# $\mathrm{MM}^{++} \Rightarrow (*)$ (part 2)

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# A slightly closer look at the theorem

Recall:

Theorem (Asperó-Schindler)
*$MM^{++}$ implies $(*)$.*

$(*)$ is the following conjunction:

- AD holds in $L(\mathbb{R})$.
- There is a $\mathbb{P}_{\max}$-generic filter $G$ over $L(\mathbb{R})$ such that $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$.

$\mathbb{P}_{\max} \in L(\mathbb{R})$ is the forcing notion I will define next.

Given $\eta \leq \omega_1$, a sequence $(\langle (M_\alpha, I_\alpha), G_\alpha, j_{\alpha,\beta} \rangle : \alpha < \beta \leq \eta)$ is a *generic iteration (of $(M_0, I_0)$)* iff

- $M_0$ is a countable transitive model of $\mathsf{ZFC}^*$ (enough of $\mathsf{ZFC}$).
- $M_0 \models$ "$I_0$ is a normal ideal on $\omega_1$".
- $j_{\alpha,\beta}$, for $\alpha < \beta \leq \eta$, is a commuting system of elementary embeddings

$$j_{\alpha,\beta} : (M_\alpha; \in, I_\alpha) \longrightarrow (M_\beta; \in, I_\beta)$$

- For each $\alpha < \eta$, $G_\alpha$ is a $\mathcal{P}(\omega_1)^{M_\alpha}/I_\alpha$-generic filter over $M_\alpha$,

$$j_{\alpha,\alpha+1} : M_\alpha \longrightarrow \mathrm{Ult}(M_\alpha, G_\alpha)$$

  is the corresponding elementary embedding, and $(M_{\alpha+1}, I_{\alpha+1}) = (\mathrm{Ult}(M_\alpha, G_\alpha), j_{\alpha,\alpha+1}(I_\alpha))$.
- If $\beta \leq \eta$ is a limit ordinal, $(M_\beta, I_\beta)$ and $j_{\alpha,\beta}$ (for $\alpha < \beta$) is the direct limit of $(\langle (M_\alpha, I_\alpha), G_\alpha, j_{\alpha,\alpha'} \rangle : \alpha < \alpha' < \beta)$.

A pair $(M, I)$ is *iterable* if the models in every generic iteration of $(M, I)$ are well-founded.

$\mathbb{P}_{\max}$ is the following forcing:

Conditions in $\mathbb{P}_{\max}$ are triples $(M, I, a)$, where

(1) $(M, I)$ is an iterable pair.

(2) $M \models \mathsf{MA}_{\omega_1}$

(3) $a \in \mathcal{P}(\omega_1)^M$ and $M \models \omega_1 = \omega_1^{L[a]}$.

Extension relation: $(M^1, I^1, a^1) \leq_{\mathbb{P}_{\max}} (M^0, I^0, a^0)$ iff $(M^0, I^0, a^0) \in M_1$ and, in $M^1$, there is a generic iteration $\mathcal{I} = (\langle (M_\alpha, I_\alpha), G_\alpha, j_{\alpha,\beta} \rangle : \alpha < \beta \leq \eta)$ of $(M^0, I^0)$ for $\eta = \omega_1^{M^1}$ such that

(a) $j_{0,\eta}(a^0) = a^1$

(b) $\mathcal{I}$ is *correct* in $(M^1, I^1)$, in the sense that $j_{0,\eta}(I^0) \subseteq I^1$ and every $I_\eta$-positive subset of $\omega_1^{M_\eta}$ $(= \omega_1^{M^1})$ in $M_\eta$ is $I^1$-positive.

The following is a very rough proof sketch of our theorem:

$\mathsf{MM}^{++}$ implies $\mathsf{AD}^{L(\mathbb{R})}$ (in fact $\mathsf{PFA}$ suffices).

Hence, we just need to show, under $\mathsf{MM}^{++}$, that $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$ for a $\mathbb{P}_{\max}$-generic filter $G$ over $L(\mathbb{R})$.
A standard fact:

**Fact**

*(Woodin) If $\mathsf{NS}_{\omega_1}$ is saturated, $MA_{\omega_1}$ holds, $\mathcal{P}(\omega_1)^\sharp$ exists, and $A \subseteq \omega_1$ is such that $\omega_1^{L[A]} = \omega_1$, then $\Gamma_A$ is a filter on $\mathbb{P}_{\max}$ and $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[\Gamma_A]$, where*

- *$\Gamma_A$ is the set of $(M, I, a) \in \mathbb{P}_{\max}$ such that there is a correct iteration of $(M, I)$ (relative to $(H_{\omega_2}, \mathsf{NS}_{\omega_1})$) sending $a$ to $A$.*

Hence, fixing $A$ as in the Fact, we only need to show that $\Gamma_A$ meets every dense $D \subseteq \mathbb{P}_{\max}$ such that $D \in L(\mathbb{R})$.

Given such a $D$, the set $X \in L(\mathbb{R})$ of reals coding a member of $D$ is universally Baire. We even have that for every cardinal $\kappa$ there is a tree $T_\kappa$ on $\omega \times 2^\kappa$ such that

$$p[T_\kappa] = X$$

and

$$\Vdash_{\mathsf{Coll}(\omega,\kappa)} \text{“ } p[T_\kappa] \text{ codes a dense subset of } \mathbb{P}_{\max}\text{”}$$

Hence, it is enough to show that there is a stationary set preserving poset $\mathcal{P}$ adding a branch $[x, b]$ through $T_{\omega_2}$ such that $x$ codes a $\mathbb{P}_{\max}$-condition $(M, I, a)$ for which there is a correct iteration sending $a$ to $A$.

The forcing $\mathcal{P}$ we constructed for this can be described as a recursively defined $\mathcal{L}$-forcing with side conditions.

$\mathcal{P}$ is $\mathcal{P}_{\omega_3}$, for a certain sequence $\langle \mathcal{P}_\lambda \,:\, \lambda \leq \omega_3 \rangle$ of forcing notions.

Given $\lambda$, $\mathcal{P}_\lambda$ consists of finite amounts of information about a certain configuration, $\mathcal{C}$, which we don't know (yet) is realized in a stat. preserving forcing extension of $V$, but which we can argue is realized in some outer model, thanks to the fact that

$$\Vdash_{\mathrm{Coll}(\omega,\,\kappa)} \text{“ } p[T_\kappa] \text{ codes a dense subset of } \mathbb{P}_{\max}\text{”}$$

The above configuration $\mathcal{C}$ can be partially described as a branch $[x, b]$ through $T_{\omega_2}$ such that $x$ codes a $(N, I, b) \in \mathbb{P}_{\max}$ for which there is a correct iteration

$$\mathcal{J} = \langle (N_\alpha, I_\alpha, a_\alpha) : \alpha \leq \omega_1 \rangle$$

with $a_{\omega_1} = A$ and such that in $N$ there is a $I$-correct iteration

$$\mathcal{I} = \langle (M_\xi, I_\xi, a_\xi) : \xi \leq \omega_1^N \rangle$$

such that the last model $(M^*, I^*)$ of $j^{\mathcal{J}}_{0,\omega_1}(\mathcal{I})$ is such that $(M^*, I^*) = (H^V_{\omega_2}, \mathrm{NS}^V_{\omega_1})$.

Moreover, in $\mathcal{C}$ there are also countable models (the side conditions) $X_{\bar{\lambda}}$, for some $\bar{\lambda} < \lambda$, such that $\mathcal{C} \cap C_{\bar{\lambda}}$ is 'generic' for $\mathcal{P}_{\bar{\lambda}}$ over $X_{\bar{\lambda}}$. The inclusion of these side conditions is crucially used in the proof that $\mathcal{P}$ preserves stationary sets and that the generic

$$\langle (N^G_\alpha, I^G_\alpha, a^G_\alpha) : \alpha \leq \omega_1 \rangle$$

added by $\mathcal{P}$ is correct in $V[G]$. $\square$

In the above proof, the forcing $\mathcal{P}$ makes $H_{\omega_2}^V$ the final model $M^*$ of a generic iteration $j : (M, \mathrm{NS}_{\omega_1}) \longrightarrow (M^*, \mathrm{NS}_{\omega_1}^V)$ of a countable $M$. It follows that $\mathcal{P}$ forces $\mathrm{cf}(\omega_2^V) = \omega$ (in fact $j``\omega_2^M$ is cofinal in $\omega_2^V$).

This can be somewhat generalized:

# Namba forcing-like outer models

Given a cardinal $\lambda \geq \omega_2$, let $\mathsf{BMM}^{++}(\mathrm{cf}(\omega))_{<\lambda}$ denote the following natural bounded form of $\mathsf{MM}^{++}$:

Let $\kappa < \lambda$, let $X \in H_{\kappa^+}$, let $\varphi(x)$ be a $\Sigma_0$ formula $\varphi(x, y)$ in the language for $(H_{\kappa^+}, \in, \mathrm{NS}_{\omega_1})$, and suppose there is some stationary preserving poset $\mathbb{P}$

(1) forcing $\mathrm{cf}(\mu) = \omega$ for every regular cardinal $\mu$ such that $\aleph_1 < \mu \leq \kappa$ and

(2) forcing $(V^{\mathbb{P}}, \in, \mathrm{NS}_{\omega_1}^{V^{\mathbb{P}}}) \models (\exists y)\varphi(X, y)$.

Then there are, in $V$, stationarily many $N \in [H_{\kappa^+}]^{\aleph_1}$ such that $X \in N$ and such that there is $Y \subseteq N$ with

$$(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1}) \models \varphi(\pi_N(X), \pi_N``Y),$$

where $\pi_N$ is the transitive collapse of $N$.

(so $\mathsf{BMM}^{++}(\mathrm{cf}(\omega))_{<\aleph_2}$ is just $\mathsf{BMM}^{++}$).

A variation of the proof of the $\mathsf{MM}^{++} \Rightarrow (*)$ theorem gives that the following enhanced form of $\mathsf{BMM}^{++}(\mathrm{cf}(\omega))_{<\aleph_{\omega_1}}$ follows from $\mathsf{MM}^{++}$.

## Theorem

*(Asperó-Schindler) Suppose $MM^{++}$ holds. Then the following holds for every uncountable $\kappa < \aleph_{\omega_1}$.*

*Let $X \in H_{\kappa^+}$, let $\varphi(x)$ be a $\Sigma_0$ formula in the language for $(H_{\kappa^+}, \in, \mathrm{NS}_{\omega_1})$, and suppose there is, in some collapse extension, an $\sharp$-closed transitive model $W$ (where $\sharp = \{\langle r, r^\sharp\rangle : r \in \mathbb{R}\}$) such that $H^V_{\kappa^+} \in W$, such that $W \models$ "there is a Woodin cardinal above $(\kappa^+)^V$", and such that*

(1) *every stationary subset of $\omega_1$ in $V$ is stationary in $W$,*
(2) *every $V$-regular cardinal $\mu$ such that $\aleph_1^V < \mu \leq \kappa$ has countable cofinality in $W$, and*
(3) $(W, \in, \mathrm{NS}^W_{\omega_1}) \models (\exists y)\varphi(X, y)$.

*Then there are, in $V$, stationarily many $N \in [H_{\kappa^+}]^{\aleph_1}$ such that $X \in N$ and such that there is $Y \subseteq N$ with $(H_{\omega_2}, \in, \mathrm{NS}_{\omega_1}) \models \varphi(\pi_N(X), \pi_N``Y)$, where $\pi_N$ is the transitive collapse of $N$.*

Given a restricted formula $\varphi(x, y)$ in the language for $(V; \in, \mathsf{NS}_{\omega_1})$ and a set $X$, the sentence $(\exists y)\varphi(X, y)$ is *honestly consistent* in case for every universally Baire set $A \subseteq \mathbb{R}$ there is an $A$-closed transitive model $W$ of $\mathsf{ZFC}^*$ in some outer model such that

(1) $H^V_{\omega_2} \in W$, $\mathsf{NS}^W_{\omega_1} \cap V = \mathsf{NS}^V_{\omega_1}$, and $X \in W$, and
(2) $(W; \in, \mathsf{NS}^W_{\omega_1}) \models (\exists y)\varphi(X, y)$.

## Definition

$\mathsf{MM}^{++,*}$ is the following statement: Let $\varphi(x, y)$ be a restricted formula in the language for $(V; \in, \mathsf{NS}_{\omega_1})$ and $X$ be a set such that $(\exists y)\varphi(X, y)$ is honestly consistent. Then there is a cardinal $\kappa$ such that $X \in H_{\kappa^+}$ and the set of $N \in [H_{\kappa^+}]^{\aleph_1}$ such that $X \in N$ and such that there is some $Y \subseteq N$ with $(H_{\omega_2}, \in, \mathsf{NS}_{\omega_1}) \models \varphi(\pi_N(X), \pi_N``Y)$, where $\pi_N$ is the transitive collapse of $N$, is stationary.

The above theorem shows that a natural fragment of $\mathsf{MM}^{++,*}$ follows from $\mathsf{MM}^{++}$.

The following is open:

Question: Is $\mathsf{MM}^{++,*}$ consistent? Is it equivalent to $\mathsf{MM}^{++}$?

## The $(*)$ (or $\mathrm{MM}^{++}$) picture vs. the CH picture

The following dichotomy was observed by Woodin.

**Theorem**

*(Woodin) Suppose $L(\mathbb{R}) \models \mathrm{AD}$ and there is a $\mathbb{P}_{\max}$-generic filter over $L(\mathbb{R})$. Then exactly one of the following holds.*

(1) $(*)$

(2) *CH*

A strong form of $(*)$:

**Definition**

(Woodin) $(*)^{++}$ is the following statement: There exists $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ and a filter $G \subseteq \mathbb{P}_{\max}$ such that

- $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$ and
- $G$ is $L(\Gamma, \mathbb{R})$-generic and $\mathcal{P}(\mathbb{R}) \in L(\Gamma, \mathbb{R})[G]$.

Woodin has proved that $(*)^{++}$ fails in all currently known models of $\mathsf{MM}^{++}$.

Question: Is $(*)^{++}$ compatible with $\mathsf{MM}^{++}$? Can $(*)^{++}$ be forced over a $\mathsf{ZFC}$ model?

Also:

**Theorem**

*(Woodin) Assume the $\Omega$ Conjecture holds and there is a proper class of Woodin cardinals. Then there is no $\Omega$-consistent axiom $A$ such that*

(1) *$A$ implies $\mathrm{MM}^{++}(\mathfrak{c})$ and*
(2) *$A$ provides, modulo forcing, a complete theory for $\underset{\sim}{\Sigma}^2_1$ sentences.*

Compare this with the well-known result, due to Woodin, that if there is a proper class of measurable Woodin cardinals, then CH provides, modulo forcing, a complete theory for $\underset{\sim}{\Sigma}^2_1$ sentences.

The following is an important open question in this context.

Question: (Steel) Is there any reasonable large cardinal hypothesis relative to which $\diamondsuit$ is maximal for $\Sigma^2_2$ sentences consistent with CH (with $\diamondsuit$) modulo forcing? I.e., is it true that if $\diamondsuit$ holds and $\sigma$ is a $\Sigma^2_2$ sentence such that $\sigma + \mathrm{CH}$ ($\sigma + \diamondsuit$) is forcible, then $\sigma$ is true?

If the answer were yes, then $\diamondsuit$ would be complete, modulo forcing, for the $\Sigma^2_2$ theory; i.e., any two forcing extensions satisfying $\diamondsuit$ would agree on $\Sigma^2_2$ sentences.

Question: Is it possible, in the presence of large cardinals, to force $\Sigma^2_2$-maximality without adding reals?

If the answer is yes, then there are no canonical inner models, as they are currently understood, for the background large cardinal hypothesis.