Sample Problems
Lecture 4: The Poisson Family of Distributions

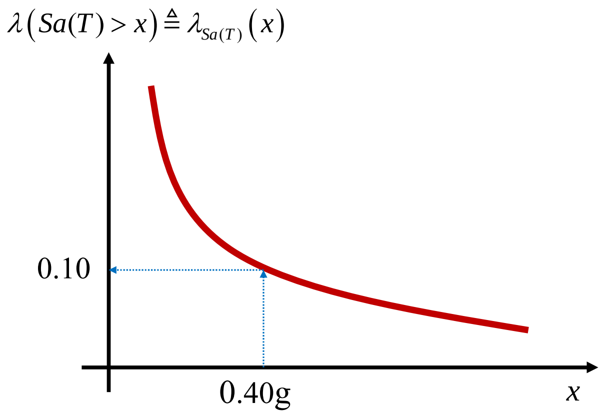
Problems prepared by: Prof. F. Jalayer and Dr. H. Ebrahimian

1. The Filtered Poisson

Suppose that the occurrence of earthquakes with $Sa(T) > 0.40g$ is described by a Poisson distribution with annual rate equal to 0.01. Moreover, suppose that a given structure with fundamental period equal to T is going to fail with probability 0.1 if it is hit by earthquakes with $Sa(T) > 0.40g$.

- **What is the probability of having two earthquakes with $Sa(T) > 0.40g$ in a year?**

$$\lambda\left(Sa(T) > 0.40g\right) = \lambda_{Sa(T)}\left(0.40g\right) = 0.01$$



$$\Rightarrow P\left(r = 2 \middle| \lambda_{Sa(T)}\left(0.40g\right) = 0.01, t = 1\right) = \frac{\left(\lambda_{Sa(T)} t\right)^r e^{-\lambda_{Sa(T)} t}}{r!} = \frac{\left(0.01\right)^2 e^{-0.01}}{2!} = 4.9502 \times 10^{-5}$$

- **What is the probability of having two earthquakes with $Sa(T) > 0.40g$ in 10 years?**

$$P\left(r = 2 \middle| \lambda_{Sa(T)}\left(0.40g\right) = 0.01, t = 10\right) = \frac{\left(0.01 \times 10\right)^2 e^{-0.01 \times 10}}{2!} = 4.5 \times 10^{-3}$$

- **What is the probability of having at least one earthquake with *Sa*(*T*) > 0.40*g* in 10 years?**

$$P\left(r \geq 1 \middle| \lambda_{Sa(T)}\left(0.40g\right) = 0.01, t = 10\right) = 1 - P\left(r = 0 \middle| \lambda_{Sa(T)}\left(0.40g\right) = 0.01, t = 10\right)$$
$$= 1 - e^{-\lambda_{Sa(T)}t} = 1 - e^{-0.01\times 10} = 0.0952 \cong 0.10$$

- **What is the expected number of events in 1000 years?**

$$\mathbb{E}\left(r \middle| \lambda_{Sa(T)} = 0.01, t = 1000\right) = \lambda_{Sa(T)}t = 10$$

- **What is the *best estimate* for the number of events in 1000 years?**

$$n_{best\,estimate} = \lambda_{Sa(T)}t + \alpha\sqrt{\lambda_{Sa(T)}t} = 10 + \sqrt{10}\alpha$$
$$\alpha = 1 \Rightarrow n_{best\,estimate} \cong 13$$
$$\alpha = 2 \Rightarrow n_{best\,estimate} \cong 16$$

- **Assuming that each time the structure fails, it is immediately repaired (to its original state), what is the annual rate of structural failure?**

$$\lambda_F = \lambda\left(Sa(T_1) > 0.40g\right)P\left[F \middle| Sa(T_1) > 0.40\,\mathrm{g}\right] = 0.01\times 0.1 = 0.001$$

- **What is the expected number of structural failures in 50 years?**

$$\mathbb{E}\left(F \middle| \lambda_F, t = 50\right) = \lambda_F t = 0.001\times 50 = 0.05$$

- **Assuming a service life of 50 years, what is the probability that the structure does not fail in its service life?**

$$P\left(F = 0 \middle| \lambda_F, t = 50\right) = e^{-\lambda_F t} = 0.9512$$

- **What is the probability of failure during the service life of the structure?**

$$P\left(F \geq 1 \middle| \lambda_F, t = 50\right) = 1 - e^{-\lambda_F t} \cong 0.05$$

2. Earthquake Hazard Assessment

Suppose that the occurrence of earthquakes with M > 4.5 is described by a Poisson distribution with annual rate equal to 0.1.

- **What is the probability of having two earthquakes with M > 4.5 in a year?**

$$\lambda\left(M>4.5\right)\triangleq\lambda_M\left(4.5\right)=0.1$$

$$\Rightarrow P\left(r=2\middle|\lambda_M\left(4.5\right)=0.1,t=1\right)=\frac{\left(\lambda_M t\right)^r e^{-\lambda_M t}}{r!}=\frac{\left(0.1\right)^2 e^{-0.1}}{2!}=0.0045$$

- **What is the probability of having two earthquakes with M > 4.5 in 20 years?**

$$\Rightarrow P\left(r=2\middle|\lambda_M\left(4.5\right)=0.1,t=20\right)=\frac{\left(0.1\times 20\right)^2 e^{-0.1\times 20}}{2!}=0.2707$$

- **Assuming that an earthquake with M > 4.5 has just taken place, what is the expected time to the next earthquake with M > 4.5?**

$$\mathbb{E}\left(IAT\middle|\lambda_M\left(4.5\right)=0.1\right)=\frac{1}{\lambda_M}=10\,\text{years}$$

- **Assuming that an earthquake with M > 4.5 has just taken place, what is the expected time to the third earthquake with M > 4.5?**

$$\mathbb{E}\left(t_3\middle|\lambda_M\right)\equiv\mathbb{E}\left(\sum_{i=1}^{3}IAT_i\middle|\lambda_M\right)=\frac{3}{\lambda_M}=30\ \text{years}$$

- **We know that in the past ten years, there has been no earthquake with M > 4.5, what is the probability of having at least one earthquake in the next year?**

  Considering the memoryless properties, if we have already waited a time τ since the last event and observed no new events (nothing has been happened), then the probability distribution of the remaining time until the next event is the same as the *IAT* distribution right after the last event.

$$P\left(IAT \le 1 \middle| \lambda_M\right) = 1 - e^{-\lambda_M \times 1} = 1 - e^{-0.1} = 0.0952$$

- **Imagine that the probability of having *PGA* > 0.5*g* given that *M* > 4.5 is equal to 0.01. What is the probability of having at least one earthquake with *PGA* > 0.50*g* in 10 years?**

$$\lambda\left(PGA > 0.50g\right) = \lambda_{PGA}\left(0.50g\right) = \lambda\left(M > 4.5\right) P\left[PGA > 0.50g \middle| M > 4.5\right]$$
$$= 0.1 \times 0.01 = 0.001$$
$$\Rightarrow P\left(r \ge 1 \middle| \lambda_{PGA}, t = 10\right) \equiv P\left(IAT \le 10 \middle| \lambda_{PGA}\right) = 1 - e^{-\lambda_{PGA} t} = 1 - e^{-0.001 \times 10} = 0.01$$

- **What is the expected number of events with *PGA* > 0.5*g* in 1000 years?**

$$\mathbb{E}\left(r \middle| \lambda_{PGA}, t = 1000\right) = \lambda_{PGA} t = 0.001 \times 1000 = 1$$

- **Given that 3 earthquakes with *M* > 4.5 have taken place, what is the probability that all of them have *PGA* > 0.5*g*?**

$$P\left(r = 3 \middle| n = 3, \pi = 0.01\right) = \binom{n}{r} \pi^r \left(1 - \pi\right)^{n-r} = \binom{3}{3} 0.01^3 \left(1 - 0.01\right)^0 = 1 \times 10^{-6}$$

3. Poisson Distribution

Suppose that in an area, the rate of having a critical storm in one year is equal to λ. We use the *Poisson* probability distribution to calculate the probability that *n* critical storms take place in an area in time *T*:

$$P\left(n|\lambda,T\right)=\frac{\left(\lambda T\right)^{n}e^{-\lambda T}}{n!}$$

- **What is the probability of having no critical storms in the next *t* years?**

$$P\left(n=0|\lambda,T\right)=\frac{\left(\lambda T\right)^{0}e^{-\lambda T}}{0!}=e^{-\lambda T}$$

- **What is the probability that there is at least one critical storm in the next t years?**

$$P\left(n\geq 1|\lambda,T\right)=1-P\left(n=0|\lambda,T\right)=1-e^{-\lambda T}$$

- **We know that the probability density function (PDF) for the parameter *X* at point *x* can be calculated as *p*(*x*) = d*F*(*x*)/d*x* where *F*(*x*) = *P*(*X* ≤ *x*) is the cumulative distribution function (CDF) for *X* at *x*. If the cumulative distribution for the time to next strong storm T_1 is calculated using the exponential probability distribution:**

  P(T_1≤*t*)=F(t)=1-exp(λt), calculate the PDF for the time to the next critical storm at t.

$$p\left(t\right)=\frac{\mathrm{d}F\left(t\right)}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(1-e^{-\lambda t}\right)=\lambda e^{-\lambda t}$$

- **What is the expected time to the next critical storm?**

  The expected value of an exponential distribution id estimated on the CLASS NOTES by using Moment Generating Function. However, the direct way to estimate it is as follows:

$$\mathbb{E}\left(T_1\right)=\int_0^{+\infty}t\left(\lambda e^{-\lambda t}\right)\mathrm{d}t=\frac{1}{\lambda}$$

- **What is the standard deviation?**

$$\mathbb{VAR}\left(T_1\right)=\mathbb{E}\left(T_1^2\right)-\left[\mathbb{E}\left(T_1\right)\right]^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$$

- **Suppose that the best estimate for parameter *T_1* equal to its expected value ± its standard deviation. What is your best estimate of the time to the next storm?**

$$best\,estimate=\mathbb{E}\left(T_1\right)+\sqrt{\mathbb{VAR}\left(T_1\right)}=\frac{1}{\lambda}+\frac{1}{\lambda}=\frac{2}{\lambda}$$

4. Seismic Risk Assessment

Suppose that the occurrence of earthquakes with M > 5.0 is described by a Poisson distribution with annual rate equal to 0.2.

- **What is the probability of having at least one earthquake with *M* > 5.0 in one year?**

  $$P\left(r \geq 1 \middle| \lambda_M\left(5.0\right) = 0.2, t = 1\right) \equiv P\left(IAT \leq 1 \middle| \lambda_M\left(5.0\right) = 0.2\right) = 1 - e^{-0.2\times1} \cong 0.18$$

- **We know that in the last five years, no earthquakes with *M* > 5.0 have happened. What is the probability that no earthquakes with *M* > 5.0 will happen in the next year?**

  $$P\left(IAT > 1 \middle| \lambda_M\left(5.0\right) = 0.2\right) = e^{-0.2\times1} = 0.82$$

- **Consider that the probability of collapse for a structure subjected to an earthquake with *M* > 5.0 is equal to 5%. What is the mean annual rate of collapse?**

  $$\lambda_C = \lambda\left(M > 5.0\right) P\left[C \middle| M > 5.0\right] = 0.2\times0.05 = 0.01$$

- **What is the probability of having at least a collapse in the life cycle of the structure (t_{life}=30 years)?**

  $$P\left(r \geq 1 \middle| \lambda_C = 0.01, t = 30\right) \equiv P\left(IAT_C \leq 30 \middle| \lambda_C = 0.01\right) = 1 - e^{-0.01\times30} \cong 0.26$$

- **We know that the structure was built 25 years ago, and it has never collapsed. What is the probability that the structure will collapse in the next five years?**

  $$P\left(IAT_C \leq 5 \middle| \lambda_C = 0.01\right) = 1 - e^{-0.01\times5} \cong 0.05$$

- **We know that the structure was built 25 years ago. 20 years ago, it was destroyed and immediately reconstructed. How many times it has to wait on average before observing the first collapse?**

$$TR = \frac{1}{\lambda_C} = 100 \Rightarrow 100 - 20 = 80 \text{ years}$$

5. Parameters of Exponential Distribution

- **Calculate the moment generating function for an Exponential distribution with probability distribution denoted by** $f_{IAT}\left(x|\lambda\right)$

The moment generating function is as follows:

$$\mathbb{E}\left(e^{tx}|\lambda\right)=\int_0^{+\infty}e^{tx}\ f_{IAT}\left(x|\lambda\right)\mathrm{d}x=\int_0^{+\infty}e^{tx}\left(\lambda e^{-\lambda x}\right)\mathrm{d}x=\lambda\int_0^{+\infty}e^{-(\lambda-t)x}\mathrm{d}x$$
$$=\frac{\lambda}{t-\lambda}e^{-(\lambda-t)x}\Bigg|_{x=0}^{x=+\infty}=\frac{\lambda}{\lambda-t}$$

- **Use the moment generating function for an Exponential distribution and calculate the expected value for the *IAT*.**

The expected value can be calculated as:

$$\mathbb{E}\left(IAT\ |\lambda\right)=\frac{\partial}{\partial t}\left[\mathbb{E}\left(e^{tx}\ |\lambda\right)\right]_{t=0}=\frac{\partial}{\partial t}\left[\frac{\lambda}{\lambda-t}\right]_{t=0}=\left[\frac{\lambda}{\left(\lambda-t\right)^2}\right]_{t=0}=\frac{1}{\lambda}\triangleq T_R$$

- **Calculate the Variance for the *IAT*.**

$$\mathbb{E}\left(IAT^2\,|\lambda\right)=\frac{\partial^2}{\partial t^2}\left[\mathbb{E}\left(e^{tx}|\lambda\right)\right]_{t=0}=\frac{\partial^2}{\partial t^2}\left[\frac{\lambda}{\lambda-t}\right]_{t=0}=\left[\frac{2\lambda\left(\lambda-t\right)}{\left(\lambda-t\right)^4}\right]_{t=0}=\frac{2}{\lambda^2}$$
$$\mathbb{VAR}\left(IAT|\lambda\right)=\mathbb{E}\left(IAT^2|\lambda\right)-\left[\mathbb{E}\left(IAT|\lambda\right)\right]^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$$

- **What is the best estimate for the IAT.**

$$Best\ Estimate=\frac{1}{\lambda}+\frac{1}{\lambda}=\frac{2}{\lambda}$$

6. Seismic Hazard From Two Seismogenic Sources

We have identified two seismogenic sources close to the site of interest. Suppose that for the first site we have identified M_{min}=4.0 and M_{max}=7.0, and that we have estimated the seismicity rate to be equal to $\lambda_1 = \lambda_1(M_{\min} \le M \le M_{\max}) = 0.60$ (assuming Poisson distribution). For the second site instead we have identified M_{min}=5.5 and M_{max}=6.5, and we have estimated the seismicity rate to be equal to $\lambda_2 = \lambda_2(M_{\min} \le M \le M_{\max}) = 0.05$ (assuming Poisson distribution). Imagine that the probability that *PGA*>0.5g for the first source is calculated as $p_1 = P(PGA > 0.5g \mid 4.0 \le M \le 7.0) = 0.001$. Moreover, the probability that *PGA*>0.5g for the second source is calculated as $p_2 = P(PGA > 0.5g \mid 5.5 \le M \le 6.5) = 0.1$.

- **Show (in a parametric manner) that the occurrence of earthquakes with *PGA*>0.5g at the site of interest can be described by a Poisson distribution with rate equal to $p_1\lambda_1 + p_2\lambda_2$ (assume that the two sites generate earthquakes in an independent manner).**

According to Eq. (31), the sum of independent Poisson random variables is Poisson. The rate of occurrence of *PGA*>0.5g in site 1 can be described with a Poisson Distribution with rate $p_1\lambda_1$, and with a Poisson Distribution with rate $p_2\lambda_2$ in site 2. Thus, the total number of events with PGA>0.5g can be expressed as a Poisson Distribution with rate equal to the sum of both rates.

- **What is the probability of having at least one earthquake with PGA>0.5g in one year?**

$$\lambda = p_1\lambda_1 + p_2\lambda_2 = 0.60 \times 0.001 + 0.05 \times 0.1 = 0.0056$$
$$P\left(r \ge 1 \middle| \lambda = 0.0056, t = 1\right) = 1 - e^{-\lambda t} = 1 - e^{-0.0056\times 1} = 0.0056$$