Citation for the fifth Hausdorff Medal awarded to David Asperó and Ralf Schindler: MM^{++} implies Axiom $(*)$

Set theory is the mathematical study of infinity. Its main goal is to develop mathematical ways of thinking about infinity that both explain our mathematical intuitions about infinity and also make the concept of infinity useful for other areas of mathematics. Regardless of its relative young age, set theory has been very successful both in answering its own deep fundamental questions and in being applied elsewhere. Applications of set theoretic ideas or methods include the solutions of Kaplansky's Conjecture by Robert Solovay and Hugh Woodin, Whitehead's Problem by Saharon Shelah, the Brown-Douglas-Fillmore Problem by Ilijas Farah, a problem of von Neumann by Matthew Foreman, Daniel Rudolph, and Benjamin Weiss and the L space problem by Justin Moore. The literature is full of such great applications of set theoretic ideas.

Historically, many of the successful set theoretic ideas were discovered while answering fundamental questions that arise within set theory, questions that *a priori* do not seem relevant to C^* algebras, infinite groups, dynamics, or topology. The work that we are celebrating today belongs to the realm of fundamental questions.

The basic axioms of set theory — the axioms of Zermelo-Fraenkel set theory with the axiom of choice (ZFC) — formalize our basic intuitions about what kind of sets exist, how they relate to one another and how to obtain new sets out of them. However, as Gödel has demonstrated, no sufficiently powerful set of axioms can decide all questions, and in particular, it can never decide the basic metamathematical question whether the system itself is consistent. Later on, Cohen's forcing technology and Gödel's constructible universe led to discovery of plethora of non-metamathematical natural statements in the language of set theory that are not decidable within ZFC.

This is the laudatio given by Grigor Sargsyan at the Hausdorff Medal ceremony on 29th August, 2022 at the eighth European Set Theory Conference in Turin.

Perhaps the best known such statement is the Continuum Hypothesis, but even simpler questions of classical descriptive set theory — such as whether all projective sets are Lebesgue measurable — are not decidable within ZFC.

In many ways, this is deeply troubling yet also deeply intriguing. Undecidability of simple questions — such as the one about Lebesgue measurability of projective sets — is deeply troubling as many of these questions seem to be the kind of question whose answer we ought to know. How can we not *know* the answer to the question whether our rudimentary set theoretic operations produce pathological sets of reals? The independence of these basic questions seem to suggest that our basic picture of the set theoretic universe is incomplete: something profound is missing. Yet, since Cohen's discovery of forcing, no *obviously true* principle of sets has been discovered, something that we simply overlooked. Most likely what we do not *know* is not *obvious*.

Modern set theory owes a great deal to the following passage from Gödel's *What is Cantor's Continuum Problem?*

> There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.

In set theory, Gödel's program is the program of removing the ambiguities in set theory in extensions of ZFC. Let us call an extension of ZFC *Gödelian* if it has the above properties. Thus, Gödel's program is to find a hierarchy of Gödelian theories that remove undecidability from set theory. Incredibly, we have discovered a hierarchy of axioms, called Large Cardinal Axioms, that are linearly ordered according to their consistency strengths. Gödel suggested that perhaps these axioms could remove ambiguity from set theory. However, it was soon shown by Lévy and Solovay that large cardinal axioms do not decide the Continuum Hypothesis.

While large cardinals themselves do not do what Gödel predicted, theories that have been derived from Large Cardinal Axioms have had enormous success in Gödel's sense. One such hugely successful theory derived from large cardinals is the Axiom of Determinacy, AD, which postulates that all two player games of length ω and of perfect information are determined. AD has interesting consequences, in that it decides essentially all natural questions about sets of reals. In particular, it implies that all sets of reals are Lebesgue measurable, have the property of Baire and have the perfect set property. It is almost a Gödelian axiom, but unfortunately it contradicts the Axiom of Choice.

AD and its extensions like Woodin's AD^+ or $\mathsf{AD}_{\mathbb{R}}$ + "Θ is a regular cardinal" have many deep consequences but since AD contradicts AC, it is easy to dismiss AD and its extensions as a wrong hierarchy of axioms. However, in a seminal work, Steel and van Wesep by forcing over a model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is a regular cardinal", obtained a model of ZFC in which the non-stationary ideal is saturated. This was the first construction of such a model albeit from a theory that at the time was not known to be consistent relative to large cardinals. Subsequent work of Martin, Steel and Woodin established the consistency of AD relative to large cardinals, and also in parallel, Woodin simplified, modified, and extended the Steel-van Wesep method of forcing into a powerful forcing technology, the $\mathbb{P}_{\mathsf{max}}$ technology, that forces a wide range of statements over a model of AD.

Woodin's explorations of $\mathbb{P}_{\mathsf{max}}$ extensions of models of AD led him to an axiom he called Axiom $(*)$ which axiomatizes the statement that the powerset of the first uncountable cardinal is contained in a $\mathbb{P}_{\mathsf{max}}$ extension of $L(\mathbb{R})$, the smallest transitive model of ZF that contains all the ordinals and reals. Axiom $(*)$ has many deep consequences and in many ways is the universal axiom for the portion of the universe coded by subsets of ω_1. Axiom $(*)$ implies that the continuum is ω_2 and decides a very large class, Π_2 class, of statements about this portion. Thus, Axiom $(*)$ — which is an axiom compatible with ZFC and is based on AD — is a Gödelian Axiom.

Yet another triumph of Gödel's program is the discovery of forcing axioms. Forcing is a method of adding to the universe of sets a new set while preserving the axioms of set theory. Forcing axioms say that large fragments of this new set can already be found in the universe. They are similar to the Baire Category Theorem.

Forcing axioms cannot hold for all partial orderings. The maximum collection of partial orderings for which they can hold is the collection of partial orderings that do not alter ω_1 too much. In technical terms, the maximal collection of posets for which forcing axioms can be true are those posets that preserve the stationary subsets of ω_1. In a seminal work, Foreman, Magidor and Shelah isolated what is now known as Martin's Maximum, the forcing axiom that works for exactly the aforementioned collection of partial orders. MM^{++} is a natural extension of Martin's Maximum. MM^{++} and other forcing axioms have far reaching consequences and solve problems in many different areas of mathematics. Just like AD and its ZFC counterpart Axiom $(*)$, forcing axioms remove pathological examples from many different contexts and solve undecidable problems according to the prevailing intuition. Set theory is rich under MM^{++}.

Both Axiom $(*)$ and MM^{++} are then compelling Gödelian axioms. From a foundational prospective, they are far apart as they are based on a completely different set of fundamental ideas. So the obvious question is: which one should we choose to add to ZFC?

The current edition of the Hausdorff prize is awarded for the paper

> David Asperó and Ralf Schindler, *Martin's Maximum^{++} implies Woodin's axiom* $(*)$, Annals of Mathematics (2) **193** (2021), no. 3, 793–835.

showing that we in fact have no choice. In this seminal work, David Asperó and Ralf Schindler, combining a vast range of sophisticated techniques from inner model theory to Jensen's L-forcing theory, showed that MM^{++} implies the Axiom $(*)$. Asperó-Schindler result is a fundamental step taken towards amalgamating two of our most Gödelian extensions of ZFC. By doing so, they construct bridges between two seemingly disjoint sub-areas of set theory, the study of determinacy axioms and the study of forcing axioms.

Further work is needed to understand if it is possible to obtain full amalgamation between models of MM^{++} and models obtained as $\mathbb{P}_{\mathsf{max}}$ extensions of models of AD. Is it possible that a forcing extension of a model of AD is a model of MM^{++}? Is it possible to find one unified technology for constructing models of determinacy and models of forcing axioms from large cardinals? Are forcing axioms and determinacy axioms one and the same exposited in different languages?

Whatever the answers to the above questions are, the Asperó-Schindler result will be in the centre of it all for many decades to come. Its authors are to be warmly thanked for delivering this beautiful piece of mathematics to us!

29th August, 2022 The Hausdorff Medal committee