Sample Problems
Lecture 5: The Normal Distribution

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1. The Tensile Test

Suppose that we have done n = 10 tests to estimate the tensile resistance f_t of a member and we have gathered the following results: D={200, 210, 195, 195, 190, 205, 205, 200, 210, 190}.

- **Calculate the sample mean and standard deviation**

$$\overline{D} = \frac{\sum_{i=1}^{10} D_i}{10} = 200$$

$$s_D = \sqrt{\frac{\sum_{i=1}^{10}\left(D_i - \overline{D}\right)^2}{10-1}} = 7.45$$

- **Calculate the mean and the standard deviation for sample average**

  Having *i.i.d.*, test data D:

$$\mathbb{E}\left(\overline{D}\right) \triangleq \mu_{\overline{D}} = \mathbb{E}\left(\frac{\sum_{i=1}^{10} D_i}{10}\right) = \frac{\sum_{i=1}^{10}\mathbb{E}\left(D_i\right)}{10} = \frac{10\times 200}{10} = 200$$

$$\mathbb{VAR}\left(\overline{D}\right) \triangleq s_{\overline{D}}^{\;2} = \mathbb{VAR}\left(\frac{\sum_{i=1}^{10} D_i}{10}\right) = \frac{\sum_{i=1}^{10}\mathbb{VAR}\left(D_i|\mathbf{I}\right)}{10^2} = \frac{10\times s_D^{\;2}}{10^2} = \frac{s_D^{\;2}}{10}$$

$$s_{\overline{D}} = \frac{s_D}{\sqrt{10}} = 2.357$$

- **Calculate the counted statistical parameters**

$$D_{\text{sorted}} = \{190, 190, 195, 195, 200, 200, 205, 205, 210, 210\}$$

$$\eta_D = p_{50} = 200$$

$$\left.\begin{matrix} p_{16} = 190 \\ p_{84} = 205 \end{matrix}\right\} \Rightarrow s_X = \frac{p_{84} - p_{16}}{2} = 7.50$$

- **Now suppose we would like to describe the test results by the standard normal distribution. What is the likelihood of observing the data based on the mean and standard deviation calculated from the data**

$$P\left(D \middle| \mu_D, \sigma_D\right) = \prod_{i=1}^{10} \phi\left(\frac{D_i - \mu_D}{\sigma_D}\right) = \phi(0)^2 \times \phi\left(\frac{5}{7.45}\right)^4 \times \phi\left(\frac{10}{7.45}\right)^4$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \times \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{5}{7.45}\right)^2}\right)^4 \times \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{10}{7.45}\right)^2}\right)^4$$

$$= 1.1296 \times 10^{-6}$$

- **Compute a 95% confidence interval for the sample mean**

$$P\left(\mu_{\overline{D}} - u_\alpha s_{\overline{D}} \le X_{\overline{D}} \le \mu_{\overline{D}} + u_\alpha s_{\overline{D}}\right) = 0.95 \Rightarrow P\left(\mu_D - u_\alpha \frac{s_D}{\sqrt{n}} \le X_{\overline{D}} \le \mu_D + u_\alpha \frac{s_D}{\sqrt{n}}\right) = 0.95$$

$$\Rightarrow \Phi(u_\alpha) - \Phi(-u_\alpha) = 0.95 \Rightarrow \Phi(u_\alpha) = \frac{1 + 0.95}{2}$$

$$\Rightarrow u_\alpha = \Phi^{-1}\left(\frac{1.95}{2}\right) = 1.96$$

$$\Rightarrow 200 - 1.96 \times \frac{7.45}{\sqrt{10}} \le X_{\overline{D}} \le 200 + 1.96 \times \frac{7.45}{\sqrt{10}} \Rightarrow 195.38 \le X_{\overline{D}} \le 204.62$$

2. The Structural Reliability Problem (Safety Ratio Formulation)

<u>General Formulation:</u> The structural reliability can be characterized by the probability of failure, P_F, in a structural member. The probability of failure can be defined as the probability that member load, denoted as *L*, exceeds member resistance *R*, i.e.:

$$P_F = P\left(R < L\right) \equiv P\left(\frac{R}{L} < 1\right) = P\left(X < 1\right) \equiv P\left(Y = \ln X < 0\right) \tag{1}$$

The term $Y = \ln X$ is the limit state (failure) function. By using the normal distribution to calculate the probability of failure:

$$P_F = P\left(\frac{Y-\mu_Y}{\sigma_Y} < \frac{0-\mu_Y}{\sigma_Y}\right) = P\left(u < -\frac{\mu_Y}{\sigma_Y}\right) \equiv \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) = \Phi\left(-\beta\right) \tag{2}$$

The ratio β is known as the *reliability index.* Thus,

$$\beta = \Phi^{-1}\left(P_F\right) \tag{3}$$

- **Suppose that the load *L* has a median equal to η_L and coefficient of variation equal to cov_L, and that resistance *R* has a median equal to η_R and coefficient of variation equal to cov_R. Moreover, the correlation coefficient between (logarithms) of load and resistance is equal to ρ, i.e. $\rho_{\ln R, \ln L} \triangleq \rho$. Suppose that load and resistance are described by lognormal probability distribution. Define the reliability index as a function of the median, coefficient of variation, and correlation between load and resistance.**

$$\mu_Y = \mu_{\ln X} = \mathbb{E}\left(\ln\frac{R}{L}\right) = \mathbb{E}\left(\ln R - \ln L\right) = \mu_{\ln R} - \mu_{\ln L} = \ln\eta_R - \ln\eta_L = \ln\frac{\eta_R}{\eta_L}$$

$$\sigma_Y^2 = \sigma_{\ln X}^2 = \mathbb{VAR}\left(\ln\frac{R}{L}\right) = \mathbb{VAR}\left(\ln R - \ln L\right) = (1)^2\,\sigma_{\ln R}^2 + (-1)^2\,\sigma_{\ln L}^2 - 2\rho\sigma_{\ln R}\sigma_{\ln L}$$

$$\cong \mathrm{cov}_R^2 + \mathrm{cov}_L^2 - 2\rho\,\mathrm{cov}_R\,\mathrm{cov}_L$$

$$\Rightarrow \beta = \frac{\mu_Y}{\sigma_Y} = \frac{\ln\left(\eta_R/\eta_L\right)}{\sqrt{\mathrm{cov}_R^2 + \mathrm{cov}_L^2 - 2\rho\,\mathrm{cov}_R\,\mathrm{cov}_L}}$$

- **Derive a formula for the probability of failure based on the median, coefficient of variation, and correlation between load and resistance.**

$$P_F = \Phi\left(\frac{\ln\left(\eta_L/\eta_R\right)}{\sqrt{\mathrm{cov}_R^2 + \mathrm{cov}_L^2 - 2\rho\,\mathrm{cov}_R\,\mathrm{cov}_L}}\right)$$

- **<u>Numerical Application:</u> Take a simple beam with length equal to *L* = 4 m, and suppose that the median ultimate moment in the beam section is equal to M_u = 2600 kg-m with a coefficient of variation equal to 15%. The load *P* acting at the beam mid-length has median equal to 2000 kg and a coefficient of variation equal to 20%. Estimate the failure probability**

$$P_F = P\left(M_u < M_s = PL/4\right) \equiv P\left(Y < 0\right) = \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) = \Phi\left(-\beta\right),\quad Y = \ln\left(\frac{M_u}{M_s}\right)$$

$$\mu_Y = \mathbb{E}\left(\ln M_u - \ln M_s\right) = \mu_{\ln M_u} - \mu_{\ln M_s} = \ln\eta_{M_u} - \ln\eta_{M_s} = \ln\frac{2600}{2000\times 4/4}$$

$$\sigma_{\ln M_u} = \sqrt{\ln\left(1+\mathrm{cov}_{M_u}^2\right)} = 0.1492 \cong \mathrm{cov}_{M_u} = 0.15$$

$$\sigma_{\ln M_s} = \sqrt{\ln\left(1+\mathrm{cov}_{M_s}^2\right)} = 0.1980 \cong \mathrm{cov}_{M_s} = 0.20$$

$$P_F = \Phi\left(\frac{\ln\left(\eta_{M_s}/\eta_{M_u}\right)}{\sqrt{\mathrm{cov}_{M_u}^2 + \mathrm{cov}_{M_s}^2 - 2\rho\,\mathrm{cov}_{M_u}\,\mathrm{cov}_{M_s}}}\right) = \Phi\left(\frac{\ln\left(2000/2600\right)}{\sqrt{0.15^2 + 0.20^2}}\right)$$

$$= \Phi\left(-1.0495\right) = 0.147 \cong 15\%$$

- **Assuming an allowable probability level of 10%, would the beam in question pass the safety-checking test?** No, because $15\% > 1 - 90\% = 10\%$

3. The Structural Reliability Problem (Safety Ratio Formulation)

Imagine a wall panel of dimensions $L \times H \times t$ in a structure made of cement bricks (see figure below). The hydrostatic pressure caused by the flood can be represented by the loading profile shown in the figure (applied perpendicular to the wall), where γ_w represents the density of the fluid (water mixed with mud and debris). Assuming that the wall is fixed at the base, the base moment can be calculated considering the loading profile shown in the Figure. Let the bending resistance of the section at the base be calculated from the following formula, assuming that the ratio of the normal forces to the ultimate normal force is equal to 0.1, that is, N/N_u=0.1:

$$M_R = 0.4t^2 L\gamma$$

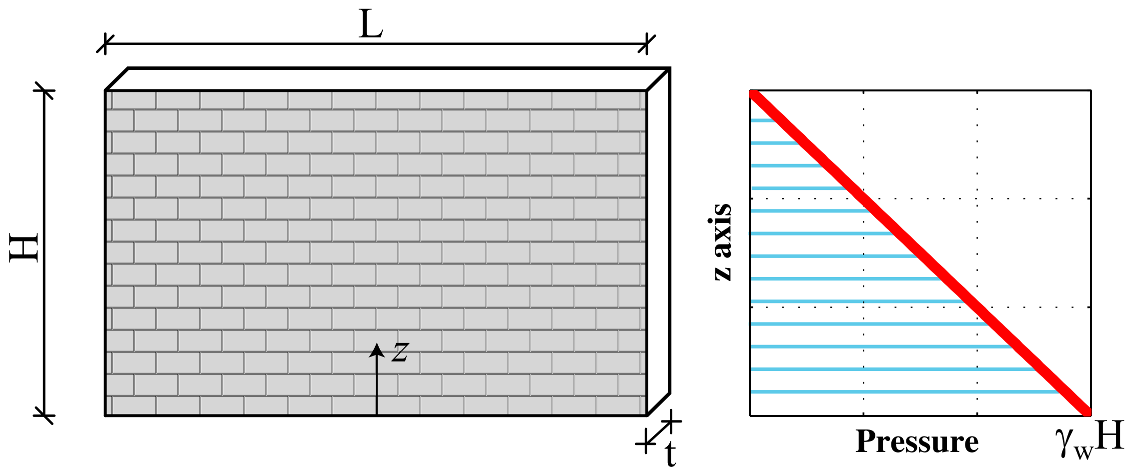

Figure 1: (left) A wall panel subjected to (right) hydrostatic pressure

Consider that γ_w is distributed as Lognormal with median equal to 11KN/m^3 and coefficient of variation equal to 10%; γ is distributed as Lognormal with median equal to 21KN/m^3 and coefficient of variation equal to 15%; t is distributed as Lognormal with median equal to 0.50m and coefficient of variation equal to 20%. γ_w, γ, and t are considered to be independent and H=1.5m.

- **Calculate the probability of failure**

$$m = \ln M_R - \ln M_L$$
$$\Rightarrow P_F = P\left(M_R < M_L\right) = P\left(\ln M_R - \ln M_L < 0\right) = P\left(m < 0\right)$$

In the Standard Normal Space, we have:

$$P_F = P\left(m<0\right) = P\left(\frac{m-\mu_m}{\sigma_m} < \frac{0-\mu_m}{\sigma_m}\right) = P\left(U < -\frac{\mu_m}{\sigma_m}\right) \triangleq \Phi\left(-\frac{\mu_m}{\sigma_m}\right) = \Phi\left(-\beta\right)$$

The marginal function can generally be written as:

$$m = a_0 + \sum_{i=1}^{n} a_i \ln X_i$$

Thus:

$$\mu_m = a_0 + \sum_{i=1}^{n} a_i \mu_{\ln X_i}$$

$$\sigma_m = \sum_{i=1}^{n} a_i^2 \sigma_{\ln X_i}^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j}}^{n} \rho_{\ln X_i,\ln X_j} a_i a_j \sigma_{\ln X_i} \sigma_{\ln X_j}$$

With reference to our problem:

$$m = \ln M_R - \ln M_L = \ln\left(0.4t^2 L\gamma\right) - \ln\left(\gamma_w HL \times \frac{H}{2} \times \frac{H}{3}\right) = \ln \frac{2.4t^2\gamma}{\gamma_w H^3}$$

$$= \ln \frac{2.4}{1.5^3} + 2\ln t + \ln\gamma - \ln\gamma_w$$

$$\mu_m = \ln \frac{2.4}{1.5^3} + 2\mu_{\ln t} + \mu_{\ln\gamma} - \mu_{\ln\gamma_w} = \ln \frac{2.4}{1.5^3} + 2\ln\eta_t + \ln\eta_\gamma - \ln\eta_{\gamma_w}$$

$$= \ln \frac{2.4}{1.5^3} + 2\ln\eta_t + \ln\eta_\gamma - \ln\eta_{\gamma_w} = -1.0806$$

$$\sigma_m^2 = 4\sigma_{\ln t}^2 + \sigma_{\ln\gamma}^2 + \sigma_{\ln\gamma_w}^2 = 0.1925$$

Finally, we get:

$$P_F = \Phi\left(-\frac{-1.0806}{\sqrt{0.1925}}\right) = \Phi\left(2.4629\right) = 0.9931$$

4. The Moment Generating Function

Recall that the moment generating function for a probability distribution denoted by $f_X(x)$ is calculated as following:

$$\mathbb{M}_X(t) = \int_{\Omega_X} e^{tx} f_X(x)\,\mathrm{d}x$$

The moment generating function for the sum of two independent random variables X and Y can be written as:

$$\mathbb{M}_{X+Y}(t) = \int_{\Omega_X}\int_{\Omega_Y} e^{t(x+y)} f_X(x) f_Y(y)\,\mathrm{d}y\,\mathrm{d}x = \mathbb{M}_X(t)\,\mathbb{M}_Y(t)$$

where $f_X(x)$ and $f_Y(y)$ are the probability density functions for x and y, respectively. The moment generating function for the Normal distribution is equal to:

$$\mathbb{M}_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

- **Using the definition of moment generating function, indicate the distribution of the sum of two independent Normal variables X and Y with mean μ_1 and μ_2 and standard deviation σ_1 and σ_2.**

$$\mathbb{M}_{X+Y}(t) = \mathbb{M}_X(t)\,\mathbb{M}_Y(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2}\, e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} = e^{(\mu_1+\mu_2)t + \frac{1}{2}\left(\sigma_1^2+\sigma_2^2\right)t^2}$$

It has a mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$

- **Calculate the third moment.**

The kth non-central moment of probability distribution $f_X(x)$ can be evaluated as:

$$\mathbb{E}\left(X^m\right) = \frac{\mathrm{d}^m \mathbb{M}_X(t)}{\mathrm{d}t^m}\Big|_{t=0}$$

Thus,

$$\mathbb{E}(X) = \frac{\mathrm{d}\mathbb{M}_X(t)}{\mathrm{d}t}\Big|_{t=0} = \left(\mu + \sigma^2 t\right) e^{\mu t + \frac{1}{2}\sigma^2 t^2}\Big|_{t=0} = \mu$$

$$\mathbb{E}\left(X^2\right)=\frac{\mathrm{d}^2\mathbb{M}_X\left(t\right)}{\mathrm{d}t^2}\Big|_{t=0}=\left(\mu+\sigma^2t\right)^2 e^{\mu t+\frac{1}{2}\sigma^2t^2}+\sigma^2e^{\mu t+\frac{1}{2}\sigma^2t^2}\Big|_{t=0}=\mu^2+\sigma^2$$

$$\mathbb{E}\left(X^3\right)=\frac{\mathrm{d}^3\mathbb{M}_X\left(t\right)}{\mathrm{d}t^3}\Big|_{t=0}=\left(\mu+\sigma^2t\right)^3 e^{\mu t+\frac{1}{2}\sigma^2t^2}+2\sigma^2\left(\mu+\sigma^2t\right)e^{\mu t+\frac{1}{2}\sigma^2t^2}+\sigma^2\left(\mu+\sigma^2t\right)e^{\mu t+\frac{1}{2}\sigma^2t^2}\Big|_{t=0}$$
$$=\mu^3+3\sigma^2\mu$$

- **Calculate the skewness for a Normal distribution.**

  The skewness is defined as:

$$\mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]=\frac{1}{\sigma^3}\mathbb{E}\left(X^3-3X^2\mu+3X\mu^2-\mu^3\right)$$
$$=\frac{1}{\sigma^3}\left[\mathbb{E}\left(X^3\right)-3\mathbb{E}\left(X^2\right)\mu+3\mathbb{E}\left(X\right)\mu^2-\mu^3\right]$$
$$=\frac{1}{\sigma^3}\left[\mu^3+3\sigma^2\mu-3\mu\left(\mu^2+\sigma^2\right)+3\mu^3-\mu^3\right]=0$$

5. The Effective Temperature

The maximum temperatures within one week are reported as follows.

$$T_{\max} = [18, 16, 19, 20, 17, 12, 16] \ {}^{\circ}C$$

In addition, the pressure on the same days as outlined as:

$$p = [10, 12, 18, 20, 25, 5, 20]$$

The effective temperature (T_e) is defined as:

$$T_e = T_{\max} + \frac{5}{9} \cdot (p - 10)$$

- **According to the given weekly data, what is the mean of effective temperature?**

Appling the equation of the effective temperature we obtain:

$$T_e = [18, 17.1, 23.4, 25.6, 25.3, 9.2, 21.6] \ {}^{\circ}C$$

The mean effective temperature is equal to:

$$\mu_{T_e} = \frac{\sum_{i=1}^{n} T_{e,i}}{n} \simeq 20^{\circ}C$$

It could be possible to arrive at the same result using the linearity of the expected value operator:

$$\mathbb{E}\left[T_{\max} + \frac{5}{9}(p - 10)\right] = \mathbb{E}[T_{\max}] + \frac{5}{9} \cdot (\mathbb{E}[p] - 10)$$

- **What is the correlation between the maximum temperature and the pressure?**

The correlation coefficient can be calculated with the following relation:

$$\rho_{T_{\max},p} = \frac{COV_{T_{\max},p}}{\sigma_{T_{\max}} \sigma_p} = \frac{\sum_{i=1}^{n} \left(T_{\max,i} - \mu_{T_{\max}}\right)\left(p - \mu_p\right)}{\sqrt{\sum_{i=1}^{n} \left(T_{\max,i} - \mu_{T_{\max}}\right)^2} \sqrt{\sum_{i=1}^{n} \left(p - \mu_p\right)^2}} \simeq 0.60$$

- **What is the standard deviation of the effective temperature?**

The standard deviation of the effective temperature is:

$$\sigma_{T_e} = \sqrt{\frac{\sum_{i=1}^{n}\left(T_{e,i} - \mu_{T_e}\right)^2}{n-1}} \simeq 5.8°C$$

It could be possible to arrive at the same result considering the following relation:

$$\sigma_{T_e} = \sqrt{\mathbb{VAR}\left[T_{\max} + \frac{5}{9}\left(p-10\right)\right]} = \sqrt{\mathbb{VAR}\left[T_{\max}\right] + \frac{25}{81}\cdot\mathbb{VAR}\left[p\right] + 2\times\frac{5}{9}\times\rho_{T_{\max},p}\sigma_{T_{\max}}\sigma_p}$$

6. Back analysis of debris-flow damage to a masonry structure

During the late evening of October 1st 2009, the village of Scaletta Zanclea (ME) was hit by a large debris flow coming from Racinazzi torrent that caused the collapse of some buildings and loss of lives. During this event the structure shown in the figure below (Fig 1a) was seriously damaged. The wall that is positioned parallel to the Racinazzi Torrent has suffered major damage in the wall panel to the left of the window in the corner with Northern wall (the detail in Fig 1b).

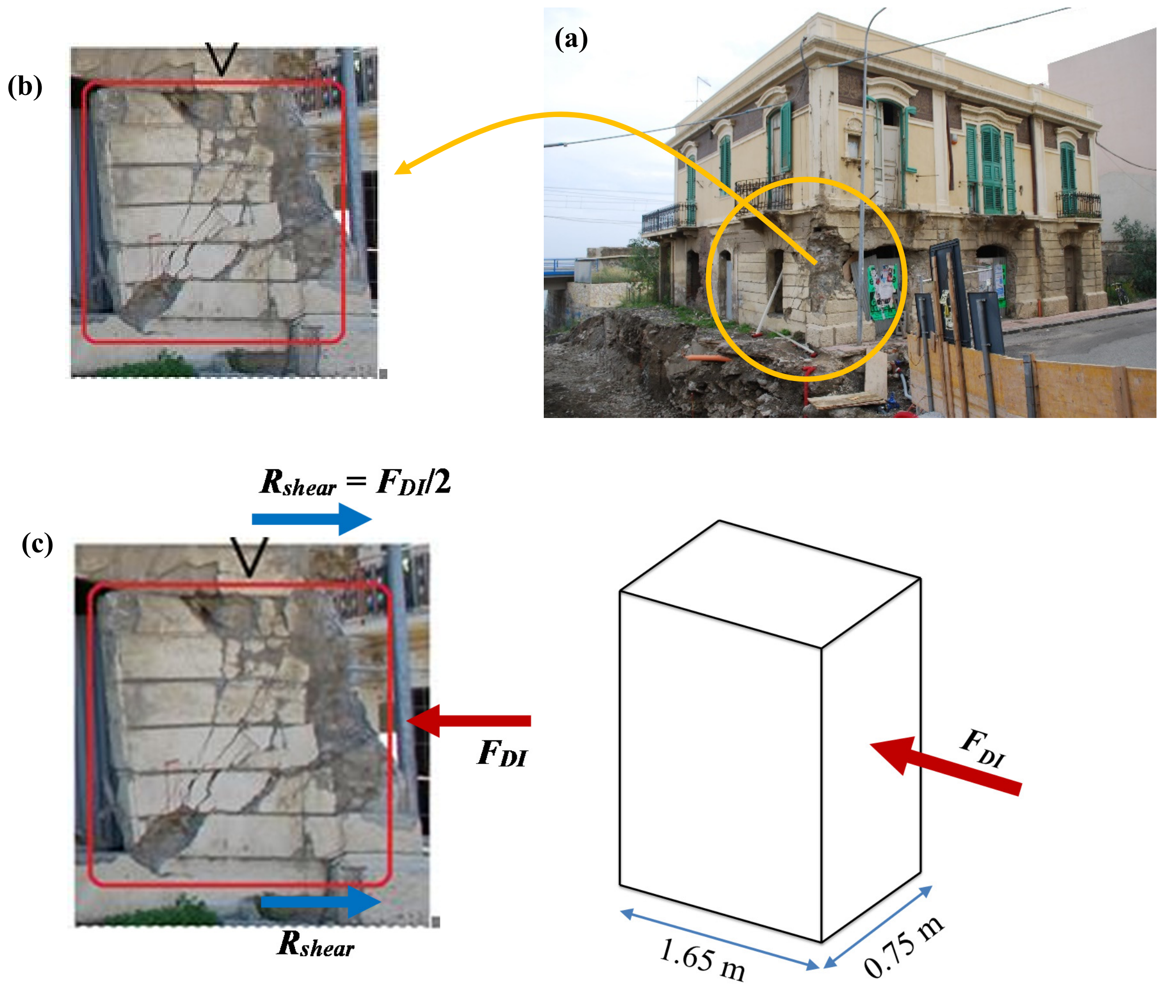

Figure 1: (a) The structure damaged by debris flow; (b) the local damage due to impact-type load; (c) the static scheme of the applied impact load and the reaction in the critical section

The signs left on the northern façade of the building (Fig 1a) provide some evidence that flow-borne objects (a large pipe in this case) have hit the building. The force resulting from impact of the flow borne objects can be calculated as follows:

$$F_{DI} = \frac{\left(W_D/g\right) \times v_D}{t} \tag{4}$$

where F_{DI} is the impact force, W_D is weight of object, v_D is velocity of object assumed, and t is the duration of impact. As it can be see in the datil in Fig 1b, the wall panel seems to suffer an in-plane diagonal shear failure. The safety ratio $Y_{in\text{-}shear}$ for the wall panel subjected to a combination of the lateral (in-plane) forces and the gravity loading is defined herein as:

$$Y_{in-shear} = \frac{R_{shear}}{R_{shear-res}} \tag{2}$$

where R_{shear} is the maximum shear force demand in the section under consideration (the section in the bottom corner of the window as shown in Fig 1c) and $R_{shear\text{-}res}$ is ultimate shear strength of section and is calculated as:

$$R_{shear-res} = \tau_{\lim} \times A \tag{6}$$

where τ_{lim} is the diagonal in-plane shear capacity and A is cross sectional area.

a) Let us assume that the debris weight W_D has a median of 160 KN and *COV* (coefficient of variation) of 0.50; that the velocity of flow-borne debris v_D has a median of 2 m/s and *COV* of 0.35; that the time of impact t has a median of 0.1 sec and a *COV* of 0.50. Moreover, the diagonal in-plane shear resistance of the masonry wall τ_{lim} is assumed to have a median of 125 KN/m^2 and a *COV* of 0.40[1] (hint: assume g=10 m/s^2; assume that the *COV* and the standard deviation of the logarithm are equal and that all these random variables are Lognormally distributed and independent)[2].

b) What is the mean and standard deviation of the logarithm of the safety ratio $Y_{in\text{-}shear}$?

ln$Y$ can be written as:

$$\ln Y = \ln\left(\frac{W_D \cdot v_D / 2gt}{\tau_{\lim} A}\right) = \ln W_D + \ln v_D - \ln(t) - \ln \tau_{\lim} - \ln(2gA)$$

Thus,

$$\mu_{\ln Y} = \ln \eta_{W_D} + \ln \eta_{v_D} - \ln \eta_t - \ln \eta_{\tau_{\lim}} - \ln(2gA) = 0.0338$$

where $\eta_{(\cdot)}$ is the median of the corresponding uncertain parameters. Accordingly,

$$\sigma_{\ln Y} = \sqrt{\sigma^2_{\ln W_D} + \sigma^2_{\ln v_D} + \sigma^2_{\ln t} + \sigma^2_{\ln \tau_{\lim}}} \cong \sqrt{V^2_{W_D} + V^2_{v_D} + V^2_t + V^2_{\tau_{\lim}}} = 0.8846$$

Since the uncertain parameters are assumed to be uncorrelated, the mutual correlation terms are equal to zero.

c) What is the expected value of $Y_{in\text{-}shear}$?

$$\mathbb{E}\left[Y \middle| \mathbf{I}\right] \equiv \mu_Y = \eta_Y e^{\frac{1}{2}\sigma^2_{\ln Y}} = e^{\mu_{\ln Y} + \frac{1}{2}\sigma^2_{\ln Y}} = 1.5296$$

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[1] It should be noted that for simplicity the hydrostatic and hydrodynamic forces applied by the debris flow have not be considered. That is, it is considered that the impact of the floating objects is the only lateral load applied to the wall panel in question.

[2] It should be noted that the values are fictitious and are not necessarily based on data available for the building in the figure.

d) Is the probability of in-plane shear failure of the wall panel more than 50%?

$$P_F = P(Y > 1) = P(\ln Y > 0) = 1 - P(\ln Y < 0) = 1 - P\left(\frac{\ln Y - \mu_{\ln Y}}{\sigma_{\ln Y}} < \frac{0 - \mu_{\ln Y}}{\sigma_{\ln Y}}\right)$$

$$= 1 - \Phi\left(-\frac{\mu_{\ln Y}}{\sigma_{\ln Y}}\right) = \Phi\left(\frac{\mu_{\ln Y}}{\sigma_{\ln Y}}\right) = \Phi(0.0382) = 0.5152$$

Since $\mu_{\ln Y}/\sigma_{\ln Y}$ is greater than zero, the standard Normal CDF, which is equal to the probability of failure, is greater than 50%.