The Inclusion-Exclusion Principle

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1 Introduction

In combinatorics, the Inclusion-Exclusion Principle - also known as the Sieve Principle - is a generalized method of counting the membership of a union of sets. For two finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cup B|$$

This statement means that the number of elements in the union of the two sets ($A \cup B$) is the sum of the elements in each set, minus the number of elements in the intersection of the two sets($A \cap B$). This formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. Possible double-counted elements would be in the intersection of the two sets. Thus, the count is corrected by subtracting the size of the intersection. Similarly, for three finite sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

In the more general case where there are n different sets, the formula becomes,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j:1 \leq i \leq j \leq n} |A_i \cap A_j| + \sum_{i,j,k:1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n| .$$

This can be more compactly written, as the Inclusion-Exclusion Principle explains.

Theorem. [The Inclusion-Exclusion Principle]

Let $A_1, A_2, ..., A_n$ be finite sets. Then,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|-1} \left|\bigcap_{i \in J} A_i\right| \tag{1}$$

2 Proof

Proof. [1. Induction]

Initial step: when n = 1

$$\left|\bigcup_{i=1}^{1}\right| = |A_1| \tag{2}$$

$$\sum_{\substack{J\subseteq\{1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| = (-1)^0\left|\bigcup_{i\in\{1\}} A_i\right| = |A_1| \tag{3}$$

Thus, the theorem holds for $n = 1$.

Induction step: suppose the the theorem holds for $n-1$

$$|A_1 \cup A_2 \cup \ldots \cup A_{n-1}| = \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| \tag{4}$$

We can find $|A_1 \cup A_2 \cup \ldots \cup A_n|$ by

$$\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= |(A_1 \cup \ldots \cup A_{n-1}) \cup A_n| &(5)\\
&= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| &(6)\\
&= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)| &(7)\\
&= \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + |A_n| - \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i \cap A_n\right| &(8)\\
&= \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + |A_n| - \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\left(\bigcap_{i\in J} A_i\right) \cap A_n\right| &(9)\\
&= \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + |A_n| - \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in(J\cup\{n\})} A_i\right| &(10)\\
&= \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + \sum_{J\subseteq\{1,2,\ldots,n-1\}} (-1)^{|J|}\left|\bigcap_{i\in(J\cup\{n\})} A_i\right| &(11)\\
&= \sum_{\substack{J\subseteq\{1,2,\ldots,n-1\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| + \sum_{\substack{J\subseteq\{1,2,\ldots,n\}\\ n\in J}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| &(12)\\
&= \sum_{\substack{J\subseteq\{1,2,\ldots,n\}\\ J\neq\emptyset}} (-1)^{|J|-1}\left|\bigcap_{i\in J} A_i\right| &(13)
\end{aligned}$$

$\therefore$ By the Principle of Mathematical Induction, the theorem is true $\forall n \in \mathbb{N}$.

$\square$

Proof. [2. Binomial Theorem]

First, suppose x belongs to exactly m sets of the $A_1, A_2, \ldots, A_n$ where $1 \leq m \leq n$. Then, x is counted exactly m times in $\sum_{i=1}^{n}|A_i|$. x is counted $\binom{m}{2}$ times in $\sum_{i<j}|A_i \cap A_j|$. x is counted $\binom{m}{3}$ times in $\sum_{i<j<k}|A_i \cap A_j \cap A_k|$. And so on.

Thus, $\binom{m}{k}$ is the number of combinations of m objects taken at k at a time. Then, we eventually reach $\sum_{i_1<i_2<\ldots<i_m}|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}|$ where x is counted exactly once, $\binom{m}{m} = 1$. If we consider the intersection of more than m sets, then x is not counted at all. Therefore, if x is contained in exactly m

of the sets, then it will be counted P times in $|A_1 \cup A_2 \cup \ldots \cup A_n|$, where

$$P = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \ldots + (-1)^{m+1}\binom{m}{m} \tag{14}$$

. To compute P, we call Binomial Theorem,

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}. \tag{15}$$

Set $x = 1$, $y = -1$, then

$$\Rightarrow \sum_{k=0}^{m} (-1)^k \binom{m}{k} = 0 \tag{16}$$

$$\Rightarrow 1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \ldots + (-1)^{m+1}\binom{m}{m} = 0 \tag{17}$$

$$\Rightarrow P = 1 \tag{18}$$

Thus, there is no multiple counting of x in eq. (1). i.e., every x contained in the union of $A_1, A_2, ..., A_n$ is counted exactly once. Therefore, the theorem holds.

□

3 Application

Application. [1. Derangements]

Derangement is a permutation in which none of the objects appear in their "natural," or original, position. We will use the Inclusion-Exclusion Principle to count the number of possible derangements of n objects, which we we will denote by D_n.

Let X denote the set of all permutations of n objects. Then, let A_i be the subset of the set of permutations such that the ith object is in its original position. Then $|A_1 \cup A_2 \cup \ldots \cup A_n|$ counts the number of permutations where at least one of the n objects will end up in it original position.

$$D_n = |X| - |A_1 \cup A_2 \cup \ldots \cup A_n| \tag{19}$$

will be the total number of derangements. Also note that

$$|A_1 \cup \ldots \cup A_n| = \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|-1} \left| \bigcap_{i \in J} A_i \right| \tag{20}$$

$$= \sum_{i_1=1}^{n} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \ldots + (-1)^{p-1} \sum_{1 \le i_1 < \ldots < i_p \le n} \left|A_{i_1} \cap \ldots \cap A_{i_p}\right| - \ldots \tag{21}$$

$$= \binom{n}{1} |A_1| - \binom{n}{2} |A_1 \cap A_2| + \ldots + (-1)^{n-1}\binom{n}{n} |A_1 \cap \ldots \cap A_n| \tag{22}$$

$$= \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)! \tag{23}$$

$$\tag{24}$$

We already know there are $n!$ possible permutations of n objects, $|X| = n!$.

$$\Rightarrow D_n = n! + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)! \tag{25}$$

Example 1. How many derangements are there of $\{1, 2, \ldots, 7\}$?

Answer: Let X be the set of all permutations of $\{1, 2, \ldots, 7\}$. Then, we need to exclude all the permutations with 1 in the first position, 2 in the second position, 3 in the third position, and so on. Thus, using the above application formula, the set of derangments will be

$$D_n = n! + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)! \tag{26}$$

$$= 7! + \sum_{i=1}^{7} (-1)^i \binom{7}{i} (7-i)! \tag{27}$$

$$= 5040 + \left[-\binom{7}{1}(6!) + \binom{7}{2}(5!) - \binom{7}{3}(4!) + \binom{7}{4}(3!) - \binom{7}{5}(2!) + \binom{7}{6}(1!) - \binom{7}{7}(0!) \right] \tag{28}$$

$$= 5040 - 3186 \tag{29}$$

$$= 1854 \tag{30}$$

Application. [2. Probability]

In probabilty, the Inclusion-Exclusion Principle for events $A_1, A_2, \ldots, A_n$ becomes, for $n = 2$

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) \tag{31}$$

for $n = 3$,

$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3) \tag{32}$$

and in general,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i,j:1 \leq i \leq j \leq n} \mathbb{P}(A_i \cap A_j) + \sum_{i,j,k:1 \leq i \leq j \leq k \leq n} \mathbb{P}(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \tag{33}$$

which can then be compactly written as

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|-1} \mathbb{P}\left(\bigcap_{i \in J} A_i\right) \tag{34}$$

If, the probability of the intersection A_I only depends on the cardinality of I, i.e., for every k in $\{1, \ldots, n\}$ with $|I| - k$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} a_k$$

due to the combinatorial interpretation of the binomial coefficient $\binom{n}{k}$.

Example 1. If 5 people get their hats returned in random order, what is the chance that at least one gets the correct hat?

Answer: Let A_i be the event that person i gets the correct hat. Then,

$$\mathbb{P}(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

Similarly,

$$\mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_k}) = \frac{(n-k)!}{n!}$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{k} (-1)^{k-1} \binom{n}{k} \frac{(n-k)!}{n!} \tag{35}$$

$$= \sum_{i=1}^{k} (-1)^{k-1} \frac{1}{k!} \tag{36}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \tag{37}$$

$$= \frac{19}{30} \tag{38}$$

Application. [3. Number of Surjective Functions]

Given finite sets A and B, we can find how many surjective functions are there from A to B by using the Inclusion-Exclusion Principle.

Suppose $A = \{1, 2 \ldots, m\}$ and $B = \{1, 2, \ldots, n\}$. We want the total number of functions, minus the number of functions which fail to take some value. Since each values in A has n choices for its image, the choices are independent, and therefore the number of functions is n^m. If we let S_i be the set of all functions which do not map to the value i in B. Then, $|S_1| = |S_2| = \ldots = |S_n| = (n-1)^m$.

Similarly, $|S_{i_1} \cap \ldots \cap S_{i_k}| = (n-k)^m$. Thus,

$$= |S| - |S_1 \cup S_2 \cup \ldots \cup S_n| \tag{39}$$

$$= n^m - \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|-1} \left| \bigcap_{i \in J} S_i \right| \tag{40}$$

$$= n^m + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)^m \tag{41}$$

is the number of surjective functions from a set of size m to a set of size n.

Example 1. In how many ways can we put 8 guests in 4 rooms so that no room stays vacant?

Answer:

$$= n^m + \sum_{i=1}^{n} (-1)^i \binom{n}{i} (n-i)^m \tag{42}$$

$$= 4^8 + \sum_{i=1}^{4} (-1)^i \binom{4}{i} (4-i)^8 \tag{43}$$

$$= 65536 + \left[-\binom{4}{1}(3^8) + \binom{4}{2}(2^8) - \binom{4}{3}(1^8) + \binom{4}{4}(0^8) \right] \tag{44}$$

$$= 40824 \tag{45}$$

References

[1] Vita Smid. Inclusion-exclusion principle: Proof by mathematical induction for dummies, 2009. Accessed: 2017-06-10.

[2] Eric W Weisstein. Inclusion-exclusion principle From MathWorld—A Wolfram Web Resource, 2009. Accessed: 2017-06-10.

[1] [2]