Algorithmic differentiation

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- Chain rule
- Dual numbers

Chain rule

$$\frac{d}{dx} f(g(h(x))) = \frac{df(u)}{du} \frac{du(x)}{dx} \qquad\qquad u(x) = g(h(x))$$

forward mode: inside to outside

backwards mode: outside to inside

Chain rule

$$\frac{d}{dx} f(g(h(x))) = \frac{df(u)}{du} \frac{du(x)}{dx} \qquad u(x) = g(h(x))$$

$$= \frac{df(u)}{du} \frac{du(w)}{dw} \frac{dw(x)}{dx} \qquad w(x) = h(x)$$

forward mode: inside to outside

backwards mode: outside to inside

Dual numbers

Usual complex numbers

$$a + b\,i \qquad\qquad i^2 = -1$$

Generalizations

- $i^2 = 1$ hyperbolic, a.k.a split-complex numbers, perplex numbers
- $i^2 = 0$ parabolic, a.k.a. dual numbers, Grassman variables

Dual numbers

Binomial formula for $x, y \in \mathbb{R}$

$$(x + i\, y)^n = x^n + n x^{n-1}\, iy$$

Polynomials of $x + iy$ are all linear in y.

Algorithmic differentiaion: Dual numbers

For analytic $f(x)$, its Taylor series

$$f(x+iy) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(x+iy)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(x^n + nx^{n-1}iy)$$

$$= f(x) + iyf'(x)$$