Ex 1:

1-

Let's first form the Lagrangian: $L(x,\lambda,\nu) = c^T x - \lambda^T x + \nu^T (Ax - b)$
$$= (A^T\nu + c - \lambda)^T x - \nu^T b$$

Dual function: $g(\lambda,\nu) = \inf_{x \in \mathbb{R}^d} \left(L(x,\lambda,\nu) \right)$

This way,

$$g(\lambda,\nu) = \begin{cases} -\nu^T b & \text{if } A^T\nu + c - \lambda = 0 \text{ (null vector)} \\ -\infty & \text{otherwise} \end{cases}$$

We then have the dual problem:
$$\max_{\nu} \; -\nu^T b$$
$$\text{subject to } A^T\nu + c - \lambda = 0, \; \lambda \succeq 0$$

Yet: $\begin{cases} A^T\nu + c - \lambda = 0 \\ \lambda \succeq 0 \end{cases} \iff -A^T\nu \preceq c$

The problem simplifies to:
$$\max_{\nu} \; -\nu^T b$$
$$\text{subject to } -A^T\nu \preceq c$$

Finally equivalent to:

$$\boxed{\begin{aligned} &\max_{\nu} \quad b^T\nu \\ &\text{subject to} \quad A^T\nu \succeq c \end{aligned} \quad (D)}$$

2- We have the Lagrangian: $L(y,\lambda) = -b^T y + \lambda^T(A^T y - c)$
$$= (A\lambda - b)^T y - \lambda^T c$$

Dual function: $g(\lambda) = \inf_{y \in \mathbb{R}^n} \left((A\lambda - b)^T y - \lambda^T c \right)$

$$= \begin{cases} -\lambda^T c & \text{if } A\lambda - b = 0 \text{ (null vector)} \\ -\infty & \text{otherwise} \end{cases}$$

The dual problem is therefore:

$$\max \; -\lambda^T c \quad \text{subject to } A\lambda - b = 0, \; \lambda \geq 0 \iff \boxed{\min_\lambda \; c^T\lambda \quad \text{subject to } A\lambda = b, \; \lambda \geq 0 \qquad (P)}$$

3- Lagrangian: $L(x,y,\lambda_1,\lambda_2,\nu) = c^T x - b^T y - \lambda_1^T x + \lambda_2^T(A^T y - c) + \nu^T(Ax - b)$
with $(x,y,\lambda_1,\lambda_2,\nu) \in \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$

Dual function:

$$g(\lambda_1,\lambda_2,\nu) = \inf_{(x,y)\in\mathbb{R}^d \times \mathbb{R}^n} \left((A^T\nu + c - \lambda_1)^T x + (A\lambda_2 - b)^T y - \lambda_2^T c - \nu^T b \right)$$

$$= \begin{cases} -\lambda_2^T c - \nu^T b & \text{if } \begin{cases} A^T\nu + c - \lambda_1 = 0 \\ A\lambda_2 - b = 0 \end{cases} \\ -\infty & \text{otherwise} \end{cases}$$

We then have the dual problem:
$$\max_{\lambda_2,\nu} \; -\lambda_2^T c - \nu^T b$$
$$\text{subject to } \begin{cases} A^T\nu + c - \lambda_1 = 0, \\ A\lambda_2 - b = 0, \\ \lambda_1 \geq 0, \; \lambda_2 \geq 0 \end{cases}$$

Yet $\begin{cases} A^T\nu + c - \lambda_1 = 0 \\ \lambda_1 \geq 0 \end{cases} \iff A^T\nu + c \geq 0$

$\iff A^T(-\nu) \leq c$

And $\left(\max_{\lambda_2, \nu} -\lambda_2^T c - \nu^T b \right) \iff \left(\min_{\lambda_2, -\nu} c^T \lambda_2 - b^T(-\nu) \right)$

Thus, with $\begin{cases} x = \lambda_2 \\ y = -\nu \end{cases}$ the dual problem can be written:

$$\min_{x, y} \quad c^T x - b^T y$$
$$\text{subject to} \quad Ax = b, \quad A^T y \leq c, \quad x \geq 0$$

We can finally conclude that <u>the problem is self-dual.</u>

4 - (Self-Dual) is feasible. This way there exist x, y such that $\begin{cases} Ax = b \\ x \geq 0 \end{cases}$ and $A^T y \leq c$.
Therefore (P) and (D) are both feasible.
Moreover (Self-Dual) is bounded, so are (P) and (D) (otherwise, by fixing x, resp. y, we could have (Self-Dual) unbounded below).
Let $E = \{ c^T x - b^T y \mid Ax = b,\ A^T y \leq c,\ x \geq 0 \}$

$$E = \{ c^T x + (-b^T y) \mid x \in \{ x \in \mathbb{R} \mid Ax = b,\ x \geq 0 \},\ y \in \{ y \in \mathbb{R} \mid A^T y \leq c \} \}$$
$$= E_1 + E_2 \quad \text{with} \quad E_1 = \{ c^T x \mid x \in \{ x \in \mathbb{R} \mid Ax = b,\ x \geq 0 \} \}$$
$$E_2 = \{ -b^T y \mid x \in \{ x \in \mathbb{R} \mid A^T y \leq c \} \}$$

Because E, E_1, E_2 are feasible and bounded, $\min(E)$, $\min(E_1)$, $\min(E_2)$ exist and $\min(E_1 + E_2) = \min(E_1) + \min(E_2)$.
Therefore solving (P) gives $\min(E_1) = p_1^*$ for optimal value x^*
solving (D) gives $\min(E_2) = p_2^*$ for optimal value y^*
and $\min(E_1 + E_2) = p^* = p_1^* + p_2^*$ for optimal value $(x^*; y^*)^T$.

$c^T x^* - b^T y^*$

$c^T x = b^T y$

Then, $Ax = b$ is affine and $x \longmapsto -x$ is convex thus (P) is convex.

Moreover, $x \neq 0$ (null vector) because

We have shown that (P) is feasible and the feasible point is not the null vector otherwise $b = Ax = 0$ that would be problem.

This way (P) is strictly feasible and (P) is strongly dual:

$$p_1^* = d_1^*$$
$$c^T x^* = b^T y^*$$

Let $p^* = p_1^* + p_2^* = c^T x^* - b^T y^* = 0$ $\boxed{\text{ccl: } p^* = 0}$

Ex 2:

1- The conjugate is defined by : $\|y\|_1^* = \sup_x (y^T x - \|x\|_1)$

We distinguish two cases :

* if $\|y\|_{1,*} = \sup_{\|x\|_1 \leq 1} x^T y \leq 1$ then the $\left(\frac{x}{\|x\|_1}\right)^T y \leq 1$

$$x^T y \leq \|x\|_1$$

ie $y^T x - \|x\|_1 \leq 0$

Thus way $\sup_x (y^T x - \|x\|_1) \leq 0$ with equality for $x = 0$

* if $\|y\|_{1,*} > 1$ ie $\sup_{\|u\| \leq 1} u^T y > 1$

then, $\exists x \ \|x\|_1 \leq 1$, $x^T y > 1$

this way $x^T y - \|x\|_1 = y^T x - \|x\|_1 > 0$

Let's now take $u = tx$, $t > 0$

We have $y^T u - \|u\|_1 = t(y^T x - \|x\|_1) \xrightarrow[t \to +\infty]{} +\infty$

And therefore $\sup [y^T x - \|x\|_1] = +\infty$

Finally, we can note that $\|y\|_{1,*} = \sup_{\|u\|_1 \leq 1} u^T y = \sup \sum u_i y_i$ with $\sum |u_i| \leq 1$

Yet $\sum u_i y_i \leq \sum |u_i| |y_i| \leq \max |y_i| \sum |u_i| = \max |y_i|$ and the value is reached by taking $u = \begin{pmatrix} \vdots \\ \text{signe of } y_j \\ \vdots \end{pmatrix}$ j-th line where $|y_j| = \max |y_i|$. This way $\|y\|_{1,*} = \|y\|_\infty$

Conclusion:

$$\boxed{\|y\|_1^* = \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}}$$

2- $\min \|Ax - b\|_2^2 + \|x\|_1$ is equivalent to: $\min \|z - b\|_2^2 + \|x\|_1$

subject to $Ax - z = 0$

We hence have the lagrangian: $L(x, z, \nu) = \|z - b\|_2^2 + \|x\|_1 + \nu^T(Ax - z)$

And the lagrange dual function:

$$g(\nu) = \min_{x, z} L(x, z, \nu)$$

$$= \min_x \left(\|x\|_1 + (A^T\nu)^T x \right) + \min_z \left(\|z - b\|_2^2 - \nu^T z \right)$$

Yet $\min_x \left(\|x\|_1 + (A^T\nu)^T x \right) = \min_x \left(\|-x\|_1 - (A^T\nu)^T(-x) \right)$

$$= \min_x \left(\|x\|_1 - (A^T\nu)^T x \right)$$

$$= -\max_x \left((A^T\nu)^T x - \|x\|_1 \right)$$

$$= -\|A^T\nu\|_1^*$$

$$= \begin{cases} 0 & \text{if } \|A^T\nu\|_\infty \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

Besides, $\min_z \|z - b\|_2^2 - \nu^T z$ is convex

and $\nabla_z \left(\|z - b\|_2^2 - \nu^T z \right) = \nabla_z \left((z^T - b^T)(z - b) - \nu^T b \right)$

$$= \nabla_z \left(z^T z - b^T z - z^T b + b^T b - \nu^T b \right)$$

$$= 2z - 2b - \nu$$

$$\nabla_z \left(\|z - b\|_2^2 - \nu^T z \right) = 0 \iff z = b + \frac{1}{2}\nu$$

And $\min_z \|z - b\|_2^2 - \nu^T z = \frac{1}{4}\|\nu\|_2^2 - \frac{1}{2}\|\nu\|_2^2 - \nu^T b = -\frac{1}{4}\|\nu\|_2^2 - \nu^T b$

This way, the dual problem can be written:

$$\max_{\nu} \; -\frac{1}{4}\|\nu\|_2^2 - \nu^T b$$

$$\text{subject to} \quad \|A^T\nu\|_{1,*} \leq 1$$

Which is equivalent to:

$$\boxed{\begin{aligned} &\min_{\nu} \; \frac{1}{4}\|\nu\|_2^2 + \nu^T b \\ &\text{subject to} \quad \|A^T\nu\|_{\infty} \leq 1 \end{aligned}}$$

Ex 3:

1 -

Let's first show that if $\tau < 0$, (Sep. 1) and (Sep. 2) are unbounded below.

* (Sep 1): Let's take $w = \begin{pmatrix} t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then:

$$\frac{1}{n}\sum_{i=1}^{n} \max(0; 1 - y_i(w^T x_i)) + \frac{\tau}{2}\|w\|_2^2 = \underbrace{\frac{1}{n}\sum_{\substack{\text{s.t.} \\ y_i t x_{i,1} < 1}} (1 - y_i t x_{i,1}) + \frac{\tau}{2} t^2}_{S}$$

and $S \xrightarrow[t \to +\infty]{} -\infty$

This way, if $\tau < 0$, $\min_{w} \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}(w, x_i, y_i) + \frac{\tau}{2}\|w\|_2^2 = -\infty$

* (Sep 2): Let's take $w = 0$ (null vector) and $\xi = \begin{pmatrix} t \\ \vdots \\ t \end{pmatrix}$

Then for $t > 1$, ξ becomes feasible.

Moreover $\frac{1}{n\tau} 1^T \xi + \frac{1}{2}\|w\|_2^2 = \frac{1}{\tau} t \xrightarrow[t \to +\infty]{} -\infty$

This way, $\min_{\xi, w} \frac{1}{n\tau} 1^T \xi + \frac{1}{2}\|w\|_2^2 = -\infty$

We can assume that $\tau > 0$ (the case $\tau = 0$ will be treated at the end)

This way, $\left(\begin{array}{ll} \min_{w, \xi} & \frac{1}{n\tau} 1^T \xi + \frac{1}{2}\|w\|_2^2 \\ \text{s.t.} & \xi_i \geq 1 - y_i(w^T x_i) \\ & \xi_i \geq 0 \end{array} \right) \iff \left(\begin{array}{ll} \min_{w, \xi} & \frac{1}{n} 1^T \xi + \frac{\tau}{2}\|w\|_2^2 \\ \text{s.t.} & \xi_i \geq 1 - y_i(w^T x_i) \\ & \xi_i \geq 0 \end{array} \right)$

Then, $\begin{cases} \xi_i \geq 1 - y_i (w^T x_i) \\ \xi_i \geq 0 \end{cases} \iff \xi_i \geq \max(0 ; 1 - y_i (w^T x_i))$

Let's now write $\left| \begin{array}{l} w_1^* \text{ and } p_1^* \text{ the optimal point and value of} \\ \xi_2^*, w_2^* \text{ and } p_2^* \end{array} \right| \begin{array}{l} \text{Sep. 1} \\ \text{Sep. 2} \end{array}$

and $f_1(w) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(w, x_i, y_i) + \frac{\tau}{2} \|w\|_2^2$ thus $p_1^* = f_1(w_1^*)$

$f_2(w, \xi) = \frac{1}{n} 1^T \xi + \frac{\tau}{2} \|w\|_2^2$ thus $p_2^* = f_2(w_2^*, \xi_2^*)$

$\forall i \quad \xi_{2,i}^* \geq \max(0; 1 - y_i (w_2^{*T}) x_i)$ so $p_2^* \geq f_1(w_2^*)$ (1)

Now, let ξ' so that $\xi_i' = \max(0; 1 - y_i (w_1^{*T}) x_i)$

This way $f_1(w_1^*) = f_2(\xi', w_1^*) \geq p_2^*$ (2) by definition of p_2^*

(1) and (2) give $f_1(w_1^*) \geq f_1(w_2^*)$

Besides, by definition of w_1^* : $f_1(w_2^*) \geq f_1(w_1^*)$

We can conclude that $f_1(w_1^*) = f_1(w_2^*)$

ie $p_1^* = f_1(w_2^*)$

Finally, solving (Sep. 2) gives w_2^* which is also an optimal point for (Sep 1).

2- Lagrangian:

$$\mathcal{L}(\xi, w, \lambda, \pi) = \frac{1}{n\tau} \mathbb{1}^T \xi + \frac{1}{2}\|w\|_2^2 + \sum_{i=1}^{n} \lambda_i (1 - y_i(w^T x_i) - \xi_i) - \pi^T \xi \quad \text{with } \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$= \left(\frac{1}{n\tau} - \pi - \lambda\right)^T \xi + \sum_{i=1}^{n} \lambda_i (1 - y_i(w^T x_i)) + \frac{1}{2}\|w\|_2^2$$

and the lagrange dual function:

$$g(\lambda, \pi) = \inf_{\xi, w} \left(\mathcal{L}(\xi, w, \lambda, \pi)\right)$$

$$= \inf_{w} \left(\inf_{\xi} \left(\left(\frac{1}{n\tau} - \pi - \lambda\right)^T \xi + \sum_{i=1}^{n} \lambda_i (1 - y_i(w^T x_i)) + \frac{1}{2}\|w\|_2^2 \right)\right)$$

$$= \inf_{w} \left(\sum_{i=1}^{n} \lambda_i (1 - y_i(w^T x_i)) + \frac{1}{2}\|w\|_2^2 + \inf_{\xi} \left(\left(\frac{1}{n\tau} - \pi - \lambda\right)^T \xi\right)\right)$$

$$= \lambda^T \mathbb{1} + \inf_{w} \Big(\underbrace{\frac{1}{2}\|w\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (w^T x_i)}_{f_1(w)} \Big) + \inf_{\xi} \Big(\underbrace{\left(\frac{1}{n\tau} - \pi - \lambda\right)^T \xi}_{f_2(\xi)} \Big)$$

* $w \mapsto f_1(w)$ is quadratic thus convex.

$$\nabla_w f_1(w) = w - \sum_{i=1}^{n} \lambda_i y_i x_i \quad \text{and} \quad \nabla_w f_1(w) = 0 \iff w = \sum_{i=1}^{n} \lambda_i y_i x_i$$

This way $\inf_{w} (f_1(w)) = \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i \left(\sum_{j=1}^{n} \lambda_j y_j x_j\right)^T x_i$

$$= \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 - \| \lambda_i y_i x_i \|_2^2$$

$$= -\frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2$$

* $\xi \mapsto f_2(\xi)$ is linear thus: $\inf_{\xi} (f_2(\xi)) = \begin{cases} 0 & \text{if } \frac{1}{n\tau} - \pi - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$

Therefore $g(\lambda, \pi) = \begin{cases} \mathbb{1}^T \lambda - \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 & \text{if } \frac{1}{n\tau} - \pi - \lambda = 0 \\ -\infty & \text{otherwise.} \end{cases}$

And the dual problem:

$$\max_{\lambda, \pi} \quad 1^T \lambda - \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2$$

$$\text{subject to} \quad \frac{1}{n\tau} - \pi - \lambda = 0$$

$$\pi \geq 0$$

$$\lambda \geq 0$$

The objective function does not depend on π, the problem can then be written:

$$\boxed{\begin{aligned} &\max_{\lambda} \quad 1^T \lambda - \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 \\ &\text{subject to} \quad \lambda \geq 0 \\ &\qquad\qquad\quad \frac{1}{n\tau} \geq \lambda \end{aligned}}$$

Ex. 4:

For a given x, $\sup_{a \in P} a^T x$ is the optimal value of:

$$\max_a \ x a^T \qquad (P_1)$$
$$\text{subject to} \quad C^T a \leq d$$

In the first exercise 1 of this homework, we have shown that such a problem has the dual:

$$\min_z \ d^T z \qquad (P_2)$$
$$\text{subject} \quad Cz = x, \ z \geq 0$$

Yet we also shown in exercise 1 that $(P) = (P_2)$ is strongly dual, hence (P_1) is also strongly dual and, by noting $\begin{vmatrix} p_1^* \\ p_2^* \end{vmatrix}$ the optimal value of $\begin{vmatrix} (P_1) \\ (P_2) \end{vmatrix}$,

$p_1^* = p_2^*$ and the robust LP can be written:
$$\min \ c^T x$$
$$\text{subject to} \ p_1^* \leq b$$

$$\iff \min \ c^T x$$
$$\text{s.t} \quad p_2^* \leq b$$

$$\iff \min \ c^T x$$
$$\text{s.t.} \quad \forall z \in \{z \mid Cz = x, \ z \geq 0\}, \ d^T z \leq b$$

$$\boxed{\begin{aligned} \iff \min \quad & c^T x \\ \text{s.t} \quad & d^T z \leq b \\ & Cz = x \\ & z \geq 0 \end{aligned}}$$

Ex 5

1- Lagrangian: $L(x, \lambda, \nu) = c^T x + \lambda^T (Ax - b) + \sum_{i=1}^{n} \nu_i x_i (1 - x_i)$

$$= c^T x + \lambda^T (Ax - b) + \nu^T x - x^T \mathrm{diag}(\nu) x$$

$$= - x^T \mathrm{diag}(\nu) x + (c + A^T \lambda + \nu)^T x - \lambda^T b$$

where $\mathrm{diag}(\nu) = \begin{pmatrix} \nu_1 & & 0 \\ & \nu_2 & \\ 0 & & \ddots \; \nu_n \end{pmatrix}$

for convenience, let's change ν to $-\nu$, we have,

$$L(x, \lambda, \nu) = x^T \mathrm{diag}(\nu) x + (c + A^T \lambda - \nu)^T x - b^T \lambda$$

L is convex iff $P \succeq 0$

iff $\nu \geq 0$

and in the case of $\nu \geq 0$, $\nabla_x L(x, \lambda, \nu) = 2 \,\mathrm{diag}(\nu) x + c + A^T \lambda - \nu$

$\nabla_x L = 0 \iff x = \frac{1}{2} \mathrm{diag}\left(\frac{1}{\nu}\right) (\nu - A^T \lambda - c)$

this way $g(\lambda, \nu) = \frac{1}{4} \mathrm{diag}\left(\frac{1}{\nu}\right) \|\nu - A^T \lambda - c\|_2^2 - \frac{1}{2} \mathrm{diag}\left(\frac{1}{\nu}\right) \|\nu - A^T \lambda - c\|_2^2 - b^T \lambda$

$$= -\frac{1}{4} \mathrm{diag}\left(\frac{1}{\nu}\right) \|\nu - A^T \lambda - c\|_2^2 - b^T \lambda$$

where $\mathrm{diag}\left(\frac{1}{\nu}\right) = \begin{pmatrix} \frac{1}{\nu_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\nu_n} \end{pmatrix}$

This way $g(\lambda, \nu) = \begin{cases} -\frac{1}{4} \mathrm{diag}\left(\frac{1}{\nu}\right) \|\nu - A^T \lambda - c\|_2^2 - b^T \lambda & \text{if } \nu \succeq 0 \\ -\infty & \end{cases}$

The dual is thus:

$$\max_{\lambda, \nu} \; -\frac{1}{4} \mathrm{diag}\left(\frac{1}{\nu}\right) \|\nu - A^T \lambda - c\|_2^2 - b^T \lambda$$

subject to $\lambda \succeq 0, \; \nu \succeq 0$

Yet $\underbrace{-\frac{1}{4}\,\mathrm{diag}\left(\frac{1}{\nu}\right)\|\nu - A^T\lambda - c\|}_{f_0(\nu)} = -\frac{1}{4}\sum_{i=1}^{n}\frac{(c_i + a_i^T\lambda_i - \nu_i)^2}{\nu_i}$ where a_i is the ith column of A.

With λ fixed, we can max. such an objective function by maximizing each of the

sum:
$$\max_{\nu} f_0(\nu) = \frac{1}{4}\sum_{i=1}^{n}\sup\left(-\frac{(c_i + a_i^T\lambda - \nu_i)^2}{\nu_i}\right)$$
$$= \sum_{i=1}^{n}\min(0;\, c_i + a_i^T\lambda)$$

This way, the dual problem can be written:

$$\boxed{\begin{aligned}&\max_{\lambda}\ \sum_{i=1}^{n}\min(0;\, c_i + a_i^T\lambda) - b^T\lambda\\ &\text{subject to}\quad \lambda \geq 0\end{aligned}}$$

2- For the LP relaxation, we have the Lagrangian:
$$L(x, \lambda_1, \lambda_2, \lambda_3) = c^T x + \lambda_1^T(Ax - b) - \lambda_2^T x + \lambda_3^T(x - 1)$$
$$= (c + A^T\lambda_1 - \lambda_2 + \lambda_3)^T x - b^T\lambda_1 - 1^T\lambda_3$$

Lagrange dual function:
$$g(\lambda_1, \lambda_2, \lambda_3) = \inf_{x}\left(L(x, \lambda_1, \lambda_2, \lambda_3)\right)$$
$$= \begin{cases} -b^T\lambda_1 - 1^T\lambda_3 & \text{if } c + A^T\lambda_1 - \lambda_2 + \lambda_3 = 0\\ -\infty & \text{otherwise}\end{cases}$$

Hence the dual problem:
$$\max_{\lambda_1, \lambda_3} -b^T\lambda_1 - 1^T\lambda_3$$
$$\text{subject to}\quad c + A^T\lambda_1 + \lambda_3 \geq 0$$
$$\lambda_1 \geq 0,\ \lambda_3 \geq 0$$

Yet let's write: $f_1(\lambda) = -b^T\lambda + \sum_{i=1}^{n} \min(0; c_i + a_i^T\lambda)$

$$f_2(\lambda_1, \lambda_3) = -b^T\lambda_1 - 1^T\lambda_3$$

and optimal points and values: $f_1(\lambda^*) = p_1^*$

$$f_2(\lambda_1^*, \lambda_3^*) = p_2^*$$

$\forall i \begin{cases} -\lambda_{3,i}^* \leq 0 \\ -\lambda_{3,i}^* \leq c^T + a_i^T\lambda_1^* \end{cases} \Rightarrow \forall i \quad -\lambda_{3,i}^* \leq \min(0; c^T + a_i^T\lambda_1^*)$

$$\Rightarrow \quad -1^T\lambda_3^* \leq \sum_{i=1}^{n} \min(0; c^T + a_i^T\lambda_1^*)$$

and then $-b^T\lambda_1^* - 1^T\lambda_3^* \leq -b^T\lambda_1^* + \sum_{i=1}^{n} \min(0; c^T + a_i^T\lambda_1^*)$

$$p_2^* \leq f_1(\lambda_1^*)$$

Yet $f_1(\lambda_1^*) \leq p_1^*$ by def of p_1^* thus $p_2^* \leq p_1^*$ (1)

Moreover,

$$p_1^* = -b^T\lambda^* - \sum_{i=1}^{n} -\min(0; c^T + a_i^T\lambda^*)$$

$$= -b^T\lambda^* - 1^T\lambda' \quad \text{with} \quad \lambda'_i = -\min(0; c^T + a_i^T\lambda_1^*)$$

$$= f_2(\lambda^*; \lambda') \leq p_2^* \text{ by def. of } p_2^*$$

This way $p_1^* \leq p_2^*$ (2)

By (1) and (2) we conclude $p_1^* = p_2^*$: the lower bounds via LP relaxation and lagrangian relaxation are the same.