Week 4 - Challenge Problems

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1 Problem 1

Due to symmetry, I will subdivide xy-plane by x-axis, y-axis, and two $45deg$ lines, resulting into 8 regions. All regions are equivalent and I will consider only one of them:

$$D = \{(x,y)|0 \leq x \leq 1 \wedge x \leq y \leq 1\}$$

This region corresponds to the indicated below surface:

Figure 1: Subregion

Therefore, the volume can be evaluated as follows:

$$A = 16\int_0^1\int_x^1 \sqrt{1+(\frac{\partial z}{\partial y})^2}dydx, z(y) = \sqrt{1-y^2}$$

$$= 16\int_0^1\int_x^1 \sqrt{1+\frac{y^2}{1-y^2}}dydx = 16\int_0^1\int_x^1 \frac{dy}{\sqrt{1-y^2}}dx$$

$$= 16\int_0^1 \arcsin(1) - \arcsin(x)dx = 16\int_0^1 \frac{\pi}{2} - \arcsin(x)dx = 16(\frac{\pi x}{2} - x\arcsin(x) - \sqrt{1-x^2})_0^1 = 16 \cdot 1$$

Answer: 16.

2 Problem 2

a) A circle with radius R is defined on $D = \{(x,y)| -R \leq x \leq R \ \wedge -\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2}\}$. Thus, the area is defined by:

$$\iint_D dA = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dydx = 2\int_{-R}^{R}\sqrt{R^2-x^2}dx$$

Use substitution $u = \frac{x}{R}, du = \frac{dx}{R}$. Thus,

$$2R^2\int_{-1}^{1}\sqrt{1-u^2}du$$

The following integral will be used frequently:

$$\int\sqrt{1-x^2}dx = \frac{1}{2}(\arcsin(x) + x\sqrt{1-x^2}) + constant, \int_{-1}^{1}\sqrt{1-x^2}dx = \frac{\pi}{2}$$

Thus,

$$2R^2\int_{-1}^{1}\sqrt{1-u^2}du = 2R^2\frac{1}{2}(\arcsin(1)+1\sqrt{1-1^2}-\arcsin(-1)-(-1)\sqrt{1-(-1)^2}) = R^2(\frac{\pi}{2}+0-\frac{-\pi}{2}-0) = \pi R^2$$

$$V_2(R) = \pi R^2$$

b) Computations are largely similar, but the domain is a 3D volume:

$$D = \{(x,y,z)| -R \leq x \leq R \wedge -\sqrt{R^2-x^2} \leq y \leq \sqrt{R^2-x^2} \wedge -\sqrt{R^2-x^2-y^2} \leq z \leq \sqrt{R^2-x^2-y^2}\}$$

Thus, the volume is defined as:

$$\iiint_D dV = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dzdydx = 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}dydx$$

Calculate iterated integrals separately:

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2-y^2}dy = \sqrt{R^2-x^2}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{1-(\frac{y}{\sqrt{R^2-x^2}})^2}dy$$

Use substituion $u = \frac{y}{\sqrt{R^2-x^2}}$. Thusly,

$$= (R^2-x^2)\int_{-1}^{1}\sqrt{1-u^2}du = \frac{(R^2-x^2)\pi}{2}$$

Calculate the outer integral:

$$2\int_{-R}^{R}\frac{(R^2-x^2)\pi}{2}dx = \pi[R^2x - x^3/3]_{-R}^{R} = \pi[R^3 - R^3/3 + R^3 - R^3/3] \Rightarrow V_3(R) = \frac{4}{3}\pi R^3$$

c) The process is totally identical, therefore I provide only mathematical steps:

$$\iiiint_D dW = \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}}\int_{-\sqrt{R^2-x^2-y^2-z^2}}^{\sqrt{R^2-x^2-y^2-z^2}} dwdzdydx$$

$$= 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}}\sqrt{R^2-x^2-y^2-z^2}dzdydx$$

First integral.

$$\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}}\sqrt{R^2-x^2-y^2-z^2}dz = (R^2-x^2-y^2)\int_{-1}^{1}\sqrt{1-u^2}du = \frac{\pi(R^2-x^2-y^2)}{2}$$

Substitute this back in the formula for the hypervolume:

$$\iiiint_D dW = \pi\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}(R^2-x^2-y^2)dydx$$

Second integral.

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}(R^2-x^2-y^2)dy = [y(R^2-x^2) - \frac{y^3}{3}]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} = 2(R^2-x^2)^{3/2} - \frac{2}{3}(R^2-x^2)^{3/2}$$

$$= \frac{4}{3}(R^2-x^2)^{3/2} \Rightarrow V_4(R) = \frac{4\pi}{3}\int_{-R}^{R}(R^2-x^2)^{3/2}dx$$

Use substitution $u = x/R$.

$$\frac{4\pi}{3}R^3\int_{-R}^{R}(1-\frac{x^2}{R^2})^{3/2}dx = \frac{4\pi}{3}R^4\int_{-1}^{1}(1-u^2)^{3/2}dx$$

Substitute $u = sin\theta, du = cos\theta d\theta$

$$\int_{-1}^{1}(1-u^2)^{3/2}du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta = ... = \frac{3\pi}{8}$$

Therefore, $V_4(R) = \frac{4\pi}{3}R^4\frac{3\pi}{8} = \frac{\pi^2}{2}R^4$

d) In order to solve this problem, I use hyperspherical coordinates that I was not able to understand and imagine but it is not necessary for this problem because of hyperspherical symmetry. I define $d\Omega_n$ to include all angular differentials in spherical coordinates. Besides, I use **gamma function** $\Gamma(n)$, defined as $\int_0^\infty t^{n-1}e^{-t}dt$.

Thus, the volume of a n-sphere is defined as follows:

$$\int \cdots \int_{||\mathbf{x}|| \leq R} d^n\mathbf{x} = \int_0^R \int_{\Omega_{n-1}} r^{n-1} dr d\Omega_{n-1}$$

$d\Omega_{n-1}$ integral has bounds from 0 to π for one angle, and for all others it is from 0 to 2π. Use Fubini's theorem to separate integrals, and define $A_{n-1} = \int_{\Omega_{n-1}} d\Omega_{n-1}$:

$$\int \cdots \int_{||\mathbf{x}|| \leq R} d^n\mathbf{x} = A_{n-1}\frac{R^n}{n} \tag{1}$$

We can calculate A_{n-1} by integrating $f(x) = e^{-||\mathbf{x}||^2}$ over $\mathbb{R}^n$. For this, evaluate the following integral:

$$\int_0^\infty t^{n-1}e^{-t^2}dt = \frac{1}{2}\int_0^\infty u^{\frac{n-2}{2}}e^{-u}du, u = t^2, du = 2tdt$$

$$\Rightarrow \int_0^\infty t^{n-1}e^{-t^2}dt = \frac{1}{2}\Gamma(\frac{n}{2})$$

Besides, it is a well-known integral: $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$

Now, we put all the pieces together and find A_{n-1}:

$$\int_0^\infty \cdots \int_0^\infty e^{-||\mathbf{x}||^2}d^n\mathbf{x} = A_{n-1}\int_0^\infty e^{-r^2}r^{n-1}dr = A_{n-1}\frac{1}{2}\Gamma(\frac{n}{2})$$

$$\int_0^\infty \cdots \int_0^\infty e^{-||\mathbf{x}||^2}d^n\mathbf{x} = \int_0^\infty e^{-x_1^2}dx_1 \cdots \int_0^\infty e^{-x_n^2}dx_n = \pi^{n/2}$$

Comparing, we have:

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

Therefore, after substituting it back into Eq. (**??**), we obtain the final formulae:

$$V_n(R) = \int \cdots \int_{||\mathbf{x}|| \leq R} d^n\mathbf{x} = A_{n-1}\frac{R^n}{n} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\frac{R^n}{n} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}\frac{R^n}{\frac{n}{2}}$$

Thus, $V_n(R) = \frac{\pi^{n/2}R^n}{\Gamma(1+\frac{n}{2})}$.

e) From computer science, I know that $O(n!) > O(c^n)$. Since for integers $\Gamma(n)$ is practically a factorial, $V_n(1)$ indeed goes to 0 as $n \to \infty$. However, I was not able to come up with a mathematical proof.

f) I was not able to find the derivative of the volume with repsect to n.

3 Problem H3

a) Since we integrate over $[0,t] \times [0,t], t \to 1^-$, we can expand using geometric series:

$$\lim_{t\to1^-}\int_0^t\int_0^t \frac{1}{1-xy}dxdy = \lim_{t\to1^-}\int_0^t\int_0^t\sum_{i=0}^\infty x^iy^idxdy = \lim_{t\to1^-}\int_0^t\sum_{i=0}^\infty y^i\int_0^t x^idxdy = \lim_{t\to1^-}\int_0^t\sum_{i=0}^\infty y^i[\frac{x^{i+1}}{i+1}]_0^tdy$$

Similarly for the dx-integral:

$$= \lim_{t\to1^-}\int_0^t\sum_{i=0}^\infty\frac{t^{i+1}y^i}{i+1}dy = \lim_{t\to1^-}\sum_{i=0}^\infty\frac{t^{i+1}}{i+1}[\frac{y^{i+1}}{i+1}]_0^t = \lim_{t\to1^-}\sum_{i=0}^\infty\frac{t^{2(i+1)}}{(i+1)^2} = \sum_{i=0}^\infty\frac{1}{(i+1)^2}$$

Make substitution $n = i + 1$. So, $i = 0$ changes to $n = 0 + 1 = 1$, and ∞ changes to $\infty + 1 = \infty$. Thus,

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dxdy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

b) To transform coordinates (x, y) to (u, v), we need to transform the boundary and the area element.

$$\det(\mathbf{J}) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1$$

Thus, the area element is $dxdy = dudv$. Now, transform boundaries.

$$x = 0 \Rightarrow u = v \text{ and } x = 1 \Rightarrow u = v + \sqrt{2}$$
$$y = 0 \Rightarrow u = -v \text{ and } y = 1 \Rightarrow u = \sqrt{2} - v$$

Figure 2: Transformed Region

Due to symmetry, we obtain:

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dxdy = 2 \int_{u=0}^{\sqrt{2}/2} \int_{v=0}^{u} \frac{1}{1 - \frac{(u+v)(u-v)}{2}} dvdu + 2 \int_{u=\sqrt{2}/2}^{\sqrt{2}} \int_{v=0}^{\sqrt{2}-u} \frac{1}{1 - \frac{(u+v)(u-v)}{2}} dvdu$$

Evaluate the first term: $2 \int_{u=0}^{\sqrt{2}/2} \int_{v=0}^{u} \frac{1}{1 - \frac{(u+v)(u-v)}{2}} dvdu$

Evaluate the inner integral.

$$\int_{v=0}^{u} \frac{2dv}{2 - u^2 + v^2} = \frac{2}{\sqrt{2 - u^2}} \arctan \frac{u}{\sqrt{2 - u^2}}$$

Evaluate outer integral.

$$2 \int_{u=0}^{\sqrt{2}/2} \frac{2}{\sqrt{2 - u^2}} \arctan \frac{u}{\sqrt{2 - u^2}} du = 4 \int_{u=0}^{\sqrt{2}/2} \frac{du}{\sqrt{2 - u^2}} \arctan \frac{u}{\sqrt{2 - u^2}}$$

Use substitution $u = \sqrt{2} \sin t$, $du = \sqrt{2} \cos(t) dt$, $t = \arcsin(u/\sqrt{2})$.

$$4 \int_{\arcsin(0/\sqrt{2})}^{\arcsin(\frac{\sqrt{2}}{2}/\sqrt{2})} \frac{\sqrt{2} \cos(t) dt}{\sqrt{2 - (\sqrt{2} \sin(t))^2}} \arctan \frac{\sqrt{2} \sin(t)}{\sqrt{2 - (\sqrt{2} \sin(t))^2}} = 4 \int_0^{\pi/6} t dt = \frac{4\pi^2}{36 \times 2} = \frac{\pi^2}{18}$$

Evaluate the second term: $2 \int_{u=\sqrt{2}/2}^{\sqrt{2}} \int_{v=0}^{\sqrt{2}-u} \frac{1}{1 - \frac{(u+v)(u-v)}{2}} dvdu$

Evaluate the inner integral.

$$\int_{v=0}^{\sqrt{2}-u} \frac{2}{2 - u^2 + v^2} dv = \frac{2}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} \Big|_0^{\sqrt{2}-u} = \frac{2}{\sqrt{2 - u^2}} \arctan \frac{\sqrt{2} - u}{\sqrt{2 - u^2}}$$

Evaluate the outer integral.

$$2 \int_{u=\sqrt{2}/2}^{\sqrt{2}} \frac{2}{\sqrt{2 - u^2}} \arctan \frac{\sqrt{2} - u}{\sqrt{2 - u^2}} du$$

Again, use substitution $u = \sqrt{2} \sin t$, $du = \sqrt{2} \cos(t) dt$, $t = \arcsin(u/\sqrt{2})$.

$$= 2 \int_{\arcsin 1/2}^{\arcsin 1} \frac{2}{\sqrt{2 - 2\sin^2 t}} \arctan \frac{\sqrt{2} - \sqrt{2} \sin(t)}{\sqrt{2 - 2\sin^2(t)}} \sqrt{2} \cos(t) dt = 4 \int_{\pi/6}^{\pi/2} \frac{\cos(t)}{\cos(t)} \arctan \frac{1 - \sin(t)}{\cos(t)} dt$$

$$= 4\int_{\pi/6}^{\pi/2} \arctan\frac{1-\sin(t)}{\cos(t)} dt$$

Perform the following trigonometric transformations:

$$\frac{1-\sin(t)}{\cos(t)} = \frac{\sin(\pi/2)-\sin(t)}{\sin(\pi/2-t)} = \frac{2\sin(\frac{\pi/2-t}{2})\cos(\frac{\pi/2+t}{2})}{2\sin\frac{\pi/2-t}{2}\cos\frac{\pi/2-t}{2}} = \frac{\cos(\pi/4+t/2)}{\cos(\pi/4-t/2)} = \frac{\sin(\pi/2-\pi/4-t/2)}{\cos(\pi/4-t/2)} = \frac{\sin(\pi/4-t/2)}{\cos(\pi/4-t/2)}$$

$$= \tan(\pi/4-t/2)$$

Thus, evaluating the intergral, we get:

$$4(\frac{\pi t}{4}-\frac{t^2}{4})_{\pi/6}^{\pi/2} = \pi t - t^2|_{\pi/6}^{\pi/2} = \frac{\pi^2}{9}$$

Therefore,

$$\int_0^1\int_0^1 \frac{1}{1-xy}dxdy = \frac{\pi^2}{9} + \frac{\pi^2}{18} = \frac{\pi^2}{6}$$

c)

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}dxdydz = \lim_{t\to 1^-}\int_0^t\int_0^t\int_0^t \frac{1}{1-xyz}dxdydz = \lim_{t\to 1^-}\int_0^t\int_0^t\int_0^t \sum_{i=0}^{\infty} x^i y^i z^i dxdydz$$

$$= \lim_{t\to 1^-}\int_0^t\int_0^t \sum_{i=0}^{\infty} y^i z^i dydz \int_0^t x^i dx = \lim_{t\to 1^-}\int_0^t\int_0^t \sum_{i=0}^{\infty} y^i z^i \frac{t^{i+1}}{i+1} dydz = ... = \lim_{t\to 1^-}\sum_{i=0}^{\infty}\frac{t^{3(i+1)}}{(i+1)^3} = \sum_{i=0}^{\infty}\lim_{t\to 1^-}\frac{t^{3(i+1)}}{(i+1)^3}$$

$$= \sum_{i=0}^{\infty}\frac{1}{(i+1)^3} = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

4 Problem H4

To go from Cartesian coordinates to spherical, we have:

$$\rho = \sqrt{x^2+y^2+z^2}, \theta = \arccos(\frac{z}{\rho}), \phi = \arctan(\frac{y}{x})$$

To derive the laplacian in spherical coordinates, we need $\rho_i, \rho_{ii}, \theta_i, \theta_{ii}, \phi_i, \phi_{ii}$, where $i \in \{x, y, z\}$. Later, I provide the table of all the derivatives and derive only θ_x in this paper. Otherwise, this paper will be too bulky.

$$\theta_x = -\frac{1}{\sqrt{1-\frac{z^2}{\rho^2}}}\frac{\partial(z/\rho)}{\partial x} = -\frac{1}{\sqrt{1-\frac{z^2}{\rho^2}}}(-z)\frac{1}{\rho^2}\frac{\partial\rho}{\partial x} = \frac{xz}{\rho^3\sqrt{1-\frac{z^2}{\rho^2}}} = \frac{xz}{\rho^2\sqrt{\rho^2-z^2}}$$

Below is the table of all the partial derivatives.

Now, find u_x and u_{xx}.

$$u_x = u_\rho\rho_x + u_\theta\theta_x + u_\phi\phi_x$$

$$u_{xx} = \partial_x(u_\rho\rho_x) + \partial_x(u_\theta\theta_x) + \partial_x(u_\phi\phi_x)$$

$$\partial_x(u_\rho\rho_x) = u_{\rho x}\rho_x + u_\rho\rho_{xx} = (u_{\rho\rho}\rho_x + u_{\rho\theta}\theta_x + u_{\rho\phi}\phi_x)\rho_x + u_\rho\rho_{xx}$$

Therefore

$$\partial_x(u_\rho\rho_x) = u_\rho\rho_{xx} + u_{\rho\rho}\rho_x^2 + u_{\rho\theta}\theta_x\rho_x + u_{\rho\phi}\phi_x\rho_x$$

$$\partial_x(u_\theta\theta_x) = u_\theta\theta_{xx} + u_{\theta\rho}\rho_x\theta_x + u_{\theta\theta}\theta_x^2 + u_{\theta\phi}\phi_x\theta_x$$

Partial Derivative	ρ	θ	ϕ
x	$\frac{x}{\rho}$	$\frac{xz}{\rho^2\sqrt{\rho^2-z^2}}$	$-\frac{y}{x^2+y^2}$
xx	$\frac{\rho^2-x^2}{\rho^3}$	$\frac{z}{\rho^2\sqrt{\rho^2-z^2}} - \frac{3x^2z}{\rho^4\sqrt{\rho^2-z^2}} - \frac{x^2z^3}{\rho^4(\rho^2-z^2)^{3/2}}$	$\frac{2xy}{(x^2+y^2)^2}$
y	$\frac{y}{\rho}$	$\frac{yz}{\rho^2\sqrt{\rho^2-z^2}}$	$\frac{x}{x^2+y^2}$
yy	$\frac{\rho^2-y^2}{\rho^3}$	$\frac{z}{\rho^2\sqrt{\rho^2-z^2}} - \frac{3y^2z}{\rho^4\sqrt{\rho^2-z^2}} - \frac{y^2z^3}{\rho^4(\rho^2-z^2)^{3/2}}$	$-\frac{2xy}{(x^2+y^2)^2}$
z	$\frac{z}{\rho}$	$\frac{z^2-\rho^2}{\rho^2\sqrt{\rho^2-z^2}}$	0
zz	$\frac{\rho^2-z^2}{\rho^3}$	$\frac{2z}{\rho^4}\sqrt{\rho^2-z^2}$	0

Table 1: Partial derivatives of spherical coordinates.

$$\partial_x(u_\phi\phi_x) = u_\phi\phi_{xx} + u_{\phi\rho}\rho_x\phi_x + u_{\phi\theta}\theta_x\phi_x + u_{\phi\phi}\phi_x^2$$

The same equations hold for y and z. Combining second-order partial derivatives back into u_{xx}, we obtain:

$$u_{xx} = (u_\rho\rho_{xx} + u_\theta\theta_{xx} + u_\phi\phi_{xx}) + 2(u_{\theta\rho}\rho_x\theta_x + u_{\rho\phi}\phi_x\rho_x + u_{\theta\phi}\theta_x\phi_x) + (u_{\rho\rho}\rho_x^2 + u_{\theta\theta}\theta_x^2 + u_{\phi\phi}\phi_x^2)$$

$$u_{yy} = (u_\rho\rho_{yy} + u_\theta\theta_{yy} + u_\phi\phi_{yy}) + 2(u_{\theta\rho}\rho_y\theta_y + u_{\rho\phi}\phi_y\rho_y + u_{\theta\phi}\theta_y\phi_y) + (u_{\rho\rho}\rho_y^2 + u_{\theta\theta}\theta_y^2 + u_{\phi\phi}\phi_y^2)$$

$$u_{zz} = (u_\rho\rho_{zz} + u_\theta\theta_{zz} + u_\phi\phi_{zz}) + 2(u_{\theta\rho}\rho_z\theta_z + u_{\rho\phi}\phi_z\rho_z + u_{\theta\phi}\theta_z\phi_z) + (u_{\rho\rho}\rho_z^2 + u_{\theta\theta}\theta_z^2 + u_{\phi\phi}\phi_z^2)$$

I wrote this in a way that the same partial derivatives of u are one under the other. When adding u_{xx}, u_{yy}, u_{zz}, take partial derivates of u out of the parenthesis, and we end up with the following pattern: $\sum u_D f(\rho_D, \theta_D, \phi_D)$, where D denotes partial derivatives. Now, I compute each $f(\rho_D, \theta_D, \phi_D)$ separately. After, I plug the values back into the pattern and eventually the laplacian.

$$\rho_{xx} + \rho_{yy} + \rho_{zz} = \frac{3\rho^2 - x^2 - y^2 - z^2}{\rho^3} = \frac{2}{\rho}$$

$$\theta_{xx} + \theta_{yy} + \theta_{zz} = \frac{2z}{\rho^4}\sqrt{\rho^2 - z^2} + \frac{2z\rho^2(\rho^2 - z^2) - (x^2 + y^2)(3z(\rho^2 - z^2) + z^3)}{\rho^4(\rho^2 - z^2)^{3/2}}$$

$$= \frac{2z}{\rho^4}\sqrt{\rho^2 - z^2} + \frac{2z\rho^4 - 2z^3\rho^2 - (\rho^2 - z^2)(3z\rho^2 - 2z^3)}{\rho^4(\rho^2 - z^2)^{3/2}} = \frac{2z}{\rho^4}\sqrt{\rho^2 - z^2} + \frac{2z\rho^4 - 2z^3\rho^2 - 3z\rho^4 + 2z^3\rho^2 + 3z^3\rho^2 - 2z^5}{\rho^4(\rho^2 - z^2)^{3/2}}$$

$$= \frac{2z}{\rho^4}\sqrt{\rho^2 - z^2} + \frac{-z\rho^4 + 3z^3\rho^2 - 2z^5}{\rho^4(\rho^2 - z^2)^{3/2}} = \frac{2z(\rho^2 - z^2)^2 - z\rho^4 + 3z^3\rho^2 - 2z^5}{\rho^4(\rho^2 - z^2)^{3/2}} = \frac{2z(\rho^4 + z^4 - 2z^2\rho^2) - z\rho^4 + 3z^3\rho^2 - 2z^5}{\rho^4(\rho^2 - z^2)^{3/2}}$$

$$= \frac{z}{\rho^2\sqrt{\rho^2 - z^2}} = \frac{cos(\theta)}{\rho^2 sin(\theta)}$$

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

$$\rho_x\theta_x + \rho_y\theta_y + \rho_z\theta_z = \cdots easy_{algebra} = 0$$

$$\phi_x\rho_x + \phi_y\rho_y + \phi_z\rho_z = 0$$

$$\theta_x\phi_x + \theta_y\phi_y + \theta_z\phi_z = 0$$

$$\rho_x^2 + \rho_y^2 + \rho_z^2 = 1$$

$$\theta_x^2 + \theta_y^2 + \theta_z^2 = \cdots = \frac{1}{\rho^2}$$

$$\phi_x^2 + \phi_y^2 + \phi_z^2 = \frac{(-y)^2}{(x^2+y^2)^2} + \frac{x^2}{(x^2+y^2)^2} = \frac{1}{\rho^2 - z^2} = \frac{1/\rho^2}{1-(z/\rho)^2} = \frac{1}{\rho^2 sin^2(\theta)}$$

Therefore,

$$\nabla^2 u = \frac{2}{\rho}u_\rho + \frac{cos(\theta)}{\rho^2 sin(\theta)}u_\theta + u_{\rho\rho} + \frac{1}{\rho^2}u_{\theta\theta} + \frac{1}{\rho^2 sin^2(\theta)}u_{\phi\phi}$$

(I think the author of the problems forgot u_θ term.) More convenient notation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{cos(\theta)}{\rho^2 sin(\theta)}\frac{\partial u}{\partial \theta} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2 sin^2(\theta)}\frac{\partial^2 u}{\partial \phi^2}$$

5 Problem H5

First, I go from Cartesian coordinates to spherical coordinates. Because of spherical symmetry, aximuthal angle is irrelevant and can be replaced the factor $\int_0^{2\pi} d\phi = 2\pi$. Orient z-axis in a way that point (x_1, y_1, z_1) has coordinates $(0, 0, r)$, where r is the distance to the point from the origin (see figure below.)

Figure 3: Sphere

If ω denotes the region of the shell and $\mathbf{x}$ is the point of interest, we have:

$$V(\mathbf{x}) = -Gm \int_\omega \frac{\rho(\mathbf{r})dV(\mathbf{r})}{||\mathbf{r}-\mathbf{x}||}$$

The sphere is uniform $\Rightarrow \rho(\mathbf{r}) = \rho$. Besides, we can expand $||\mathbf{r}-\mathbf{x}|| = \sqrt{(\mathbf{r}-\mathbf{x})\cdot(\mathbf{r}-\mathbf{x})} = \sqrt{\mathbf{r}^2 + \mathbf{x}^2 - 2\mathbf{r}\cdot\mathbf{x}}$. Therefore, $||\mathbf{r}-\mathbf{x}|| = \sqrt{r^2 + x^2 - 2rx\cos(\theta)}$, where x is the distance from the origin to the point of interest. Therefore, we can re-write the integral as follows:

$$V(\mathbf{x}) = -Gm2\pi\rho \int_{\rho_1}^{\rho_2} \int_0^\pi \frac{r^2 \sin(\theta)d\theta dr}{\sqrt{r^2 + x^2 - 2rx\cos(\theta)}}$$

Evaluate the inner integral by substituting $u = r^2 + x^2 - 2rx\cos(\theta)$ and $du = 2rx\sin(\theta)d\theta$:

$$\int_0^\pi \frac{r^2\sin(\theta)d\theta}{\sqrt{r^2+x^2-2rx\cos(\theta)}} = \frac{r}{2x}\int_{r^2+x^2-2rx}^{r^2+x^2+2rx}\frac{du}{\sqrt{u}} = \frac{r}{x}[\sqrt{u}]_{r^2+x^2-2rx}^{r^2+x^2+2rx} = r\frac{\sqrt{r^2+x^2+2rx} - \sqrt{r^2+x^2-2rx}}{x}$$

$$x < \rho_1 < r \Rightarrow r\frac{\sqrt{r^2+x^2+2rx} - \sqrt{r^2+x^2-2rx}}{x} = \frac{(r+x)-(r-x)}{x} = \frac{2x}{x} = 2r$$

Evaluate outer integral:

$$\int_{\rho_1}^{\rho_2} 2rdr = \rho_2^2 - \rho_1^2$$

Thus,

$$V(\mathbf{x}) = -2Gm\pi\rho(\rho_2^2 - \rho_1^2)$$

Dimensional analysis does indeed yield *Joules* for $V(\mathbf{x})$.