Lecture 3: Continuous Probability Distributions

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1. Continuous Probability Distributions

Consider the following statements:

$A \equiv X \leq a$

$B \equiv X \leq b$

The probability $P(A|\mathbf{I})$ and $P(B|\mathbf{I})$ can be illustrated as follows:

$$\begin{aligned} P\left(A|\mathbf{I}\right) &= P\left(X \leq a|\mathbf{I}\right) \triangleq F_X\left(a|\mathbf{I}\right) \\ P\left(B|\mathbf{I}\right) &= P\left(X \leq b|\mathbf{I}\right) \triangleq F_X\left(b|\mathbf{I}\right) \end{aligned} \tag{12}$$

The expression $F_X(x)$ generally denotes the CDF of the uncertain parameter X for a specific realization x. Moreover, the Complementary CDF (CCDF), denoted herein as G_X, can be stated as:

$$\begin{aligned} G_X\left(a|\mathbf{I}\right) &\triangleq P\left(\overline{A}|\mathbf{I}\right) = P\left(X > a|\mathbf{I}\right) = 1 - P\left(X \leq a|\mathbf{I}\right) = 1 - F_X\left(a|\mathbf{I}\right) \\ G_X\left(b|\mathbf{I}\right) &\triangleq P\left(\overline{B}|\mathbf{I}\right) = P\left(X > b|\mathbf{I}\right) = 1 - P\left(X \leq b|\mathbf{I}\right) = 1 - F_X\left(b|\mathbf{I}\right) \end{aligned} \tag{13}$$

Now, consider the statement:

$Q \equiv a < X \leq b$

The probability $P(Q|\mathbf{I})$ can be derived as:

$$\begin{aligned} P\left(Q|\mathbf{I}\right) &= P\left(a < X \leq b|\mathbf{I}\right) = P\left((X > a).(X \leq b)|\mathbf{I}\right) \equiv P\left(\overline{A}.B|\mathbf{I}\right) \\ &= P\left(\overline{A+\overline{B}}|\mathbf{I}\right) = 1 - P\left(A+\overline{B}|\mathbf{I}\right) = 1 - P\left(A|\mathbf{I}\right) - P\left(\overline{B}|\mathbf{I}\right) = P\left(B|\mathbf{I}\right) - P\left(A|\mathbf{I}\right) \\ &= P\left(X \leq b|\mathbf{I}\right) - P\left(X \leq a|\mathbf{I}\right) = F_X\left(b|\mathbf{I}\right) - F_X\left(a|\mathbf{I}\right) \end{aligned} \tag{14}$$

The above expression is derived based on the fact that A and $\overline{B}$ are ME. The probability $P(Q|\mathbf{I})$ can also be obtained by considering the fact that both statements A and Q are ME. Thus,

$$
\begin{aligned}
&P(B|\mathbf{I}) = P(A+Q|\mathbf{I}) = P(A|\mathbf{I}) + P(Q|\mathbf{I}) \\
&\Rightarrow P(Q|\mathbf{I}) = P(B|\mathbf{I}) - P(A|\mathbf{I}) = P(X \le b|\mathbf{I}) - P(X \le a|\mathbf{I}) = F_X(b|\mathbf{I}) - F_X(a|\mathbf{I})
\end{aligned}
\tag{15}
$$

. Now, consider the following statements:

$Q_i \equiv x_i < X \le x_i + \Delta x,\ x_i \in [a,b]$

Thus (note that $x_1 = a,\ x_n = b - \Delta x$),

$$
\begin{aligned}
P(Q|\mathbf{I}) &= P(a < X \le b|\mathbf{I}) = P(Q_1 + Q_2 + \cdots + Q_n|\mathbf{I}) = \sum_{i=1}^{n} P(Q_i|\mathbf{I}) = \sum_{i=1}^{n} P(x_i < X \le x_i + \Delta x|\mathbf{I}) \\
&= \sum_{i=1}^{n} \left(F_X(x_i + \Delta x|\mathbf{I}) - F_X(x_i|\mathbf{I})\right) = \sum_{i=1}^{n} \left(\frac{F_X(x_i + \Delta x|\mathbf{I}) - F_X(x_i|\mathbf{I})}{\Delta x}\right)\Delta x \triangleq \sum_{i=1}^{n} f_X(x_i|\mathbf{I})\Delta x
\end{aligned}
\tag{16}
$$

where the term f_X is generally denoted as the Probability Density Function (PDF). The concept of PDF is somehow different from PMF, as shown in Figure 6. PDF is not a probability by itself; nonetheless, it provides an idea of (i.e., density of) the probability at a point. Strictly speaking, the area under the PDF ($f_X \times \Delta x$) is probability $P(x < X \le x + \Delta x|\mathbf{I})$, as illustrated in Figure 4.

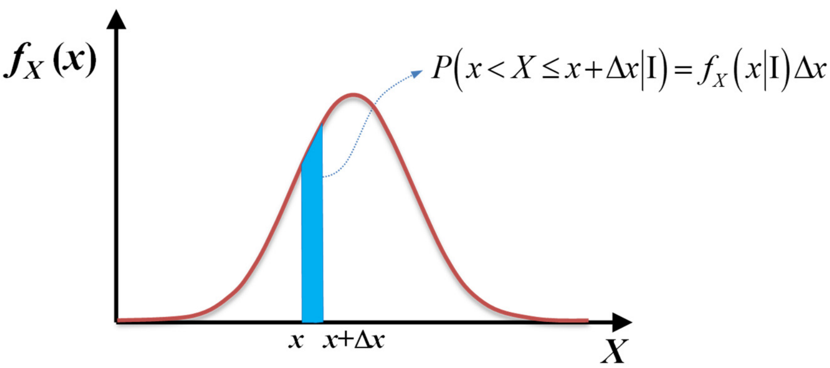

Figure 6: The definition of PDF

Therefore, f_X can be defined as follows (see also Eq. 16):

$$
\begin{aligned}
f_X(x|\mathrm{I}) &= \lim_{\Delta x \to 0} P(x \le X \le x+\Delta x|\mathrm{I}) = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x|\mathrm{I}) - F_X(x|\mathrm{I})}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta F_X(x|\mathrm{I})}{\Delta x} \\
&= \frac{\mathrm{d}F_X(x|\mathrm{I})}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[1 - G_X(x|\mathrm{I})\right] = -\frac{\mathrm{d}}{\mathrm{d}x} G_X(x|\mathrm{I}) = \left|\frac{\mathrm{d}G_X(x|\mathrm{I})}{\mathrm{d}x}\right|
\end{aligned} \tag{17}
$$

Based on the general rule of integration, Eq. (17) can further be re-written as:

$$
F_X(b|\mathrm{I}) - F_X(a|\mathrm{I}) = \int_a^b \frac{\mathrm{d}F_X(x|\mathrm{I})}{\mathrm{d}x}\mathrm{d}x \triangleq \int_a^b f_X(x|\mathrm{I})\,\mathrm{d}x \tag{18}
$$

Considering the domain of all possible values of x to be Ω_X, i.e. $x \in \Omega_X$ the following expression holds (showing the sum over MECE events, see Figure 6):

$$
\int_{\Omega_X} f_X(x|\mathrm{I})\,\mathrm{d}x = \sum_{\Omega_X} P(x < X \le x+\Delta x|\mathrm{I}) = 1 \tag{19}
$$

Moreover, the CDF and CCDF, as shown in Figure 7, can be defined as:

$$
\begin{aligned}
F_X(x|\mathrm{I}) &= \int_{0 \text{ or } -\infty}^{x} f_X(y|\mathrm{I})\,\mathrm{d}y \\
G_X(x|\mathrm{I}) &= 1 - F_X(x|\mathrm{I})
\end{aligned} \tag{20}
$$

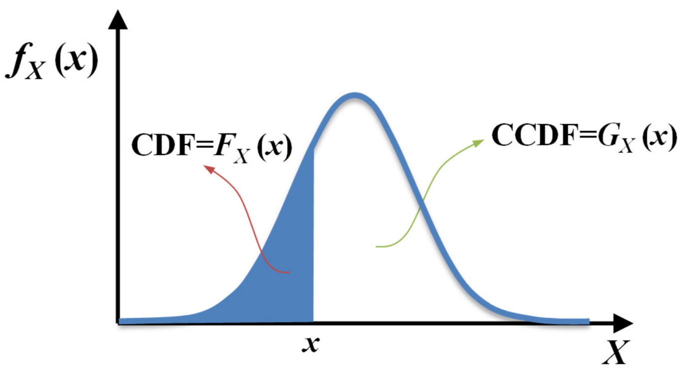

Figure 7: The definition of CDF and CCDF

Example 1: The continuous *Uniform* probability distribution

$\mathbf{I} \equiv a < X \leq b$

The PDF is uniform in the interval from a to b with a density equal to $f_X\left(x|\mathrm{I}\right) = 1/\left(b-a\right)$ as the area under the PDF according to Eq. 19 must be equal to 1. Accordingly, the CDF is a line starting from zero at a up to 1 at b. Both PDF and CDF of a Uniform distribution is schematically illustrated in Figure 8.

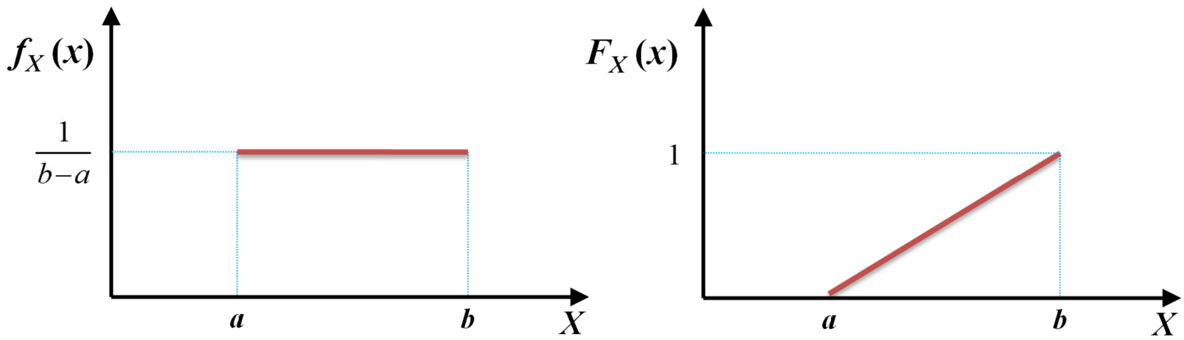

Figure 8: The Uniform probability distribution (a) PDF, and (b) CDF

1.1. *Transformation of PDF and CDF*

An additional key feature of the PDF compared to CDF is that the PDF is going to be scaled based on the change in the associated argument. To prove this statement, consider that we want to estimate the PDF of a one-by-one function of x, say $u(x)$:

$$f_U\left(u(x)|\mathrm{I}\right) = \frac{\mathrm{d}F_U\left(u(x)|\mathrm{I}\right)}{\mathrm{d}u(x)} = \frac{\mathrm{d}F_U\left(u(x)|\mathrm{I}\right)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}u(x)} \tag{21}$$

According to Eq. (12), we have:

$$F_U\left(u(x)|\mathbf{I}\right) = P\left(U(X) \leq u(x)|\mathbf{I}\right) \equiv P\left(X \leq x|\mathbf{I}\right) = F_X\left(x|\mathbf{I}\right) \tag{22}$$

It is to note that in a mapping, probability (i.e., CDF) does not change since the probability content is not changing. Hence, Eq. (17) can be re-written as:

$$f_U\left(u(x)\middle|\mathrm{I}\right)=\frac{\mathrm{d}F_X\left(x\middle|\mathrm{I}\right)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}u(x)}=\frac{f_X\left(x\middle|\mathrm{I}\right)}{u'\left(x\right)} \tag{23}$$

As a result, PDF is going to be scaled by the factor $1/u'\left(x\right)$, while CDF is not scaled.

2. Joint Probability Distribution

With reference to Eq. (18), the PDF f_X can be rewritten as follows assuming that A is a TRUE statement:

$$f_X\left(x\middle|\mathrm{I}\right)=\frac{\mathrm{d}F_X\left(x\middle|\mathrm{I}\right)}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}P\left(X\le x\middle|\mathrm{I}\right)=\frac{\mathrm{d}}{\mathrm{d}x}P\left[\left(X\le x\right)\cdot\mathrm{A}\middle|\mathrm{I}\right] \tag{24}$$

Statement **A** can be defined as a set of MECE events as:

$$\mathbf{A}\equiv\mathrm{A}_1+\mathrm{A}_2+\cdots+\mathrm{A}_n\equiv\mathrm{TRUE}$$

$$\mathrm{A}_i\equiv\left\{y_i<Y\le y_i+\Delta y\middle|y_i\in\Omega_Y\right\}$$

$$\mathrm{B}\equiv X\le x$$

Thus, by applying the Total Probability Theorem:

$$\begin{aligned}P\left(X\le x\middle|\mathbf{I}\right)&=P\left((X\le x)\cdot\mathbf{A}\middle|\mathbf{I}\right)=P\left(\mathrm{B}\cdot\mathbf{A}\middle|\mathbf{I}\right)=P\left(\mathrm{B}\cdot\mathrm{A}_1+\mathrm{B}\cdot\mathrm{A}_2+\cdots+\mathrm{B}\cdot\mathrm{A}_n\middle|\mathbf{I}\right)\\&=\sum_{i=1}^{n}P\left(\mathrm{B}\cdot\mathrm{A}_i\middle|\mathbf{I}\right)=\sum_{i=1}^{n}P\left(\mathrm{B}\middle|\mathrm{A}_i\mathbf{I}\right)P\left(\mathrm{A}_i\middle|\mathbf{I}\right)\\&=\lim_{\Delta y\to 0}\sum_{\Omega_Y}P\left[X\le x\middle|y_i<Y\le y_i+\Delta y,\mathrm{I}\right]P\left[y_i<Y\le y_i+\Delta y\middle|\mathrm{I}\right]\\&\equiv\int_{\Omega_Y}P\left[X\le x\middle|Y=y,\mathrm{I}\right]f_Y\left(y\middle|\mathrm{I}\right)\mathrm{d}y=\int_{\Omega_Y}F_{X|Y}\left(x\middle|y,\mathrm{I}\right)f_Y\left(y\middle|\mathrm{I}\right)\mathrm{d}y\end{aligned} \tag{25}$$

where $F_{X|Y}$ is the *conditional* CDF. Accordingly, Eq. (24) can be re-written as:

$$f_X\left(x\middle|\mathrm{I}\right)=\int_{\Omega_Y}\frac{\mathrm{d}}{\mathrm{d}x}F_{X|Y}\left(x\middle|y,\mathrm{I}\right)f_Y\left(y\right)\mathrm{d}y=\int_{\Omega_Y}f_{X|Y}\left(x\middle|y,\mathrm{I}\right)f_Y\left(y\middle|\mathrm{I}\right)\mathrm{d}y\triangleq\int_{\Omega_Y}f_{XY}\left(x,y\middle|\mathrm{I}\right)\mathrm{d}y \tag{26}$$

where f_{XY} is defined as the joint PDF of X and Y. Similarly,

$$f_Y\left(y|\mathrm{I}\right)=\int_{\Omega_X} f_{XY}\left(x,y|\mathrm{I}\right)\mathrm{d}x \tag{27}$$

Now, consider the following derivation:

$$\begin{aligned}
&f_{XY}\left(x,y|\mathrm{I}\right)\Delta x\,\Delta y \triangleq P\left[\left(x<X\le x+\Delta x\right)\cdot\left(y<Y\le y+\Delta y\right)|\mathbf{I}\right]\\
&=P\left[\left(x<X\le x+\Delta x\right)|\left(y<Y\le y+\Delta y\right),\mathbf{I}\right]P\left[y<Y\le y+\Delta y|\mathbf{I}\right]\\
&\equiv\left[f_{X|Y}\left(x|y,\mathrm{I}\right)\Delta x\right]\left[f_Y\left(y|\mathrm{I}\right)\Delta y\right]
\end{aligned} \tag{28}$$

Hence, the PDF version of the product rule holds as well. Moreover, if the random variables X and Y are independent, the following relation holds:

$$f_{XY}\left(x,y|\mathrm{I}\right)=f_X\left(x|\mathrm{I}\right)f_Y\left(y|\mathrm{I}\right) \tag{29}$$

The PDF's f_X and f_Y obtained from the joint PDF are called *marginal* distributions. General properties of f_{XY} can be outlined as:

$$\begin{aligned}
&\int_{\Omega_X}\int_{\Omega_Y} f_{XY}\left(x,y|\mathrm{I}\right)\mathrm{d}x\,\mathrm{d}y=1\\
&f_{XY}\left(x,y|\mathrm{I}\right)=\frac{\partial^2}{\partial x\partial y}F_{XY}\left(x,y|\mathrm{I}\right)\\
&F_{XY}\left(x,y|\mathrm{I}\right)=\int_{0\text{ or }-\infty}^{y}\int_{0\text{ or }-\infty}^{x} f_{XY}\left(u,v|\mathrm{I}\right)\mathrm{d}u\,\mathrm{d}v
\end{aligned} \tag{30}$$