^\infty dr_1\left(\frac{1}{\beta}e^{-2\beta r_1}r_1^3 + \frac{1}{\beta^2}e^{-2\beta r_1}r_1^2\right) \\
&= -\frac{1}{\beta^2}\int_0^\infty e^{-2\beta r_1}r_1^2dr_1 - \frac{2}{\beta^3}\int_0^\infty e^{-2\beta r_1}r_1dr_1 + \frac{2}{\beta^3}\int_0^\infty e^{-\beta r_1}r_1dr_1 \qquad \text{(by (12), (13))} \\
&= -\frac{1}{\beta^2}\frac{2}{(2\beta)^3} - \frac{2}{\beta^3}\frac{1}{(2\beta)^2} + \frac{2}{\beta^3}\frac{1}{\beta^2} \\
&= \frac{5}{4\beta^2} \\
&= \frac{5}{128}\frac{a_0^5}{Z^5} \qquad (16)
\end{aligned}
$$

Plug back to (10), we eventually get

$$
\begin{aligned}
\Delta_{(1s)^2} &= \frac{Z^6e^2}{\pi^2a_0^6}(4\pi)^2\frac{5}{128}\frac{a_0^5}{Z^5} \\
&= \frac{5}{4}\frac{e^2}{a_0} \qquad (17)
\end{aligned}
$$