Cheat Sheet

Proposition 0.1. *A functor is an equivalence if and only if it is full, faithful, and essentially surjective on objects.*

The naturality axiom of adjoints has two parts:

$$\overline{\left(F(A) \xrightarrow{g} B \xrightarrow{q} B'\right)} = \left(A \xrightarrow{\bar{g}} G(B) \xrightarrow{G(q)} G(B')\right) \tag{1}$$

(that is, $\overline{q \circ g} = G(q) \circ \bar{g}$) for all g and q, and

$$\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B)\right)} = \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B\right) \tag{2}$$

for all p and f.

Forgetful functors between categories of algebraic structures usually have left adjoints.

Example 0.1. Initial and terminal objects can be described as adjoints. Let $\boldsymbol{\mathcal{A}}$ be a category. There is precisely one functor $\boldsymbol{\mathcal{A}} \to \mathbf{1}$. Also, a functor $\mathbf{1} \to \boldsymbol{\mathcal{A}}$ is essentially just an object of $\boldsymbol{\mathcal{A}}$. Viewing functors $\mathbf{1} \to \boldsymbol{\mathcal{A}}$ as objects of $\boldsymbol{\mathcal{A}}$, a left adjoint to $\boldsymbol{\mathcal{A}} \to \mathbf{1}$ is exactly an initial object of $\boldsymbol{\mathcal{A}}$.

Similarly, a right adjoint to the unique functor $\boldsymbol{\mathcal{A}} \to \mathbf{1}$ is exactly a terminal object of $\boldsymbol{\mathcal{A}}$.

For each $A \in \boldsymbol{\mathcal{A}}$, we have a map

$$\left(A \xrightarrow{\eta_A} GF(A)\right) = \overline{\left(F(A) \xrightarrow{1} F(A)\right)}. \tag{3}$$

Dually, for each $B \in \boldsymbol{\mathcal{B}}$, we have a map

$$\left(FG(B) \xrightarrow{\epsilon_B} B\right) = \overline{\left(G(B) \xrightarrow{1} G(B)\right)}. \tag{4}$$

These define natural transformations

$$\eta : 1_{\boldsymbol{\mathcal{A}}} \to G \circ F, \qquad \epsilon : F \circ G \to 1_{\boldsymbol{\mathcal{B}}},$$

called the **unit** and **counit** of the adjunction, respectively.

Lemma 0.1. *Given an adjunction $F \dashv G$ with unit η and counit ϵ, the triangles*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

commute.

Remark. These are called the **triangle identities**. An equivalent statement is that the triangles

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "1_{F(A)}"'] & FGF(A) \arrow[d, "\epsilon_{F(A)}"] \\ & F(A) \end{tikzcd} \qquad \begin{tikzcd} G(B) \arrow[r, "\eta_{G(B)}"] \arrow[dr, "1_{G(B)}"'] & GFG(B) \arrow[d, "G(\epsilon_B)"] \\ & G(B) \end{tikzcd} \tag{5}$$

commute for all $A \in \boldsymbol{\mathcal{A}}$ and $B \in \boldsymbol{\mathcal{B}}$.

Definition 0.1. Given categories and functors

$$\begin{tikzcd} & \boldsymbol{\mathcal{B}} \arrow[d, "Q"] \\ \boldsymbol{\mathcal{A}} \arrow[r, "P"'] & \boldsymbol{\mathcal{C}}, \end{tikzcd}$$

the **comma category** $(P \Rightarrow Q)$ (often written as $(P \downarrow Q)$) is the category defined as follows:

- objects are triples (A, h, B) with $A \in \boldsymbol{\mathcal{A}}, B \in \boldsymbol{\mathcal{B}}$, and $h : P(A) \to Q(B)$ in $\boldsymbol{\mathcal{C}}$;
- maps $(A, h, B) \to (A', h', B')$ are pairs $(f : A \to A', g : B \to B')$ of maps such that the square

$$\begin{tikzcd} P(A) \arrow[r, "P(f)"] \arrow[d, "h"'] & P(A') \arrow[d, "h'"] \\ Q(B) \arrow[r, "Q(g)"'] & Q(B') \end{tikzcd}$$

  commutes.

Example 0.2. Let $\boldsymbol{\mathcal{A}}$ be a category and $A \in \boldsymbol{\mathcal{A}}$. The **slice category** of $\boldsymbol{\mathcal{A}}$ over A, denoted by $\boldsymbol{\mathcal{A}}/A$, is the category whose objects are maps into A and whose maps are commutative triangles. More precisely, an object is a pair (X, h) with $X \in \boldsymbol{\mathcal{A}}$ and $h : X \to A$ in $\boldsymbol{\mathcal{A}}$, and a map $(X, h) \to (X', h')$ in $\boldsymbol{\mathcal{A}}/A$ is a map $f : X \to X'$ in $\boldsymbol{\mathcal{A}}$ making the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\ & A & \end{tikzcd}$$

commute.

Slice categories are a special case of comma categories. Functors $\mathbf{1} \to \boldsymbol{\mathcal{A}}$ are just objects of $\boldsymbol{\mathcal{A}}$. Now, given an object A of $\boldsymbol{\mathcal{A}}$, consider the comma category $(1_{\boldsymbol{\mathcal{A}}} \Rightarrow A)$, as in the diagram

$$\begin{tikzcd} & \mathbf{1} \arrow[d, "A"] \\ \boldsymbol{\mathcal{A}} \arrow[r, "1_{\boldsymbol{\mathcal{A}}}"'] & \boldsymbol{\mathcal{A}}. \end{tikzcd}$$

An object of $(1_{\boldsymbol{\mathcal{A}}} \Rightarrow A)$ is in principle a triple (X, h, B), with $X \in \boldsymbol{\mathcal{A}}$, $B \in \mathbf{1}$, and $h : X \to A$ in $\boldsymbol{\mathcal{A}}$; but $\mathbf{1}$ has only one object, so it is essentially just a pair (X, h). Hence the comma category $(1_{\boldsymbol{\mathcal{A}}} \Rightarrow A)$ has the same objects as the slice category $\boldsymbol{\mathcal{A}}/A$ and one can check that is has the same maps too, so $\boldsymbol{\mathcal{A}}/A \cong (1_{\boldsymbol{\mathcal{A}}} \Rightarrow A)$.

Lemma 0.2. *Take an adjunction* $\begin{tikzcd} \boldsymbol{\mathcal{A}} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \boldsymbol{\mathcal{B}} \arrow[l, shift left, "G"] \end{tikzcd}$ *and an object* $A \in \boldsymbol{\mathcal{A}}$*. Then the unit map* $\eta_A : A \to GF(A)$ *is an initial object of* $(A \Rightarrow G)$*.*

Corollary 0.1. *Let* $G : \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}$ *be a functor. Then* G *has a left adjoint if and only if for each* $A \in \boldsymbol{\mathcal{A}}$*, the category* $(A \Rightarrow G)$ *has an initial object.*

Definition 0.2. Let $\boldsymbol{\mathcal{A}}$ be a locally small category and $A \in \boldsymbol{\mathcal{A}}$. We define a functor

$$H^A = \boldsymbol{\mathcal{A}}(A, -) : \boldsymbol{\mathcal{A}} \to \mathbf{Set}$$

as follows:

- for objects $B \in \boldsymbol{\mathcal{A}}$, put $H^A(B) = \boldsymbol{\mathcal{A}}(A, B)$;
- for maps $B \xrightarrow{g} B'$ in $\boldsymbol{\mathcal{A}}$, define

$$H^A(g) = \boldsymbol{\mathcal{A}}(A, g) : \boldsymbol{\mathcal{A}}(A, B) \to \boldsymbol{\mathcal{A}}(A, B')$$

  by

$$p \mapsto g \circ p$$

  for all $p : A \to B$.

Definition 0.3. Let $\boldsymbol{\mathcal{A}}$ be a locally small category. A functor $X : \boldsymbol{\mathcal{A}} \to \mathbf{Set}$ is **representable** if $X \cong H^A$ for some $A \in \boldsymbol{\mathcal{A}}$. A **representation** of X is a choice of an object $A \in \boldsymbol{\mathcal{A}}$ and an isomorphism between H^A and X.

Proposition 0.2. *Any set-valued functor with left adjoint is representable.*

Definition 0.4. Let $\boldsymbol{\mathcal{A}}$ be a locally small category and $A \in \boldsymbol{\mathcal{A}}$. We define a functor

$$H_A = \boldsymbol{\mathcal{A}}(-, A) : \boldsymbol{\mathcal{A}}^{\mathrm{op}} \to \mathbf{Set}$$

as follows:

- for objects $B \in \boldsymbol{\mathcal{A}}$, put $H_A(B) = \boldsymbol{\mathcal{A}}(B, A)$;
- for maps $B' \xrightarrow{g} B$ in $\boldsymbol{\mathcal{A}}$, define

  $$H_A(g) = \boldsymbol{\mathcal{A}}(g, A) : \boldsymbol{\mathcal{A}}(B, A) \to \boldsymbol{\mathcal{A}}(B', A)$$

  by

  $$p \mapsto p \circ g$$

  for all $p : B \to A$.

Definition 0.5. Let $\boldsymbol{\mathcal{A}}$ be a locally small category. A functor $X : \boldsymbol{\mathcal{A}}^{\mathrm{op}} \to \mathbf{Set}$ is **representable** if $X \cong H_A$ for some $A \in \boldsymbol{\mathcal{A}}$. A **representation** of X is a choice of an object $A \in \boldsymbol{\mathcal{A}}$ and an isomorphism between H_A and X.

Any map $A \xrightarrow{f} A'$ in $\boldsymbol{\mathcal{A}}$ induces a natural transformation

$$\begin{tikzcd} \boldsymbol{\mathcal{A}}^{\mathrm{op}} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "H_{A'}"'] \arrow[r, phantom, "\Downarrow H_f"] & \mathbf{Set} \end{tikzcd}$$

whose component at an object $B \in \boldsymbol{\mathcal{A}}$ is

$$\begin{array}{ccc} H_A(B) = \boldsymbol{\mathcal{A}}(B, A) & \to & H_{A'}(B) = \boldsymbol{\mathcal{A}}(B, A') \\ p & \mapsto & f \circ p. \end{array}$$

Definition 0.6. Let $\boldsymbol{\mathcal{A}}$ be a locally small category. The **Yoneda embedding** of $\boldsymbol{\mathcal{A}}$ is the functor

$$H_\bullet : \boldsymbol{\mathcal{A}} \to [\boldsymbol{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}]$$

defined on objects A by $H_\bullet(A) = H_A$ and on maps f by $H_\bullet(f) = H_f$.

Here is a summary of the definitions so far.

For each $A \in \boldsymbol{\mathcal{A}}$, we have a functor	$\boldsymbol{\mathcal{A}} \xrightarrow{H^A} \mathbf{Set}.$
Putting them all together gives a functor	$\boldsymbol{\mathcal{A}}^{\mathrm{op}} \xrightarrow{H^\bullet} [\boldsymbol{\mathcal{A}}, \mathbf{Set}].$
For each $A \in \boldsymbol{\mathcal{A}}$, we have a functor	$\boldsymbol{\mathcal{A}}^{\mathrm{op}} \xrightarrow{H_A} \mathbf{Set}.$
Putting them all together gives a functor	$\boldsymbol{\mathcal{A}} \xrightarrow{H_\bullet} [\boldsymbol{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}].$

Fix a small category $\boldsymbol{\mathcal{A}}$. Take an object $A \in \boldsymbol{\mathcal{A}}$ and a functor $X : \boldsymbol{\mathcal{A}}^{\mathrm{op}} \to \mathbf{Set}$. The object A gives rise to another functor $H_A = \boldsymbol{\mathcal{A}}(-, A) : \boldsymbol{\mathcal{A}}^{\mathrm{op}} \to \mathbf{Set}$. The question is: what are the maps $H_A \to X$? Since H_A and X are both objects of the presheaf category $[\boldsymbol{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}]$, the 'maps' concerned are maps in $[\boldsymbol{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}]$. So, we are asking what natural transformations

$$\begin{tikzcd} \boldsymbol{\mathcal{A}}^{\mathrm{op}} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{Set} \end{tikzcd}$$

there are. The set of such natural transformations is called

$$[\boldsymbol{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}](H_A, X).$$

Are there any other ways to construct a set from the same input data (A, X)? Yes: simply take the set $X(A)$.

Theorem 0.1 (Yoneda)**.** *Let $\mathcal{A}$ be a locally small category. Then*

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \cong X(A) \tag{6}$$

naturally in $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

Proposition 0.3. *Let $\mathcal{A}$ be a category.*

(a) If $\mathcal{A}$ has all products and equalizers then $\mathcal{A}$ has all limits.

(b) If $\mathcal{A}$ has binary products, a terminal object and equalizers then $\mathcal{A}$ has finite limits.

Lemma 0.3. *A map $X \xrightarrow{f} Y$ is monic if and only if the square*

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback.

Example 0.3. Take sets and functions $X \overset{s}{\underset{t}{\rightrightarrows}} Y$. To find the coequalizer of s and t, we must construct in some canonical way a set C and a function $p : Y \to C$ such that $p(s(x)) = p(t(x))$ for all $x \in X$. Let $\sim$ be the equivalence relation on Y generated by $s(x) \sim t(x)$ for all $x \in X$. Take the quotient map $p : Y \to Y/\sim$. The maps $Y/\sim \to B$ correspond one-to-one with maps $f : Y \to B$, so p is indeed the coequalizer of s and t.

Definition 0.7. (a) Let $\mathbf{I}$ be a small category. A functor $F : \mathcal{A} \to \mathcal{B}$ **preserves limits of shape I** if for all diagrams $D : \mathbf{I} \to \mathcal{A}$ and all cones $\left(A \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{I}}$ on D,

$$\begin{aligned} &\left(A \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{I}} \text{ is a limit cone on } D \text{ in } \mathcal{A} \\ \Rightarrow &\left(F(A) \xrightarrow{Fp_I} FD(I)\right)_{I \in \mathbf{I}} \text{ is a limit cone on } F \circ D \text{ in } \mathcal{B}. \end{aligned}$$

(b) A functor $F : \mathcal{A} \to \mathcal{B}$ **preserves limits** if it preserves limits of shape $\mathbf{I}$ for all small categories $\mathbf{I}$.

(c) **Reflection** of limits is defined as in (a), but with $\Leftarrow$ instead of $\Rightarrow$.

Definition 0.8. A functor $F : \mathcal{A} \to \mathcal{B}$ **creates limits (of shape I)** if whenever $D : \mathbf{I} \to \mathcal{A}$ is a diagram in $\mathcal{A}$,

- for any limit cone $\left(B \xrightarrow{q_I} FD(I)\right)_{I \in \mathbf{I}}$ on the diagram $F \circ D$, there is a unique cone $\left(A \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{I}}$ on D such that $F(A) = B$ and $F(p_I) = q_I$ for all $I \in \mathbf{I}$;
- this cone $\left(A \xrightarrow{p_I} D(I)\right)_{I \in \mathbf{I}}$ is a limit cone on D.

The forgetful functors from $\mathbf{Grp}, \mathbf{Ring}, \ldots$ to $\mathbf{Set}$ all create limits.

Lemma 0.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and $\mathbf{I}$ a small category. Suppose that $\mathcal{B}$ has, and F creates, limits of shape $\mathbf{I}$. Then $\mathcal{A}$ has, and F preserves limits of shape $\mathbf{I}$.*

0.1 Bonus

$$\mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B) \cong \mathrm{Hom}(X, A \times B)$$

$$\mathrm{Hom}(A, X) \times \mathrm{Hom}(B, X) \cong \mathrm{Hom}(A + B, X)$$

$$\mathrm{Hom}(X \times A, B) \cong \mathrm{Hom}(X, B^A)$$

$$HA \cong HB \Rightarrow A \cong B$$