Machine Learning Assignment 52

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53-1

(A) $Cov[X, X^2]$

$$\begin{aligned}
Cov[X, X^2] &= E[(X - E[X])(X^2 - E[X^2])] \\
&= \int_0^1 \left[\left(X - \frac{1}{2}\right)\left(X^2 - \frac{1}{3}\right)\right] \cdot (1)\, dX \\
&= \int_0^1 X^3 - \frac{X}{3} - \frac{X^2}{2} + \frac{1}{6}\, dX \\
&= \left[\frac{X^4}{4} - \frac{X^2}{6} - \frac{X^3}{6} + \frac{X}{6}\right]_0^1 \\
&= \left[\frac{[1]^4}{4} - \frac{[1]^2}{6} - \frac{[1]^3}{6} + \frac{[1]}{6}\right] - [0] \\
&= \frac{1}{4} - \frac{1}{6} \\
&= \frac{9}{36} - \frac{6}{36} \\
&= \frac{3}{36} \\
&= \frac{1}{12}
\end{aligned}$$

(B) $Cov[X_1, X_2]$

$$\begin{aligned}
Cov[X, X^2] &= E[(X - E[X])(X^2 - E[X^2])] \\
&= \int_0^1 \int_0^1 \left[\left(X_1 - \frac{1}{2}\right)\left(X_2 - \frac{1}{2}\right)\right] \cdot (1)\, dX_1\, dX_2 \\
&= \int_0^1 \int_0^1 X_1 \cdot X_2 - \frac{X_1}{2} - \frac{X_2}{2} + \frac{1}{4}\, dX_1\, dX_2 \\
&= \int_0^1 \left[\frac{X_1^2 \cdot X_2}{2} - \frac{X_1^2}{4} - \frac{X_1 \cdot X_2}{2} + \frac{X_1}{4}\right]_0^1 dX_2 \\
&= \int_0^1 \frac{X_2}{2} - \frac{1}{4} - \frac{X_2}{2} + \frac{1}{4}\, dX_2 \\
&= \int_0^1 0\, dX_2 \\
&= 0
\end{aligned}$$

(C) $Var[X_1 + X_2] = Var[X_1] + Var[X_2] + 2Cov[X_1, X_2]$

$$\begin{aligned}
Var[X_1 + X_2] &= E[(X_1 + X_2)^2] - E[X_1 + X_2]^2 \\
&= E[X_1^2 + 2X_1 \cdot X_2 + X_2^2] - (E[X_1] + E[X_2])^2 \\
&= E[X_1^2] + 2 \cdot (E[X_1 \cdot X_2]) + E[X_2^2] - E[X_1]^2 - 2 \cdot (E[X_1] \cdot E[X_2]) - E[X_2]^2 \\
&= [E[X_1^2] - E[X_1]^2] + [2 \cdot (E[X_1 \cdot X_2] - E[X_1] \cdot E[X_2])] + [E[X_2^2] - E[X_2]^2] \\
&= [Var[X_1]] + [2 \cdot Cov[X_1, X_2]] + [Var[X_2]
\end{aligned}$$

(D) $Cov[X_1, X_2] = E[X_1 X_2] E[X_1] E[X_2]$

$$\begin{aligned}
Cov[X_1, X_2] &= E[(X_1 - E[X_1])(X_2 - E[X_2])] \\
&= E[(X_1 \cdot X_2 - E[X_1]E[X_2] - E[X_1]E[X_2] + E[X_1]E[X_2]] \\
&= E[X_1 \cdot X_2 + E[X_1]E[X_2]] \\
&= E[X_1 \cdot X_2] + E[X_1]E[X_2]
\end{aligned}$$