Machine Learning Assignment 53

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Problem 1

(a) *Given that* $X \sim U[0,1]$*, compute* $\boldsymbol{Cov}[X, X^2]$.

$$\begin{aligned}
E[X] &= \frac{1}{b-a} = 1 \\
E[X^2] &= \frac{a^2+ab+b^2}{3} = \frac{1}{3} \\
\mathrm{Cov}[X, X^2] &= (x_1 - \overline{x_1})(x_2 - \overline{x_2})\mathrm{d}x \\
&= \int_0^1 (x-1)(x^2 - \frac{1}{3})\mathrm{d}x \\
&= \int_0^1 x^3 - x^2 - x + 1\mathrm{d}x \\
&= \frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{6} + \frac{x}{3}\bigg|_0^1 \\
&= \frac{1}{12}
\end{aligned}$$

(b) *Given that* $X_1, X_2 \sim U[0,1]$*, compute* $\boldsymbol{Cov}[X_1, X_2]$.

$$
\begin{aligned}
E[X_1] &= \frac{a+b}{2} = \frac{1}{2} \\
E[X_2] &= \frac{a+b}{2} = \frac{1}{2} \\
\mathrm{Cov}[X, X^2] &= \iint_{[0,1]\times[0,1]} (x_1 - \overline{x_1})(x_2 - \overline{x_2})\mathrm{d}x \\
&= \iint_{[0,1]\times[0,1]} (x_1 - \frac{1}{2})(x_2 - \frac{1}{2})\mathrm{d}x \\
&= \iint_{[0,1]\times[0,1]} x_1x_2 - 0.5x_1 - 0.5x_2 + 1\mathrm{d}x \\
&= \left.\frac{(x_1^2 - x)(x_2^2 - x_2}{4}\right|_0^1 \\
&= 0
\end{aligned}
$$

(c) *Prove that* $Var[X_1 + X_2] = Var[X_1] + Var[X_2] + 2Cov[X_1, X_2]$.

$$
\begin{aligned}
Var[X_1 + X_2] &= E[((X_1 + X_2) - E[(X_1 + X_2)])^2] \\
&= E[(X_1 + X_2 - E[X_1] - E[X_2])^2] \\
&= E[((X_1 - E[X_1]) + (X_2 - E[X_2])^2] \\
&= E[(X_1 - E[X_1])^2 + 2(X_1 - E[X_1])(X_2 - E[X_2]) + (X_2 - E[X_2])^2] \\
&= E[(X_1 - E[X_1])^2] + E[2(X_1 - E[X_1])(X_2 - E[X_2])] + E[(X_2 - E[X_2])^2] \\
&= Var[X_1] + Var[X_2] + 2\mathrm{Cov}[X_1, X_2]
\end{aligned}
$$

(d) *Prove that* $Cov[X_1, X_2] = E[X_1X_2]E[X_1]E[X_2]..$

$$
\begin{aligned}
\mathrm{Cov}[X_1, X_2] &= E[(X_1 - E[X_1])(X_2 - E[X_2])] \\
&= E[(X_1X_2) - X_1E[X_2] - X_2E[X_1] + E[X_1]E[X_2]] \\
&= E[X_1X_2] - E[X_1]E[X_2] - E[X_2]E[X_1] + E[X_1]E[X_2] \\
&= E[X_1X_2] - E[X_1]E[X_2]
\end{aligned}
$$