Oscillations

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Abstract

This experiment answers the question: What is the relation between the damping factor and the resonance frequency in a damped driven oscillator system? The natural frequency of an undamped oscillator is measured, damping is introduced by a magnet and a driving force is supplied by a rotary motor. The amplitude and phase are recorded at different driving frequencies and used to construct amplitude-frequency and phase-frequency graphs. The hypothesis that is tested is the relation between the damping factor and resonance frequency $\omega_{res} = \sqrt{\omega^2 - 2\gamma^2}$.

Contents

1 Introduction

A damped driven oscillatory system has a peak amplitude and the phase abruptly breaks at a specific resonance frequency. This resonance frequency depends on intrinsic constant natural angular frequency of the oscillations, and the damping factor. Because driven damped oscillatory systems are prevalent in modern physical and chemical models and characteristic resonance frequency successfully explains many physical and chemical phenomena, relation between the resonance frequency and the damping factor is essential in research. Therefore, this report aims to find the relation between the damping factor and the resonance frequency in such a system. The hypothesis of this report is the mathematical relation $\omega_{res} = \sqrt{\omega^2 - 2\gamma^2}$ which is used to compare damping factors of the same system found in two distinct ways, thus providing emperical grounds to the hypothesis. It also discusses the sources of errors, and relevant improvements needed to make more accurate numerical support for the hypothesis.

2 Theory

In a closed damped driven oscillatory system, three forces are acting on a point mass: restoring force ($F_{restoring} = -kx$), damping force ($F_{damping} = -b\dot{x}$), and driving force ($F_{driving} = F_0 \cos(\omega_d t)$). By the Second Newton's law, resulting motion is modelled by:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_d t)$$

More commonly, this differential equation is written as

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = \frac{F_0}{m}\cos(\omega_d t) \tag{1}$$

where $\gamma = \frac{b}{2m}$ is the damping factor, $\omega = \sqrt{\frac{k}{m}}$ is natural angular frequency of the oscillations.

Depending on the value of $\Omega^2 = \gamma^2 - \omega^2$, there are three cases to consider: **underdamping**, **overdamping**, and **critical damping**. In this experiment, the damping factor was chosen in a way that the oscillatory system is underdamped ($\Omega^2 < 0$). The homogeneous solution to Eq. (**??**) becomes:

$$x(t) = e^{-\gamma t}\cos(\omega t + \phi) \tag{2}$$

Thus, after long enough time, the initial natural oscillations will die out, and the system will oscillate with some amplitude A at the driving frequency ω_d but shifted by some phase ϕ. The amplitude can be found to be:

$$A = \frac{F_d/m}{\sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}} \tag{3}$$

The phase is:

$$\tan(\phi) = \frac{2\gamma\omega_d}{\omega^2 - \omega_d^2} \tag{4}$$

There is one frequency when the amplitude from Eq. (**??**) is maximized. This particular frequency is called **resonance frequency** (ω_{res}). The behavior of the damped driven oscillatory system at resonance frequency is further addressed in appendix **??**.

In this experiment, the system is modelled using above equations since the restoring force comes from the springs, driven force is harmonic, and the damping force depends on the change of magnetic flux, which in our case depends on the speed. The only difference is that displacements ($x(t)$) and speeds are angular.

3 Methods

The main objective of this experiment was to investigate the oscillatory motion of a damped harmonic oscillator that was driven by an electric motor. The experimental setup depicted in Figure 1 consists of an oscillation disc at the top, that is equipped with a rotation sensor to measure the angular displacement. A magnet that adds a damping term is located next to the disc. An electric motor at the bottom of the setup delivers a driving force. A second rotary motion sensor is located above the motor to measure the frequency and phase of the driving force.

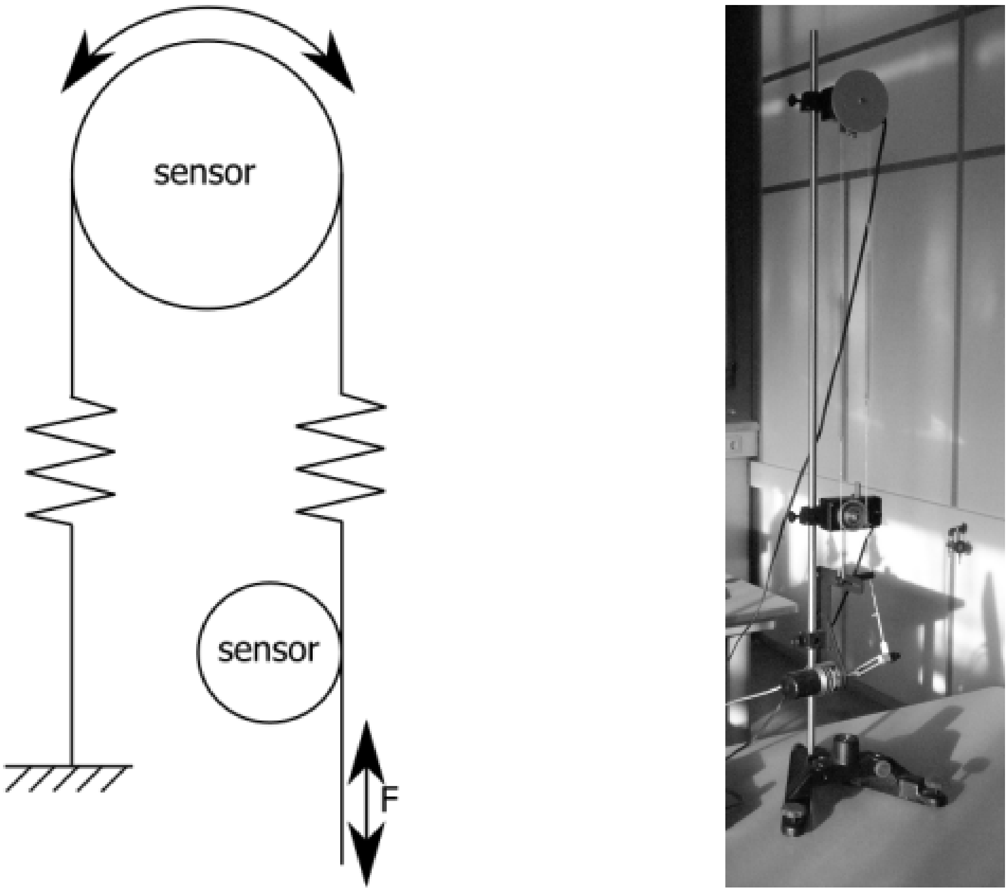

Figure 1: The experimental setup

During the first phase of the experiment, the goal was to determine the natural frequency of the oscillator. To accomplish this, the magnet was first removed and there was no driving force being supplied. Instead, the disc was manually rotated and its oscillations were recorded. After these measurements the magnet was reinstalled to introduce a damping term, the magnet's position was adjusted so the oscillations would be damped after about 10 periods.

In the second phase, a driving force was supplied by the electrical motor. First, a couple of measurements of the amplitude of the oscillations and the frequency of the driving force were taken to gauge the voltage range that is to be used in finding the peak amplitude. After this, two measurements were taken at seventeen different voltage levels. Most of these measurements were conducted around the resonance frequency to get a more precise determination of the peak amplitude.

4 Results

4.1 Direct calculation of the damping factor.

The oscillations produced without added damping factor (i.e. the magnet) are displayed in Figure ??. The assumption was made that the string to which the springs were attached did not slip from the pulley.

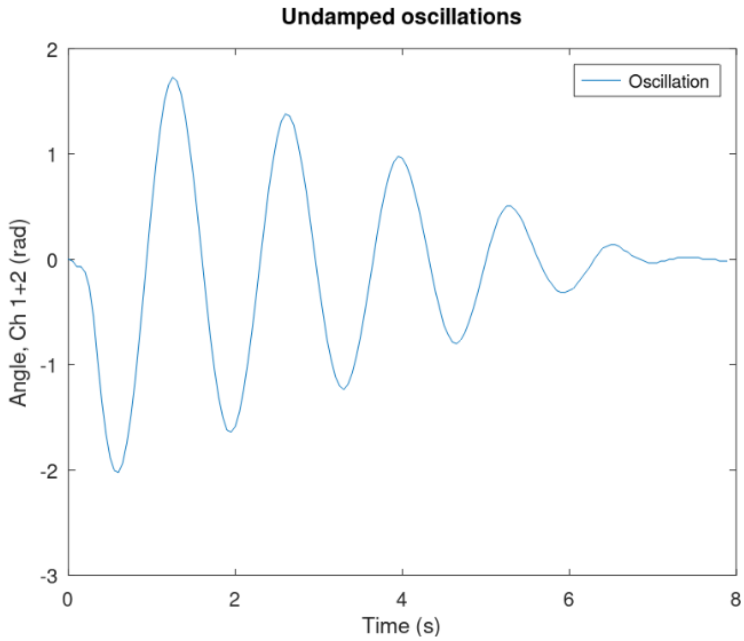

Figure 2: Oscillations without damping

From the measurements, the coordinates of the maxima were determined and from this the natural frequency was determined to be $\omega = 4.8 \pm 0.1 (rad/s)$, which was constant throughout the measurements.

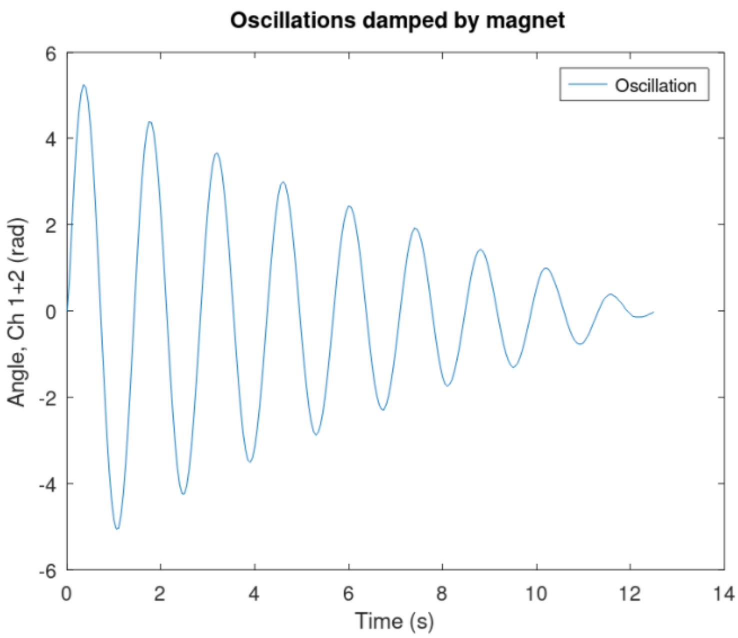

Figure 3: Decrease in amplitude by an added damping term

A damping term was added by placing the magnet back in its original position, the oscillations are displayed in Figure **??**. The coordinates of nine of the maxima were determined and were plotted in Figure **??**.

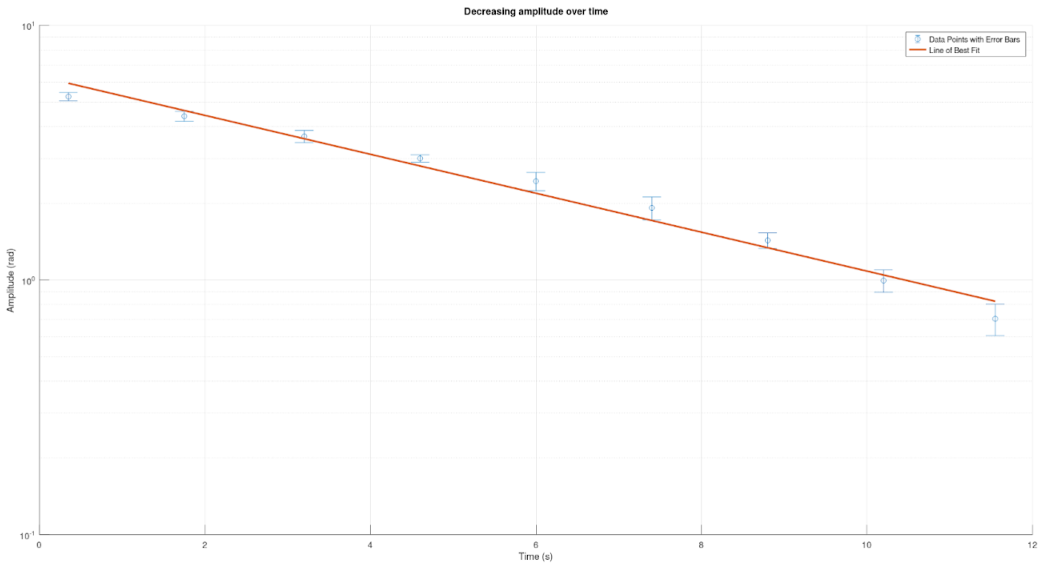

Figure 4: Semi-logaritmic plot of the amplitude over time

The displacement amplitude $x(t)$ decreases as shown in Eq. (**??**). The slope of the line of best-fit m is equal to the $-\gamma$ term in Eq. (**??**) ($|m| = \gamma$). By this a damping factor of $\gamma_{direct} = 1.3 \pm 0.2(rad/s)$ was determined.

4.2 Indirect calculation of the damping factor.

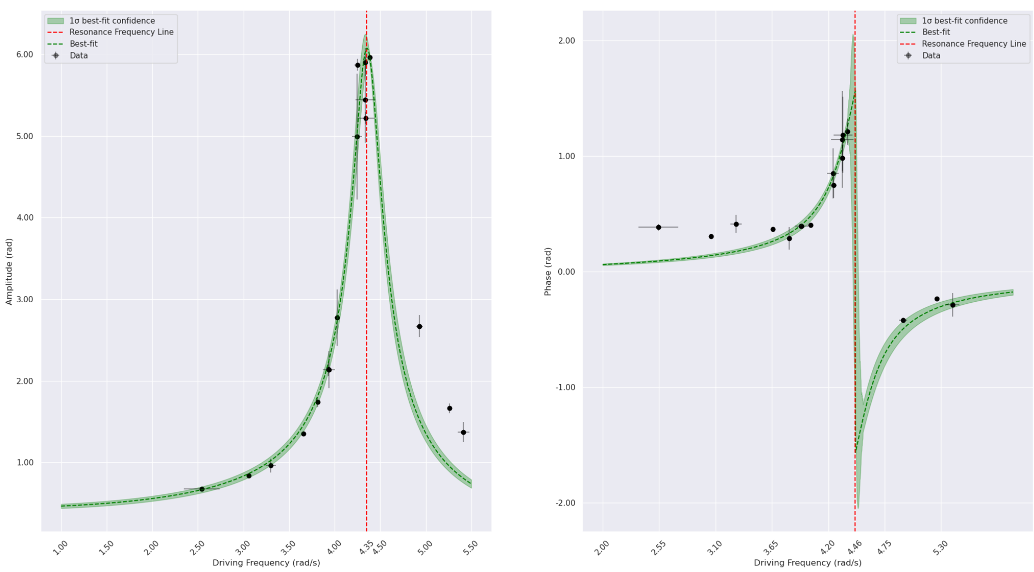

Figure 5: The amplitude and phase difference of the oscillations plotted against driving frequency.

The graph of the amplitude and phase difference of the oscillations against driving frequencies with best-fit curves and correspoding uncertanties are depicted in Figure.**??**. Tabular data with respective errors are depicted in Table **??** in appendix **??**.

Resonance frequency from the amplitude-frequency graph is estimated to be $\omega_{amplitude} = 4.35 \pm 0.09(rad/s)$. Resonance frequency from the phase-frequency graph is estimated to be $\omega_{phase} = 4.46 \pm 0.02(rad/s)$. Combined resonance frequency is

$$\omega_{res} = \frac{\omega_{amplitude} + \omega_{phase}}{2} \pm \sqrt{\Delta\omega^2_{amplitude} + \Delta\omega^2_{phase}} = 4.4 \pm 0.1(rad/s)$$

Damping factor is again evaluated from Eq. **??**.

$$\gamma_{indirect} = \sqrt{\frac{\omega^2 - \omega^2_{res}}{2}} = \sqrt{\frac{4.8^2 - 4.4^2}{2}} = 1.3 \pm 0.8(rad/s)$$

This damping factor matches the damping factor previously calculated, γ_{direct}.

Error analysis for the resonance frequencies and the damping factor is located in Appendix **??**.

5 Discussion

5.1 Interpretation

Figure **??** shows that experimental data from Table **??** follows the model given in Theory (section **??**) by Eq. (**??**) and Eq. (**??**). There is a frequency ($\langle\omega_{res}\rangle \approx 4.4(rad/s)$), at which the amplitude is maximized; whereas the phase abruptly changes from $\pi/2$ to $-\pi/2$. According to the Theory (section **??**), this particular frequency is called resonance frequency. Calculating this frequency from the two graphs yields $\omega_{amplitude} = 4.35 \pm 0.09(rad/s)$ and $\omega_{phase} = 4.46 \pm 0.02(rad/s)$ respectively. Averaging the result and calculating the error gives an actual resonance frequency of the system: $\omega_{res} = 4.4 \pm 0.1(rad/s)$.

According to the hypothesis, damping factor can be evaluated from Eq. (**??**): $\gamma_{indirect} = 1.32 \pm 0.8(rad/s)$. This value coincides with the damping factor calculated directly (γ_{direct}).

Therefore, our hypothesis is correct and the relation between the damping factor and the resonance frequency is indeed $\omega_{res} = \sqrt{\omega^2 - 2\gamma^2}$, as given by Eq. (**??**).

5.2 Errors and Improvements

In Figure **??**, some data points do not perfectly lie on the best-fit curves, or have large uncertainties. This behaviour has several sources.

Firstly, no-slipping assumption was made. However, during the experiment, the string slipped drastically for voltages greater than $4(V)$, which disrupted the expected oscillatory motion leading to the uncertainties and deviations from the best-fit curves.

Besides, the springs underwent circular motion around the axis of oscillatory motion. This angular motion prevented the string from perfectly staying on the rotary motor, causing less friction and, hence, more slipping.

Also, the experiment was performed on a table that was connected with other teams. Sometimes, the table was slightly distorted from its position, causing additional (usually, insignificant) influence on the oscillating system.

Lastly, the string was made of the material that is easily deformed by large enough forces. Measurements with higher voltages ($\geq 4(V)$) deformed the string, causing the system to deviate from the desired ideal oscillations.

Above mentioned experimental errors resulted in waveforms that are slightly assymetrical and that had tendency to shift away from the time axis. Besides, extrapolating data from those waveforms presented uncertainties, that are shown in green in Figure **??**. Phase was calculated by finding the maximum cross-correlation between the two waves (system oscillations and the driving force), which resulted in more uncertainties because of imperfect waves and insufficient accuracy of data obtained from motary sensors.

More accurate results could be obtained by following the next improvements. If the system was standing on a more stable base (table), no superfluous motion would disrupt the oscillatory motion. Also, rotary sensors can be replaced by more accurate devices. The strings and springs should ideally have only one degree of freedom: angular motion of the springs and strings should be restricted by using materials with a larger moment of inertia or adding low-friction walls around the strings, springs and the disk so that the slipping does not cause significant motion distortions.

6 Conclusions

To conclude, experimentally calculated damping factors, resonance frequency, and natural frequency followed the hypothesis. Future improvements of the setup are the performance of the experiment on a more stable base, usage of rotary sensors with a higher time-resolution and more precise measurements of displacements and derivatives. Slipping of the strings should be minimized by restricting the angular motion of the springs and strings.

A Preparatory Exercises

A.1 Question 1

A change in γ will cause a change in amplitude and a shift in resonance frequency. When γ increases, the amplitude will decrease, because the amplitude is inversely proportional to γ (see Eq. (**??**)). The peak will be shifted to the left, since an increase in γ causes a decrease in ω_{res}, as seen in Eq. (**??**). A decrease in γ will have an inverse effect, the amplitude will increase and the peak will shift to the right.

A.2 Question 2

The amplitude was given by Eq. (**??**). The resonance frequency occurs when the amplitude A is maximized, this occurs when the denominator in Eq. (**??**) is minimized. For this we take the derivative of the denominator with respect to ω_d and set it to zero. The resonance frequency is then determined to be:

$$\omega_{\text{res}} = \sqrt{\omega^2 - 2\gamma^2} \tag{5}$$

From this follows that $\omega_{\text{res}} = \omega$ when $\gamma = 0$, this occurs when no damping is taking place.

A.3 Question 3

Note, that ϕ in Eq. (**??**) can be re-written as follows:

$$\phi = \arctan(\frac{2\gamma\omega_d}{\omega^2 - \omega_d^2}) = \arctan(\gamma(\frac{1}{\omega - \omega_d} - \frac{1}{\omega + \omega_d}))$$

Then, taking both half-limits of ϕ as $\omega_d \to \omega$ gives discontinuity [1]:

$$\arctan \lim_{\omega_d \to \omega^+} \gamma(\frac{1}{\omega - \omega_d} - \frac{1}{\omega + \omega_d}) \asymp \arctan(-\infty) = -\frac{\pi}{2}$$

$$\arctan \lim_{\omega_d \to \omega^-} \gamma(\frac{1}{\omega - \omega_d} - \frac{1}{\omega + \omega_d}) \asymp \arctan(\infty) = \frac{\pi}{2}$$

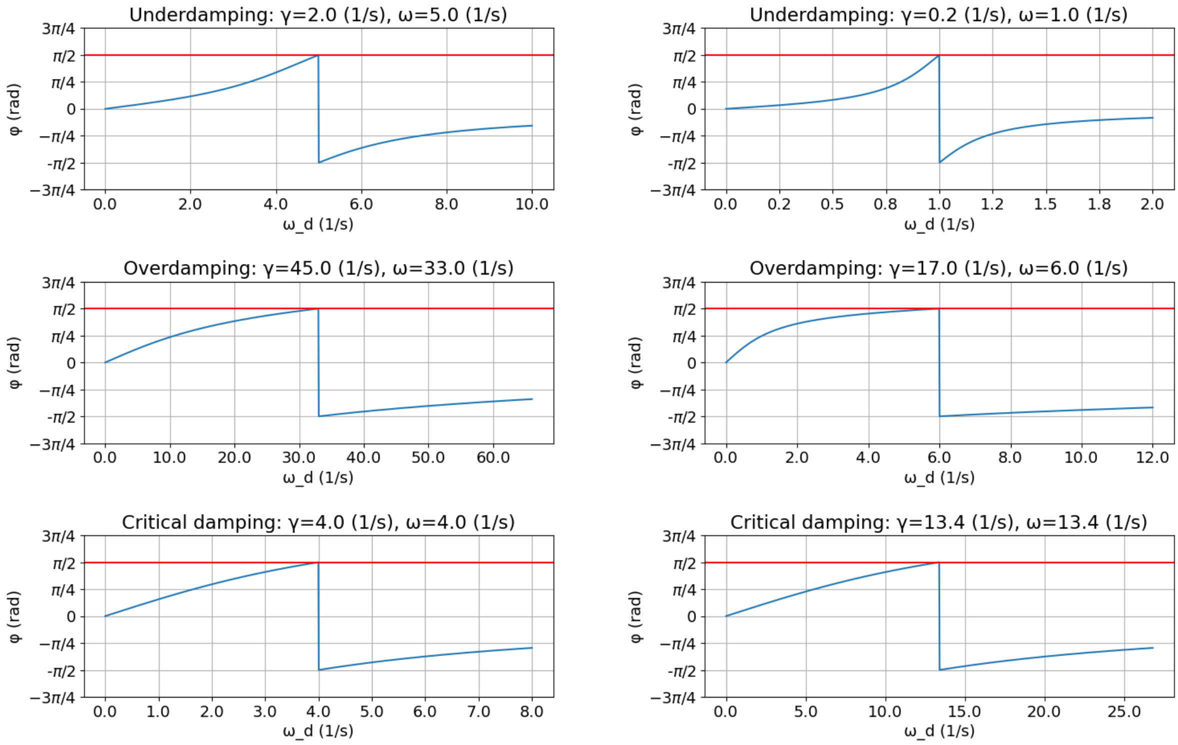

Figure 6: Plots of the phase ϕ for different values of the damping factor γ and natural frequency ω.

In Fig. **??**, this discontinuity is clear for different initial conditions (γ and ω). If the driving frequency ω_d approaches natural frequency ω from the negative side, the phase ϕ approaches $\pi/2$ radians. In other words, the driving force becomes more out of phase with the natural oscillations. Identical reasoning applies to the positive half limit.

B Tabular Data

Two sets of measurements were taken for each experiment. The resulting data was calculated by taking the mean, and the error by taking the half absolute difference.

[1] $arctan(\theta)$ is continuous on $(-\frac{\pi}{2}; \frac{\pi}{2})$, so the limit operator can be brought inside $arctan(\theta)$.

Table 1: Experimental Data

Input Voltage (V)	Amplitude (rad)	Driving Frequency (rad/s)	Phase (rad)
3.00	0.68 ± 0.03	2.5 ± 0.2	0.38 ± 0.03
3.30	0.84 ± 0.01	3.06 ± 0.02	0.307 ± 0.002
3.50	0.96 ± 0.08	3.30 ± 0.06	0.41 ± 0.08
3.79	1.36 ± 0.02	3.66 ± 0.01	0.367 ± 0.001
4.09	2.8 ± 0.3	4.03 ± 0.03	0.404 ± 0.003
4.34	5.9 ± 0.2	4.34 ± 0.01	1.0 ± 0.1
4.04	2.1 ± 0.2	3.94 ± 0.07	0.395 ± 0.007
3.95	1.74 ± 0.06	3.82 ± 0.02	0.29 ± 0.1
4.00	2.14 ± 0.08	3.931 ± 0.004	0.3949 ± 0.0004
5.00	2.7 ± 0.1	4.93 ± 0.04	-0.42 ± 0.02
4.40	5.22 ± 0.07	4.34 ± 0.09	1.2 ± 0.3
4.32	5.0 ± 0.8	4.25 ± 0.06	0.8 ± 0.2
4.39	5.965 ± 0.002	4.39 ± 0.01	1.2 ± 0.1
4.41	5.4 ± 0.5	4.3 ± 0.1	1.1 ± 0.4
5.20	1.66 ± 0.06	5.26 ± 0.02	-0.24 ± 0.01
5.40	1.4 ± 0.1	5.41 ± 0.06	-0.3 ± 0.1
4.44	5.87 ± 0.07	4.25 ± 0.03	0.7 ± 0.1

C Errors

C.1 Natural Frequency from the Amplitudes

The uncertanty in T_i is given by:

$$\Delta T = \frac{\Delta t}{\sqrt{N}} \tag{6}$$

The error in the natural frequency gets propagated as:

$$\Delta f_n = f_n^2 \Delta T \tag{7}$$

C.2 The damping factor from an underdamped oscillator

In this case, $\Delta\gamma \propto \Delta t + \Delta m + \Delta A_i$. Here Δt is the assigned error of $0.05(s)$, this being the measurement intervals. The line of best-fit is in the form $y = mx + c$, the uncertainty in the slope is given by:

$$\Delta m = \sqrt{\frac{\Delta y^2}{\sum(t_i - \bar{t_i})^2}}$$

Where Δy^2 denotes the variance in A_i, with Δy^2 being equal to:

$$\Delta y^2 = \sqrt{\sum_{i=1}^{N} (t_i - \bar{t_i})^2 f(t_i)}$$

The uncertainty in A_i is calculated by:

$$s = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (t_i - \bar{t_i})^2}$$

Combining the previous stated uncertanties leads to the value of $\Delta\gamma$ that is determined by:

$$\Delta\gamma = \sqrt{(\Delta m)^2 + \sum_{i=1}^{N}\left[\left(\frac{\Delta A_i}{t_i}\right)^2 + \left(\gamma \cdot \frac{\Delta t}{t_i}\right)^2\right]}$$

C.3 Resonance Frequency from the Amplitude Graph

In this case, resonance frequency was estimated by finding the peak value of the best-fit curve $A(x, a, b, c) = \frac{a}{\sqrt{(b-x^2)^2+cx^2}}$. This is done by setting $\frac{\partial A}{\partial x} = 0$. Therefore,

$$\frac{\partial}{\partial x}A(x, a, b, c) = \frac{ax(2b - c - 2x^2)}{(b^2 + x^2(-2b + c + x^2))^{3/2}} = 0 \Rightarrow x = \sqrt{b - c/2}$$

Values for a, b, and c were obtained using standard statistical optimization techniques that also yielded the covarience matrix for a, b, and c. The values of interest are:

$$a = 8.40, b = 19.02, c = 0.10, \Delta b = 0.18, \Delta c = 0.02$$

Therefore, $x = \omega_{amplitude} = \sqrt{19.02 - 0.10/2} \approx 4.35(rad/s)$. Error is propagated using:

$$\frac{\Delta x}{x} = \sqrt{\left(\frac{\partial x}{\partial b}\Delta b\right)^2 + \left(\frac{\partial x}{\partial c}\Delta c\right)^2} = \sqrt{\left(\frac{1}{2\sqrt{b - c/2}}\Delta b\right)^2 + \left(\frac{1}{-4\sqrt{b - c/2}}\Delta c\right)^2}$$

Plugging the values in, error in the resonance frequency can be computed to be $\Delta x = 0.09(rad/s)$.

Therefore, $\omega_{amplitude} = 4.41 \pm 0.09(rad/s)$.

C.4 Resonance Frequency from the Phase Graph

In this case, resonance frequency was estimated by finding the point of discontinuity of the best-fit curve $\phi(x, a, b) = \arctan \frac{ax}{b-x^2}$. For this function, the point of discontinuity coincides with the peak value, so the same statistical optimization techniques were utilized as in section **??**. The values and errors of the parameters are:

$$a = 0.48, b = 19.93, \Delta b = 0.18$$

Resonance frequency is located where $\phi = \pi/2$, thus in the peak-discontinuity point. This equality holds if $b - x^2 = 0$. Therefore, $x = \sqrt{b} > 0, x = 4.46(rad/s)$. The error is computed as follows:

$$\frac{\Delta x}{x} = \frac{\partial x}{\partial b}\Delta b \Rightarrow \Delta x = \frac{\Delta b}{2\sqrt{b}} = 0.02$$

Therefore, $\omega_{phase} = 4.46 \pm 0.02(rad/s)$.

C.5 Damping Factor in terms of Resonance Frequency

The relation between the damping factor γ and natural frequency ω and the resonance frequency ω_{res} is

$$\gamma = \sqrt{\frac{\omega^2 - \omega_{res}^2}{2}} = 1.32(rad/s)$$

The error is

$$\left(\frac{\Delta\gamma}{\gamma}\right)^2 = \left(\frac{\partial\gamma}{\partial\omega}\Delta\omega\right)^2 + \left(\frac{\partial\gamma}{\partial\omega_{res}}\Delta\omega_{res}\right)^2 = \frac{\omega^2\Delta\omega}{2(\omega^2-\omega_{res}^2)} + \frac{\omega_{res}^2\Delta\omega_{res}}{2(\omega^2-\omega_{res}^2)}$$

$$\Delta\gamma = \gamma\sqrt{\frac{\omega^2\Delta\omega}{2(\omega^2-\omega_{res}^2)} + \frac{\omega_{res}^2\Delta\omega_{res}}{2(\omega^2-\omega_{res}^2)}} = \gamma\sqrt{\frac{\omega^2\Delta\omega + \omega_{res}^2\Delta\omega_{res}}{2(\omega^2-\omega_{res}^2)}}$$

$$= 1.32\sqrt{\frac{4.78^2 \times 0.02 + 4.41^2 \times 0.09}{2(4.78^2 - 4.41^2)}} = 0.8$$

Therefore,

$$\gamma = 1.3 \pm 0.8(rad/s)$$