Numerical Methods I

Root finding

Graziano Giuliani

/afs/ictp/public/g/ggiulian/WORLD

The Abdus Salam International Centre for Theoretical Physics

Solve equations

- Babylonian mathematicians, as early as 2000 BC, could solve problems relating the areas and sides of rectangles. The generic formula for solving the quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with negative and irrational numbers was first codified by Abu Kamil [أبو كامل] in the 10th century.

- Higher order equations solving formulas were treasured and high value secrets (Girolamo Cardano)

- Generic formulas generally do not exist

The Abdus Salam International Centre for Theoretical Physics

The root finding problem

Find a root of an equation of the form

$$f(x)=0$$

- A root of this equation is called a zero of the function f
- Numerical methods can help us find the solution if...
  - The function has at least one continuous derivative
  - We have some estimate of the position of the root
- Numerical methods help us compute a sequence of increasingly accurate estimates of the root
  - Iteration methods

Bisection method: when

When does it work?

- The root is expected to be inside the finite interval [a,b]
- The function f is continuous in [a,b]
- The sign of the function f(a) is opposite from the sign of the function f(b)

Bisection method: how

How does it work?

- Start with initial estimate of a,b such that

$$f(a)\times f(b)<0$$

- Compute the mid point of the interval [a,b]

$$c=\frac{a+b}{2}$$

- Replace **a with c** if $\quad f(c)\times f(b)<0$
- Replace **b with c** if $\quad f(a)\times f(c)<0$
- Iterate the procedure until the distance between a and b is less than the required precision:

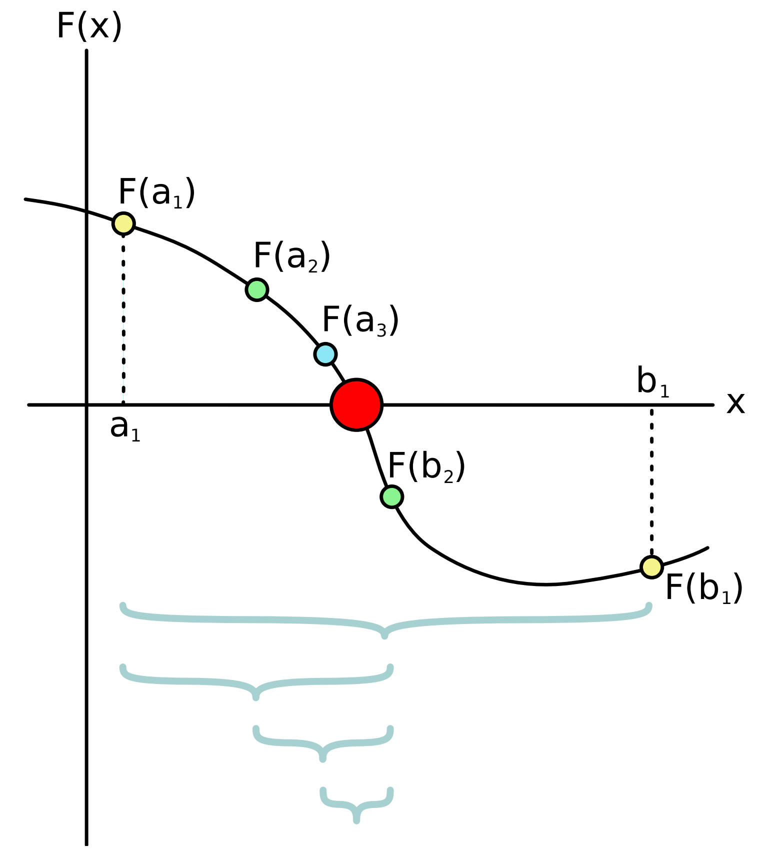
$$|(a-b)|<eps$$

The Abdus Salam
International Centre
for Theoretical Physics

Bisection algorithm

Given:

- The function f
- The interval [a, b] with a < b
- The error tolerance $\varepsilon > 0$



1) Define $c = (a+b)/2$

2) If $(b - c \leq \varepsilon)$, then accept c as the root and stop.

3) If $(\text{sign}[f(b)] \cdot \text{sign}[f(c)]) \leq 0$, then set $a = c$.

4) Otherwise, set $b = c$. Return to step 1.

QUESTION: Why the sign?

The Abdus Salam International Centre for Theoretical Physics

Pros and Cons

The advantages of this method are that

- It always find the root (if the initial assumptions are true)
- This is the only method in which you know beforehand the number of iterations you need to find the root using the predefined precision (converges linearly)

However, there are also some disadvantages which are

- It requires two initial guesses which bracket the root
- It converges slowly compared to other methods
- It is difficult to extend the method to multiple dimensions

The Abdus Salam International Centre for Theoretical Physics

A refined version

To refine the bisection method algorithm, we can use a different estimate for the point c:

$$c=\frac{f(b)\times a-f(a)\times b}{b-a}$$

The algorithm stays the same. The form for the estimate of c is a linear interpolation. We interpolate from both sides until the required precision for the estimate is obtained.

The formula?

We find the point (c, 0) where the secant line L joining the points (a, f(a)) and (b, f(b)) crosses the x-axis. To find the value c, we write down two versions of the slope m of the line L:

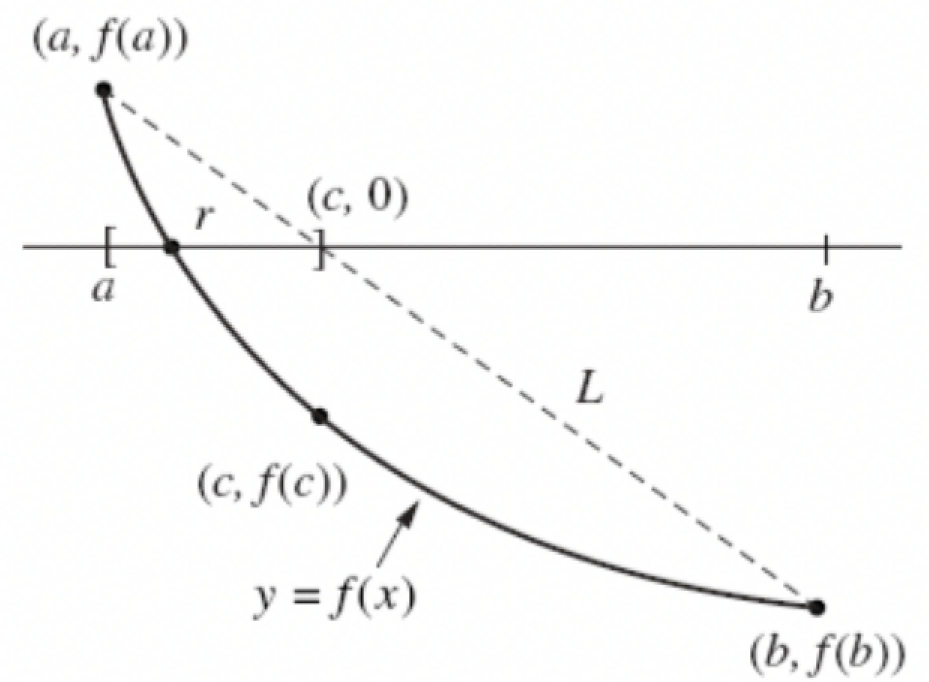
$$m=\frac{f(b)-f(a)}{b-a}$$

We calculate the same for the points (c, 0) and(b,f(b)):

$$m=\frac{0-f(b)}{c-b}$$

Equating the slopes, it is easy to get for c:

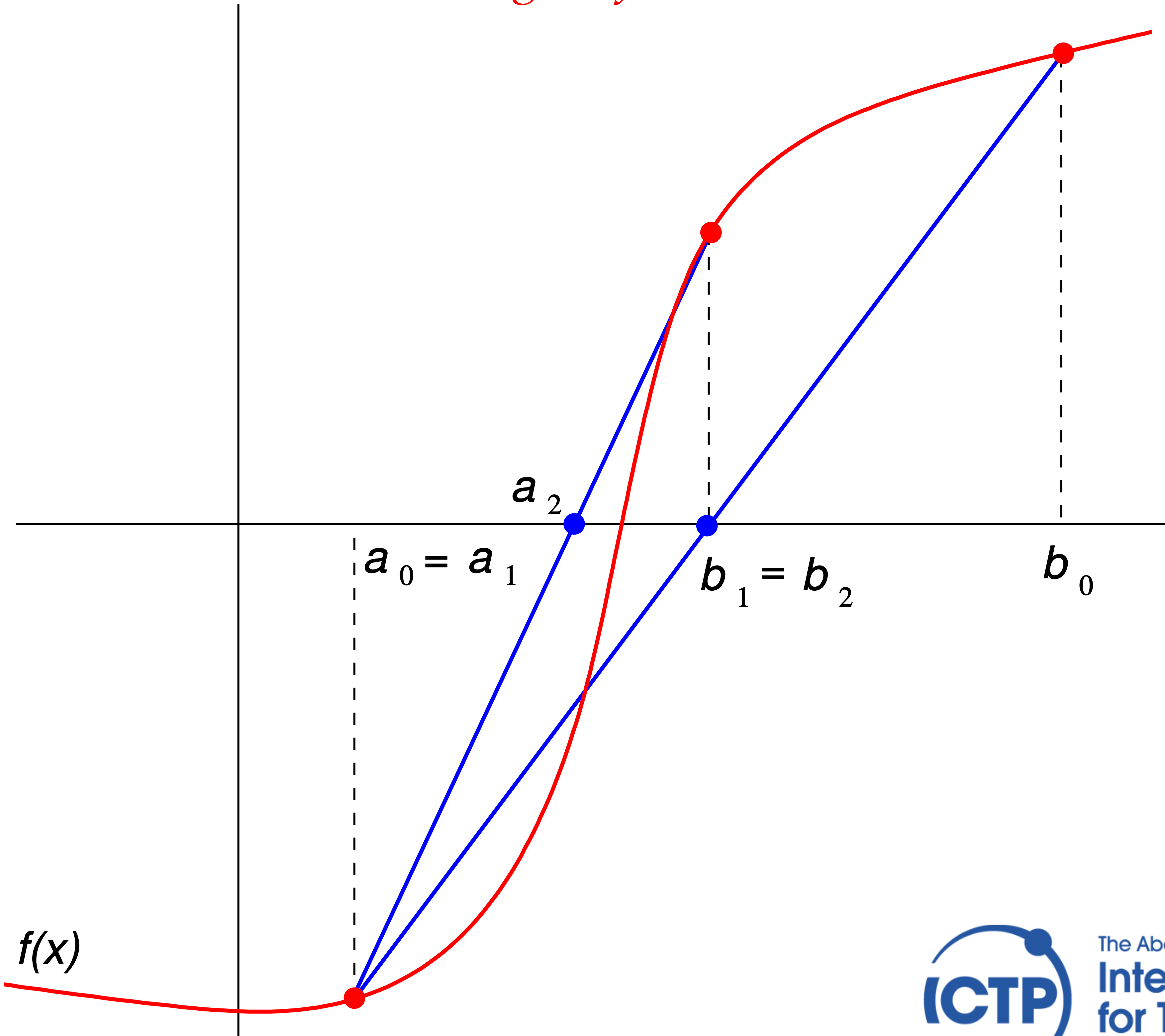
$$c=b-\frac{f(b)(b-a)}{f(b)-f(a)}=\frac{f(b)a-f(a)b}{b-a}$$



The Abdus Salam International Centre for Theoretical Physics

The false position

The method is called the *regula falsi*



The Abdus Salam
International Centre
for Theoretical Physics