Matemáticas para las Ciencias Aplicadas III
Tarea 2

5 de octubre de 2018

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Sección 14.2

15-18 Evaluate the double integral in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region.

18.

$\iint\limits_R y\ \mathrm{d}A$; R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

19-24. Evaluate the double integral in two ways using iterated integrals:

22. $\iint\limits_R x\ \mathrm{d}A$; R is the region enclosed by $y = sin^{-1}x, x = \frac{1}{\sqrt{2}}$, and $y = 0$.

<u>Solución:</u>

Entonces $0 \le y \le \frac{\pi}{4}$ y x va desde $x = \sin y$ a $x = \frac{1}{\sqrt{2}}$ por lo que se tiene

$$\iint\limits_R x\ \mathrm{d}A;\ R = \int_0^{\frac{\pi}{4}} \int_{siny}^{\frac{1}{\sqrt{2}}} xdxdy$$

$$= \int_0^{\frac{\pi}{4}} \frac{x^2}{2}\Big|_{siny}^{\frac{1}{\sqrt{2}}} dy$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{2} - \frac{(siny)^2}{2} \right) dy$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4} - \frac{sin^2 y}{2} dy$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{4}} 1 - 2sin^2 ydy$$

$$
\begin{aligned}
&= \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos 2y dy \\
&= \frac{1}{4}\left(\frac{1}{2}\sin 2y\Big|_0^{\frac{\pi}{4}}\right) \\
&= \frac{1}{8}\left(\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right) \\
&= \frac{1}{8}(1-0) \\
&= \frac{1}{8}
\end{aligned}
$$

28.

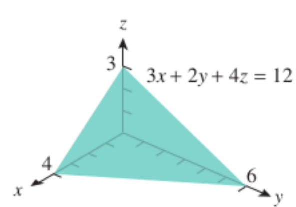
(a) By hand or with the help of a graphing utility, make a sketch of the region R enclosed between the curves $y = 4x^3 - x^4$ and $y = 3 - 4x + 4x^2$.

(b) Find the intersections of the curves in part (a).

(c) Find $\iint\limits_R x \, \mathrm{d}A$

37-38 Use double integration to find the volume of the solid.

37.



57. Try to evaluate the integral with a CAS using the stated order of integration, and then by reversing the order of integration.

Utilizaré Wolfram Mathematica para evaluar las integrales

(a) $\int\limits_{0}^{4} \int\limits_{\sqrt{x}}^{2} \sin \pi y^3 \ \mathrm{d}x\mathrm{d}y$

```
Problema 57.nb
WOLFRAM MATHEMATICA | EDICIÓN PARA ESTUDIANTES    Demostraciones | MathWorld | Comunidad Wolfram | Hel

In[6]:= (* 57 a) *)
        (* Evaluando con el orden establecido*)
        Integrate[Sin[Pi * y^3], {x, 0, 4}, {y, √x, 2}]
Out[6]= 0

        (*Evaluando con el orden invertido*)
        Integrate[Sin[Pi * y^3], {y, 0, 2}, {x, 0, y^2}]
Out[4]= 0
```

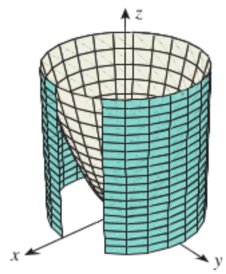
Obtuvimos que la integral siguiendo el orden establecido es 0 al igual que si invertimos el orden de la integral.

(b) $\int\limits_{0}^{1} \int\limits_{sin^{-1}y}^{\frac{\pi}{2}} \sec \cos x^2 \ \mathrm{d}x\mathrm{d}y$

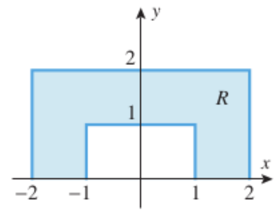
```
In[15]:= (* 57 b) *)
         (* Evaluando con el orden establecido *)
         Integrate[Sec[Cos[x]]^2, {y, 0, 1}, {x, ArcSin[y], Pi/2}]
Out[15]= Sec[Cos[x]]^2

         (* Evaluando con el orden invertido *)
         Integrate[Sec[Cos[x]]^2, {x, 0, Pi/2}, {y, 0, Sin[x]}]
Out[12]= Tan[1]
```

La integral siguiendo el orden establecido no se puede calcular en Mathematica, y la integral invirtiendo el orden es tan(1)

59. Evaluate $\iint\limits_{R} xy^2 \ \mathrm{d}A$ over the region R shown in the accompanying figure.



▲ **Figure Ex-58**



▲ **Figure Ex-59**

<u>Solución:</u>

Como podemos ver en la figura la podemos dividir en 3 partes

$$1.\ -2 \leq x \leq -1,\ 0 \leq y \leq 2$$

$$2.\ -1 \leq x \leq 1,\ 1 \leq y \leq 2$$

$$3.\ 1 \leq x \leq 2,\ 0 \leq y \leq 2$$

Entonces

$$\iint\limits_R xy^2 \,\mathrm{d}A = \int_{-2}^{-1}\int_0^2 xy^2\,\mathrm{d}y\,\mathrm{d}x + \int_{-1}^{1}\int_1^2 xy^2\,\mathrm{d}y\,\mathrm{d}x + \int_1^2\int_0^2 xy^2\,\mathrm{d}y\,\mathrm{d}x \tag{1}$$

Ahora calcularemos cada una de las integrales por separado

- Calculemos la integral en $-2 \leq x \leq -1$, $0 \leq y \leq 2$

$$\begin{aligned}
\int_{-2}^{-1}\int_0^2 xy^2\,\mathrm{d}y\,\mathrm{d}x &= \int_{-2}^{-1}\left(\frac{xy^3}{3}\right)\Big|_0^2\,\mathrm{d}x \\
&= \int_{-2}^{-1}\frac{8x}{3}\,\mathrm{d}x \\
&= \frac{8}{3}\int_{-2}^{-1} x\,\mathrm{d}x \\
&= \frac{8}{3}\left(\frac{x^2}{2}\right)\Big|_{-2}^{-1} \\
&= \frac{8}{\cancel{3}}\left(-\frac{\cancel{3}}{2}\right) \\
&= -4
\end{aligned}$$

- Calculemos la integral en $-1 \leq x \leq 1$, $1 \leq y \leq 2$

$$\begin{aligned}
\int_{-1}^{1}\int_{1}^{2} xy^2 \,\mathrm{d}y\,\mathrm{d}x &= \int_{-1}^{1} \left(\frac{xy^3}{3}\right)\Big|_{1}^{2} \mathrm{d}x \\
&= \int_{-2}^{-1} \left(\frac{8x}{3} - \frac{x}{3}\right) \mathrm{d}x \\
&= \frac{7x}{3}\int_{-2}^{-1} x\,\mathrm{d}x \\
&= \frac{7x}{3}\left(\frac{x^2}{2}\right)\Big|_{-1}^{1} \\
&= \frac{7x}{3}\left(\frac{(1)^2}{2} - \frac{(-1)^2}{2}\right) \\
&= \frac{7x}{3}(0) \\
&= 0
\end{aligned}$$

- Calculemos la integral en $1 \leq x \leq 2$, $0 \leq y \leq 2$

$$\begin{aligned}
\int_{1}^{2}\int_{0}^{2} xy^2 \,\mathrm{d}y\,\mathrm{d}x &= \int_{1}^{2} \left(\frac{xy^3}{3}\right)\Big|_{0}^{2} \mathrm{d}x \\
&= \int_{1}^{2} \frac{8x}{3}\,\mathrm{d}x \\
&= \frac{8}{3}\int_{1}^{2} x\,\mathrm{d}x \\
&= \frac{8}{3}\left(\frac{x^2}{2}\right)\Big|_{1}^{2} \\
&= \frac{8}{\cancel{3}}\left(\frac{\cancel{3}}{2}\right) \\
&= 4
\end{aligned}$$

Entonces sustituyendo los valores en (1) se tiene

$$\iint\limits_{R} xy^2 \;\mathrm{d}A = \cancel{-4} + 0 + \cancel{4} = 0$$

63. Suppose that the temperature in degrees Celsius at a point (x, y) on a flat metal plate is $T(x, y) = 5xy + x^2$, where x and y are in meters. Find the average temperature of the diamond-shaped portion of the plate for which $|2x + y| \leq 4$ and $|2x - y| \leq 4$.

Sección 14.5

12. $\displaystyle\iiint\limits_G \cos\tfrac{z}{y}\, \mathrm{d}V$, where G is the solid defined by the inequalities $\frac{\pi}{6} \leq y \leq \frac{\pi}{2}, y \leq x \leq \frac{\pi}{2}, 0 \leq z \leq xy$.

37. Let G be the tetrahedron in the first octant bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, (a > 0, b > 0, c > 0)$$

(a) List six different iterated integrals that represent the volume of G.
<u>Solución:</u>Para este plano los trazos estan en $x = a$, $y = b$ y $z = c$ y permutando el orden de integración, podemos escribir la integral de las siguientes 6 formas:

$$1.\,\text{Volume of G} = \int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dx\, dy\, dz$$

$$2.\,\text{Volume of G} = \int_0^b \int_0^{c\left(1-\frac{y}{b}\right)} \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dx\, dz\, dy$$

$$3.\,\text{Volume of G} = \int_0^a \int_0^{c\left(1-\frac{x}{a}\right)} \int_0^{b\left(1-\frac{x}{a}-\frac{z}{c}\right)} dy\, dz\, dx$$

$$4.\,\text{Volume of G} = \int_0^c \int_0^{a\left(1-\frac{z}{c}\right)} \int_0^{b\left(1-\frac{x}{a}-\frac{z}{c}\right)} dy\, dx\, dz$$

$$5.\,\text{Volume of G} = \int_0^b \int_0^{a\left(1-\frac{y}{b}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dx\, dy$$

$$6.\,\text{Volume of G} = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

(b) Evaluate any one of the six to show that the volume of G is $\frac{1}{6}abc$.
<u>Solución:</u>Calculemos el volumen de G con la siguiente integral

$$
\begin{aligned}
\text{Volume of G} &= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx \\
&= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx \\
&= c\left(\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} 1-\frac{x}{a}-\frac{y}{b} dy\,dx\right) \\
&= c\left(\int_0^a \left(y-\frac{xy}{a}-\frac{y^2}{2b}\Big|_0^{b\left(1-\frac{x}{a}\right)}\right) dx\right) \\
&= c\left(\int_0^a \left(\left(b\left(1-\frac{x}{a}\right)\right)-\frac{x\left(b\left(1-\frac{x}{a}\right)\right)}{a}-\frac{\left(b\left(1-\frac{x}{a}\right)\right)^2}{2b}\right) dx\right) \\
&= c\left(\int_0^a \frac{b(a-x)}{a}-\frac{bx(a-x)}{a^2}-\frac{b(a-x)^2}{2a^2}\,dx\right) \\
&= c\left(\int_0^a \frac{b(a-x)^2}{2a^2}\,dx\right) \\
&= c\left(\int_0^a \frac{b}{2}-\frac{bx}{a}+\frac{bx^2}{2a^2}\,dx\right) \\
&= c\left(\frac{bx}{2}-\frac{bx^2}{2a}+\frac{bx^3}{6a^2}\right)\Big|_0^a \\
&= c\left(\frac{ab}{2}-\frac{ab}{2}+\frac{ab}{6}\right) \\
&= \frac{abc}{2}-\frac{abc}{2}+\frac{abc}{6} \\
&= \frac{abc}{6}
\end{aligned}
$$

38. Use a triple integral to derive the formula for the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Hughes-Hallet

Sección 16.2

35.

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\,\mathrm{d}x\,\mathrm{d}y$$

37.

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\,\mathrm{d}x\,\mathrm{d}y$$

Solución:

Los límites de integración son $0 \leq y \leq 1$ y $e^y \leq x \leq e$, por lo que se tiene que $x = e^y - > (1)$, entonces para conseguir el límite de integración repecto a y debemos despejarlo de (1)

$$x = e^y$$

$$\ln x = \ln e^y$$

$$\ln x = y$$

Entonces el límite de integración respecto a y es $0 \leq y \leq \ln x$ y el límite de integración respecto a x es $0 \leq x \leq e$

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\,\mathrm{d}x\,\mathrm{d}y = \int_1^e \int_0^{\ln x} \frac{x}{\ln x}\,\mathrm{d}y\,\mathrm{d}x = \int_1^e \frac{x}{\ln x}\cdot y\Big|_0^{\ln x}\,\mathrm{d}x = \int_1^e \frac{x}{\ln x}\cdot(\ln x - 0)\,\mathrm{d}x$$

$$= \int_1^e \frac{x}{\cancel{\ln x}}\cdot\cancel{\ln x}\,\mathrm{d}x = \int_1^e x\,\mathrm{d}x = \frac{x^2}{2}\Big|_1^e = \frac{e^2}{2} - \frac{1^2}{2} = \frac{e^2-1}{2}$$

60. Show that for a right triangle the average distance from any point in the triangle to one of the legs is one-third the length of the other leg. (The legs of a right triangle are the two sides that are not the hypotenuse.)

62. Find the area of the crescent-moon shape with circular arcs as edges and the dimensions shown in Figure 16.22.

Sección 16.3

In Problems 14–18, decide whether the integrals are positive,negative, or zero. Let S be the solid sphere $x^2+y^2+z^2 \leq 1$, and T be the top half of this sphere (with $z \geq 0$), and B be the bottom half (with $z \leq 0$), and R be the right half of the sphere (with $x \geq 0$), and L be the left half (with $x \leq 0$).

14.

$$\int_T e^z \, \mathrm{d}V$$

15.

$$\int_B e^z \, \mathrm{d}V$$

16.

$$\int_S \sin z \, \mathrm{d}V$$

17.

$$\int_T \sin z \, \mathrm{d}V$$

18.

$$\int_R \sin z \, \mathrm{d}V$$

31. A trough with triangular cross-section lies along the x-axis for $0 \leq x \leq 10$. The slanted sides are given by $z = y$ and $z = -y$ for $0 \leq z \leq 1$ and the ends by $x = 0$ and $x = 10$, where x, y, z are in meters. The trough contains a sludge whose density at the point (x, y, z) is $\delta = e^{-3x}$ kg per m^3.

a) Express the total mass of sludge in the trough in terms of triple integrals.

<u>Solución:</u>

Encontremos la masa de lodo de un lado

$$\int_0^{10} \int_0^1 \int_z^1 e^{-3x} \, dy \, dz \, dx$$

Como la depresión es simétrica con el plano xz, podemos encontrar la masa de lado y duplicarla, por lo que la masa total esta dada por

$$\text{Total mass of sludge} = 2 \int_0^{10} \int_0^1 \int_z^1 e^{-3x} \, dy \, dz \, dx$$

b) Express the total mass of sludge in the trough in terms of triple integrals.

<u>Solución:</u>

Evaluemos la ecuación encontrada en el inciso anterior

$$\begin{aligned}\text{Total mass of sludge} &= 2 \int_0^{10} \int_0^1 \int_z^1 e^{-3x} \, dy \, dz \, dx = 2 \int_0^{10} \int_0^1 \left(ye^{-3x} \right) \Big|_z^1 dz \, dx \\ &= 2 \int_0^{10} \int_0^1 e^{-3x} - e^{-3x} z \, dz \, dx = 2 \int_0^{10} e^{-3x} z - \frac{e^{-3x} z^2}{2} \Big|_0^1 dz \\ &= 2 \int_0^{10} e^{-3x} - \frac{e^{-3x}}{2} \, dz = 2 \int_0^{10} \frac{e^{-3x}}{2} \, dz\end{aligned}$$

Sea u = -3x se tiene que

$$= 2\int_0^{30} -\frac{e^u}{6}\,du = 2\left(-\frac{e^{-3x}}{6}\right)\Big|_0^{10} = 2\left(-\frac{e^{-30}}{6}+1\right) = \frac{1-e^{-30}}{3}$$

Problems 54−56 refer to Figure 16.27, which shows triangular portions of the planes $2x + 4y + z = 4$, $3x - 2y = 0$, $z = 2$, and the three coordinate planes $x = 0$, $y = 0$, and $z = 0$. For each solid region E, write down an iterated integral for the triple integral

$$\int_E f(x,y,z)dV$$

.

55. E is the region bounded by $x = 0$, $y = 0$, $z = 0$, $z = 2$, and $2x+4y+z = 4$.
<u>Solución:</u>

Integraremos en dirección x. La parte trasera de E será descrita por el plano $x = 0$, ahora encontraremos la parte delantera de E

Despejemos a x en $2x + 4y + z = 4$

$$2x = 4 - 4y - z$$
$$x = 2 - 2y\frac{z}{2}$$

Entonces a parte delantera de E será descrita por el plano $x = 2-2y\frac{z}{2}$. Ahora integremos en dirección y. Para ello tenemos que encontrar los limites de integración, proyectando E en el plano yz se tiene que $y = 0$ y en $4y + z = 4$ depejamos y, entonces $y = 1 - \frac{z}{4}$ y como z va de 0 a 2 la integral iterada es

$$\int_0^2 \int_0^{1-\frac{z}{4}} \int_0^{2-2y\frac{z}{2}} f(x,y,z)\,dx\,dy\,dz$$

57. Figure 16.28 shows part of a spherical ball of radius 5 cm. Write an iterated triple integral which represents the volume of this region.
<u>Solución:</u>

La ecuación de la esfera es $x^2+y^2+z^2 = 5^2$, queremos encontrar el volumen entre los planos $z = 2$ y $z = 3$, sustituyamos el valor de z en $x^2+y^2+z^2 = 5^2$ para encontar el círculo en el que $z = 3$ corta a la esfera.

$$x^2 + y^2 + 9 = 25$$

$$x^2 + y^2 = 25 - 9$$
$$x^2 + y^2 = 16$$

Ahora encontremos los limites de integración, $-4 \leq x \leq 4$, $-\sqrt{16-x^2} \leq y \leq \sqrt{16-x^2}$ y $3 \leq z \leq \sqrt{25-x^2-y^2}$ Por lo que integral triple iterada es:

$$\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}}$$

66. Find the center of mass of the tetrahedron that is bounded by the xy, yz, xz planes and the plane $x + 2y + 3z = 1$. Assume the density is 1 gm/cm^3 and x, y, z are in centimeters.

Problems 67–69 concern a rotating solid body and its *moment of inertia* about an axis; this moment relates angular acceleration to torque (an analogue of force). For a body of constant density and mass m occupying a region W of volume V , the moments of inertia about the coordinate axes are

$$I_x = \frac{m}{V} \int_W (y^2 + z^2)\, \mathrm{d}V$$

$$I_y = \frac{m}{V} \int_W (x^2 + z^2)\, \mathrm{d}V$$

$$I_z = \frac{m}{V} \int_W (x^2 + y^2)\, \mathrm{d}V$$

67. Find the moment of inertia about the z-axis of the rectangular solid of mass m given by $0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3$.

<u>Solución:</u>

Tenemos que calcular

$$I_z = \frac{m}{V} \int_W (x^2 + y^2)\, \mathrm{d}V \qquad (1)$$

para ello primero encontraremos el valor de V

$$V = 1 \cdot 2 \cdot 3 = 6$$

Sustituyendo el valor de V en (1) se tiene

$$I_z = \frac{m}{6}\int_W (x^2+y^2)\,\mathrm{d}V$$

Sabemos por La integral triple como una integral iterada que

$$\int_W f\,\mathrm{d}V = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z)dx\right) dy\right) dz$$

Entonces se tiene que

$$\begin{aligned}
\frac{m}{6}\int_W (x^2+y^2)\,\mathrm{d}V &= \frac{m}{6}\int_0^3 \left(\int_0^2 \left(\int_0^1 x^2+y^2 dx\right) dy\right) dz \\
&= \frac{m}{6}\int_0^3 \left(\int_0^2 \left(\frac{x^3}{3}+xy^2\Big|_0^1\right) dy\right) dz \\
&= \frac{m}{6}\int_0^3 \left(\int_0^2 \left(\frac{(1)^3}{3}+(1)y^2\Big|_0^1\right) dy\right) dz \\
&= \frac{m}{6}\int_0^3 \left(\int_0^2 \frac{1}{3}+y^2 dy\right) dz \\
&= \frac{m}{6}\int_0^3 \left(\frac{y}{3}+\frac{y^3}{3}\right)\Big|_0^2 dz \\
&= \frac{m}{6}\int_0^3 \left(\frac{2}{3}+\frac{2^3}{3}\right) dz \\
&= \frac{m}{6}\left(\frac{10}{3}z\right)\Big|_0^3 \\
&= \frac{m}{6}\left(\frac{10}{\cancel{3}}\cancel{3}\right) \\
&= \frac{5m}{3}
\end{aligned}$$

Are the statements in Problems 74–83 true or false? Give reasons for your answer.

75. The region of integration of the triple iterated integral $\int_0^1\int_0^1\int_0^x f dzdydx$ lies above a square in the xy-plane and below a plane.

78. The iterated integrals $\int_{-1}^{1}\int_{0}^{1}\int_{0}^{1-x^2} f dzdydx$ and $\int_{0}^{1}\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}} f dxdydz$ are equal.

<u>Solución:</u>

Los limites de la primera integral $-1 \leq x \leq 1$, $0 \leq y \leq 1$ y $0 \leq z \leq 1 - x^2$

Los limites de la segunda integral $0 \leq z \leq 1$, $0 \leq y \leq 1$ y $-\sqrt{1-z} \leq x \leq \sqrt{1-z}$

Como podemos ver ambas integrales se encuentran debajo del cílindro parabólico debido a que $0 \leq z \leq 1 - x^2$ en la integral 1 y $-\sqrt{1-z} \leq x \leq \sqrt{1-z}$ en la integral 2, entonces $z = 1 - x^2$, además describen el rectángulo $-1 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$

Por lo tanto, es verdadera

80. If W is the unit cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ and $\int_W f \, \mathrm{d}V = 0$, then $f = 0$ everywhere in the unit.