# Ex 1 :

**1)** $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i=1,\dots,n\}$

$$= \bigcap_{i=1}^{n} \left( \{x \in \mathbb{R}^n \mid \alpha_i \leq x_i\} \cap \{x \in \mathbb{R}^n \mid x_i \leq \beta_i\} \right)$$

Yet, $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i\} = \{x \in \mathbb{R}^n \mid a^T x \leq b\}$

with $a = \begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ i^{th} \text{ row} \\ \\ \\ \\ \end{array} \end{array}$ and $b = -\alpha_i$.

Likewise, $\{x \in \mathbb{R}^n \mid x_i \leq \beta_i\} = \{x \in \mathbb{R}^n \mid a^T x \leq b\}$

with $a = \begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ i^{th} \text{ row} \\ \\ \\ \\ \end{array} \end{array}$ and $b = \beta_i$

Therefore,

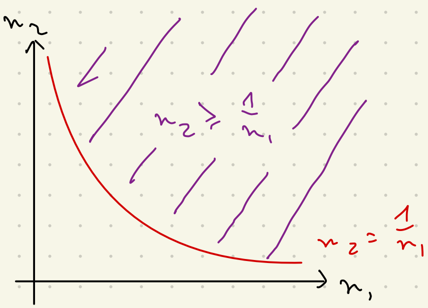
$\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i\}$ and $\{x \in \mathbb{R}^n \mid x_i \leq \beta_i\}$ are half-spaces, hence convex.

This way, a rectangle is the intersection of convex sets, and therefore is convex itself.

2) First of all, $\{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\} = \{x \in \mathbb{R}_+^2 \mid x_1 \geq \frac{1}{x_2}\}$

(Indeed, we have necessarily $x_2 > 0$)

This way, $\{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$ can be seen as the epigraph of the inverse function on $\mathbb{R}_+^*$:




The inverse function is obviously convex on $]0; +\infty[$, therefore its epigraph is convex. We can conclude that our set is convex.

3) Let's set an element $y \in S$

Then for all $x \in \mathbb{R}^n$

$$\|x - x_0\|_2 \leq \|x - y\|_2 \iff \|x - x_0\|_2^2 \leq \|x - y\|_2^2$$

$$\iff (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y)$$

$$\iff x^T x - x^T x_0 - x_0^T x + x_0^T x_0 \leq x^T x - x^T y - y^T x + y^T y$$

$$\iff (-x_0 - x_0^T + y + y^T)\, x \leq y^T y - x_0^T x_0$$

$$\iff \underbrace{\left(y + y^T - (x_0 + x_0^T)\right)^T}_{a^T} x \leq \underbrace{y^T y - x_0^T x_0}_{b}$$

with $a^T \in \mathbb{R}^n, b \in \mathbb{R}$

$$\iff a^T x \leq b$$

This way, with $y \in S$ set, $\{x \mid \|x - x_0\|_2 \leq \|x - y\|\}$ is a half-space and thus convex.

Finally,

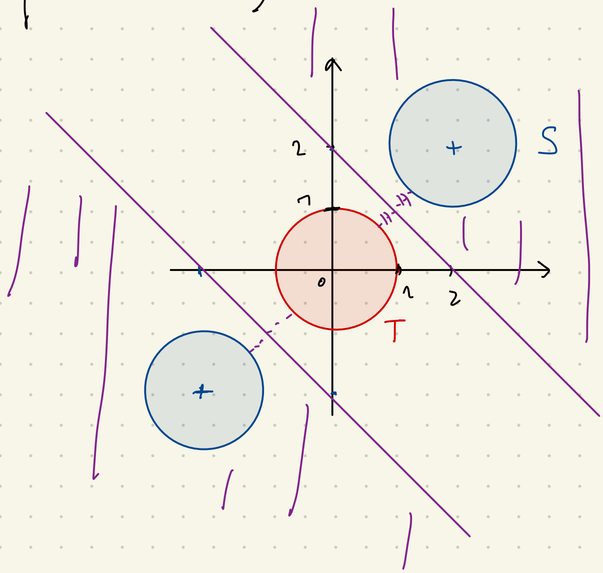
$$\{x \mid \|x - x_0\|_2 \leq \|x - y\| \text{ for all } y \in S\} = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

is convex as it is an intersection of convex sets.

h) Let's take $S = \mathcal{B}'((-2,2),1) \cap \mathcal{B}'((1,1),1)$

and $T = \mathcal{B}'((0,0),1)$

($\mathcal{B}'((x,y),r)$ is the closed ball in $\mathbb{R}^2$ centered in $(x,y)$ and of radius $r$)




$\{x \mid dist(x,S) \leq dist(x,T)\}$

$= \{(x,y) \in \mathbb{R}^2 \mid y \geq -2x+2\} \cup \{(x,y) \in \mathbb{R}^2 \mid y \leq -2x+2\}$

which is clearly not convex (see above drawing).

This way, $\{x \mid dist(x,S) \leq dist(x,T)\}$ is not necessarily convex.

5) Let $x_1, x_2 \in \{x \mid x + S_2 \subseteq S_1\}$

then $\forall y \in S_2 \quad x_1 + y, \; x_2 + y \in S_1$ and because $S_1$ is convex:

$$\forall \theta \in [0,1], \; \forall y \in S_2 \qquad \theta(x_1 + y) + (1-\theta)(x_2 + y) \in S_1$$

$$\iff \quad \theta x_1 + (1-\theta) x_2 + y \in S_1$$

In other words, $\forall \theta \in [0,1] \quad \theta x_1 + (1-\theta) x_2 \in \{x \mid x + S_2 \subseteq S_1\}$

We can conclude that $\{x \mid x + S_2 \subseteq S_1\}$ is convex.

Ex 2

1) First of all, $\forall (x_1,x_2) \in \mathbb{R}_{++}^2$ $\nabla f(x_1,x_2) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$

and $\nabla^2 f(x_1,x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and for all $X=(x,y) \in \mathbb{R}^2$, then $X^T \nabla^2 f(x_1,x_2) X = 2xy$ which is not necessarily positive. Therefore, $\nabla^2 f(x,y) \not\succcurlyeq 0$ and f is not convex.

Likewise, let $g = -f$. Then $\nabla^2 g(x_1,x_2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

and for all $X=(x,y) \in \mathbb{R}^2$, $X^T \nabla^2 g(x_1,x_2) X = -2xy$ which is not necessarily positive. Therefore $\nabla^2 g(x_1,x_2) \not\succcurlyeq 0$ and f is not concave.

Then, let $\alpha \in \mathbb{R}$ and we now consider $S = \{x \in \mathbb{R}_{++}^2 \mid g(x_1,x_2) \leq \alpha\}$.
For all $x = (x_1,x_2) \in S$

$$g(x_1,x_2) \leq \alpha \iff -x_1x_2 \leq \alpha$$
$$\iff x_1x_2 \geq -\alpha$$

If $-\alpha \leq 0$, then $S = \mathbb{R}_{++}^2$ because for all $(x_1,x_2) \in \mathbb{R}_{++}^2$, $x_1x_2 > 0$ and S is convex as $\mathbb{R}_{++}^2$ is the intersection of half-spaces.

If $-\alpha > 0$, then $-\alpha = \sqrt{-\alpha} \times \sqrt{-\alpha}$

and $x_1x_2 \geq -\alpha \iff x_1x_2 \geq \sqrt{-\alpha} \times \sqrt{-\alpha}$

$$\iff \frac{x_1}{\sqrt{-\alpha}} \times \frac{x_2}{\sqrt{-\alpha}} \geq 1$$

$$\iff X_1 \times X_2 \geq 1 \quad \text{with} \quad \begin{cases} X_1 = \frac{x_1}{\sqrt{-\alpha}} \\ X_2 = \frac{x_2}{\sqrt{-\alpha}} \end{cases}$$

and, as $\left\{\left(\frac{x_1}{\sqrt{-\alpha}}, \frac{x_2}{\sqrt{-\alpha}}\right) \text{ for } x_1, x_2 \text{ in } \mathbb{R}_{++}\right\} = \mathbb{R}^2_{++}$, we can rewrite $S = \left\{x \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq 1\right\}$ which is convex as seen in Ex 1.

We can conclude that $g = -f$ is quasiconvex, f is quasiconcave

2) For all $x = (x_1, x_2) \in \mathbb{R}^2_{++}$

$$\nabla f(x_1, x_2) = \begin{pmatrix} -\frac{1}{x_1^2 x_2} \\ -\frac{1}{x_1 x_2^2} \end{pmatrix}$$

and
$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{pmatrix}$$

$$= \frac{1}{x_1 x_2} \begin{pmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{pmatrix}$$

Then, let $x = (x, y) \in \mathbb{R}^2$,

$$x^T \nabla^2 f(x_1, x_2) x = \frac{1}{x_1 x_2} \left( \frac{2x^2}{x_1^2} + \frac{2xy}{x_1 x_2} + \frac{2y^2}{x_2^2} \right)$$

$$= \frac{1}{x_1 x_2} \left( \left(\frac{x}{x_1}\right)^2 + \left(\frac{y}{x_2}\right)^2 + \left(\frac{x}{x_1} + \frac{y}{y_1}\right)^2 \right) \geq 0$$

This way, $\nabla^2 f(x_1, x_2) \succeq 0$ and f is convex (and quasi convex).

3) For all $x = (x_1, x_2) \in \mathbb{R}_{++}^2$

$$\nabla f(x_1,x_2) = \begin{pmatrix} \frac{1}{x_2} \\ -\frac{x_1}{x_2^2} \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x_1,x_2) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

Let $(x,y) \in \mathbb{R}^2$, then: $x^T \nabla^2 f(x_1,x_2) x = \frac{1}{x_2^2}\left(2 \frac{x_1}{x_2} y^2 - 2xy\right)$

Let's take $(1,1) \in \operatorname{dom} f$, then $x^T \nabla f(x_1,x_2) x = 2y^2 - 2xy$

Now let's take $\begin{cases} x = \frac{1}{2}, \ y = 1, \ \text{then } x^T \nabla f(x_1,x_2) x = 1 > 0 \\ x = 2, \ y = 1, \ \text{then } x^T \nabla f(x_1,x_2) x = -2 < 0 \end{cases}$

This way, $\begin{cases} \nabla f(x_1,x_2) \not\preceq 0 \\ \nabla f(x_1,x_2) \not\succeq 0 \end{cases}$ and $f$ is neither convex nor concave

Now, let's consider $\alpha \in \mathbb{R}$ and $S_\alpha = \{x \in \mathbb{R}_{++}^2 \mid f(x) \leq \alpha\}$

$$\begin{aligned} S_\alpha &= \left\{(x_1,x_2) \in \mathbb{R}_{++}^2 \mid \frac{x_1}{x_2} \leq \alpha\right\} \\ &= \left\{(x_1,x_2) \in \mathbb{R}_{++}^2 \mid x_1 - x_2\alpha \leq 0\right\} \\ &= \left\{(x_1,x_2) \in \mathbb{R}_{++}^2 \mid a^T x \leq b\right\} \end{aligned}$$

with $a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $b = 0 \in \mathbb{R}$.

This way, $S_\alpha$ is an halfspace, therefore convex and $f$ is quasiconvex

With the same reasoning on $S_\alpha = \{(x_1;x_2) \in \mathbb{R}_{++}^2 \mid -\frac{x_1}{x_2} \leq \alpha\}$

with $a = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $b = 0$, we show that $f$ is also quasiconcave.

h) For all $(x_1, x_2) \in \mathbb{R}_{++}^2$

$$\nabla f(x_1,x_2) = \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{pmatrix}$$

and
$$\nabla^2 f(x_1,x_2) = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{pmatrix}$$

$$= \alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{pmatrix} -\frac{1}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & -\frac{1}{x_2^2} \end{pmatrix}$$

Let $x = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ then

$$x^T \begin{pmatrix} -\frac{1}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & -\frac{1}{x_2^2} \end{pmatrix} x = -\frac{x^2}{x_1^2} + 2\frac{xy}{x_1 x_2} - \frac{y^2}{x_2^2}$$

$$= -\left(\frac{x}{x_1} - \frac{y}{x_2}\right)^2 \leq 0$$

This way $\nabla^2 f(x_1, x_2) \preccurlyeq 0$ and f is concave (and quasiconcave)

Ex 3:

1) Let's decompose $X \in S^n_{++}$ as $X = Z + tV$ with $\begin{cases} t \in \mathbb{R} \\ Z \in S^n_{++} \\ V \in S^n \end{cases}$

Indeed, $h: \mathbb{R} \times S^n_{++} \times S^n \longrightarrow S_n^{++}$ is injective as we can
$(t, Z, V) \longmapsto Z + tV$

for all $X \in S^n_{++}$ take $t = 1$ and $V = X - Z$

We now consider $g(t) = f(Z + tV)$

$$= \mathrm{tr}\left((Z + tV)^{-1}\right)$$

Yet $Z \in S^n_+$ thus $\exists!\, Z^{\frac{1}{2}} \in S^n_+ \quad (Z^{\frac{1}{2}})^2 = Z$ and $Z^{\frac{1}{2}} \in S^n_+$ so it is invertible

this way,

$$g(t) = \mathrm{tr}\left(\left(Z^{\frac{1}{2}}\left(Z^{\frac{1}{2}} + tZ^{-\frac{1}{2}}V\right)\right)^{-1}\right)$$

$$= \mathrm{tr}\left(\left[Z^{\frac{1}{2}}\left(I_n + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}\right)Z^{\frac{1}{2}}\right]^{-1}\right)$$

$$= \mathrm{tr}\left(Z^{-\frac{1}{2}}\left(I_n + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}\right)^{-1}Z^{-\frac{1}{2}}\right)$$

$$= \mathrm{tr}\left(Z^{-\frac{1}{2}}Z^{-\frac{1}{2}}\left(I_n + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}\right)^{-1}\right)$$

$$= \mathrm{tr}\left(\left(Z^{\frac{1}{2}}Z^{\frac{1}{2}}\right)^{-1}\left(I_n + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}\right)^{-1}\right)$$

$$= \mathrm{tr}\left(Z^{-1}\left(I_n + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}}\right)^{-1}\right)$$

Since $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}} \in S^n$, $\exists Q$ orthogonal and $\Sigma$ diagonal with values being eigen values of $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$, $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}} = Q\Sigma Q^T$

this way, $g(t) = \mathrm{tr}\left(Z^{-1}\left(I_n + tQ\Sigma Q^T\right)^{-1}\right)$

$$= \mathrm{tr}\left(Z^{-1}\left(QQ^T + tQ\Sigma Q^T\right)^{-1}\right)$$

$$= \mathrm{tr}\left(Z^{-1}Q^T\left(I_n + t\Sigma\right)^{-1}Q\right)$$

$$= \mathrm{tr}\left(QZ^{-1}Q^T\left(I_n + t\Sigma\right)^{-1}\right)$$

Naming $d_{ii}$ the diagonal entries of $QZ^{-1}Q^T$, we get:

$$g(t) = \sum_{i=1}^{n} d_{ii} \frac{1}{1+\lambda_i t}$$

with $\lambda_i$ the $i$-th eigen value of $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$ in descending order.

Yet for all $x \in \mathbb{R}^n \setminus \{0\}$ $\quad x^T Q Z^{-1} Q^T x = (x^T Q) Z^{-1} (Q^T x)$

and $\begin{cases} x^T Q \neq 0 \\ x Q^T \neq 0 \end{cases}$ because $Q$ is orthogonal and $x \neq 0$

this way, as $Z^{-1} \in S^n_{++}$, $x^T Q Z^{-1} Q^T x > 0$

and thus is positive definite.

For this reason $d_{ii} > 0$ for all $i$.

For any $x \in \mathbb{R}^n \setminus \{0\}$, then

$$x^T \left( I_n + t Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}} \right) x =$$

$$x^T Z^{-\frac{1}{2}} (Z + tV) Z^{-\frac{1}{2}} x = (Z^{-\frac{1}{2}} x)^T X Z^{-\frac{1}{2}} x$$

Yet $Z^{-\frac{1}{2}} x \neq 0$ because $Z^{-\frac{1}{2}}$ is full-rank and $x \neq 0$

This way, because $X \in S^n_{++}$, $(Z^{-\frac{1}{2}} x)^T X Z^{-\frac{1}{2}} x > 0$

For this reason, $I_n + t Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}} \in S^n_{++}$ and thus its eigen values are strictly positive: $\forall i \;\; 1 + t\lambda_i > 0$

Then $\quad g''(t) = \sum_{i=1}^{n} d_{ii} \frac{2\lambda_i^2}{(1+t\lambda_i)^3} > 0$ because $\begin{cases} d_{ii} > 0 \\ 1 + t\lambda_i > 0 \end{cases}$

We can finally conclude that for all $t \in \mathbb{R}$ and $Z \in S^n_{++}$, $V \in S^n$, $g$ is convex and therefore <u>$f$ is itself convex</u>

2) Let's first show that:

$$\forall x,y \in \mathbb{R}^n, \forall X \in S_{++}^n \qquad \frac{1}{2}\left(y^T X^{-1} y + x^T X x\right) \geq x.y$$

Let $X \in S_{++}^n$, $x,y \in \mathbb{R}^n$

$\exists R \in S_{++}^n$, $X = R^2$

Let's note $\tilde{x} = R^{-1}x$ and $\tilde{y} = Ry$

This way $\tilde{x}.\tilde{y} = (R^{-1}x)^T . Ry$

$= x^T (R^{-1})^T . Ry$

$= x^T (R^T)^{-1} . Ry$

$= x^T R^{-1} R y$

$= x^T y$

$= x.y$

Yet $(\tilde{x}-\tilde{y}).(\tilde{x}-\tilde{y}) = \tilde{x}.\tilde{x} - 2\tilde{x}.\tilde{y} + \tilde{y}.\tilde{y} \geq 0$

$\Rightarrow \quad \frac{1}{2}(\tilde{x}.\tilde{x} + \tilde{y}.\tilde{y}) \geq \tilde{x}.\tilde{y}$

$\Leftrightarrow \quad \frac{1}{2}\left((R^{-1}x)^T R^{-1}x + (Ry)^T Ry\right) \geq x.y$

$\Leftrightarrow \quad \frac{1}{2}\left(x^T R^{-1} R^{-1} x + y^T R R y\right) \geq x.y$

$\Leftrightarrow \quad \frac{1}{2}\left(x^T (R^2)^{-1} x + y^T R^2 y\right) \geq x.y$

$\Leftrightarrow \quad \frac{1}{2}\left(x^T X^{-1} x + y^T X y\right) \geq x^T y$

Secondly, we have this way: $\forall x,y \in \mathbb{R}^2, \forall X \in S_{++}^n \quad y^T x - \frac{1}{2} x^T X x \leq \frac{1}{2} y^T X^{-1} y$

and for $x = X^{-1}y \quad y^T X^{-1} y - \frac{1}{2}(X^{-1}y) X X^{-1} y =$

$y^T X^{-1} y - \frac{1}{2} y^T X^{-1} y = \frac{1}{2} y^T X^{-1} y$

We can conclude: $\sup_{x \in \mathbb{R}^n} \left(y^T x - \frac{1}{2} x^T X x\right) = \frac{1}{2} y^T X^{-1} y$

This way, $f(X,y) = \frac{1}{2} y^T X^{-1} y = \sup_{x \in \mathbb{R}^n} \left(y^T x - \frac{1}{2} x^T X x\right) = f^*(X,y)$

As $f^*$ is the dual of $f$, it is convex and so is $f$!

3) Let $A \in S^n$ and $A = X\Sigma Y^T$ the singular value decomposition of $A$.

Let $U, V \in O_n$.

$$U^T A V = U^T X \Sigma Y^T V$$
$$= \tilde{U}^T \Sigma \tilde{V} \quad \text{with} \quad \begin{cases} \tilde{U} = X^T U \\ \tilde{V} = Y^T V \end{cases}$$

Moreover, $\tilde{U}^T \tilde{U} = U^T \underbrace{X X^T}_{I_n \text{ because } X \in O_n} U = U^T U = I_n$

$\tilde{V}^T \tilde{V} = V^T \underbrace{Y Y^T}_{I_n \text{ because } Y \in O_n} V = V^T V = I_n$

$$\operatorname{tr}(U^T A V) = \operatorname{tr}(\tilde{U}^T \Sigma \tilde{V})$$
$$= \sum_{i=1}^n (\tilde{U}^T \Sigma \tilde{V})_{ii} \quad \text{where } (\tilde{U}^T \Sigma \tilde{V})_{ii} \text{ is the } i\text{-th diagonal coef}$$
$$= \sum_{i=1}^n \left( \sum_{k=1}^n \sum_{l=1}^n \tilde{U}_{ki} \Sigma_{kl} \tilde{V}_{li} \right)$$
$$= \sum_{i=1}^n \sum_{k=1}^n \tilde{U}_{ki} \sigma_k(A) \tilde{V}_{ki} \qquad \Sigma_{kl} = \begin{cases} \sigma_k(A) & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}$$
$$= \sum_{k=1}^n \sum_{i=1}^n \tilde{U}_{ki} \sigma_k(A) \tilde{V}_{ki}$$

with $\sigma_k(A)$ the $k$-th entry of the diagonal of $A$, being its $k$-th singular value.

$$= \sum_{k=1}^n \left[ \sigma_k(A) \left( \sum_{i=1}^n \tilde{U}_{ki} \tilde{V}_{ki} \right) \right.$$
$$= \sum_{k=1}^n \sigma_k(A) (\tilde{U} \tilde{V}^T)_{kk}$$

Then $|\operatorname{tr}(U^T A V)| \leq \sum_{k=1}^n \sigma_k(A) \, |(\tilde{U}\tilde{V}^T)_{kk}|$ (the $\sigma_k(A) \geq 0$)

Yet $|(\tilde{U}\tilde{V}^T)_{kk}| = |\sum_{j=1}^{n} \tilde{U}_{kj} \tilde{V}_{kj}| \leq (\sum_{j=1}^{n} \tilde{U}_{\ell j})^{\frac{1}{2}} (\sum_{j=1}^{n} \tilde{V}_{kj})^{\frac{1}{2}}$

Because $U, V \in \mathcal{O}_n$, we have $(\sum_{j=1}^{n} \tilde{U}_{\ell j})^{\frac{1}{2}} (\sum_{j=1}^{n} \tilde{V}_{kj})^{\frac{1}{2}} = 1^{\frac{1}{2}} \times 1^{\frac{1}{2}} = 1$

And this way: $|tr(U^T A V)| \leq \sum_{k=1}^{n} \sigma_k(A)$

Now by taking $U = X$ and $V = Y$, we have:

$$U^T A V = X^T X \Sigma Y^T Y$$
$$= \Sigma$$

and thus $|tr(U^T A V)| = \sum_{k=1}^{n} \sigma_k(A)$

By adding that $\forall U, V \in \mathcal{O}_n \quad tr(U^T A V) \leq |tr(U^T A V)| \leq \sum_{k=1}^{n} \sigma_k(A)$

and that for $U = X, V = Y$, $tr(U^T A V) = \sum_{k=1}^{n} \sigma_k(A)$

We can conclude that $\sum_{k=1}^{n} \sigma_k(A) = \max_{\substack{U \in \mathcal{O}_n \\ V \in \mathcal{O}_n}} tr(U^T A V)$

Finally, for all $U \in \mathcal{O}_n, V \in \mathcal{O}_n$, $A \longmapsto tr(U^T A V)$ is a linear function.

This way, $X \mapsto f(X) = \sum_{i=1}^{n} \sigma_i(X)$ is the supremum (maximum) of a family of linear (hence convex) functions and is therefore itself convex!

Ex 4:

a) $K_{m+} = \{ x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0 \}$

$$= \{ x \in \mathbb{R}^n \mid x_n \geq 0 \} \cap \left( \bigcap_{i=1}^{n-1} \{ x \in \mathbb{R}^n \mid x_i \geq x_{i+1} \} \right)$$

$$= \{ x \in \mathbb{R}^n \mid x_n \geq 0 \} \cap \left( \bigcap_{i=1}^{n} \{ x \in \mathbb{R}^n \mid x_i - x_{i+1} \geq 0 \} \right)$$

Yet $\{ x \in \mathbb{R}^n \mid x_n \geq 0 \} = \{ x \in \mathbb{R}^n \mid a^T x \leq b \}$ with $a = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ and $b = 0$

and for all $i \in [\![1; n-1]\!]$ $\{ x \in \mathbb{R}^n \mid x_i - x_{i+1} \geq 0 \} = \{ x \in \mathbb{R}^n \mid a^T x \leq b \}$

with $a = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ --- $i$-th row and $b = 0$.

This way $K_{n+}$ is an intersection of half-spaces, therefore convex.

Now, let $(x^{(k)})_{k \in \mathbb{N}} = (x_1^{(k)}; \dots; x_n^{(k)})$ a sequence of elements of $K_{n+}$ with $\forall i \in [\![1, n]\!]$ $x_i^{(k)} \xrightarrow[k \to +\infty]{} l_i$.

Then, for all $k \in \mathbb{N}$ $x_1^{(k)} \geq x_2^{(k)} \geq \dots \geq x_n^{(k)} \geq 0$

and thus $l_1 \geq l_2 \geq \dots \geq l_n \geq 0$ when going to limits.

This way $l = (l_1, \dots, l_n) \in K_{n+}$ and $K_{n+}$ is closed.

Its interior is nonempty as $x = (n; n-1; \dots; 1)$ verifies the inequalities strictly.

Finally, if $\begin{cases} x \in K_{n+} \\ -x \in K_{n+} \end{cases}$, then $\forall i \in [\![1; n]\!]$ $\begin{cases} x_i \geq x_{i+1} \\ x_{i+1} \geq x_i \end{cases}$

which implies that $x = (0; \dots; 0)$ and therefore $K_{n+}$ is pointed

As a conclusion, we showed that $K_{n+}$ is a proper cone!

b) The dual cone of $K_{n+}$ is defined by: $K_{n+}^* = \{y \mid y^T x \geq 0 \text{ for } x \in K_{n+}\}$

Yet $y^T x = \sum_{i=1}^n y_i x_i = (x_1 - x_2) y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)$

$+ \dots + (x_{n-1} - x_n)(y_1 + \dots + y_{n-1}) + x_n (y_1 + \dots + y_n)$

which can be easily shown by recurrence.

This way:

$y \in K_{n+}^* \iff \forall x \in K_{n+} \quad y^T x \geq 0$

$\iff \forall x \in \mathbb{R}^n, \; x_1 \geq x_2 \geq \dots \geq x_n \geq 0 \; , \; \sum_{i=1}^n x_i y_i \geq 0$

$\iff \begin{cases} y_1 \geq 0 \\ y_1 + y_2 \geq 0 \\ \vdots \\ y_1 + \dots + y_n \geq 0 \end{cases}$

(Indeed, if one of those inequalities was $< 0$, we could find $x \in \mathbb{R}^n$ with $x_1, \dots, x_n$ big enough so that $(x_i - x_{i+1})(y_1 + \dots + y_i)$ becomes bigger that the others factors and thus $y^T x < 0$)

$$\boxed{\text{We can conclude that } K_{n+}^* = \left\{ y \;\middle|\; \sum_{i=1}^{\ell} y_i \geq 0, \; \forall \ell \in [\![1, n]\!] \right\}}$$

## Ex 5:

1) $f^*(y) = \sup_{x \in \mathbb{R}^n} \left( y^T x - f(x) \right) = \sup_{x \in \mathbb{R}^n} \left( y^T x - \max_{i \in [\![1,n]\!]} x_i \right)$

- Lets first suppose that $\exists i \in [\![1,n]\!] \mid y_i < 0$

Then, by taking $x = \begin{pmatrix} 0 \\ \vdots \\ -\alpha \\ \vdots \\ 0 \end{pmatrix}$ $i^{th}$ row with $\alpha > 0$, we get $y^T x = -\alpha y_i$

Yet $-\alpha y_i \xrightarrow[\alpha \to +\infty]{} +\infty$ and then $\sup_{x \in \mathbb{R}^n} \left( y^T x - \max_{i \in [\![1,n]\!]} x_i \right) = +\infty$

and thus $f^*(y)$ would not be defined.

We can deduce that $\operatorname{dom} f^* = \{ y \in \mathbb{R}^n \mid \forall i \in [\![1,n]\!] \; y_i \geq 0 \}$

- Secondly, lets take $y \in \mathbb{R}^n$ with $\forall i \in [\![1,n]\!]$ $y_i \geq 0$.

  * if $\sum_{i=1}^n y_i > 1$ and by taking $x = \begin{pmatrix} \alpha \\ \vdots \\ \alpha \end{pmatrix}$, we get:

$$y^T x - \max x_i = \sum_{i=1}^n \alpha y_i - \alpha$$

$$= \alpha \left( \sum_{i=1}^n y_i - 1 \right) \xrightarrow[\alpha \to +\infty]{} +\infty$$ making $f^*(y)$ not defined!

  * if $\sum_{i=1}^n y_i < 1$ and by taking $x = -\begin{pmatrix} \alpha \\ \vdots \\ \alpha \end{pmatrix}$, we get:

$$y^T x - \max x_i = -\sum_{i=1}^n \alpha y_i + \alpha$$

$$= \alpha \left( 1 - \sum_{i=1}^n y_i \right) \xrightarrow[\alpha \to +\infty]{} +\infty$$ making $f^*(y)$ not defined!

* if $\sum_{i=1}^{n} y_i = 1$, we have $y^T x = \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} (\max x_i)\, y_i = \max x_i$

and this way: $\forall x \in \mathbb{R}^n,\ y^T x - \max x_i \leq 0$

for $x = 0$ $\quad y^T x - \max x_i = 0$ $\quad \Rightarrow \sup_{x \in \mathbb{R}} (y^T x - \max x_i) = 0$

We can conclude that:

$$f^*(y) = \begin{cases} 0 & \text{if } \begin{cases} \forall i \in [\![1, n]\!]\ y_i \geq 0 \\ \sum_{i=1}^{n} y_i = 1 \end{cases} \\ +\infty & \text{otherwise} \end{cases}$$

2) Same as in previous question, • let's suppose $\exists i \in [\![1, n]\!] \mid y_i < 0$

Then by taking $x = \begin{pmatrix} 0 \\ \vdots \\ -\alpha \\ \vdots \\ 0 \end{pmatrix}$ $i^{\text{th}}$ row, with $\alpha > 0$

we get $y^T x - f(x) = -\alpha y_i \xrightarrow[\alpha \to +\infty]{} +\infty$

• let's now suppose $\exists i \in [\![1, n]\!] \mid y_i > 1$

Then by taking $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\alpha > 0$

we get $y^T x - f(x) = \alpha y_i - \alpha$

$= \alpha (y_i - 1) \xrightarrow[\alpha \to +\infty]{} +\infty$ making $f^*(y)$ not defined!!

• Finally, let's suppose $\sum_{i=1}^{n} y_i \neq 1$, then by taking $x = \begin{pmatrix} \alpha \\ \vdots \\ \alpha \end{pmatrix}$, $\alpha > 0$

we get $y^T x - f(x) = \sum_{i=1}^{n} \alpha y_i - \alpha \cdot 1$

$= \alpha \left( \sum_{i=1}^{n} y_i - 1 \right) \longrightarrow +\infty$

$\alpha \to +\infty$ if $\sum_{i=1}^{n} y_i > 1$

$\alpha \to -\infty$ if $\sum_{i=1}^{n} y_i < 1$

Now if $\sum_{i=1}^{n} y_i = r$ and $\forall i \in [\![1, n]\!]$ $\quad 0 \leq y_i \leq 1$

then $\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i$

$$\leq \sum_{i=1}^{n} x_{[i]} = f(x)$$

thus $\forall x \in \mathbb{R}^n$ $\quad y^T x - f(x) \leq 0$

for $x = 0$ $\quad y^T x - f(x) = 0$

$$\Longrightarrow \sup_{x \in \mathbb{R}^n} \left( y^T x - f(x) \right) = 0$$

We can conclude that:

$$\boxed{f^*(y) = \begin{cases} 0 & \text{if } \begin{cases} \forall i \in [\![1, n]\!] \quad 0 \leq y_i \leq 1 \\ \sum_{i=1}^{n} y_i = r \end{cases} \\ +\infty & \text{otherwise} \end{cases}}$$