Numerical Integration II

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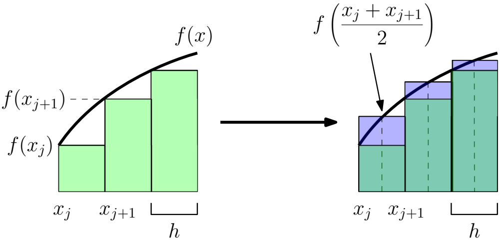
Numerical Methods I

Midpoint method

Rectangle method can be improved:

compute function at the midpoints of the intervals



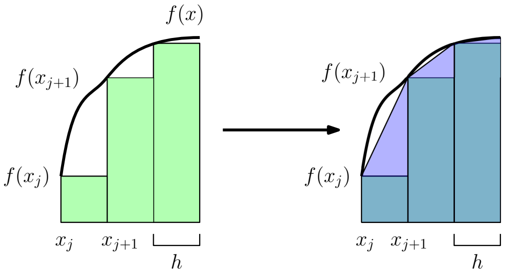
Better coverage
of area under the curve

$$\int_a^b f(x)\, dx \simeq I_M \equiv \sum_{k=0}^{N-1} f\left(\frac{x_k + x_{k+1}}{2}\right) \cdot h \qquad \begin{cases} h \equiv \dfrac{b-a}{N} \\ x_k \equiv a + k \cdot h \\ k = 0, \ldots, N \end{cases}$$

Higher accuracy for given h: $\int_a^b f(x)\, dx = I_M + O\left(h^2\right)$

Trapezoidal method I

Another way of improving the rectangle method:
approximate the area below the curve with trapezes



Area of trapeze starting in $x = x_j$:

$$A_j = \frac{h}{2}\left(f(x_j) + f(x_{j+1})\right)$$

Summing all trapezes together:

$$\int_a^b f(x)\,dx \simeq I_{T1} \equiv \sum_{k=0}^{N-1} \left[f\left(x_k\right) + f\left(x_{k+1}\right)\right] \cdot \frac{h}{2} \qquad \begin{cases} h \equiv \dfrac{b-a}{N} \\ x_k \equiv a + k \cdot h \\ k = 0, \ldots, N \end{cases}$$

Trapezoidal method II

Equivalent formulation: sum the trapezes beforehand

$$\frac{h}{2}\left[\underbrace{f(x_0)+f(x_1)}_{A_0}+\underbrace{f(x_1)+f(x_2)}_{A_1}+\underbrace{f(x_2)+f(x_3)}_{A_2}+\ldots\right]$$

In the end,

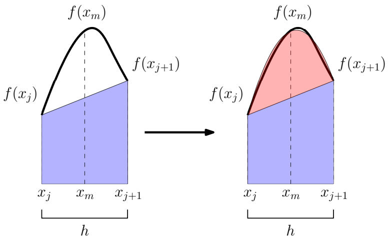
$$\boxed{\int_a^b f(x)\,dx \simeq I_{T2} \equiv \frac{h}{2}\left[f(x_0)+\sum_{k=1}^{N-1} 2f\left(x_k\right)+f\left(x_N\right)\right]}$$

Equivalent but **preferred: each $f(x_j)$ evaluated once instead of twice**
(evaluation may be numerically expensive for complicated f)

Same precision as midpoint rule: $\boxed{\int_a^b f(x)\,dx = I_{T(1,2)} + O\left(h^2\right)}$

Simpson Method

2nd-order improvement (higher order also possible): **use parabolic arcs**



Each arc passes for

- $(x_j, f(x_j))$
- $\left(\frac{x_j + x_{j+1}}{2}, f\left(\frac{x_j + x_{j+1}}{2}\right)\right)$
- $(x_{j+1}, f(x_{j+1}))$

Integrate exactly each arc, then sum: algebra yields

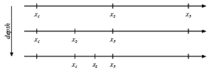
$$\int_a^b f(x)\,dx \simeq I_S \equiv \frac{h}{3}\left[f(x_0) + 2\sum_{\text{even } k}^{k>0} f(x_k) + 4\sum_{\text{odd } k'} f(x_{k'}) + f(x_N)\right] \qquad \begin{cases} h \equiv \dfrac{b-a}{N} \\ x_k \equiv a + k \cdot h \\ k = 0, \ldots, N \end{cases}$$

High accuracy: $\boxed{\int_a^b f(x)\,dx = I_S + O\left(h^4\right)}$

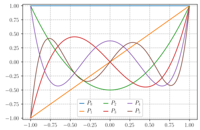
Advanced integration methods

Beyond the basic approaches,
many advanced techniques available, such as:

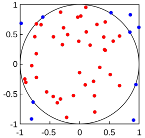
Adaptive integration:
thicker grid where function changes faster



Gauss quadrature:
mathematically optimal grid



Monte Carlo integration (Alex):
randomized grid, best in high dimension



Assignment

Write a program to computes the integral of $f(x) = e^x$

Steps:

- Ask for integration interval bound $[a, b]$ and the desired precision ϵ
- Integrate with the rectangle method with increasing N until the precision is achieved
- Output the result at each step with its number of points, and the final result once more (more readability)
- Compute the integral $\int_0^1 e^x \, dx$ with precision 10^{-5}
- Do the same for midpoint, trapeze, and Simpson methods
- Comment (in the email) about the required number of points for each method

Hints:

- You can recycle the previous assignment, adding new functions

Submit to umorzan@ictp.it by wednesday
Filename: `<your surname>-assignment-int-2.f90`