Lecture 2: Discrete Probability Distributions

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1. Discrete Probability Distribution

1.1. *Discrete Uniform Distribution*

Consider the following n logical statements:

$A_i \equiv X$ is equal to x_i, where i=1:n

In this case, we map the statement A_i to the parameter X. The parameter X can be defined as an *uncertain parameter* (also known as *random* or *aleatory variable*). In order to calculate $P(A_i|\mathbf{I})$, let us assume that the background information $\mathbf{I}$ is un-informative (i.e., has not enough information to distinguish between different statements A_i). Thus, $P(A_i|\mathbf{I})$ can easily be written as:

$$P\left(A_i|\mathbf{I}\right)=P\left(X=x_i|\mathbf{I}\right)=\frac{1}{n} \tag{1}$$

This probability expression $P(X|\mathbf{I})$ is called a *Uniform* distribution which is actually a *discrete* probability distribution of an uncertain parameter X that takes the values x_i, i=1:n, with an equal probability of 1/n. The *discrete Uniform* probability distribution is schematically illustrated in Figure 1. This kind of probability distribution is generally known as the Probability Mass Function (PMF) as the probability is shown in discrete values.

Accordingly, the probability that X is smaller than a specified value is called the Cumulative Density (Distribution) Function (CDF). The CDF of a discrete Uniform distribution can be calculated assuming that the statements A_i are ME:

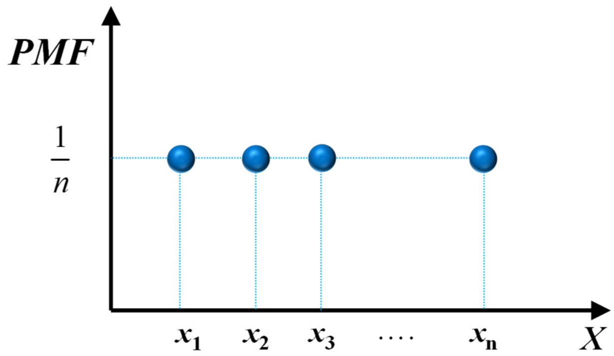


Figure 1: The discrete Uniform probability mass function (PMF)

$$
\begin{aligned}
P\left(X \le x_k \middle| \mathbf{I}\right) &= P\left[\left(X = x_1 \middle| \mathbf{I}\right) + \left(X = x_2 \middle| \mathbf{I}\right) + \cdots + \left(X = x_k \middle| \mathbf{I}\right)\right] \\
&= P\left(X = x_1 \middle| \mathbf{I}\right) + P\left(X = x_2 \middle| \mathbf{I}\right) + \cdots + P\left(X = x_k \middle| \mathbf{I}\right) = \frac{k}{n}
\end{aligned} \tag{2}
$$

On the other hand, the Complementary CDF (a.k.a. CCDF) can be expressed as:

$$
P\left(X > x_k \middle| \mathbf{I}\right) = 1 - P\left(X \le x_k \middle| \mathbf{I}\right) = 1 - \frac{k}{n} \tag{3}
$$

The discrete Uniform CDF and CCDF are shown in Fig. 2.

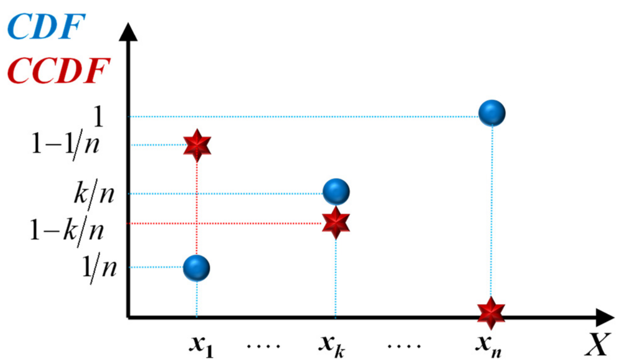


Figure 2: The discrete Uniform cumulative density function (CDF) with blue cycles, and the complementary CDF (CCDF) with red asterisk

1.2. *Bernoulli (Binary) Distribution*

Consider the following two logical statements:

$A \equiv$ having success in an experiment $\equiv X$ is equal to 1

$\overline{A} \equiv$ having failure (no success) $\equiv X$ is equal to 0

$\mathbf{I} \equiv$ the probability of having success is π

In this case, we have mapped the statement A and its negation to the parameter X. The parameter X can be defined as an *uncertain parameter* (also known as *random* or *aleatory variable*). Let the probability term $P(A|\mathbf{I})$ be equal to:

$$P\left(A|\mathbf{I}\right) = P\left(X = 1|\mathbf{I}\right) = \pi \tag{4}$$

Consequently,

$$P\left(\overline{A}|\mathbf{I}\right) = P\left(X = 0|\mathbf{I}\right) = 1 - \pi \tag{5}$$

The probability expression $P(X|\mathbf{I})$ is called a *Bernoulli* distribution which is actually the *discrete* probability distribution of an uncertain parameter which takes value 1 with probability equal to π *(the so-called success)*, and value 0 with probability equal to 1-π *(the so-called failure)*. Thus, there is no intermediate value between 0 and 1. The *Bernoulli* probability mass function (PMF) is schematically illustrated in Figure 3.

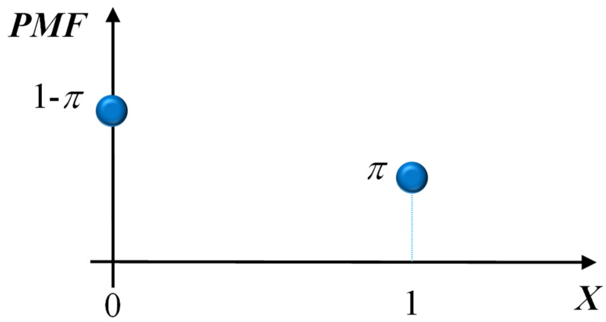


Figure 3: The Bernoulli probability mass function (PMF)

Accordingly, the probability that X is smaller than a specified value is called the Cumulative Density (Distribution) Function (CDF). The CDF of a Bernoulli distribution can be calculated as:

$$
\begin{aligned}
&P(X \le 0|\mathbf{I}) = P(X=0|\mathbf{I}) = 1-\pi \\
&P(X \le 1|\mathbf{I}) = P(X=0|\mathbf{I}) + P(X=1|\mathbf{I}) = (1-\pi)+\pi = 1
\end{aligned}
\tag{6}
$$

The second statement in Eq. (6) reveals a TRUE statement, and it is obvious to be equal to 1. The Bernoulli CDF and CCDF are shown in Fig. 4.

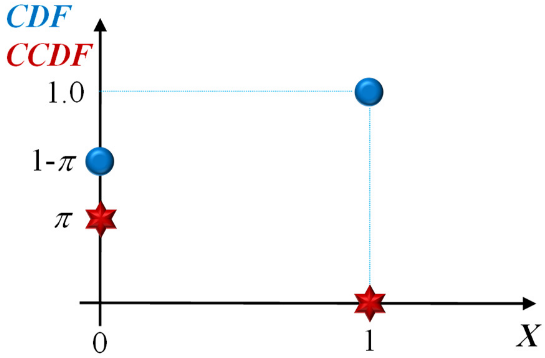


Figure 4: The Bernoulli cumulative density function (CDF) with blue cycles, and the complementary CDF (CCDF) with red asterisk

A number of examples are illustrated in order to gain a better perspective of this important and basic PMF.

Example 1: The toss of a coin:

B ≡ having "heads" ≡ Y is equal to 1

$\bar{B}$ ≡ having "tails" ≡ Y is equal to 0

$\mathbf{I}$ ≡ *the coin has no imperfections*

The Bernoulli variable Y can be defined to have a PMF as follows:

$$P(Y=1|\mathbf{I}) = 1/2$$

$$P(Y=0|\mathbf{I}) = 1/2$$

Example 2: Concrete cube test with compression strength f_{ck}:

$B \equiv f_{ck} > 20\text{MPa} \equiv Z$ is equal to 1

$\bar{B} \equiv f_{ck} \leq 20\text{MPa} \equiv Z$ is equal to 0

$\mathbf{I} \equiv$ the probability that Z=1 is π

The Bernoulli variable Z with success probability of π can be defined to have a PMF as follows:

$$P\left(Z=1\middle|\mathbf{I}\right)=\pi$$

$$P\left(Z=0\middle|\mathbf{I}\right)=1-\pi$$

Example 3: Decision-making in the aftermath of a major ground motion:

$D \equiv$ re-occupation of the building $\equiv X$ is equal to 0

$\bar{D} \equiv$ no re-occupation of the building $\equiv X$ is equal to 1

$\mathbf{I} \equiv$ probability that X=1 is π

The Bernoulli variable X with success probability of π can be defined to have a PMF as follows:

$$P\left(X=1\middle|\mathbf{I}\right)=\pi$$

$$P\left(X=0\middle|\mathbf{I}\right)=1-\pi$$

1.3. *Binomial Distribution*

We want to calculate the probability of having r success out of n independent trials (experiments) knowing also that the probability of success of a single trial is π. The associated logical statements are symbolized as follows:

$A \equiv$ having r successes

$\mathbf{I} \equiv$ having n independent (Bernoulli) trials with the success probability of a single trial to be π

In order to estimate $P(A|\mathbf{I})$, let us define the set of ME statements $\{A_i, i=1:N\}$, where N is equal to the number of ways (combinations) in which one can arrange (order does not matter) r successes out of n experiments. Thus,

$A_i \equiv$ having r success given a specific configuration i of r successes out of n trials

In order to calculate N, which is also known as the *binomial coefficient*, consider that there are n free places to put r success; hence, the first success has n possibilities to be placed, the second one has (n-1) possibilities, while the rth success has (n-r+1) possibilities. It is to note that since the order between successful cases does not matter, we have to divide the whole value by r! (we can permute them in r! ways). As a result, N can be derived as:

$$N = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} \triangleq \binom{n}{r} \tag{7}$$

Now, getting back to calculation of $P(A|\mathbf{I})$,

$$P(A|\mathbf{I}) \triangleq P(r|n,\pi) = P(A_1 + A_1 + \cdots + A_N |\mathbf{I}) = \sum_{i=1}^{N} P(A_i |\mathbf{I}) \tag{8}$$

Therefore, for a given configuration i:

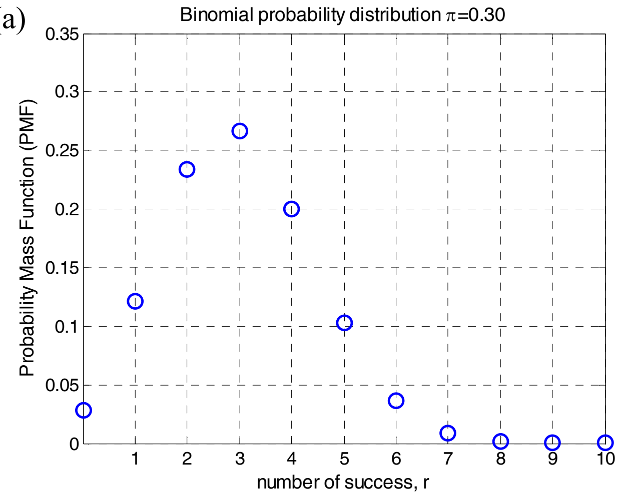
$$
\begin{aligned}
P\left(A_i\middle|\mathbf{I}\right) &= P\left(\overbrace{(X_1=1)\cdot(X_2=1)\cdots(X_r=1)}^{r\ \text{terms}}\cdot\overbrace{(X_{r+1}=0)\cdots(X_n=0)}^{(n-r)\ \text{terms}}\middle|\mathbf{I}\right)\\
&= P\left(X_1=1\middle|\mathbf{I}\right)P\left(X_2=1\middle|\mathbf{I}\right)\cdots P\left(X_r=1\middle|\mathbf{I}\right)P\left(X_{r+1}=0\middle|\mathbf{I}\right)\cdots P\left(X_n=0\middle|\mathbf{I}\right)\\
&= \pi^r\left(1-\pi\right)^{n-r}
\end{aligned}
\tag{9}
$$

where X_i's are *independent and identically distributed* (*i.i.d*) uncertain parameters. Arguing that $P\left(A_i\middle|\mathbf{I}\right)=\pi^r(1-\pi)^{n-r}$ for any configuration i, one can have the PMF for the *Binomial* distribution (Eq. 8) as follows:

$$
P\left(A\middle|\mathbf{I}\right)=P\left(r\middle|n,\pi\right)=\binom{n}{r}\pi^r\left(1-\pi\right)^{n-r}
\tag{10}
$$

As the boundary conditions, when r=0, (i.e., no success in n trials), it is apparent that $P\left(0\middle|n,\pi\right)=\left(1-\pi\right)^n$; if r=n (i.e., all trials lead to success), it can easily be concluded that $P\left(n\middle|n,\pi\right)=\pi^n$. Fig. 5(a) illustrates the binomial Probability Mass Function (PMF) with n=10 and π=0.30.

(a)



(b)

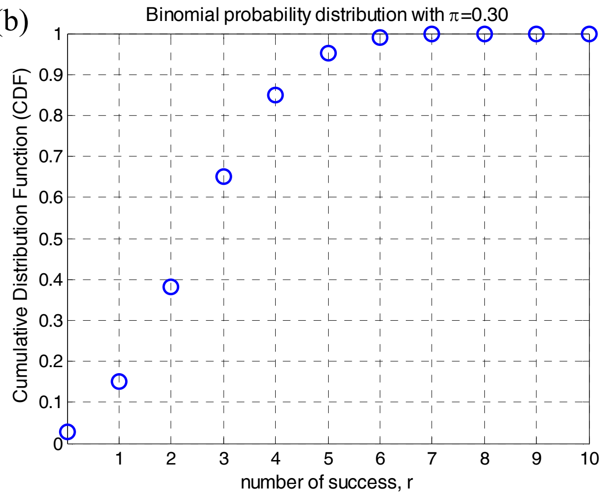


Figure 5: (a) The binomial probability mass function (PMF), (b) The binomial cumulative distribution function (CDF)

The expression of PMF in Eq. (10) reveals the probability that exactly $n_r = r$ success takes place within n individual experiments (n_r, herein, denotes the uncertain parameter which in this case has a specific value equal to r). However, there may be cases where one is interested to estimate the probability that n_r is less than a certain value r (i.e., $n_r \leq r$). In this case, considering the statements to be ME,

$$\begin{aligned} P\left(n_r \leq r \middle| n,\pi\right) &= P\left((n_r=0)+(n_r=1)+\cdots+(n_r=r)\middle| n,\pi\right) \\ &= \sum_{n_r=0}^{r} P\left(n_r \middle| n,\pi\right) = \sum_{n_r=0}^{r} \binom{n}{n_r} \pi^{n_r}\left(1-\pi\right)^{n-n_r} \end{aligned} \tag{11}$$

This expression is called the CDF for the *Binomial* distribution, and is shown in Fig. 5(b). It can be seen that as n_r increase, the CDF yields to 1.