$$\varphi \equiv \cancel{\forall x}\, \exists u \left(p(u,y) \to \exists y \exists z \left(\cancel{\forall x}\, . \, q(y,z) \to \exists v . p(v,z)\right)\right)$$

① volné

$$\equiv \exists u \left(p(u,y) \to \exists y \exists z \left(q(y,z) \to \exists v . p(v,z)\right)\right)$$

$$\equiv \exists u \left(\overbrace{p(u,y)}^{B} \to \exists \bar{y} \overbrace{\exists z \left(q(\bar{y},z) \to \exists v . p(v,z)\right)}^{A}\right)$$

$$B \to (Q_x A) \iff Q_x (B \to A)$$

$$\equiv \exists u \exists \bar{y} \left(p(u,y) \to \exists z \left(q(\bar{y},z) \to \exists v . p(v,z)\right)\right)$$

$$\equiv \exists u \exists \bar{y} \exists z \left(p(u,y) \to \left(q(\bar{y},z) \to \exists v . p(v,z)\right)\right)$$

$$\equiv \exists u \exists \bar{y} \exists z \left(p(u,y) \to \exists v . \left(q(\bar{y},z) \to p(v,z)\right)\right)$$

$$\equiv \exists u \exists \bar{y} \exists z \exists v . \left(p(u,y) \to \left(q(\bar{y},z) \to p(v,z)\right)\right)$$

$$\varphi \equiv \forall x \forall y . \left(x > y \leftrightarrow \exists z . \left(z > 0 \wedge y + z = x \right) \right) \qquad (Q_x A) \to B \Leftrightarrow \overline{Q}_x (A \to$$

$$\equiv \forall x \forall y \left(\left((x > y) \to \exists z . \left(z > 0 \wedge y + z = x \right) \right) \wedge \left(\exists z . \left(z > 0 \wedge y + z = x \right) \right) \to x > y \right)$$

$$\equiv \forall x \forall y \left(\left((x > y) \to \exists \bar{z} . \left(\bar{z} > 0 \wedge y + \bar{z} = x \right) \right) \wedge \left(\exists z . \left(z > 0 \wedge y + z = x \right) \right) \to x > y \right)$$

$$\equiv \forall x \forall y \exists \bar{z} \left(\left((x > y) \to \left(\bar{z} > 0 \wedge y + \bar{z} = x \right) \right) \wedge \left(\left(\exists z . \left(z > 0 \wedge y + z = x \right) \right) \to x > y \right) \right)$$

$$\equiv \forall x \forall y \exists \bar{z} \forall z \left(\left((x > y) \to \left(\bar{z} > 0 \wedge y + \bar{z} = x \right) \right) \wedge \left(\left(z > 0 \wedge y + z = x \right) \to x > y \right) \right)$$

1) schemata výrokových axiomů

A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$

A2) $(\varphi \rightarrow (\psi \rightarrow \eta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \eta))$

A3) $((\neg \varphi) \rightarrow (\neg \psi)) \rightarrow (\psi \rightarrow \varphi)$

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2) AK) • $(\forall x (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \forall x \psi)$

x nemá volný výskyt ve φ

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3) AS) • $(\forall x \varphi) \rightarrow \varphi_x[t]$

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4) AR)

$x_1 = y_1 \rightarrow (x_2 = y_2 \rightarrow (\dots (x_n = y_n \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n))$

$x_1 = y_1 \rightarrow (x_2 = y_2 \rightarrow (\dots (x_n = y_n \rightarrow (p(x_1, \dots, x_n) \rightarrow p(y_1, \dots, y_n))$

---

5) MP $\dfrac{\varphi, \varphi \rightarrow \psi}{\psi}$ generalizace: $\dfrac{\varphi}{\forall x \varphi}$

dokažte, že $\vdash \varphi_x[t] \to (\exists x . \varphi)$

$\vdash \varphi_x[t] \to (\neg \forall x . \neg \varphi)$

1) $\vdash (\forall x (\neg \varphi)) \to \neg \varphi_x[t]$ (A5)

2) $\vdash \neg\neg (\forall x . (\neg \varphi)) \to \forall x (\neg \varphi)$ $(\neg\neg A \to A)$

$A \mapsto \forall x . \neg \varphi$

3) $\neg\neg (\forall x (\neg \varphi)) \vdash \forall x (\neg \varphi)$ (2), VD

4) $\neg\neg (\forall x (\neg \varphi)) \vdash \neg \varphi_x[t]$ (1), (3), MP

5) $\vdash (\neg\neg \forall x (\neg \varphi)) \to \neg \varphi_x[t]$ (4), VD

6) $\vdash ((\neg\neg \forall x (\neg \varphi)) \to \neg \varphi_x[t]) \to (\varphi_x[t] \to \neg \forall x (\neg \varphi))$ (A3)

7) $\vdash \varphi_x[t] \to \neg \forall x (\neg \varphi)$ (5), (6), MP

pokud $\vdash \varphi \rightarrow \psi$ a x nemá volný výskyt ve φ

potom $\vdash \varphi \rightarrow (\forall x\, \psi)$

1) $\vdash \varphi \rightarrow \psi$ (předpoklad)

2) $\vdash \forall x (\varphi \rightarrow \psi)$ (1), G

3) $\vdash (\forall x (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \forall x\, \psi)$ (AK)

4) $\vdash \varphi \rightarrow (\forall x\, \psi)$ (2), (3), MP

(pravidlo $\forall$)

pokud $\vdash \varphi \to \psi$ a x nemá volný výskyt v ψ,
potom $\vdash (\exists x \varphi) \to \psi$ (pravidlo $\exists$)

$\vdash (\neg \forall x . (\neg \varphi)) \to \psi$

1) $\vdash \varphi \to \psi$ (předpoklad)

$(A \to B) \to (\neg B \to \neg A)$

2) $\vdash (\varphi \to \psi) \to (\neg \psi \to \neg \varphi)$

3) $\vdash \neg \psi \to \neg \varphi$ (1), (2), MP

4) $\vdash \neg \psi \to \forall x (\neg \varphi)$ (pravidlo $\forall$)

5) $\vdash (\neg \psi \to \forall x (\neg \varphi)) \to ((\neg \forall x (\neg \varphi)) \to \neg\neg \psi)$ (z VL)

6) $\vdash (\neg \forall x (\neg \varphi)) \to \neg\neg \psi$ (4), (5), MP

7) $\cancel{\vdash} (\neg \forall x (\neg \varphi)) \vdash \neg\neg \psi$ (6), VD

8) $\vdash \neg\neg \psi \to \psi$ z VL: $\neg\neg A \to A$

9) $(\neg \forall x (\neg \varphi)) \vdash \psi$ (7), (8), MP

10) $\vdash (\neg \forall x (\neg \varphi)) \to \psi$ (9), VD

pokud $\vdash \varphi \to \psi$, potom $\vdash (\forall x\, \varphi) \to (\forall x\, \psi)$

1) $\vdash \varphi \to \psi$ (předpoklad)

2) $\vdash (\forall x\, \varphi) \to \varphi$ (A5)

3) $(\forall x\, \varphi) \vdash \varphi$ (2)(VD)

4) $(\forall x\, \varphi) \vdash \psi$ (1), (3), MP

5) $\vdash (\forall x\, \varphi) \to \psi$ (4), VD

6) $\vdash (\forall x\, \varphi) \to (\forall x\, \psi)$ (pravidlo $\forall$)

(distribuce kvantifikátoru)

1) $\vdash \varphi \to \neg\neg\varphi$ (taut.)

2) $\vdash (\forall x\, \varphi) \to (\forall x.\, \neg\neg\varphi)$ (distr. $\forall$)

3) $\vdash ((\forall x\, \varphi) \to (\forall x.\, \neg\neg\varphi)) \to (\neg(\forall x\, \neg\neg\varphi) \to \neg(\forall x\, \varphi))$

4) $\vdash \neg(\forall x\, \neg\neg\varphi) \to \neg(\forall x\, \varphi)$ (2), (3), MP

5) $\vdash \neg(\forall x\, \varphi) \to (\forall x\, \varphi \to \psi)$ (taut.)

6) $\vdash \neg(\forall x\, \neg\neg\varphi) \to (\forall x\, \varphi \to \psi)$ (s.i.)

7) $\vdash (\exists x\, \neg\varphi) \to (\forall x.\, \varphi \to \psi)$ ($\forall \to \exists$)

1) $T \vdash \forall x (p(x,y) \rightarrow q(y))$ (β)

2) $\vdash (\forall x (p(x,y) \rightarrow q(y))) \rightarrow (p(x,y) \rightarrow q(y))$ (AS)

3) $T \vdash p(x,y) \rightarrow q(y)$ (1),(2), MP

4) $T \vdash \boxed{p(x,y)}$ (α)

5) $T \vdash q(y)$ (3),(4), MP

6) $T \vdash \forall y\, q(y)$ (5)(zobecnění)

$\vdash 1+1=2$

$\vdash S(0)+S(0)=S(S(0))$

1) $PA \vdash \forall x \forall y.\ x+S(y)=S(x+y)$

2) $\vdash (\forall x(\forall y.\ x+S(y)=S(x+y))) \to (\forall y.\ S(0)+S(y)=S(S(0)+y))$ (AS) $x \mapsto S(0)$

3) $PA \vdash \forall y.\ S(0)+S(y)=S(S(0)+y)$ (1),(2), MP

4) $\vdash (\forall y.\ S(0)+S(y)=S(S(0)+y)) \to (S(0)+S(0)=S(S(0)+0))$ (AS) $y \mapsto 0$

5) $PA \vdash S(0)+S(0)=S(S(0)+0)$ (3),(4), MP

6) $PA \vdash S(0)+0=S(0)$ (PA ax +AS)

7) $\vdash S(0)+0=S(0) \to (S(S(0)+0)=S(S(0)))$ (AR)

8) $PA \vdash S(S(0)+0)=S(S(0))$ (6),(7) MP

9) $\vdash (S(0)+S(0)=S(S(0)+0)) \to ((S(S(0)+0)=S(S(0))) \to (S(0)+S(0)=S(S(0))))$ Lema 5.12 (iii)

10) $PA \vdash S(0)+S(0)=S(S(0))$ (5),(8),(9), MP×2