Lecture 5: The Poisson Family of Distributions

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1. Poisson Probability Distributions

Consider a binomial distribution (defined previously) where the number of independent trials (experiments), n, is very large and approaches to infinity (i.e. $n \to +\infty$), and the probability p takes small values, (i.e., $p \to 0$). As a result, the expected number of success (mean of the distribution), which is denoted as s herein and is equal to np, is a finite number.

If the number of success, r, compared to trials, n, is low, the Binomial coefficient can be approximated as:

$$\lim_{n\to\infty}\binom{n}{r} = \lim_{n\to\infty}\frac{n!}{r!(n-r)!} = \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \cong \frac{n^r}{r!} \tag{1}$$

Hence, the Binomial PMF can be re-written as:

$$P(r|n,p) = \lim_{n\to\infty}\frac{n^r}{r!}p^r(1-p)^{n-r} = \frac{(s)^r}{r!}\lim_{n\to\infty}(1-p)^n \tag{2}$$

Based on the binomial series as well as definition of the exponential function,

$$\lim_{n\to\infty}(1-p)^n = \lim_{n\to\infty}\left(1-\frac{s}{n}\right)^n \cong e^{-s} \tag{3}$$

Thus, Eq. (2) can be written as:

$$P(r|n,\mathrm{p}) = P(r|s) = \frac{(s)^r e^{-s}}{r!} \tag{4}$$

As a result, the Binomial distribution with two parameters n and p has become a distribution with only one parameter, $s = np$, called the *Poisson distribution*. Strictly speaking, for situation in which n is large and p is very small, the Poisson distribution can be used to approximate the binomial distribution. The larger the n and the smaller the p, the better will be the approximation. The parameter s is the *shape parameter* which indicates actually the average number of success. The Poisson distribution is a *discrete* probability distribution which counts events that occur randomly in a given interval of time or space. For instance, the temporal occurrence of earthquakes is most commonly described by a Poisson model, i.e., the probability of occurring a prescribed number of events during a given time interval or in a specified spatial region follows a Poisson process. Considering s as the average (expected) number of occurrences in the considered time interval denoted as t, it can be stated that:

$$s = \lambda t \tag{5}$$

where $\lambda = s/t$ stands for the average (mean) rate of occurrence of the events (success) in unit time (its dimension is T^{-1}). As a result, the probability of a random variable *r,* representing the number of occurrences of a particular event during a given time interval is given by:

$$P\left(r \middle| s = \lambda t\right) = \frac{\left(\lambda t\right)^r e^{-\lambda t}}{r!} \tag{6}$$

Eq. (6) expresses the PMF of a Poisson distribution. It is noteworthy that the probability of occurrence of at least one event or two events in a period of time T is respectively given by:

$$\begin{aligned} P\left(r \geq 1 \middle| \lambda t\right) &= \sum_{k=1}^{+\infty} P\left(r = k \middle| \lambda, t\right) = 1 - P\left(r = 0 \middle| \lambda, t\right) = 1 - \frac{\left(\lambda t\right)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t} \\ P\left(r \geq 2 \middle| \lambda t\right) &= 1 - P\left(r = 0 \middle| \lambda t\right) - P\left(r = 1 \middle| \lambda t\right) = 1 - e^{-\lambda t} - \frac{\left(\lambda t\right)^1 e^{-\lambda t}}{1!} = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} \end{aligned} \tag{7}$$

In case of having $\lambda \ll 1$ describing rare events, and based on the definition of Exponential function:

$$P\left(r \geq 1 \middle| \lambda t\right) = 1 - e^{-\lambda t} = 1 - \sum_{n=0}^{+\infty} \frac{\left(-\lambda t\right)^n}{n!} = 1 - \left[1 - \lambda t + \frac{\left(\lambda t\right)^2}{2!} - \cdots\right] \cong \lambda t \tag{8}$$

As a result, if t is equal to a time interval of one year, the probability is equal to the rate for rare events, i.e.:

$$P\left(r \geq 1 \middle| \lambda, t = 1\right) \cong \lambda \tag{9}$$

That is why in estimating of the risk for rare events (e.g., seismic risk), we can see that rate of occurrence λ and the probability of having at least one event in unit time with the rate λ are interchangeably used. Moreover, it is important to note another important aspect within the occurrences of an event called the *Return Period* and denoted generally as T_R. It is a statistical measurement typically based on historic data denoting the average recurrence interval over an extended period of time, and is usually used for risk analysis. In the estimations related to the Poisson process, $T_R = 1/\lambda$. For instance, the probability of having no event (i.e. no success) during the return period is defined as follows:

$$\begin{aligned} &P\left(r = 0 \middle| \lambda, \mathrm{t} = T_R = \frac{1}{\lambda}\right) = \frac{\left(\lambda T_R\right)^0 e^{-\lambda T_R}}{0!} = e^{-1} = 0.37 \\ &\Rightarrow P\left(r \geq 1 \middle| \lambda, \mathrm{t} = T_R = \frac{1}{\lambda}\right) = 1 - e^{-1} = 0.63 \end{aligned} \tag{10}$$

It reveals that the probability zero occurrences during the return period is equal around 33%. Subsequently, the probability of having at least one event in the time interval T_R is around 66%. The properties of the Poisson probability distribution can be summarized as follows:

(1) The sum of the Poisson distribution (a probability mass function, PMF) over all r values is equal to unity, i.e.

$$\sum_{r=0}^{+\infty} P\left(r\middle|s\right) = \sum_{r=0}^{+\infty} \frac{\left(s\right)^r e^{-s}}{r!} = e^{-s} \sum_{r=0}^{+\infty} \frac{\left(s\right)^r}{r!} = e^{-s} e^{s} = 1 \tag{11}$$

(2) The expected value is equal to s:

$$\mathbb{E}\left(r\middle|s\right) = \sum_{r=0}^{+\infty} r \frac{\left(s\right)^r e^{-s}}{r!} = \sum_{r=1}^{+\infty} r \frac{\left(s\right)^r e^{-s}}{r!} = s \sum_{r=1}^{+\infty} \frac{\left(s\right)^{r-1} e^{-s}}{\left(r-1\right)!} = s = \lambda t \tag{12}$$

This conclusion can directly be obtained by knowing the fact that for a Binomial distribution, the expected value is $np \approx s = \lambda t$.

(3) The variance is also equal to $s = \lambda t$:

$$\begin{aligned}
\mathbb{E}\left(r^2\middle|s\right) &= \sum_{r=0}^{+\infty} r^2 \frac{\left(s\right)^r e^{-s}}{r!} = \sum_{r=1}^{+\infty} r^2 \frac{\left(s\right)^r e^{-s}}{r!} = \sum_{r=1}^{+\infty} r \frac{\left(s\right)^r e^{-s}}{\left(r-1\right)!} = \sum_{r=1}^{+\infty} \left(r-1+1\right) \frac{\left(s\right)^r e^{-s}}{\left(r-1\right)!} \\
&= \sum_{r=1}^{+\infty} \left(r-1\right) \frac{\left(s\right)^r e^{-s}}{\left(r-1\right)!} + \sum_{r=1}^{+\infty} \frac{\left(s\right)^r e^{-s}}{\left(r-1\right)!} = s^2 \sum_{r=2}^{+\infty} \frac{\left(s\right)^{r-2} e^{-s}}{\left(r-2\right)!} + s \sum_{r=1}^{+\infty} \frac{\left(s\right)^{r-1} e^{-s}}{\left(r-1\right)!} \\
&= s^2 + s
\end{aligned} \tag{13}$$

$$\mathbb{VAR}\left(r\middle|s\right) = \mathbb{E}\left(r^2\middle|s\right) - \left[\mathbb{E}\left(r\middle|s\right)\right]^2 = \left(s^2 + s\right) - s^2 = s = \lambda t \tag{14}$$

This conclusion can also be extracted from the variance of a Binomial distribution, where

$$\mathbb{VAR}\left(r\middle|n, p\right) = \lim_{n\to+\infty, p\to 0} np\left(1-p\right) = np \approx s = \lambda t \tag{15}$$

As a result, one of the key features of the Poisson distribution is that its expected value and variance are identical and equal to $s = \lambda t$.

2. Exponential Probability Distributions

Instead of defining the Poisson probability distribution in terms of the probability of occurrence of events, we can equivalently define it in terms of the time between occurrences by defining the distribution of *Inter-Arrival Time* (*IAT*). This distribution is called the *Exponential* distribution. Accordingly, let the statement A be defined as follows:

$A \equiv IAT$ is greater than a desired time interval t (i.e., $IAT > t$) ≡ No events occur during the time interval t

$\mathbf{I} \equiv$ the mean rate of occurrence of the event is λ

Hence, the probability expression, $P(A|\mathbf{I})$, can be defined based on the Poisson probability distribution as:

$$P(A|\mathbf{I}) = P(IAT > t|\lambda) \,\square\, G_{IAT}(t|\mathbf{I}) \equiv P(r = 0|\lambda t) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t} \tag{16}$$

This probability expression is the CCDF of Exponential distribution. Hence, the CDF of the Exponential distribution can be derived as:

$$F_{IAT}(t|\mathbf{I}) = P(IAT \le t|\lambda) = 1 - P(IAT > t|\lambda) \equiv P(r \ge 1|\lambda t) = 1 - e^{-\lambda t} \tag{17}$$

It is apparent that the Exponential distribution is a *continuous* distribution. Accordingly, the associated PDF can be defined as:

$$f_{IAT}(t|\lambda) = \frac{\mathrm{d}F_{IAT}(t|\lambda)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - e^{-\lambda t}\right) = \lambda e^{-\lambda t} \tag{18}$$

It is revealed that both the Poisson and Exponential distributions are describing the same phenomenon; the former in terms of occurrence of events (success) in a *discrete* manner and the latter in terms of *IAT* in a *continuous* form.

The existence of exponential *IAT*'s leads to an important feature of simple Poisson processes. that the inter-arrival time from any point in time until the next occurrence time is independent of the time since the last event, or of any other events activity in the past. This particular characteristic of Poisson processes is known as the *memory-less* property. To derive it mathematically, suppose that (see also Figure 1):

$A \equiv IAT > t + \tau$

$B \equiv IAT > \tau$

$\mathbf{I}$ ≡ Time is measured from the occurrence of the last event, and the mean rate of occurrence of the events is λ

Using the product rule:

$$\begin{aligned} P(A|B,\mathbf{I}) &= \frac{P(A \cdot B|\mathbf{I})}{P(B|\mathbf{I})} \equiv \frac{P\left[(IAT > t+\tau).(IAT > \tau)|\lambda\right]}{P(IAT > \tau|\lambda)} = \frac{P\left[(IAT > t+\tau)|\lambda\right]}{P(IAT > \tau|\lambda)} \\ &= \frac{e^{-\lambda(t+\tau)}}{e^{-\lambda\tau}} = e^{-\lambda t} \equiv P(IAT > t|\lambda) \end{aligned} \tag{19}$$

Therefore, if we have already waited a time *t* since the last event and observed no new events (nothing has been happened), then the probability distribution of the remaining time until the next event is the same as the *IAT* distribution right after the last event (see Figure 1). Although the distribution of any increment does not depend on past activities, it has also been verified that the same holds for the distribution of the time you have to wait until the next event.

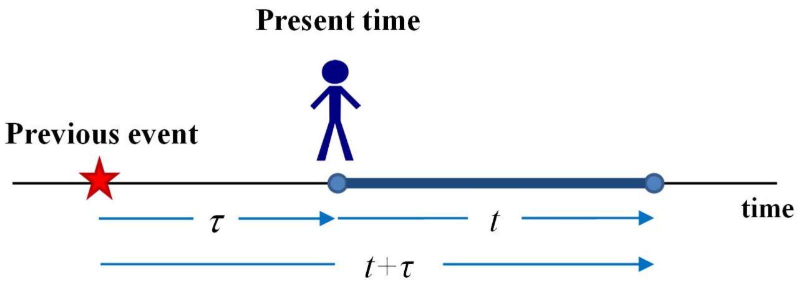

Figure 1: Random process in terms of the arrival time