2D Poisson equation

Dirichlet BV problem

Dirichlet problem for the 2D Poisson equation

Let $u(x)$ is defined for $x = (x_1, x_2) \in G = [0,1] \times [0,1]$.

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = f(x) \tag{1}$$

Boundary conditions

$$u(x) = \mu(x) \qquad \text{for } x \in \Gamma$$

Grids and grid functions

Consider a 2D grid

$$\Omega_h = \{x_{kl} = (x_1^k, x_2^l)\}$$

with

$$\begin{aligned} x_1^k &= h_1 k, \qquad k = 0, \cdots, K \qquad (h_1 K = 1) \\ x_2^l &= h_1 l, \qquad l = 0, \cdots, L \qquad (h_2 L = 1) \end{aligned}$$

Call ω_h the set of internal nodes and γ_h the set of boundary nodes.

Define a *grid function* $y_{kl} \equiv y(x_1^k, x_2^l)$.

The five-point stencil

Using the five-point stencil

Eq.(1) generates at $x_{kl} \in \omega_h$

$$\frac{y_{k+1,l} - 2y_{kl} + y_{k-1,l}}{h_1^2} + \frac{y_{k,l+1} - 2y_{kl} + y_{k,l-1}}{h_2^2} = f_{kl} \qquad (2)$$

and

$$y_{kl} = \mu(x_{kl}) \qquad \text{for } x_{kl} \in \gamma_h$$

The scheme approximates Eq. (1) with $O(h_1^2 + h_2^2)$.

The matrix form of the FDE

Rewrite the FDE (2) in the matrix form.

Define the flat index

$$j = k\,(L+1) + l$$

$$y_j \equiv y_{kl} \qquad \text{with} \quad k = 0, 1, \cdots, K \qquad l = 0, 1, \cdots, L$$

The matrix form of the FDE

$$\frac{y_{k+1,l} - 2y_{kl} + y_{k-1,l}}{h_1^2} + \frac{y_{k,l+1} - 2y_{kl} + y_{k,l-1}}{h_2^2} = f_{kl}$$

Indices:

$$(k, l \pm 1) \Rightarrow k(L+1) + (l \pm 1) = j \pm 1$$
$$(k \pm 1, l) \Rightarrow (k \pm 1)(L+1) + l = j \pm (L+1)$$

$$\frac{y_{j+L+1} - 2y_j + y_{j-(L+1)}}{h_1^2} + \frac{y_{j+1} - 2y_j + y_{j-1}}{h_2^2} = f_j$$

The matrix form of the FDE

Taking, for simplicity, $\mu(x) = 0$,

$$\mathbf{A}\mathbf{y} = \mathbf{f}$$

with $\mathbf{A}$ being "tridiagonal with fringes" of $(L-1) \times (L-1)$ blocks

The matrix form of the FDE

- The size of the $\mathbf{A}$ matrix is $O(L^2) = O(h^{-2})$
- Each row of $\mathbf{A}$ has at most five $\neq 0$ elements, the sparsity is

$$\sim 5/L^2 = O(h^{-2})$$

- Direct solve with e.g. Gauss elimination requires

$$O((L^2)^3) = O(L^6)$$

  flops.
  $\Longrightarrow$ Need a sparse solver.

Iterative methods for the 2D Poisson equation

Iterative methods, recap

The canonic form of the two-step iterative methods (a.k.a. *relaxation*)

$$\mathbf{P}\,\frac{\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}}{\tau} + \mathbf{A}\mathbf{x}^{(n)} = \mathbf{b}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \longrightarrow \qquad \mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$$

lower diagonal upper

Iterative methods, recap

- Jacobi iteration: $\mathbf{D}\mathbf{x}^{(n+1)} + (\mathbf{L} + \mathbf{U})\mathbf{x}^{(n)} = \mathbf{b}$

- Seidel iteration: $\mathbf{D}\mathbf{x}^{(n+1)} + \mathbf{L}\mathbf{x}^{(n+1)} + \mathbf{U}\mathbf{x}^{(n)} = \mathbf{b}$

Jacobi method for the 2D Poisson equation

Take $\mu(x) = 0$ for clarity.

$$\frac{y^n_{k+1,l} - 2y^{n+1}_{kl} + y^n_{k-1,l}}{h_1^2} + \frac{y^n_{k,l+1} - 2y^{n+1}_{kl} + y^n_{k,l-1}}{h_2^2} = f_{kl}\,, \quad x_{kl} \in \omega_h$$

$$y^n_{kl} = 0\,, \quad x_{kl} \in \gamma_h$$

Here $n = 0, 1, \cdots$

The convergence rate is $O(h^{-2})$.

Seidel iteration in components

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ & & \cdots & & \end{pmatrix}$$

$$a_{11}x_1^{(n+1)} + \qquad a_{12}x_2^{(n)} + a_{13}x_3^{(n)} + \cdots + a_{1m}x_m^{(n)} = b_1$$

$$a_{22}x_2^{(n+1)} + a_{21}x_1^{(n+1)} + \qquad a_{23}x_3^{(n)} + \cdots + a_{2m}x_m^{(n)} = b_2$$

$$\cdots$$

At each iteration, sweep down the system of equations; at each step use $\mathbf{x}^{(n+1)}$-s from previous steps: for $x_i^{(n+1)}$ use $x_1^{(n+1)} \cdots x_{i-1}^{(n+1)}$(Jacobi iteration uses $x_1^{(n)} \cdots x_{i-1}^{(n)}$)

Seidel method for the 2D Poisson equation

Take $\mu(x) = 0$ for clarity.

$$\frac{y^{n}_{k+1,l} - 2y^{n+1}_{kl} + y^{n+1}_{k-1,l}}{h_1^2} + \frac{y^{n}_{k,l+1} - 2y^{n+1}_{kl} + y^{n+1}_{k,l-1}}{h_2^2} = f_{kl}\,, \quad x_{kl} \in \omega_h$$

$$y^{n+1}_{kl} = 0\,, \quad x_{kl} \in \gamma_h$$

Here $n = 0, 1, \cdots$

The convergence rate is $O(h^{-2})$.

Relaxation methods

- Seidel iteration converges faster then Jacobi's (symmetric $\mathbf{A}$)
- The convergence rate is still $O(h^{-2})$
- S.O.R. and/or Chebyshev iterations: can push to $O(h^{-1})$
- More specialized methods: can push to $O(h^{-1/2})$

see, e.g. Part III Chap 5 in S.G.

Spectral approach to the 2D Poisson FDE

1D fast Fourier transform + tridiagonal solve

Dirichlet problem for 2D Poisson equation

Consider

$$\frac{y_{k+1,l} - 2y_{kl} + y_{k-1,l}}{h_1^2} + \frac{y_{k,l+1} - 2y_{kl} + y_{k,l-1}}{h_2^2} = f_{kl} \qquad (3)$$

$$y_{kl} = 0 \qquad \text{for } x_{kl} \in \gamma_h$$

Here

$$\begin{aligned}\Omega_h = \{x_{kl} = (x_1^k, x_2^l) \ : \quad & x_1^k = h_1 k, x_2^l = h_2 l \\ & k = 0, 1, \cdots, N_1\,, \quad l = 0, 1, \cdots, N_2 \\ & N_\alpha h_\alpha = a_\alpha\,, \alpha = 1, 2\}\end{aligned}$$

1D eigenvalue problem

Consider the 1D eigenvalue problem

$$\frac{g(l+1) - 2g(l) + g(l-1)}{h_2^2} + \lambda g(l) = 0$$
$$g(0) = g(N_2) = 0$$

$l = 1, 2, \cdots, N_2 - 1$ and $l_2 N_2 = a_2$. Here $g(l) \equiv g(x_2^l)$ for brevity.

1D eigenvalue problem

Consider the 1D eigenvalue problem

$$\frac{g(l+1)-2g(l)+g(l-1)}{h_2^2}+\lambda g(l)=0$$
$$g(0)=g(N_2)=0$$

$l=1,2,\cdots,N_2-1$ and $l_2N_2=a_2$. Here $g(l)\equiv g(x_2^l)$ for brevity.

The eigenvalues and eigenfunctions are

$$g_s(l)=\sqrt{\frac{2}{a_2}}\sin\frac{\pi s x_2^l}{a_2}$$
$$\lambda_s=\frac{4}{h_2^2}\sin^2\frac{\pi s h_2}{2a_2}\,,\qquad s=1,\cdots N_2-1$$

Fourier expansion over the x_2-eigenfunctions

At a fixed value of k, $0 < k < N_1$, expand

$$y_{kl} = \sum_{s=1}^{N_2-1} c_s(k) g_s(l)$$
$$f_{kl} = \sum_{s=1}^{N_2-1} \widehat{f_s}(k) g_s(l)$$

NB: for each value of k, there are $N_2 - 1$ coefficients $c_s(k)$.

The Fourier coefficients of f_{kl} are $\widehat{f_s}(k) = h_2 \sum_{l=1}^{N_2-1} f_{kl}\, g_s(l)$

Fourier expansion over the x_2-eigenfunctions

Substitute the expansions into the FDE, use the completeness of the eigensystem:

$$\frac{c_s(k+1) - 2c_s(k) + c_s(k-1)}{h_1^2} - \lambda_s c_s(k) = \widehat{f}_s(k)$$
$$c_s(0) = c_s(N_1) = 0$$

$k = 1, 2, \cdots, N_1 - 1.$

This is a tridiagonal system *at each value of* s.

The spectral algorithm

1. Compute the Fourier coefficients of the r.h.s., $\widehat{f}_s(k)$.

   For each k, $s = 1, \cdots, N_2 - 1$. Using the discrete FFT, the complexity is $O(N_2 \ln N_2)$ per k value; the total complexity is $O(N_1 N_2 \ln N_2)$

2. Solve for $c_s(k)$.

   For each s, $s = 1, \cdots, N_1 - 1$, use a tridiagonal solver. The total complexity is $O(N_1 N_2)$

3. Restore y_{kl} via

$$y_{kl} = \sum_{s=1}^{N_2-1} c_s(k) g_s(l)$$

   With FFT, the complexity is $O(N_1 N_2 \ln N_2)$