Numeric Linear Algebra

Eigenvalues and eigenvectors

Matrix decompositions

1. Lower-upper(LU)
2. Cholesky
3. QR
4. Eigen decomposition (Schur)
5. Singular value decomposition(SVD)

Eigenvalue Problem

Let $\mathbf{A}$ is a real-valued square $m \times m$ matrix.

λ is an *eigenvalue* of $\mathbf{A}$ and $\mathbf{x} \neq \mathbf{0}$ is an *eigenvector* if

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Equivalently,

$$(\mathbf{A} - \lambda\widehat{\mathbf{1}})\mathbf{x} = 0$$

This (homogenous) system has a non-null solution iff

$$\det(\mathbf{A} - \lambda\widehat{\mathbf{1}}) = 0$$

Eigenvalue Problem

The l.h.s. of the *secular equation*

$$\det(\mathbf{A} - \lambda\widehat{\mathbf{1}}) = 0$$

is a *characteristic polynomial* (C.P.) of $\mathbf{A}$.

- The eigenvalues are the roots of the C.P.
- C.P. has degree m and therefore has m roots
- For $m > 4$ there are no explicit formulas for λ in terms of a_{ij}

  $\Longrightarrow$ need numerics.

Eigenvalue Problem

Formally, the eigenvalue problem is equivalent to the root-finding problem for the C.P.

In practice, the latter is poorly conditioned. E.g. compare the roots of

$$(x-1)^{32} = 0 \qquad \text{and} \qquad (x-1)^{32} = 10^{-16}$$

Need specialized methods.

Detour: a companion matrix of a polynomial

Consider an n-by-n matrix

$$\mathbf{A}_n = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & \ddots & & -a_2 \\ & \ddots & \ddots & 0 & \vdots \\ & & 0 & 1 & -a_{n-1} \end{pmatrix}$$

Its secular equation

$$0 = \det(x\mathbf{1} - \mathbf{A}_n) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Root-finding $\iff$ eigenvalue problem for $\mathbf{A}_n$

Numerical eigenvalue problem

Two classes of approaches

- All eigenvalues

  e.g. eigenmodes of a system

- A select subset

  e.g. PageRank, effective low-energy theory etc

Localization of eigenvalues

Localization of eigenvalues

Given a matrix, what is the distribution of λ_k in $\mathbb{R}$?

Let r_k is the sum of the off-diag elements of the k-th row of $\mathbf{A}$:

$$r_k = \sum_{j=1, j \neq k}^{m} |a_{kj}|$$

Gershgorin circle: a circle with center a_{kk} & the radius r_k:

$$\mathcal{S}_k = \{z \in \mathbb{C} : |z - a_{kk}| \leqslant r_k\}$$

Gerschgorin theorem

Theorem: (Gerschgorin)

All eigenvalues of $\mathbf{A}$ belong to the union of $\mathcal{S}_k$, $k = 1, \cdots, m$.

Proof

Consider an arbitary eigenvalue λ of $\mathbf{A}$, and its eigenvector $\vec{x}$.

Let x_k is the $\max \mathrm{abs}$ component of $\vec{x} = (x_1, x_2, \cdots, x_k, \cdots, x_m)$:

$$|x_k| \geqslant |x_j|, \qquad j \neq k$$

Gerschgorin theorem

Consider the linear system $(\mathbf{A} - \lambda\widehat{\mathbf{1}})\vec{x} = 0$, write out its k-th equation:

$$(a_{kk} - \lambda)x_k = -\sum_{j \neq k} a_{kj} x_j$$

Then,

$$|a_{kk} - \lambda| \leqslant \frac{\left|\sum_{j \neq k} a_{kj} x_j\right|}{|x_k|} \leqslant \sum_{j \neq k} |a_{kj}| \cdot \underbrace{\left|\frac{x_j}{x_k}\right|}_{\substack{\leqslant 1 \\ \text{by assumption}}} \leqslant r_k$$

Gerschgorin 2nd theorem

Theorem: (Gerschgorin 2)

If s Gerschgorin circles form a closed area $\overline{G}$, which is isolated from other Gerschgorin circles, then $\overline{G}$ has exactly s eigenvalues (including multiplicity).

Corollary: An isolated Gerschgorin circle contains exactly one eigenvalue.