1. Reminder: We have shown in homework 2:

$$\boxed{\min_x \|Ax - b\|_2^2 + \|x\|_1 \text{ has the dual: } \min_\nu \frac{1}{4}\nu^T\nu + b^T\nu \quad \text{subject to: } \|A^T\nu\|_\infty \leq 1}$$

Yet $\min_w \frac{1}{2}\|Xw - y\|_2^2 + \lambda\|w\|_1 \iff \min_w \left\| \frac{1}{\sqrt{2}}Xw - \frac{1}{\sqrt{2}}y \right\|_2^2 + \|\lambda w\|_1$

$$\iff \min_x \|Ax - b\|_2^2 + \|x\|_1$$

with $\begin{cases} x = \lambda w \\ A = \dfrac{1}{\lambda\sqrt{2}} X \\ b = \dfrac{1}{\sqrt{2}} y \end{cases}$

This way, the dual of the problem at hand can be written:

$$\min_\nu \frac{1}{4}\nu^T\nu + \frac{1}{\sqrt{2}}y^T\nu$$

$$\text{subject to } \left\| \frac{1}{\lambda\sqrt{2}} X^T\nu \right\|_\infty \leq 1$$

Yet $\left\| \frac{1}{\lambda\sqrt{2}} X^T\nu \right\|_\infty \leq 1 \iff \max_i \left| \frac{1}{\lambda\sqrt{2}} \sum_j X_{ji}\nu_j \right| \leq 1$

$$\iff \max_i \left| \sum_j X_{ji}\nu_j \right| \leq \lambda\sqrt{2}$$

$$\iff \forall i \quad \left| \sum_j X_{ji}\nu_j \right| \leq \lambda\sqrt{2}$$

$$\iff \forall i \quad -\lambda\sqrt{2} \leq \sum_j X_{ji}\nu_j \leq \lambda\sqrt{2}$$

$$\iff \forall i \quad \begin{cases} \sum_j X_{ji}\dfrac{\nu_j}{\sqrt{2}} \leq \lambda \\ \sum_j (-X_{ji})\dfrac{\nu_j}{\sqrt{2}} \leq \lambda \end{cases}$$

Now, let $A \in \mathbb{R}^{2d \times n}$ so that $A = \begin{bmatrix} X^T \\ -X^T \end{bmatrix}$

We then have: $\| \frac{1}{2\sqrt{2}} X^T \nu \|_\infty \leq 1 \iff A\left(\frac{1}{\sqrt{2}}\nu\right) \preceq b$ with $\begin{cases} A = \begin{bmatrix} X^T \\ -X^T \end{bmatrix} \\ b = \lambda \mathbb{1}_{2d,1} \end{cases}$

This way, the dual can be written:

$$\min_\nu \left(\frac{1}{2}\nu\right)^T \left(\frac{1}{2}\nu\right) + y^T \left(\frac{1}{\sqrt{2}}\nu\right)$$

$$\text{subject to: } \quad A\left(\frac{1}{\sqrt{2}}\nu\right) \preceq b$$

which is equivalent by noting $v = \frac{1}{\sqrt{2}}\nu$

$$\boxed{\begin{aligned} &\min_v \quad v^T Q v + p^T v \\ &\text{subject to} \quad Av \preceq b \\ &\text{with: } Q = \frac{1}{2} I_n \\ &\qquad p = y \\ &\qquad b = \lambda \mathbb{1}_{2d,1} \\ &\qquad A = \begin{bmatrix} X^T \\ -X^T \end{bmatrix} \end{aligned}}$$

Moreover $Q \succeq 0$ hence the problem is convex. Since $\lambda > 0$, we have for the null vector $Av = 0 < \lambda$ and this way the problem is strictly feasible. Thanks to Slater constraint qualification, strong duality holds: we can solve (LASSO) by solving (QP)

Notes for Barrier method implementation:

Centering: Given $t>0$, solve: $v^*(t)$

$$\min \; t f_0(v) + \varphi(v)$$

$$f(v) = t f_0(v) + \varphi(v) = t\,(v^T Q v + y^T v) - \sum_{i=1}^{2d} \log(-A_i v + \lambda 1_{2d,1})$$

where A_i is the i-th line of A.

Yet $* \; \nabla f(v) = t\,(Qv + Q^T v + y) - \sum_{i=1}^{2d} \frac{-A_i^T}{-A_i v + \lambda}$

$$\boxed{\nabla f(v) = t\,(2Qv + y) + \sum_{i=1}^{2d} A_i^T \frac{1}{b_i - A_i v}}$$

$*$ for $(l, l') \in [\![1, n]\!]^2$

$$\frac{\partial^2 f(v)}{\partial l' \partial l} = \frac{\partial\,(t v_l + y_l)}{\partial l'} + \frac{\partial}{\partial v_{l'}} \left(\sum_{i=1}^{2d} \frac{a_{il}}{\lambda - \sum_{j=1}^{n} a_{ij} v_j} \right)$$

$$= \begin{cases} t & \text{if } l = l' \\ 0 & \text{otherwise} \end{cases} + \sum_{i=1}^{2d} \frac{a_{il}\, a_{il'}}{[\lambda - \sum_{j=1}^{n} a_{ij} v_j]^2}$$

This way, $\boxed{\nabla^2 f(v) = 2tQ + \sum_{i=1}^{2d} \frac{1}{\lambda - A_i^T v} A_i^T A_i}$

HW3_notebook

November 21, 2023

[1]:
```python
# imports
import numpy as np
from matplotlib import pyplot as plt
import cvxpy as cp
# from matplotlib_inline.backend_inline import set_matplotlib_formats
# set_matplotlib_formats('svg')
```

1 Solving (QP) using Barrier method

[2]:
```python
class NotFeasible(Exception):
    pass

class QP:
    def __init__(self, Q, A, p, b, t):
        """
        Initialises QP problem parameters.
        """
        self._Q = Q
        self._A = A
        self._p = p
        self._b = b
        self._t = t

    def set_t(self, t):
        self._t = t

    def is_feasible(self, v):
        """
        Test whether point v is feasible.
        """
        return np.all(self._A @ v <= self._b)

    def is_strictly_feasible(self, v):
        """
        Test whether point v is stricly feasible.
        """
        return np.all(self._A @ v < self._b)
```

```
    def f0(self, v):
        """
        Returns objectif fonction evaluated at point v.
        """
        if not self.is_feasible(v):
            raise NotFeasible("The point is not feasible.")

        return (v @ self._Q) @ v + self._p @ v

    def f(self, v):
        """
        Return redifined objectif function evaluated at point v.
        """
        if not self.is_strictly_feasible(v):
            raise NotFeasible("The point is not feasible.")

        return self._t * self.f0(v) - np.sum(np.log(- self._A @ v + self._b))

    def grad_f(self, v):
        """
        Gradient of the redifined objective function evaluated at point v.
        """
        return self._t * (2 * self._Q @ v + self._p) + np.sum(self._A / (- self.
 ↪_A @ v + self._b)[:, np.newaxis], axis=0)

    def hessian_f(self, v):
        """
        Hessian of the redifined objective function evaluated at point v.
        """
        return 2 * self._t * self._Q + np.sum(self._A[:, :, np.newaxis] * self.
 ↪_A[:, np.newaxis, :] * (1/(- self._A @ v + self._b)**2)[:, np.newaxis, np.
 ↪newaxis], axis=0)
```

1.1 Solving the centering step

[3]:
```
def backtracking(f, grad_f, v, search_dir, alpha=0.4, beta=0.7):
    step_size = 1

    while True:
        try:
            gap =  f(v + step_size * search_dir) - f(v) + alpha * step_size *␣
 ↪grad_f(v) @ search_dir
            if gap < 0:
                break
            else:
                step_size *= beta
```

```
        except NotFeasible: # point not feasible
            step_size *= beta

    return step_size
```

[4]:
```
def centering_step(Q, p, A, b, t, v0, eps):
    # defining the QP problem
    qp = QP(Q, A, p, b, t)
    v = np.copy(v0)

    # keeping track of the progression
    v_seq, grad_seq = [], []

    while True:
        grad, hess = qp.grad_f(v), qp.hessian_f(v)
        grad_seq.append(grad)
        v_seq.append(np.copy(v))

        newton_step = np.linalg.solve(hess, - grad)
        newton_decrement = - grad @ newton_step

        if newton_decrement / 2 <= eps:
            break

        step_size = backtracking(qp.f, qp.grad_f, np.copy(v), newton_step)

        v += step_size * newton_step

    return v_seq, grad_seq
```

1.2 Implementing barrier method

[5]:
```
def barr_method(Q, p, A, b, v0, eps, t=0.5, mu=1.5, verbose=False):
    # defining the QP problem
    qp = QP(Q, A, p, b, t)
    v = np.copy(v0)

    # keeping track of progress
    v_seq = []
    n_inner = []
    prec_crit = []

    # making shure starting point is strictly feasible
    assert qp.is_strictly_feasible(v0), "Starting point is not strictly␣
↪feasible"

    while True:
```

```
        prec_crit.append(A.shape[0] / t)
        v_seq.append(v)
        v_seq_inner, _ = centering_step(Q, p, A, b, t, v, eps)
        v = np.copy(v_seq_inner[-1])
        n_inner.append(len(v_seq_inner))

        if verbose:
            print(f"Iteration {len(n_inner)} after {n_inner[-1]} newton␣
↪iterations")

        if A.shape[0] / t < eps:
            return v_seq, n_inner, prec_crit

        t *= mu
        qp.set_t(t)
```

2 Running tests

[6]:
```
# dimesions
d = 50
n = 100

# generate random data
np.random.seed(seed=10)
X = np.random.randn(n, d)
y = np.random.randn(n, )

# regularization parameter
lambd = 10

# QP optimization problem
Q = np.identity(n) * 0.5
p = y
A = np.vstack((X.T,-X.T))
b = lambd * np.ones((2 * d,))

# Barrier method parameters
eps = 1e-6
v0 = np.zeros(n)
```

[7]:
```
v0 = np.zeros(n)
eps = 1e-3
v_seq, _, _ = barr_method(Q, p, A, b, v0, eps)
qp = QP(Q, A, p, b, 0.5)
print(f"optimal value: {qp.f0(v_seq[-1])}")
```

```
optimal value: -48.36035915447788
```

2.1 Verifying results using cvxpy library

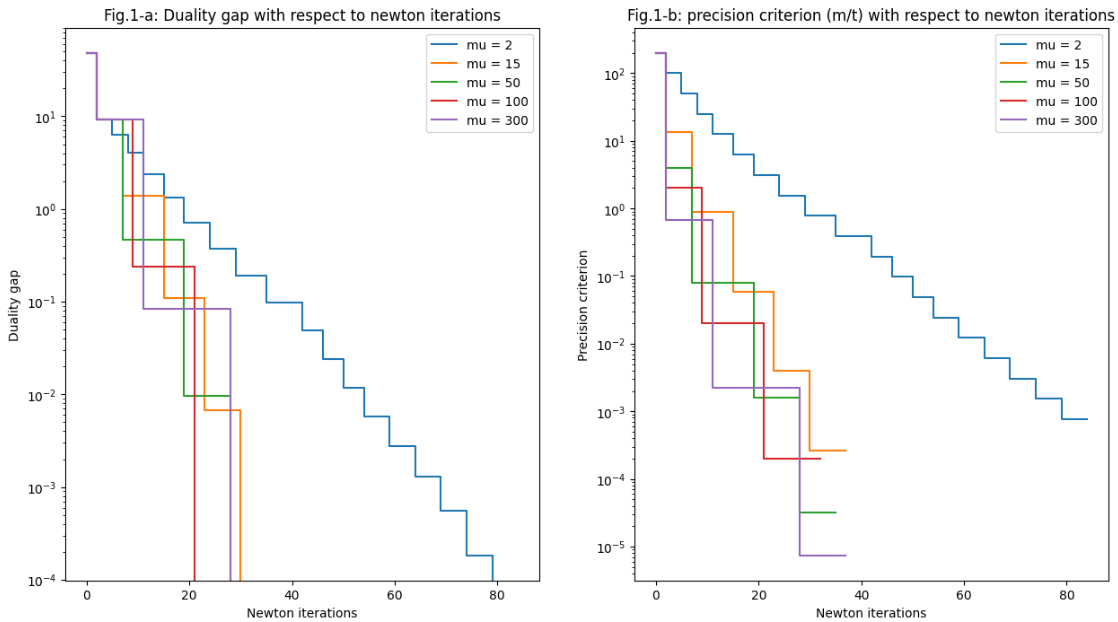
[8]:
```
# Posing the dual problem
x = cp.Variable(n)
dual_prob = cp.Problem(
    cp.Minimize(
        cp.quad_form(x, Q) + p.T @ x
        ),
    [A @ x <= b]
)

# Solving the problem
dual_prob.solve()

# Printing the optimal solution
print(f"cvxpy optimal value: {dual_prob.value}")
```

```
cvxpy optimal value: -48.36048396249347
```

2.2 Duality gap with respect to newton iterations

[9]:
```
fig, axs = plt.subplots(1, 2, figsize=(15, 8))

for mu in [2, 15, 50, 100, 300]:

    v_seq, n_inner, crit_prec = barr_method(Q, p, A, b, v0, eps, mu=mu)
    v_star = v_seq[-1]
    f_star = qp.f0(v_star)

    values_v = [qp.f0(v) - f_star for v, num in zip(v_seq, n_inner) for _ in␣
 ↪range(num)]
    axs[0].step(range(len(values_v)), values_v, label='mu = '+str(mu))
    axs[0].semilogy()
    axs[0].legend()
    axs[0].set_title(f"Fig.1-a: Duality gap with respect to newton iterations")
    axs[0].set_xlabel('Newton iterations')
    axs[0].set_ylabel('Duality gap')


    values_crit_prec = [v for v, num in zip(crit_prec, n_inner) for _ in␣
 ↪range(num)]
    axs[1].step(range(len(values_crit_prec)), values_crit_prec, label='mu =␣
 ↪'+str(mu))
    axs[1].semilogy()
    axs[1].legend()
    axs[1].set_title(f"Fig.1-b: precision criterion (m/t) with respect to␣
 ↪newton iterations")
    axs[1].set_xlabel('Newton iterations')
```

```
    axs[1].set_ylabel('Precision criterion')
```

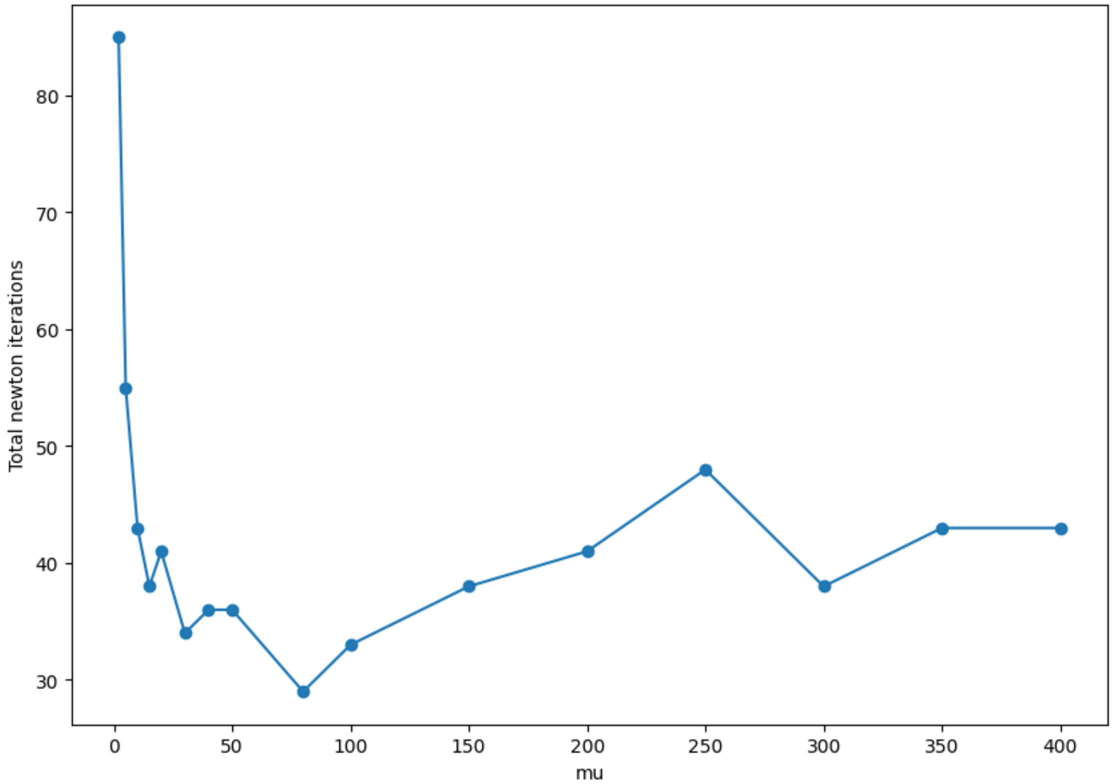


2.3 Total newton iterations with respect to mu

[10]:
```
plt.figure(figsize=(10, 7))

mu_l = [2, 5, 10, 15, 20, 30, 40, 50, 80, 100, 150, 200, 250, 300, 350, 400]
n_inner_l = []

for mu in mu_l:
    _, n_inner_eps, _ = barr_method(Q, p, A, b, v0, eps, mu=mu)
    n_inner_l.append(sum(n_inner_eps))

plt.plot(mu_l, n_inner_l, 'o-')
plt.xlabel('mu')
plt.ylabel('Total newton iterations')
plt.title("Fig.2: Total newton iterations with respect to mu")
```

[10]:
```
Text(0.5, 1.0, 'Fig.2: Total newton iterations with respect to mu')
```

Fig.2: Total newton iterations with respect to mu



Comment

It seems that a wise choice is to set mu to a sufficiently large value, as we observe a significant drop in the total number of Newton steps (Fig. 2) from mu=2 to mu=20. Indeed, we can distinguish two linear tendencies in Fig. 1, where, from mu=15, we observe fewer steps with a more substantial reduction in the dual gap.

After that, the difference in the number of inner Newton iterations becomes less significant, even though it appears to show an increasing tendency. Here, mu=100 appears to be the most appropriate choice among the values tested