Notes on Category Theory

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1 Introduction

The most important concept in Category Theory is that of universal property. Universal properties state how the object being described relates to the entire universe in which it lives.

Example 1.1. Let 1 denote a set with one element. The 1 has the following property:

for all sets X, there exists a unique map from X to 1.

Indeed, let X be a set. There *exists* a map $X \to 1$, because we can define $f : X \to 1$ by taking $f(x)$ to be the single element of 1 for each $x \in X$. This is the *unique* map $X \to 1$, because there is no choice in the matter: any map $X \to 1$ must send each elemet of X to the single element of 1.

There can be only *one* object satisfying a universal property in the sense that if two objects satisfy the property, then they are isomorphic.

2 Categories, functors, and natural transformations

Categories are systems of related objects and they are *themselves* mathematical objects. The maps between categories are called functors and maps between functors are called natural transformations.

2.1 Categories

Definition 2.1. A **category** $\mathcal{A}$ consists of:

- a collection ob($\mathcal{A}$) of **objects**;

- for each $A, B \in$ ob($\mathcal{A}$), a collection $\mathcal{A}(A, B)$ of **maps** or **arrows** or **morphisms** from A to B;

- for each $A, B, C \in$ ob($\mathcal{A}$), a function

$$\mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$$
$$(g, f) \mapsto g \circ f,$$

  called **composition**;

- for each $A \in$ ob($\mathcal{A}$), an element 1_A of $\mathcal{A}(A, A)$, called the **identity** on A,

  satisfying the following axioms:

- **associativity**: for each $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$ and $h \in \mathcal{A}(C, D)$, we have

$$(h \circ g) \circ f = h \circ (g \circ f);$$

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- **identity laws**: for each $f \in \mathcal{A}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Remark.
- We often write:

$$A \in \mathcal{A} \text{ to mean } A \in \mathrm{ob}(\mathcal{A});$$

$$f : A \to B \text{ or } A \xrightarrow{f} B \text{ to mean } f \in \mathcal{A}(A, B);$$

$$gf \text{ to mean } g \circ f.$$

- We often speak of commutative diagrams. Given objects and maps as in

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\
C & \xrightarrow{\;j\;} & D
\end{array}
$$

  we say that the diagram commutes if $g \circ f = j \circ h$.

- If $f \in \mathcal{A}(A, B)$, we call A the **domain** and B the **codomain** of f.

Example 2.1 (Categories of mathematical structures)**.**

(a) There is a category **Set** described as follows. Its objects are sets, and an arrow from set A to set B is an ordinary function from A to B. Composition in the category is the ordinary composition of functions, and the identity maps are the identity functions.

(b) There is a category **Grp** of groups, whose objects are groups and arrows are homomorphisms.

(c) For a field k, there is a category **Vect**$_k$ of vector spaces over k and linear maps between them.

(d) There is a category **Top** of topological spaces and continuous maps.

Definition 2.2. A map $f : A \to B$ in a category $\mathcal{A}$ is an **isomorphism** if there exists a map $g : B \to A$ in $\mathcal{A}$ such that $gf = 1_A$ and $fg = 1_B$.

We call g the inverse of f and write $g = f^{-1}$. If there exists an isomorphism from A to B, we say that A and B are isomorphic and write $A \cong B$.

Example 2.2 (Categories)**.**

(a) The isomorphisms in **Set** are the bijections.

(b) The isomorphisms in **Grp** are the isomorphisms of groups.

(c) The isomorphisms in **Top** are the homeomorphisms. A bijective map in **Top** is not necessarily an isomorphism.

In a general category the objects need not be sets with added structure, and the maps need not behave like functions.

Some categories contain no maps at all apart from the identity maps. Such categories are called **discrete categories**.

A group is the same thing as a category with one object in which all the maps are isomorphisms. Assume the object is called A and the category is $\mathcal{A}$. Then the set $\mathcal{A}(A, A)$ corresponds to the elements of the group G. The identity arrow 1_A correponds to the identity element in the group. And the fact that every arrow is an isomorphism corresponds to the fact that there

exists an inverse for every element in G. Composition correponds to the binary operation of G. The category $\mathcal{A}$ looks like:
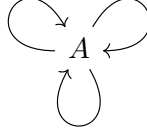
A **monoid** is a set equipped with an associative binary operation and a two-sided unit element, i.e., a group without inverses. A category with one object is essentially the same thing as a monoid.

A **preorder** is a reflexive transitive binary relation. A **preordered set** $(S, \leq)$ is a set S together with a preorder $\leq$ on it. Examples are: the ordering of $\mathbb{R}$ or inclusion of sets. A preordered set can be regarded as a category $\mathcal{A}$ where for each $A, B \in \mathcal{A}$ the set $\mathcal{A}(A, B)$ has at most one element. In this case we write $A \leq B$ to mean that there exists a map $A \rightarrow B$. It is easy to check that the composition of arrows corresponds to transitivity, and identity arrows correspond to reflexivity. Associativity and identity axioms follow immediately.

A **poset** is an antisymmetric preorder.

Every category $\mathcal{A}$ has an **opposite** or **dual** category $\mathcal{A}^{\mathrm{op}}$, defined by reversing the arrows. Formally, $\mathrm{ob}(\mathcal{A}^{\mathrm{op}})=\mathrm{ob}(\mathcal{A})$ and $\mathcal{A}^{\mathrm{op}}(B, A) = \mathcal{A}(A, B)$ for all objects A and B.

Remark. The **principle of duality** is fundamental to category theory. It states that every categorical definition, theorem, and proof has a **dual** which is obtained by reversing all the arrows. This can save work as given any theorem, one can produce a dual theorem simply by reversing the arrows and applying the same proof to $\mathcal{A}^{\mathrm{op}}$.

Exercise 2.1. Show that a map in a category can have at most one inverse.

Solution. Suppose that $f : A \rightarrow B$ has inverses g_1 and g_2. Then

$$g_1 = 1_A g_1 = (g_2 f) g_1 = g_2 (f g_1) = g_2 1_B = g_2.$$

2.2 Functors

A functor is a map between two categories that preserves composition and identities.

Definition 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **functor** $F : \mathcal{A} \rightarrow \mathcal{B}$ consists of:

- a function
$$\mathrm{ob}(\mathcal{A}) \rightarrow \mathrm{ob}(\mathcal{B}),$$
written as $A \mapsto F(A)$;

- for each $A, A' \in \mathcal{A}$, a function
$$\mathcal{A}(A, A') \rightarrow \mathcal{B}(F(A), F(A')),$$
written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{A}$;

- $F(1_A) = 1_{F(A)}$ whenever $A \in \mathcal{A}$.

Example 2.3 (The Forgetful Functor)**.**

(a) There is a functor $U:$ **Grp** $\to$ **Set** defined as follows: if G is a group then $U(G)$ is the underlying set of G, and if $f : G \to H$. is a group homomorphism, then $U(f)$ is the function f itself. So U forgets some structure of G.

(b) Forgetful functors do not have to forget all the structure. E.g., there is a forgetful functor **Ring** $\to$ **Ab** from rings to abelian groups, that forgets the multiplicative structure, but preserves the additive group.

Example 2.4 (The Free Functor)**.**

(a) Given a set S, one can build the **free group** $F(S)$ on S. This group is obtained by S by adding just enough elements so that it becomes a group.
Any map of sets $f : S \to S'$ gives rise to a homomorphism of groups $F(f) : F(S) \to F(S')$.

(b) We can also construct the free vector space on a set. Fix a field k. The free functor $F :$ **Set** $\to$ **Vect**$_k$ is defined on objects by taking $F(S)$ to be a vector space with basis S. Think of $F(S)$ as the set of all linear combinations of elements in S. More precisely, we can define $F(S)$ to be the set of all functions $\lambda : S \to k$ such that $\{s \in S | \lambda(s) \neq 0\}$ is finite (this restriction of finiteness is imposed because one can only take finite sums in a vector space). For each $\lambda, \mu \in F(S)$ the sum $\lambda + \mu \in F(S)$ is given by

$$(\lambda + \mu)(s) = \lambda(s) + \mu(s)$$

($s \in S$). Similarly, the scalar multiplication is given by $(c \cdot \lambda)(s) = c \cdot \lambda(s)$ ($c \in k, \lambda \in F(S), s \in S$).

(c) Given two monoids, regarded as one-object categories: a functor between them is determined by its effects on maps. Preservation of composition, means that the functor is just a homomorphism between the monoids.

(d) When A and B are (pre)ordered sets, a functor between the corresponding categories is an **order-preserving map**, i.e., a function $f : A \to B$ such that $a \leq a' \implies f(a) \leq f(a')$.

Sometimes we meet a functor-like behaving operation that reverses all the arrows, with a map $A \to A'$ giving rise to a map $F(A) \leftarrow F(A')$ in $\mathcal{B}$. Such operations are called contravariant functors.

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **contravariant functor** from $\mathcal{A}$ to $\mathcal{B}$ is a functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$.

Contravariant functors whose codomain is **Set** are important and have special names.

Definition 2.5. Let $\mathcal{A}$ be a category. A **presheaf** on $\mathcal{A}$ is a functor $\mathcal{A}^{\mathrm{op}} \to$ **Set**.

Let X be a topological space. Write $\mathcal{O}(X)$ for the poset of open subsets of X ordered by inclusion, and view it as a category. A **presheaf** on the space X is a presheaf on the category $\mathcal{O}(X)$. For example, given any space X, there is a presheaf F on X defined by

$$F(U) = \{\text{continuous functions } U \to \mathbb{R}\}$$

for $U \in \mathcal{O}(X)$, and whenever $U \subseteq U'$ are open subsets of X, by taking the map $F(U') \to F(U)$ to be restriction.

Definition 2.6. A functor $F : \mathcal{A} \to \mathcal{B}$ is **faithful** (respectively, **full**) if for each $A, A' \in \mathcal{A}$, the function

$$\mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$$
$$f \mapsto F(f)$$

is injective (respectively, surjective).

Remark. Faithfulness does **not** say that if f_1 and f_2 are distinct maps in $\mathcal{A}$ then $F(f_1) \neq F(f_2)$. F is faithful (respectively, full) if for each $A, A' \in \mathcal{A}$ and for each $g \in \mathcal{B}(F(A), F(A'))$, there is at most (respectively, at least) one arrow that F sends to g.

Exercise 2.2. Find an example of a functor $F : \mathcal{A} \to \mathcal{B}$ such that F is faithful but there exist distinct maps f_1 and f_2 in $\mathcal{A}$ with $F(f_1) = F(f_2)$.

Solution.
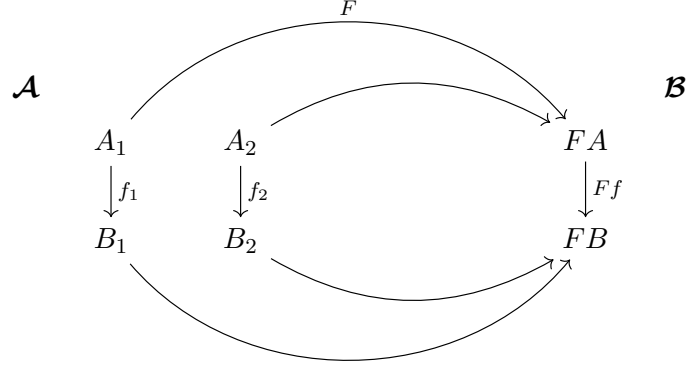
Definition 2.7. Let $\mathcal{A}$ be a category. A **subcategory** $\mathcal{S}$ of $\mathcal{A}$ consists of a subclass $\mathrm{ob}(\mathcal{S})$ of $\mathrm{ob}(\mathcal{A})$ together with, for each $S, S' \in \mathrm{ob}(\mathcal{S})$, a subclass $\mathcal{S}(S, S')$ of $\mathcal{A}(S, S')$, such that $\mathcal{S}$ is closed under composition and identities. It is a **full** subcategory if $\mathcal{S}(S, S') = \mathcal{A}(S, S')$ for all $S, S' \in \mathrm{ob}(\mathcal{S})$.

So, a full subcategory can be specified simply by saying what its objects are. For example, **Ab** is a full subcategory of **Grp**.

Whenever $\mathcal{S}$ is a subcategory of a category $\mathcal{A}$, there is an inclusion functor $I : \mathcal{S} \to \mathcal{A}$ defined by $I(S) = S$ and $I(f) = f$. It is automatically faithful, and it is full if and only if $\mathcal{S}$ is a full subcategory.

Exercise 2.3.   (a) What are the subcategories of an ordered set? Which are full?

(b) What are the subcategories of a group? Which are full?

Solution.

(a) A subcategory of an ordered set P is another ordered set Q, such that $\mathrm{ob}(Q) \subseteq \mathrm{ob}(P)$ and for all $a, b \in Q$ we have

$$\text{if } a \leq_Q b \text{ then } a \leq_P b.$$

In adition, a full subcategory will satisfy

$$a \leq_Q b \Longleftrightarrow a \leq_P b.$$

(b) Subcategories of a group are all of its submonoids. The full subcategory is the group itself.

Exercise 2.4. Show that functors preserve isomorphisms.

Solution. Assume that $F : \mathcal{A} \to \mathcal{B}$ is a functor and $A, A' \in \mathcal{A}$ with $A \cong A'$. This means that there exists $f : A \to A'$ and $g : A' \to A$, such that $gf = 1_A$ and $fg = 1_{A'}$. Because the functor preserves composition and identities, we have $F(g)F(f) = F(gf) = F(1_A) = 1_{F(A)}$ and $F(f)F(g) = F(fg) = F(1_{A'}) = 1_{F(A')}$. I.e, $F(A) \cong F(A')$.

Definition 2.8. Two categories $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic**, written as $\mathcal{A} \cong \mathcal{B}$, if they are isomorphic as objects of **CAT**. Equivalently, if there are functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ with $F(G) = 1_B$ and $G(F) = 1_A$ (where the composition of functors is just a composition on objects and a composition on maps).

2.3   Natural transformations

Maps between functors are called natural transformations. This notion applies only when the functors have the same domain and codomain.

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}.$$

Let's consider a special case. Let $\mathcal{A}$ be a discrete category whose objects are the natural numbers. A functor F from $\mathcal{A}$ to another category $\mathcal{B}$ is a sequence of $F_0, F_1, ...,$ of objects in $\mathcal{B}$. Take another functor G from $\mathcal{A}$ to $\mathcal{B}$ which consists of another sequence $G_0, G_1, ...$ of objects. It is reasonable to define a 'map' from F to G as

$$(F_0 \xrightarrow{a_0} G_0, F_1 \xrightarrow{a_1} G_1, ...)$$

of maps in $\mathcal{B}$. This suggests that in the general case, a natural transformation between functors $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ should consist of maps $\alpha_A : F(A) \to G(A)$, one for each $A \in \mathcal{A}$.

Definition 2.9. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ be functors. A **natural transformation** $\alpha : F \to G$ is a family $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathcal{A}}$ of maps in $\mathcal{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathcal{A}$, the square

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(A') \\
\downarrow{\alpha_A} & & \downarrow{\alpha_{A'}} \\
G(A) & \xrightarrow{G(f)} & G(A')
\end{array}
$$

commutes. The maps α_i are called the **components** of α.

We write

$$\mathcal{A} \overset{F}{\underset{G}{\Downarrow \alpha}} \mathcal{B}$$

to mean that α is a natural transformation from F to G.

Example 2.5. Fix a natural number n. For any ring R, the $n \times n$ matrices with entries in R form a monoid $M_n(R)$ under multiplication. Any ring homomorphism $R \to S$ induces a monoid homomorphism $M_n(R) \to M_n(S)$. This defines a functor $M_n : \mathbf{CRing} \to \mathbf{Mon}$ from the category of commutative rings to the category of monoids.

Also, the elements of any ring R form a monoid $U(R)$ under multiplication, giving another functor $U : \mathbf{CRing} \to \mathbf{Mon}$.

Now, every $n \times n$ matrix X over a commutative ring R has a determinant $\det_R(X)$, which is an element of R. Familiar properties of the determinant -

$$\det_R(XY) = \det_R(X)\det_R(Y), \qquad \det_R(I) = 1$$

- tell us that for each R, the function $\det_R : M_n(R) \to U(R)$ is a monoid homomorphism. So, we have a family of maps

$$\left(M_n(R) \xrightarrow{\det_R} U(R)\right)_{R \in \mathbf{CRing}}$$

which define a natural transformation. That the naturality squares commute reflects the fact that determinant is defined in the same way for all rings.

Natural transformations can be composed. Given natural transformations

$$
\mathcal{A}
\begin{array}{c}
F \\
\Downarrow \alpha \\
\xrightarrow{\quad G \quad} \\
\Downarrow \beta \\
H
\end{array}
\mathcal{B}
$$

there is a composite natural transformation

$$
\mathcal{A}
\begin{array}{c}
F \\
\Downarrow \beta \circ \alpha \\
H
\end{array}
\mathcal{B}
$$

defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for all $A \in \mathcal{A}$. There is also an identity natural transformation

$$
\mathcal{A}
\begin{array}{c}
F \\
\Downarrow 1_F \\
F
\end{array}
\mathcal{B}
$$

on any functor F, defined by $(1_F)_A = 1_{F(A)}$. So for any two categories $\mathcal{A}$ and $\mathcal{B}$, there is a category whose objects are the functors from $\mathcal{A}$ to $\mathcal{B}$ and whose maps are the natural transformations between them. This is called the **functor category** from $\mathcal{A}$ to $\mathcal{B}$, and written as $[\, \mathcal{A}, \mathcal{B} \,]$ or $\mathcal{B}^{\mathcal{A}}$.

Example 2.6. Take ordered sets A and B, viewed as categories. Given order-preserving maps $\mathcal{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{B}$ viewed as functors, there is at most one natural transformation

$$
\mathcal{A}
\begin{array}{c}
f \\
\Downarrow \\
g
\end{array}
\mathcal{B}
$$

and there is one if and only if $f(a) \leq g(a)$ for all $a \in A$.

Definition 2.10. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A **natural isomorphism** between functors from $\mathcal{A}$ to $\mathcal{B}$ is an isomorphism in $[\, \mathcal{A}, \mathcal{B} \,]$.

Equivalently,

Lemma 2.1. *Let* $\mathcal{A} \begin{array}{c} F \\ \Downarrow \alpha \\ G \end{array} \mathcal{B}$ *be a natural transformation. Then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathcal{A}$.*

Definition 2.11. Given functors $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$, we say that

$$F(A) \cong G(A) \textbf{ naturally in } A$$

if F and G are naturally isomorphic.

We may sometimes use the word **canonical** to refer to a decision without making arbitrary choices, i.e., something that arises naturally. E.g., for any two sets A and B there is a canonical bijection $A \times B \to B \times A$ defined by $(a, b) \to (b, a)$.

The concept of natural isomorphism leads to the equivalence of categories. In set theory we say two objects are the same if they are equal. In category theory, sameness refers to isomorphisms. Two objects are the same if they are isomorphic. Two functors are the same if they are naturally isomorphic.

However, our definition of two isomorphic categories is strict. Given $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$, we say that $A \cong B$ if there are functors such that

$$G \circ F = 1_{\mathcal{A}} \qquad \text{and} \qquad F \circ G = 1_{\mathcal{B}}.$$

It is more useful to speak of equivalence between categories. In this case we simply replace the strict equalities by isomorphisms. This gives

$$G \circ F \cong 1_{\mathcal{A}} \qquad \text{and} \qquad F \circ G \cong 1_{\mathcal{B}}.$$

Definition 2.12. An **equivalence** between categories $\mathcal{A}$ and $\mathcal{B}$ consists of a pair of functors together with natural isomorphisms

$$\eta : 1_{\mathcal{A}} \to G \circ F, \qquad \epsilon : F \circ G \to 1_{\mathcal{B}}.$$

If there exists an equivalence between $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ and $\mathcal{B}$ are **equivalent**, and write $\mathcal{A} \simeq \mathcal{B}$.

Definition 2.13. A functor $F : \mathcal{A} \to \mathcal{B}$ is **essentially surjective on objects** if for all $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $F(A) \cong B$.

Proposition 2.1. *A functor is an equivalence if and only if it is full, faithful, and essentially surjective on objects.*

This result is useful because it allows us to show that a functor is an equivalence without constructing an 'inverse'.

Corollary 2.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful functor. Then $\mathcal{C}$ is equivalent to the full subcategory $\mathcal{C}'$ of $\mathcal{D}$ whose objectts are those of the form $F(C)$ for some $C \in \mathcal{C}$.*

Proof. Trivial.

Example 2.7. Let **FinSet** be the category of finite sets and functions between them, For each natural number n, choose a set **n** with n elements, and let $\mathcal{B}$ be the full subcategory of **FinSet** with objects $\mathbf{0}, \mathbf{1}, \dots$. Then $\mathcal{B} \simeq$ **FinSet**.

Example 2.8. An equivalence of the form $\mathcal{A}^{\mathrm{op}} \simeq \mathcal{B}$ is sometimes called a **duality** between $\mathcal{A}$ and $\mathcal{B}$. One says that $\mathcal{A}$ is **dual** to $\mathcal{B}$.

The composition of natural transformations we defined
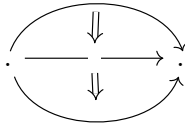
is sometimes called a **vertical composition**. There is also **horizonal compositon**, which takes natural transformations
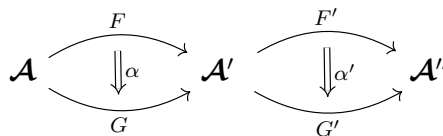
and produces a natural transformation

$$
\mathcal{A} \underset{G' \circ G}{\overset{F' \circ F}{\Downarrow}} \mathcal{A}''
$$

traditionally written as $\alpha' * \alpha$.

Vertical and horizonal compositon interact well: natural transformations

$$
\mathcal{A} \xrightarrow{G} \mathcal{A}' \xrightarrow{G'} \mathcal{A}''
$$

obey the **interchange law**,

$$
(\beta' \circ \alpha') * (\beta \circ \alpha) = (\beta' * \beta) \circ (\alpha' * \alpha) : F' \circ F \to H' \circ H.
$$

3 Adjoints

3.1 Definition and examples

Consider a pair of functors in opposite directions, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$. Roughly speaking, F is said to be a left adjoint to G if, whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$, maps $F(A) \to B$ are essentially the same as maps $A \to G(B)$.

Definition 3.1. Let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ be categories and functors. We say that F is **left adjoint** to G, and G is **right adjoint** to F, and write $F \dashv G$, if

$$
\mathcal{B}(F(A), B) \cong \mathcal{A}(A, G(B)) \tag{1}
$$

naturally in $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Naturally in $A \in \mathcal{A}$ and $B \in \mathcal{B}$ means that there is a specified bijection for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and that it satisfies a naturality axiom. To state it, we need some notation. Given objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the correspondence between maps $F(A) \to B$ and $A \to G(B)$ is denoted by a horizontal bar, in both directions:

$$
\big(F(A) \xrightarrow{g} B\big) \mapsto \big(A \xrightarrow{\bar{g}} G(B)\big),
$$

$$
\big(F(A) \xrightarrow{\bar{f}} B\big) \leftarrow\!\shortmid \big(A \xrightarrow{f} G(B)\big).
$$

So $\bar{\bar{f}} = f$ and $\bar{\bar{g}} = g$. We call $\bar{f}$ the **transpose** of f, and similarly for g. The naturality axiom has two parts:

$$
\overline{\big(F(A) \xrightarrow{g} B \xrightarrow{q} B'\big)} = \big(A \xrightarrow{\bar{g}} G(B) \xrightarrow{G(q)} G(B')\big) \tag{2}
$$

(that is, $\overline{q \circ g} = G(q) \circ \bar{g}$) for all g and q, and

$$
\overline{\big(A' \xrightarrow{p} A \xrightarrow{f} G(B)\big)} = \big(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B\big) \tag{3}
$$

for all p and f.

A given functor G may or may not have a left adjoint, but if it does, it is unique up to isomorphism. Same goes for right adjoints.

Example 3.1. Forgetful functors between categories of algebraic structures usually have left adjoints. For instance:

Let k be a field. There is an adjunction

$$\mathbf{Vect}_k$$
$$F \uparrow \dashv \downarrow U$$
$$\mathbf{Set},$$

where U is the forgetful functor and F is the free functor. Adjointness says that given a set S and a vector space V, a linear map $F(S) \to V$ is essentially the same thing as a function $S \to U(V)$.

Fix a set S and a vector space V. Given a linear map $g : F(S) \to V$, we may define a map of sets $\bar{g} : S \to U(V)$ by $\bar{g}(s) = g(s)$ for all $s \in S$. This gives a function

$$\mathbf{Vect}_k(F(S), V) \to \mathbf{Set}(S, U(V))$$
$$g \mapsto \bar{g}.$$

In the other direction, given a map of sets $f : S \to U(V)$, we may define a linear map $\bar{f} : F(S) \to V$ by $\bar{f}\left(\sum_{s \in S} \lambda_s s\right) = \sum_{s \in S} \lambda_s f(s)$ for all formal linear combinations $\sum \lambda_s s \in F(S)$. This gives a function

$$\mathbf{Set}(S, U(V)) \to \mathbf{Vect}_k(F(S), V)$$
$$f \mapsto \bar{f}.$$

These two functions 'bar' are mutually inverse: for any linear map $g : F(S) \to V$, we have

$$\bar{\bar{g}}\left(\sum_{s \in S} \lambda_s s\right) = \sum_{s \in S} \lambda_s \bar{g}(s) = \sum_{s \in S} \lambda_s g(s) = g\left(\sum_{s \in S} \lambda_s s\right)$$

for all $\sum \lambda_s s \in F(S)$, so $\bar{\bar{g}} = g$, and for any map of sets $f : S \to U(V)$, we have

$$\bar{\bar{f}}(s) = \bar{f}(s) = f(s)$$

for all $s \in S$, so $\bar{\bar{f}} = f$. We therefore have a canonical bijection between $\mathbf{Vect}_k(F(S), V)$ and $\mathbf{Set}(S, U(V))$ for each $S \in \mathbf{Set}$ and $V \in \mathbf{Vect}_k$, as required.

Remark. Note that we are using $(\bar{\cdot})$ for both directions. With a little care, the notation becomes clear and useful.

Example 3.2. There are adjunctions

$$\mathbf{Top}$$
$$D \uparrow \dashv\ U\ \dashv \uparrow I$$
$$\mathbf{Set}$$

where U sends a space to its set of points, D equips a set with the discrete topology, and I equips a set with the indiscrete topology.

Example 3.3. Given sets A and B, we can form their (cartesian) product $A \times B$. We can also form the set B^A of functions from A to B. This is the same as the set $\mathbf{Set}(A, B)$, but we tend to use the notattion B^A when we want to emphasize that it is an object of the same category as A and B.

Now fix a set B. Taking the product with B defines a functor

$$\_ \times B : \mathbf{Set} \to \mathbf{Set}$$
$$A \mapsto A \times B.$$

There is also a functor

$$(\_)^B : \mathbf{Set} \to \mathbf{Set}$$
$$C \mapsto C^B.$$

Moreover, there is a canonical bijection

$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$$

for any sets A and C. It is defined by simply changing the punctuation: given a map $g : A \times B \to C$, define $\bar{g} : A \to C^B$ by

$$(\bar{g}(a))(b) = g(a, b)$$

($a \in A, b \in B$), and in the other direction, given $f : A \to C^B$, define $\bar{f} : A \times B \to C$ by

$$\bar{f}(a, b) = (f(a))(b)$$

($a \in A, b \in B$). Putting all this together, we obtain an adjunction

$$
\mathbf{Set}
$$
$$
{\scriptstyle \_\times B} \Big\uparrow \dashv \Big\downarrow {\scriptstyle (\_)^B}
$$
$$
\mathbf{Set}
$$

for every set B.

Definition 3.2. Let $\mathcal{A}$ be a category. An object $I \in \mathcal{A}$ is **initial** if for every $A \in \mathcal{A}$, there is exactly one map $I \to A$. An object $T \in \mathcal{A}$ is **terminal** if for every $A \in \mathcal{A}$, there is exactly one map $A \to T$.

For example, the empty set is initial in **Set**, the trivial group is initial in **Grp**, and $\mathbb{Z}$ is initial in **Ring**. The one-element set is terminal in **Set**, the trivial group is terminal (as well as initial) in **Grp**, and the trivial (one-element) ring is terminal in **Ring**.

A category need not have an initial object, but if it does have one, it is unique up to isomorphism.

Lemma 3.1. *Let I and I' be initial objects of a category. Then there is a unique isomorphism $I \to I'$. In particular, $I \cong I'$.*

Proof. Since I is initial, there is a unique map $f : I \to I'$. Since I' is initial, there is a unique map $f' : I' \to I$. Now $f' \circ f$ and 1_I are both maps $I \to I$, and I is initial, so $f' \circ f = 1_I$. Similarly, $f \circ f' = 1_{I'}$. Hence, f is an isomorphism. $\square$

Example 3.4. Initial and terminal objects can be described as adjoints. Let $\mathcal{A}$ be a category. There is precisely one functor $\mathcal{A} \to \mathbf{1}$. Also, a functor $\mathbf{1} \to \mathcal{A}$ is essentially just an object of $\mathcal{A}$. Viewing functors $\mathbf{1} \to \mathcal{A}$ as objects of $\mathcal{A}$, a left adjoint to $\mathcal{A} \to \mathbf{1}$ is exactly an initial object of $\mathcal{A}$.

Similarly, a right adjoint to the unique functor $\mathcal{A} \to \mathbf{1}$ is exactly a terminal object of $\mathcal{A}$.

Remark. Adjunctions can be composed. Take adjunctions

$$\mathcal{A} \xrightarrow[\substack{\perp \\ \xleftarrow{\;\;\;\;\;\;} \\ G}]{\xrightarrow{F}} \mathcal{A}' \xrightarrow[\substack{\perp \\ \xleftarrow{\;\;\;\;\;\;} \\ G'}]{\xrightarrow{F'}} \mathcal{A}''.$$

Then we obtain an adjunction

$$\mathcal{A} \xrightarrow[\substack{\perp \\ \xleftarrow{\;\;\;\;\;\;} \\ G \circ G'}]{\xrightarrow{F' \circ F}} \mathcal{A}''.$$

3.2 Adjunctions via units and counits

To start building the theory of adjoint functors, we have to take seriously the naturality reqirement. Suppose we have maps

$$F(A) \xrightarrow{\;g\;} B \xrightarrow{\;q\;} B'$$

in $\mathcal{B}$. We can either compose and take the transpose, which produces a map $\overline{q \circ g} : A \to G(B')$, or take the trasnpose of g then compose it with $G(q)$, which produces $G(q) \circ \bar{g} : A \to G(B')$. The first naturality equation says that they are equal.

For each $A \in \mathcal{A}$, we have a map

$$\left(A \xrightarrow{\eta_A} GF(A)\right) = \overline{\left(F(A) \xrightarrow{1} F(A)\right)}. \tag{4}$$

Dually, for each $B \in \mathcal{B}$, we have a map

$$\left(FG(B) \xrightarrow{\epsilon_B} B\right) = \overline{\left(G(B) \xrightarrow{1} G(B)\right)}. \tag{5}$$

These define natural transformations

$$\eta : 1_{\mathcal{A}} \to G \circ F, \qquad \epsilon : F \circ G \to 1_{\mathcal{B}},$$

called the **unit** and **counit** of the adjunction, respectively.

Example 3.5. Take the usual adjunction $\mathbf{Vect}_k \xrightarrow[\substack{\top \\ \xleftarrow{\;\;\;\;} \\ F}]{\xrightarrow{U}} \mathbf{Set}$. Its unit $\eta : 1_{\mathbf{Set}} \to U \circ F$ has components

$$\eta_S : S \to UF(S) = \{\text{formal } k\text{-linear sums} \sum_{s \in S} \lambda_s s\}$$

$$s \mapsto \quad s$$

($S \in \mathbf{Set}$). The component of the counit ϵ at a vector space V is the linear map

$$\epsilon_V : FU(V) \to V$$

that sends a *formal* linear sum $\sum_{v \in V} \lambda_v v$ to its *actual* value in V.

The vector space $FU(V)$ is enormous. For instance, if $k = \mathbb{R}$ and V is the vector space $\mathbb{R}^2$, then $U(V)$ is the set $\mathbb{R}^2$ and $FU(V)$ is a vector space with one basis element for every element of $\mathbb{R}^2$; thus, it is uncountably infinite-dimensional. Then ϵ_V is a map from this infinite-dimensional space to the 2-dimensional space V.

Lemma 3.2. *Given an adjunction $F \dashv G$ with unit η and counit ϵ, the triangles*

$$
\begin{array}{ccc}
F & \xrightarrow{F\eta} & FGF \\
& {\scriptstyle 1_F} \searrow & \downarrow {\scriptstyle \epsilon F} \\
& & F
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{\eta G} & GFG \\
& {\scriptstyle 1_G} \searrow & \downarrow {\scriptstyle G\epsilon} \\
& & G
\end{array}
$$

commute.

Remark. These are called the **triangle identities**. An equivalent statement is that the triangles

$$
\begin{array}{ccc}
F(A) \xrightarrow{F(\eta_A)} FGF(A) & \qquad & G(B) \xrightarrow{\eta_{G(B)}} GFG(B) \\
\searrow^{1_{F(A)}} \quad \downarrow^{\epsilon_{F(A)}} & & \searrow^{1_{G(B)}} \quad \downarrow^{G(\epsilon_B)} \\
F(A) & & G(B)
\end{array}
\tag{6}
$$

commute for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proof of Lemma 3.2. We prove that the triangles (6) commute. Let $A \in \mathcal{A}$. From (5) we know that $\overline{1_{GF(A)}} = \epsilon_{F(A)}$ and from (4) we know that $\eta_A = \overline{1_{F(A)}}$, i.e., $\overline{\eta_A} = \overline{\overline{1_{F(A)}}} = 1_{F(A)}$. We can express $\overline{\left(A \xrightarrow{\eta_A} GF(A)\right)}$ as $\overline{\left(A \xrightarrow{\eta_A} GF(A) \xrightarrow{1} GF(A)\right)}$ where we just extended $GF(A)$ by the identity arrow. Then from the second part of the naturality axiom (3) we get

$$
\overline{\left(A \xrightarrow{\eta_A} GF(A) \xrightarrow{1} GF(A)\right)} = \left(F(A) \xrightarrow{F(\eta_A)} FGF(A) \xrightarrow{\epsilon_{F(A)}} F(A)\right).
$$

But the left-hand side is $\overline{\eta_A} = 1_{F(A)}$, proving the first identity. The second follows by duality. $\qquad\square$

Lemma 3.3. *Let* $\mathcal{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$ *be an adjunction, with unit η and counit ϵ. Then*

$$
\bar{g} = G(g) \circ \eta_A
$$

for any $g : F(A) \to B$, and

$$
\bar{f} = \epsilon_B \circ F(f)
$$

for any $f : A \to G(B)$.

Proof. Let $g : F(A) \to B$. By using the naturality axiom (2), we have

$$
\overline{\left(F(A) \xrightarrow{g} B\right)} = \overline{\left(F(A) \xrightarrow{1} F(A) \xrightarrow{g} B\right)}
$$
$$
= \left(A \xrightarrow{\eta} GF(A) \xrightarrow{G(g)} G(B)\right)
$$

which proves the first statement. The second follows by duality. $\qquad\square$

Theorem 3.1. *Take categories and functors* $\mathcal{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$ *. There is a one-to-one correspondence between:*

(a) *adjunctions between F and G (with F on the left and G on the right)*

(b) *pairs $\left(1_\mathcal{A} \xrightarrow{\eta} GF, FG \xrightarrow{\epsilon} 1_\mathcal{B}\right)$ of natural transformations satisfying the triangle identities.*

Example 3.6. An adjunction between two ordered sets consists of order-preserving maps $A \underset{g}{\overset{f}{\rightleftarrows}} B$ such that

$$
\forall a \in A, \forall b \in B, \qquad f(a) \leq b \iff a \leq g(b).
\tag{7}
$$

This is because each of the homsets in the definition of adjoints (1) contains at most one element and for the isomorphism to hold, both homsets should either be empty or both contain a single arrow. Naturality axioms are trivially satisfied since any two maps with the same domain and codomain are equal in **PreOSet**.

The unit of the adjunction says that $a \leq gf(a)$ for all $a \in A$, and the counit says that $fg(b) \leq b$ for all $b \in B$. Theorem (3.1) states that (7) is equivalent to:

$$
\forall a \in A, a \leq gf(a) \quad \text{and} \quad \forall b \in B, fg(b) \leq b.
$$

Example 3.7. Let X be a topological space. Take the set $\mathcal{C}(X)$ of closed subsets of X and the set $\mathcal{P}(X)$ of all subsets of X, both ordered by $\subseteq$. There are order-preserving maps

$$\mathcal{P}(X) \underset{i}{\overset{\text{Cl}}{\rightleftarrows}} \mathcal{C}(X)$$

where i is the inclusion map and Cl is the closure. This is an adjunction, with Cl left adjoint of i as can be seen by

$$\text{Cl}(A) \subseteq B \iff A \subseteq B$$

for all $A \subseteq X$ and closed $B \subseteq X$.

Remark. An equivalence of categories is not necessarily an adjunction.

The idea of an adjoint functor is best understood as an approximation of a possibly non-existent inverse. Any pair of adjoint functors, however, restricts to an equivalence of categories on subcategories. These subcategories are sometimes known as the **center of the adjunction**, and their objects are known as **fixed points** of the adjunction.

The equivalences of categories that arise from fixed points of adjunctions this way are often known as **dualities**.

Definition 3.3. Let $\mathcal{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$ be a pair of adjoint functors. Say that an object $A \in \mathcal{A}$ is a fixed point of the adjunction if its adjunction unit is an isomorphism

$$A \overset{\eta_A}{\underset{\cong}{\longrightarrow}} GF(A)$$

and write $\mathcal{A}_{\text{fix}}$ for the full subcategory on these fixed objects. Similarly, an object $B \in \mathcal{B}$ is a fixed point of the adjunction if its adjunction unit is an isomorphism

$$FG(B) \overset{\epsilon_B}{\underset{\cong}{\longrightarrow}} B$$

and write $\mathcal{B}_{\text{fix}}$ for the full subcategory on these fixed objects.

Proposition 3.1. *The adjunction* (F, G, η, ϵ) *restricts to an adjoint equivalence* $(F', G', \eta', \epsilon')$ *on these full subcategories of fixed points*

$$\mathcal{A}_{\text{fix}} \underset{G'}{\overset{F'}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}_{\text{fix}}$$

Proof. The restricted adjunction unit/counit are isomorphisms by definition. So in order to show that the functors exhibit an adjoint equivalence, it is sufficient to see that the functors restrict as claimed.

Hence we need to show that $F'(\mathcal{A}_{\text{fix}}) \subset \mathcal{B}_{\text{fix}}$ and $G'(\mathcal{B}_{\text{fix}}) \subset \mathcal{A}_{\text{fix}}$. Consider the first of these.

Given some $A \in \mathcal{A}_{\text{fix}}$ we need to show that $F'(A) \in \mathcal{B}_{\text{fix}}$, in other words we need to show that $\epsilon'_{F'(A)}$ is an isomorphism. But because $A \in \mathcal{A}_{\text{fix}}$, then η'_A is an isomorphism. Functors preserve isomorphisms so $F'(\eta'_A)$ must be an isomorphism. Because inverses are unique and $\epsilon'_{F'(A)} \circ F'(\eta'_A) = 1_{F'(A)}$ by the triangle identities (6), the inverse of $F'(\eta'_A)$ must be $\epsilon'_{F'(A)}$ which means that $\epsilon'_{F'(A)}$ is itself an isomorphism.

The second statement follows by duality. $\qquad\square$

If the adjunction is idempotent, then the fixed objects in $\mathcal{A}$ are precisely those of the form $G(B)$, and dually the fixed objects in $\mathcal{B}$ are those of the form $F(A)$. This is essentially the definition of an idempotent adjunction.

3.3 Adjunctions via initial objects

This third formulation of adjointness is probably the most common.

Consider the adjunction

$$\mathbf{Vect}_k$$
$$F \uparrow \dashv \downarrow U$$
$$\mathbf{Set}.$$

Let S be a set. The universal property of $F(S)$, the vector space whose basis is S, is most commonly stated like this:

given a vector space V, any function $f : S \to V$ extends uniquely to a linear map $\bar{f} : F(S) \to V$.

Forgetful functors are often forgotten: in this statement '$f : S \to V$' should strictly speaking be '$f : S \to U(V)$'. The word 'extends' refers to the embedding

$$\eta_S : \quad S \to UF(S)$$
$$s \mapsto s.$$

In precise language the statement reads:

for any $V \in \mathbf{Vect}_k$ and $f \in \mathbf{Set}(S, U(V))$, there is a unique $\bar{f} \in \mathbf{Vect}_k(F(S), V)$ such that the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\eta_S} & U(F(S)) \\
& {\scriptstyle f} \searrow & \downarrow {\scriptstyle U(\bar{f})} \\
& & U(V)
\end{array}
\tag{8}
$$

commutes.

In this section we show that this statement is equivalent to the statement that F is left adjoint to U with unit η.

Definition 3.4. Given categories and functors

$$
\begin{array}{ccc}
& & \mathcal{B} \\
& & \downarrow {\scriptstyle Q} \\
\mathcal{A} & \xrightarrow{P} & \mathcal{C},
\end{array}
$$

the **comma category** $(P \Rightarrow Q)$ (often written as $(P \downarrow Q)$) is the category defined as follows:

- objects are triples (A, h, B) with $A \in \mathcal{A}, B \in \mathcal{B}$, and $h : P(A) \to Q(B)$ in $\mathcal{C}$;

- maps $(A, h, B) \to (A', h', B')$ are pairs $(f : A \to A', g : B \to B')$ of maps such that the square

$$
\begin{array}{ccc}
P(A) & \xrightarrow{P(f)} & P(A') \\
{\scriptstyle h} \downarrow & & \downarrow {\scriptstyle h'} \\
Q(B) & \xrightarrow{Q(g)} & Q(B')
\end{array}
$$

commutes.

Example 3.8. Let $\mathcal{A}$ be a category and $A \in \mathcal{A}$. The **slice category** of $\mathcal{A}$ over A, denoted by $\mathcal{A}/A$, is the category whose objects are maps into A and whose maps are commutative triangles. More precisely, an object is a pair (X, h) with $X \in \mathcal{A}$ and $h : X \to A$ in $\mathcal{A}$, and a map $(X, h) \to (X', h')$ in $\mathcal{A}/A$ is a map $f : X \to X'$ in $\mathcal{A}$ making the triangle

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ f\ \ } & X' \\
& \searrow_{h} \quad \swarrow_{h'} & \\
& A &
\end{array}
$$

commute.

Slice categories are a special case of comma categories. Functors $\mathbf{1} \to \mathcal{A}$ are just objects of $\mathcal{A}$. Now, given an object A of $\mathcal{A}$, consider the comma category $(1_{\mathcal{A}} \Rightarrow A)$, as in the diagram

$$
\begin{array}{ccc}
& & \mathbf{1} \\
& & \downarrow{\scriptstyle A} \\
\mathcal{A} & \xrightarrow{\ 1_{\mathcal{A}}\ } & \mathcal{A}.
\end{array}
$$

An object of $(1_{\mathcal{A}} \Rightarrow A)$ is in principle a triple (X, h, B), with $X \in \mathcal{A}$, $B \in \mathbf{1}$, and $h : X \to A$ in $\mathcal{A}$; but $\mathbf{1}$ has only one object, so it is essentially just a pair (X, h). Hence the comma category $(1_{\mathcal{A}} \Rightarrow A)$ has the same objects as the slice category $\mathcal{A}/A$ and one can check that is has the same maps too, so $\mathcal{A}/A \cong (1_{\mathcal{A}} \Rightarrow A)$.

Dually, there is **coslice category** $A/\mathcal{A} \cong (A \Rightarrow 1_{\mathcal{A}})$, whose objects are the maps out of A.

Example 3.9. Let $G : \mathcal{B} \to \mathcal{A}$ be a functors and let $A \in \mathcal{A}$. We can form the comma category $(A \Rightarrow G)$, as in the diagram

$$
\begin{array}{ccc}
& & \mathcal{B} \\
& & \downarrow{\scriptstyle G} \\
\mathbf{1} & \xrightarrow{\ A\ } & \mathcal{A}.
\end{array}
$$

Its objects are pairs $(B \in \mathcal{B}, f : A \to G(B))$. A map $(B, f) \to (B', f')$ in $(A \Rightarrow G)$ is a map $q : B \to B'$ in $\mathcal{B}$ making the triangle

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & G(B) \\
& \searrow_{f'} & \downarrow{\scriptstyle G(q)} \\
& & G(B')
\end{array}
$$

commute.

We now make the connecton between comma categories and adjunctions.

Lemma 3.4. *Take an adjunction* $\mathcal{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$ *and an object* $A \in \mathcal{A}$. *Then the unit map* $\eta_A : A \to GF(A)$ *is an initial object of* $(A \Rightarrow G)$.

Proof. Take $(B, f) \in (A \Rightarrow G)$, where $B \in \mathcal{B}$ and $f : A \to G(B)$. We want to show that there exists a single map from $(F(A), \eta_A)$ to (B, f). Such a map in $(A \Rightarrow G)$ is a map q in $\mathcal{B}$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{\ \eta_A\ } & GF(A) \\
& \searrow_{f} & \downarrow{\scriptstyle G(q)} \\
& & G(B)
\end{array}
$$

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commutes. By Lemma 3.3, we know that $G(q) \circ \eta_A = \bar{g}$. So the triangle above commutes if and only if $\bar{g} = f$ if and only if $g = \bar{f}$. Hence $\bar{f}$ is the unique map from $(F(A), \eta_A)$ to (B, f). $\qquad \square$

This leads us the the third formulation of adjointness.

Theorem 3.2. *Take categories and functors* $\mathbf{\mathcal{A}} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbf{\mathcal{B}}$ *. There is a one-to-one correspondence between:*

(a) *adjunctions between F and G;*

(b) *natural transformations $\eta : 1_{\mathbf{\mathcal{A}}} \to GF$ such that $\eta_A : A \to GF(A)$ is initial in $(A \Rightarrow G)$ for every $A \in \mathbf{\mathcal{A}}$.*

Proof. Omitted. $\qquad \square$

Corollary 3.1. *Let $G : \mathbf{\mathcal{B}} \to \mathbf{\mathcal{A}}$ be a functor. Then G has a left adjoint if and only if for each $A \in \mathbf{\mathcal{A}}$, the category $(A \Rightarrow G)$ has an initial object.*

Proof. We have already shown the 'only if' part from Lemma 3.4. Now choose for each $A \in \mathbf{\mathcal{A}}$ an initial object of $(A \Rightarrow G)$ and call it $\big(F(A), \eta_A : A \to GF(A)\big)$. (Here $F(A)$ and η_A are names we chose.) For each map $f : A \to A'$ in $\mathbf{\mathcal{A}}$, let $F(f) : F(A) \to F(A')$ be the unique map such that

$$
\begin{array}{ccc}
A & \xrightarrow{\ \eta_A\ } & G(F(A)) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(F(f))} \\
A' & \xrightarrow[\ \eta_{A'}\ ]{} & G(F(A'))
\end{array}
$$

commutes. It is easily checked that F is a functor $\mathbf{\mathcal{A}} \to \mathbf{\mathcal{B}}$, and the diagram tells us that η is a natural transformation $1 \to GF$. By Theorem 3.2, F is left adjoint to G. $\qquad \square$

4 Interlude on sets

4.1 Constructions with sets

Sets and functions form a category, denoted by **Set**. The empty set, $\emptyset$, is an initial object of **Set** and the one-element set, 1, is a terminal object. Any two sets have a **product**, $A \times B$, and a **sum** $A + B$, also known as disjoint union written as $\sqcup$. For any two sets A and B, we can form the set A^B of functions from B to A.

Let 2 be the set $1 + 1$ (two elements). Write the elements of 2 as `true` and `false`.

Let A be a set. Given a subset S of A, we obtain a function $\chi_S : A \to 2$ (the **characteristic function** of $S \subseteq A$), where

$$
\chi_S(a) = \begin{cases} \texttt{true} & \text{if } a \in S, \\ \texttt{false} & \text{if } a \notin S \end{cases}
$$

$(a \in A)$. Conversely, given a function $f : A \to 2$, we obtain a subset

$$
f^{-1}\{\texttt{true}\} = \{a \in A \mid f(a) = \texttt{true}\}
$$

of A. These two processes are mutually inverse; that is, χ_S is the unique function $f : A \to 2$ such that $f^{-1}\{\texttt{true}\} = S$. Hence:

Subsets of A correspond one-to-one with functions $A \to 2$.

Hence we can think of 2^A as the set of all subset of A, and call it the **power set** of A and write it as $\mathcal{P}(A)$.

Equalizers: Given sets and functions $A \underset{g}{\overset{f}{\rightrightarrows}} B$, there is a set

$$\{a \in A \mid f(a) = g(a)\}.$$

This set is called the **equalizer** of f and g, since it is the part of A on which the two functions are equal.

Quotients: Let A be a set and $\sim$ an equivalence relation on A. There is a set $A/\sim$, the **quotient of A by $\sim$**, whose elements are the equivalence classes. There is also a canonical map

$$p : A \to A/\sim,$$

sending an element of A to its equivalence class. It is surjective, and has the property that $p(a) = p(a') \iff a \sim a'$. In fact, it has a universal property: any function $f : A \to B$ such that

$$\forall a, a' \in A, \quad a \sim a' \Rightarrow f(a) = f(a') \tag{9}$$

factorizes uniquely through p, as in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ p\ } & A/\sim \\
 & f \searrow & \downarrow \bar{f} \\
 & & B
\end{array}
$$

Thus, for any set B, the functions $A/\sim \; \to B$ correspond one-to-one with the functions $f : A \to B$ satisfying (9).

Natural numbers: A function with domain $\mathbb{N}$ is usually called a **sequence**. A crucial property of $\mathbb{N}$ is that sequences can be defined recursively: given a set X, an element $a \in X$, and a function $r : X \to X$, there is a unique sequence $(x_n)_{n=0}^{\infty}$ of elements of X such that

$$x_0 = a, \quad x_{n+1} = r(x_n) \text{ for all } n \in \mathbb{N}.$$

This property is related to two pieces of structure on $\mathbb{N}$: the element 0, and the function $s : \mathbb{N} \to \mathbb{N}$ defined by $s(n) = n + 1$. Reformulated in terms of functions, and writing $x_n = x(n)$, the property is this: for any set X, element $a \in X$, and function $r : X \to X$, there is a unique function $x : \mathbb{N} \to X$ such that $x(0) = a$ and $x \circ s = r \circ x$. This is a universal property of $\mathbb{N}, 0$ and s (for more, see Peano Category).

Choice: Let $f : A \to B$ be a map in a category $\mathcal{A}$. A **section** (or **right inverse**) of f is a map $i : B \to A$ in $\mathcal{A}$ such that $f \circ i = 1_B$.

In the category of sets, any map with a section is certainly surjective. The converse statement is called the **axiom of choice**:

Every surjection has a section.

It is called 'choice' because specifying a section of $f : A \to B$ amounts to choosing, for each $b \in B$, an element of the nonempty set $\{a \in A \mid f(a) = b\}$.

4.2 Small and large categories

Given sets A and B, write $|A| \leq |B|$ if there exists an injection $A \to B$. Since identity maps are injective, $|A| \leq |A|$ for all sets A.

Theorem 4.1 (Cantor-Bernstein)**.** *Let A and B be sets. If $|A| \leq |B| \leq |A|$ then $A \cong B$.*

These observations tell us that $\leq$ is a preorder. We write $|A| = |B|$, and say that A and B have the same **cardinality** if $A \cong B$.

Theorem 4.2. *Let A be a set. Then $|A| < |\mathcal{P}(A)|$.*

Corollary 4.1. *For every set A, there is a set B such that $|A| < |B|$.*

In other words, there is no biggest set.

We use the word **class** informally to mean any collection of mathematical objects. All sets are classes, but some classes are too big to be sets. A class will be called **small** if it is a set, and **large** otherwise.

A category $\mathcal{A}$ is **small** if the class or collection of all maps in $\mathcal{A}$ is small, and **large** otherwise. If $\mathcal{A}$ is small then the class of objects of $\mathcal{A}$ is small too, since objects correspond one-to-one with identity maps.

A category $\mathcal{A}$ is **locally small** if for eah $A, B \in \mathcal{A}$, the class $\mathcal{A}(A, B)$ is small. The class $\mathcal{A}(A, B)$ is often called the **hom-set** from A to B.

Example 4.1. Set, $\mathbf{Vect}_k$, **Grp**, **Ab**, **Ring** and **Top** are all locally small but not small.

A category is small if and only if it is locally small and its class of objects is small.

A category is **essentially small** if it is equivalent to some small category. For example, the category of finite sets is essentially small as it is equivalent to the category whose objects are natural numbers, which form a set.

Proposition 4.1. Set *is not essentially small.*

Definition 4.1. We denote by **Cat** the category of small categories and functors between them.

5   Representables

5.1   Definitions and examples

Definition 5.1. Let $\mathcal{A}$ be a locally small category and $A \in \mathcal{A}$. We define a functor

$$H^A = \mathcal{A}(A, -) : \mathcal{A} \to \mathbf{Set}$$

as follows:

- for objects $B \in \mathcal{A}$, put $H^A(B) = \mathcal{A}(A, B)$;

- for maps $B \xrightarrow{g} B'$ in $\mathcal{A}$, define

$$H^A(g) = \mathcal{A}(A, g) : \mathcal{A}(A, B) \to \mathcal{A}(A, B')$$

   by

$$p \mapsto g \circ p$$

   for all $p : A \to B$.

Definition 5.2. Let $\mathcal{A}$ be a locally small category. A functor $X : \mathcal{A} \to \mathbf{Set}$ is **representable** if $X \cong H^A$ for some $A \in \mathcal{A}$. A **representation** of X is a choice of an object $A \in \mathcal{A}$ and an isomorphism between H^A and X.

Example 5.1. Consider $H^1 : \mathbf{Set} \to \mathbf{Set}$ where 1 is the singleton. H^1 sees elements of a set S in **Set**. We have

$$H^1(S) \cong S$$

for each $S \in \mathbf{Set}$. It can be verified that this isomorphism is natural in S, so H^1 is isomorphic to the identity functor $1_{\mathbf{Set}}$. Hence $1_{\mathbf{Set}}$ is representable.

Example 5.2. All the 'seeing' functors are representable. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is isomorphic to $H^1 = \mathbf{Top}(1, -)$, and the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ is isomorphic to $\mathbf{Grp}(\mathbb{Z}, -)$.

Example 5.3. There is a functor ob:$\mathbf{Cat} \to \mathbf{Set}$ sending a small category to its set of objects. It is representable. Indeed, consider the terminal category $\mathbf{1}$. A functor from $\mathbf{1}$ to a category $\mathcal{B}$ simply picks out an object of $\mathcal{B}$. Thus,

$$H^1(\mathcal{B}) \cong \mathrm{ob}\mathcal{B}.$$

Again, it is easily verified that this isomorphism is natural in $\mathcal{B}$; hence ob$\cong \mathbf{Cat}(\mathbf{1}, -)$.

Lemma 5.1. *Let* $\mathcal{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$ *be locally small categories, and let $A \in \mathcal{A}$. Then the functor*

$$\mathcal{A}(A, G(-)) : \mathcal{B} \to \mathbf{Set}$$

(that is, the composite $\mathcal{B} \xrightarrow{G} \mathcal{A} \xrightarrow{H^A} \mathbf{Set}$) is representable.

Proof. We have

$$\mathcal{A}(A, G(B)) \cong \mathcal{B}(F(A), B)$$

for each $B \in \mathcal{B}$. This isomorphism is natural in B, so $\mathcal{A}(A, G(-))$ is isomorphic to $H^{F(A)}$ and is therefore representable. $\qquad\square$

Proposition 5.1. *Any set-valued functor with left adjoint is representable.*

Proof. Let $G : \mathcal{A} \to \mathbf{Set}$ be a functor with a left adjoint F. If 1 is the singleton, consider

$$\mathcal{A} \xrightarrow{G} \mathbf{Set} \xrightarrow{H^1} \mathbf{Set}.$$

We have

$$G(A) \cong \mathbf{Set}(1, G(A))$$

naturally in $A \in \mathcal{A}$, i.e., $G \cong \mathbf{Set}(1, G(-))$. By Lemma 5.1, we have $\mathbf{Set}(1, G(-)) \cong H^{F(1)}$, hence, $G \cong H^{F(1)}$. $\qquad\square$

Example 5.4. The forgetful functor $U : \mathbf{Vect}_k \to \mathbf{Set}$ is representable, since it has a left adjoint. Indeed, if F is the left ajoint then $F(1)$ is the 1-dimensional vector space k, so $U \cong H^k$.

A map $A' \xrightarrow{f} A$ induces a natural transformation

whose B-component (for $B \in \mathcal{A}$) is the function

$$H^A(B) = \mathcal{A}(A, B) \to H^{A'}(B) = \mathcal{A}(A', B)$$
$$p \qquad \mapsto \qquad p \circ f.$$

Note the reversal of direction. Each functor H^A is covariant, but they come together to form a *contravariant* functor.

Definition 5.3. Let $\mathcal{A}$ be a locally small category. The functor

$$H^{\bullet} : \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}, \mathbf{Set}]$$

is defined on objects A by $H^{\bullet}(A) = H^A$ and on maps f by $H^{\bullet}(f) = H^f$.

All the definitions presented so far can be dualized.

Definition 5.4. Let $\mathcal{A}$ be a locally small category and $A \in \mathcal{A}$. We define a functor

$$H_A = \mathcal{A}(-, A) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$$

as follows:

- for objects $B \in \mathcal{A}$, put $H_A(B) = \mathcal{A}(B, A)$;

- for maps $B' \xrightarrow{g} B$ in $\mathcal{A}$, define

$$H_A(g) = \mathcal{A}(g, A) : \mathcal{A}(B, A) \to \mathcal{A}(B', A)$$

 by

$$p \mapsto p \circ g$$

 for all $p : B \to A$.

Definition 5.5. Let $\mathcal{A}$ be a locally small category. A functor $X : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is **representable** if $X \cong H_A$ for some $A \in \mathcal{A}$. A **representation** of X is a choice of an object $A \in \mathcal{A}$ and an isomorphism between H_A and X.

Any map $A \xrightarrow{f} A'$ in $\mathcal{A}$ induces a natural transformation

whose component at an object $B \in \mathcal{A}$ is

$$H_A(B) = \mathcal{A}(B, A) \to H_{A'}(B) = \mathcal{A}(B, A')$$
$$p \qquad \mapsto \qquad f \circ p.$$

Definition 5.6. Let $\mathcal{A}$ be a locally small category. The **Yoneda embedding** of $\mathcal{A}$ is the functor

$$H_{\bullet} : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$$

defined on objects A by $H_{\bullet}(A) = H_A$ and on maps f by $H_{\bullet}(f) = H_f$.

Here is a summary of the definitions so far.

For each $A \in \mathcal{A}$, we have a functor	$\mathcal{A} \xrightarrow{H^A} \mathbf{Set}$.
Putting them all together gives a functor	$\mathcal{A}^{\mathrm{op}} \xrightarrow{H^{\bullet}} [\mathcal{A}, \mathbf{Set}]$.

For each $A \in \mathcal{A}$, we have a functor	$\mathcal{A}^{\mathrm{op}} \xrightarrow{H_A} \mathbf{Set}$.
Putting them all together gives a functor	$\mathcal{A} \xrightarrow{H_{\bullet}} [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

Definition 5.7. Let $\mathcal{A}$ be a locally small category. The functor

$$\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$$

is defined by

$$
\begin{array}{ccc}
(A, B) & \mapsto & \mathcal{A}(A, B) \\
f \Big\uparrow \ \Big\downarrow g & \mapsto & \Big\downarrow g \circ - \circ f \\
(A', B') & \mapsto & \mathcal{A}(A', B').
\end{array}
$$

In other words, $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathcal{A}(A, B)$ and $(\mathrm{Hom}_{\mathcal{A}}(f, g))(p) = g \circ p \circ f$, whenever $A' \xrightarrow{f} A \xrightarrow{p} B \xrightarrow{g} B'$.

Remark. We can now explain the naturality in the definition of adjunction. Take categories and functors $\mathcal{A} \ \overset{F}{\underset{G}{\rightleftarrows}} \ \mathcal{B}$. They give rise to functors

$$
\begin{array}{ccc}
\mathcal{A}^{\mathrm{op}} \times \mathcal{B} & \xrightarrow{\ 1 \times G\ } & \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \\
{\scriptstyle F^{\mathrm{op}} \times 1} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}} \\
\mathcal{B}^{\mathrm{op}} \times \mathcal{B} & \xrightarrow[\ \mathrm{Hom}_{\mathcal{B}}\ ]{} & \mathbf{Set}.
\end{array}
$$

The composite functor $\downarrow_{\rightarrow}$ sends (A, B) to $\mathcal{B}(F(A), B)$; it can be written as $\mathcal{B}(F(-), -)$. The composite $^{\rightarrow}\downarrow$ sends (A, B) to $\mathcal{A}(A, G(B))$. Two functors

$$\mathcal{B}(F(-), -), \ \mathcal{A}(-, G(-)) : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$$

are naturally isomorphic if and only if F and G are adjoint.

Definition 5.8. Let A be an object of a category. A **generalized element** of A is a map with codomain A. A map $S \to A$ is a generalized element of A of **shape** S.

'Generalized element' is nothing more than a synonym of 'map', but sometimes it is useful to think of maps as generalized elements. For example, when A is a set, a generalized element of A of shape 1 is an ordinary element of A, and a generalized element of A of shape $\mathbb{N}$ is a sequence in A.

5.2 The Yoneda lemma

Functors from $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ are sometimes called 'presheaves' on $\mathcal{A}$. So for each $A \in \mathcal{A}$ we have a representable presheaf H_A, and we are asking how the rest of the presheaf category $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ looks from the viewpoint of H_A. I.e., if X is another presheaf, what are the maps $H_A \to X$?

Fix a small category $\mathcal{A}$. Take an object $A \in \mathcal{A}$ and a functor $X : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. The object A gives rise to another functor $H_A = \mathcal{A}(-, A) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. The question is: what are the maps $H_A \to X$? Since H_A and X are both objects of the presheaf category $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, the 'maps' concerned are maps in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. So, we are asking what natural transformations

$$
\mathcal{A}^{\mathrm{op}} \ \overset{H_A}{\underset{X}{\Downarrow}} \ \mathbf{Set}
$$

there are. The set of such natural transformations is called

$$[\boldsymbol{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}](H_A, X).$$

Are there any other ways to construct a set from the same input data (A, X)? Yes: simply take the set $X(A)$.

Theorem 5.1 (Yoneda)**.** *Let $\boldsymbol{\mathcal{A}}$ be a locally small category. Then*

$$[\boldsymbol{\mathcal{A}}^{op}, \mathbf{Set}](H_A, X) \cong X(A) \tag{10}$$

naturally in $A \in \boldsymbol{\mathcal{A}}$ and $X \in [\boldsymbol{\mathcal{A}}^{op}, \mathbf{Set}]$.