Numerical Integration I

Uriel Morzan

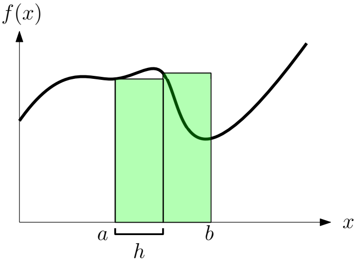
umorzan@ictp.it

Integration

We want to compute **integrals**:
definition via the **integral sum**

$$\int_a^b f(x)\,dx \equiv \lim_{N\to+\infty}\left[\sum_{k=0}^{N-1} f\left(x_k\right)\cdot h\right] \qquad \begin{cases} h \equiv \dfrac{b-a}{N} \\ x_k \equiv a + k\cdot h \\ k = 0,\ldots,N-1 \end{cases}$$

Geometrical interpretation: **area under the curve**
Approximation for finite N (improves for growing N, exact for $N \to \infty$)



Analytical integration

Analytically you use the **fundamental theorem of calculus**:

$$F'(x) = f(x) \Rightarrow \int_a^b f(x)\, dx = F(b) - F(a)$$

So to integrate you look for a **primitive** F of your function f by inverting the derivative

Many times not possible:
primitive may be hard to write or not known
e.g., $\int_{-\infty}^{+\infty} e^{-x^2}\, dx$

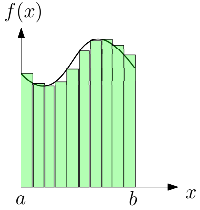
Sometimes you can find tricks: $\int_{\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$
otherwise, **numerical integration**

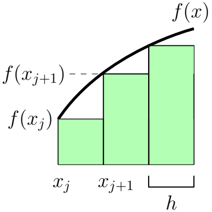
Rectangle method

Simplest numerical integration technique:
left Riemann sum (example of **rectangle method**)

$$\int_a^b f(x)\,dx \simeq I_L \equiv \sum_{k=0}^{N-1} f(x_k)\cdot h$$

$$\begin{cases} h \equiv \dfrac{b-a}{N} \\ x_k \equiv a + k\cdot h \\ k = 0, \ldots N \end{cases}$$





The function is approximated by **rectangles**
from below (if $f(x)$ is increasing) or above (if increasing)

Approximation worse where the function changes faster

Many rectangle methods (right Riemann sum, midpoint rule...):
this one is a good starting point

Error and precision

Precision estimation

How does the error depend on the choice of h?

For a given h, $\boxed{\int_a^b f(x)\,dx = I_L + O(h^1)}$

(error decreases **faster or as fast as h^1** when decreasing h,
higher exponent is **better**)

Integration precision

How do we make sure the error is small enough?

1. Choose a (small) **threshold** ϵ
2. Compute I_L for a value $h = h_1$ of choice
3. Compute I_L for $h = h_2 \ll h_1$ (e.g., $h_2 = h_1/2$)
4. If $|I_L(h_2) - I_L(h_1)| \leq \epsilon$ stop,
   otherwise go to (3) with $h_1 \leftarrow h_2$ and $I_L(h_1) \leftarrow I_L(h_2)$

Assignment

Write a program to computes the integral of $f(x) = e^x$

Steps:

- Ask for integration interval bound $[a, b]$ and the desired precision ϵ
- Integrate with the rectangle method with increasing N until the precision is achieved
- Output the result at each step with its number of points, and the final result once more (more readability)
- Compute the integral $\int_0^1 e^x \, dx$ with precision 10^{-5}

Hints:

- Create separate functions for $f(x)$ and integration
- Increase N significantly at each step (e.g., $N \leftarrow 2N$)
- No arrays are required

Submit to umorzan@ictp.it

Filename: `<your surname>-assignment-int-1.f90`