Deriving free and clamped cubic spline interpolation and their algorithms

Given $n+1$ knot points x_i , with $i=0,\dots,n$,and the values y_i at these knot points the interpolating cubic spline is defined by:

$$S(x)=S_i(x),\ \ x_i \le x \le x_{i+1},\ \ i=0,\dots n-1 \tag{1}$$

$$S_i(x)=a_i+b_i(x-x_i)+c_i(x-x_i)^2+d_i(x-x_i)^3 \tag{2}$$

The conditions for the cubic spline are:

$$S_i(x_i)=y_i,\ \ i=0,\dots n-1 \ \text{ and } \ S_{n-1}(x_n)=y_n \tag{3}$$

$$S_{i+1}(x_{i+1})=S_i(x_{i+1}),\ \ i=0,\dots,n-2 \tag{4}$$

$$S'_{i+1}(x_{i+1})=S'_i(x_{i+1}),\ \ i=0,\dots,n-2 \tag{5}$$

$$S''_{i+1}(x_{i+1})=S''_i(x_{i+1}),\ \ i=0,\dots,n-2 \tag{6}$$

The boundary conditions depend on whether it is a free or clamped spline:

$$\text{Free:} \quad S''_0(x_0)=S''_{n-1}(x_n)=0 \tag{7}$$

$$\text{Clamped:} \quad S'_0(x_0)=u,\ \ S'_{n-1}(x_n)=v \tag{8}$$

The aim is to find the coefficients: a_i, b_i, c_i and d_i

a) Deriving the set of equations that describe a_i, b_i, c_i and d_i

From the definition of the spline function (2) we derive:

$$S_i(x_i)=a_i,\ \ i=0,\dots,n-1 \tag{10}$$

$$S'_i(x_i)=b_i,\ \ i=0,\dots,n-1 \tag{11}$$

$$S''_i(x_i)=2c_i,\ \ i=0,\dots,n-1 \tag{12}$$

From (3) and (10) we easily find:

$$a_i=y_i,\ \ i=0,\dots,n-1 \tag{13}$$

We now define:

$$h_i=x_{i+1}-x_i,\ \ i=0,\dots,n-1 \tag{14}$$

This means we can write $S_i(x_{i+1})$, $S'_i(x_{i+1})$ and $S''_i(x_{i+1})$ as:

$$S_i(x_{i+1})=a_i+b_i h_i+c_i h_i^2+d_i h_i^3 \tag{15}$$

$$S'_i(x_{i+1})=b_i+2c_i h_i+3d_i h_i^2 \tag{16}$$

$$S''_i(x_{i+1})=2c_i+6d_i h_i \tag{17}$$

Combining (4), (5), (6) with (10), (11), (12) and (15), (16), (17), we get:

$$a_{i+1}=a_i+b_i h_i+c_i h_i^2+d_i h_i^3 \tag{15}$$

$$b_{i+1}=b_i+2c_i h_i+3d_i h_i^2 \tag{16}$$

$$c_{i+1}=c_i+3d_i h_i \tag{17}$$

We can rewrite this as:

$$d_i=\frac{c_{i+1}-c_i}{3h_i} \tag{18}$$

$$b_i=b_{i-1}+h_{i-1}(c_{i-1}+c_i) \tag{19}$$

$$b_i=\frac{a_{i+1}-a_i}{h_i}-\frac{h_i}{3}(2c_i+c_{i+1}) \tag{20}$$

Substituting (20) for i and $i-1$ into (19) yields:

$$\frac{a_{i+1}-a_i}{h_i}-\frac{h_i}{3}(2c_i+c_{i+1})=\frac{a_i-a_{i-1}}{h_{i-1}}-\frac{h_{i-1}}{3}(2c_{i-1}+c_i)+h_{i-1}(c_{i-1}+c_i) \tag{21}$$

Which can be rewritten as:

$$\frac{3}{h_i}(a_{i+1}-a_i)-\frac{3}{h_{i-1}}(a_i-a_{i-1})=h_{i-1}c_{i-1}+2(h_i+h_{i-1})c_i+h_i c_{i+1} \tag{22}$$

Because everything on the left hand side is known we can rename it as:

$$m_i=\frac{3}{h_i}(a_{i+1}-a_i)-\frac{3}{h_{i-1}}(a_i-a_{i-1}) \tag{23}$$

This means the set of equations we have to solve is:

$$m_i = h_{i-1} c_{i-1} + 2(h_i + h_{i-1}) c_i + h_i c_{i+1}, \quad i = 1, \dots, n-1 \qquad (24)$$

with c_i as the unknowns.

Once c_i is known then d_i and b_i can be found with (18) and (20)

The boundary conditions for c_0 and c_n come from (7) – for a free or natural spline – or from (8) – for a clamped spline.

b) Free or natural spline

From (7), (6) and (12) you easily find that:

$$c_0=0 \quad \text{and} \quad c_n=0 \tag{25}$$

We can now write the set of equations (24) in matrix form:

$$LC=M \tag{26}$$

or

$$\begin{bmatrix} 1 & 0 & & & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & & \\ 0 & h_1 & 2(h_1+h_2) & h_2 & & \\ \vdots & & & \ddots & & \\ 0 & \cdots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ m_1 \\ \vdots \\ 0 \end{bmatrix} \tag{27}$$

c) Clamped spline

From (8) and (11) we get:

$$b_0=u \quad \text{and} \quad b_n=v \tag{28}$$

Substituting b_0 in (20) we get:

$$u=\frac{a_1-a_0}{h_0}-\frac{h_0}{3}(2c_0+c_1) \tag{29}$$

Substituting b_n in (19) and then use (20) for b_{n-1} :

$$v=\frac{a_n-a_{n-1}}{h_{n-1}}-\frac{h_{n-1}}{3}(2c_{n-1}+c_n)+h_{n-1}(c_{n-1}+c_n) \tag{30}$$

rewriting (29) and (30) we get:

$$2h_0c_0+h_0c_1=\frac{3}{h_0}(a_1-a_0)-3u \tag{31}$$

$$h_{n-1}c_{n-1}+2h_{n-1}c_n=3v-\frac{3}{h_{n-1}}(a_n-a_{n-1}) \tag{32}$$

If we define $m_0=(3/h_0)(a_1-a_0)-3u$ and $m_n=3v-(3/h_{n-1})(a_n-a_{n-1})$, we can also write the set of equations (24) for a clamped spline in matrix form:

$$LC=M \tag{33}$$

or

$$\begin{bmatrix} 2h_0 & h_0 & & & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & & \\ 0 & h_1 & 2(h_1+h_2) & h_2 & & \\ \vdots & & & \ddots & & \\ 0 & \cdots & & & h_{n-1} & 2h_{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{bmatrix} \tag{34}$$

c) Solving the tridiagonal set of equations

Both for the free (27) and the clamped (34) spline the set of equations that need to be solved can be written in a tridiagonal matrix form $LC=M$, with the matrix elements $l_{i,j}, \; i,j=0,\ldots,n$ and $m_i, \; i=0,\ldots,n$ known.

To solve the set of equations we use the tridiagonal matrix algorithm (TDMA), also known as the Thomas algorithm.

Because L is tridiagonal we can define:

$$r_i=l_{i-1,i}, \; i=1,\ldots,n \text{ and } r_0=0 \tag{35}$$

$$s_i=l_{i,i}, \; i=0,\ldots,n \tag{36}$$

$$t_i=l_{i+1,i}, \; i=0,\ldots,n-1 \text{ and } t_n=0 \tag{37}$$

For each i the equation then becomes:

$$r_i c_{i-1}+s_i c_i+t_i c_{i+1}=m_i \tag{38}$$

We can now recursively change these equations, in order to eliminate the r_i coefficients. The first equation remains the same ($r_0=0$ anyway):

$$s_0 c_0+t_0 c_1=m_0 \tag{39}$$

we multiply the second equation with s_0 and subtract the first equation multiplied by r_1 :

$$s_0(r_1 c_0+s_1 c_1+t_1 c_2)-r_1(s_0 c_0+t_0 c_1)=s_0 m_1-r_1 m_0$$

$$(s_0 s_1-r_1 t_0) c_1+s_0 t_1 c_2=s_0 m_1-r_1 m_0 \tag{40}$$

If we define new coefficients $s'_1=s_0 s_1-r_1 t_0$, $t'_1=s_0 t_1$ and $m'_1=s_0 m_1-r_1 m_0$, the second equation is rewritten as:

$$s'_1 c_1+t'_1 c_2=m'_1 \tag{41}$$

We can now apply the same trick with the new second equation and the third equation, resulting in:

$$s'_2 c_2+t'_2 c_3=m'_2 \tag{42}$$

with $s'_2=s'_1 s_2-r_2 t'_1$, $t'_2=s'_1 t_2$ and $m'_2=s'_1 m_2-r_2 m'_1$
and again for the 4th equation et cetera.

Hence by recursively defining the new coefficients:

$$s'_i=s'_{i-1} s_i-r_i t'_{i-1}, \; i=1,\ldots,n \text{ and } s'_0=s_0 \tag{43}$$

$$t'_i=s'_{i-1} t_i, \; i=1,\ldots,n \text{ and } t'_0=t_0 \tag{44}$$

$$m'_i=s'_{i-1} m_i-r_i m'_{i-1}, \; i=1,\ldots,n \text{ and } m'_0=m_0 \tag{45}$$

we have simplified the set of equations to:

$$s'_i c_i+t'_i c_{i+1}=m'_i \tag{46}$$

Because $t_n=0$, also $t'_n=0$ and the last equation has the trivial solution:

$$c_n=\frac{m'_n}{s'_n} \tag{47}$$

Backwards recursively, we find the solutions for c_i :

$$c_i=\frac{m'_i-t'_i c_{i+1}}{s'_i} \tag{48}$$