Linear systems with banded left-hand sides

A matrix $\mathbf{A}$ is *banded* if $a_{ij} = 0$ for $|i - j| > p$. Then p is the *bandwidth*.

For banded matrices, constructing the LU factorization is $O(m)$.

Memory requirements for storing banded matrices is also $O(m)$.

Tridiagonal systems: Thomas algorithm

$$\begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{m-1} & b_{m-1} & c_{m-1} \\ & & & a_m & b_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_{m-1} \\ x_m \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ d_{m-1} \\ d_m \end{pmatrix}$$

Forward sweep: bidiagonalization

row 1: $b_1 x_1 + c_1 x_2 = d_1 \qquad \Rightarrow x_1 = \alpha_1 x_2 + \beta_1$

row2: $a_2 x_1 + b_2 x_2 + c_2 x_3 = d_2 \qquad \Rightarrow x_2 = \alpha_2 x_3 + \beta_2$

$\cdots$

Tridiagonal systems: Thomas algorithm

Forward sweep: bidiagonalization

$$\begin{pmatrix} \times & \times & & & \\ & \times & \times & & \\ & & \ddots & \ddots & \\ & & & \times & \times \\ & & & & \times \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{pmatrix} = \begin{pmatrix} \times \\ \times \\ \vdots \\ \times \\ \times \end{pmatrix}$$

Then do backsubstitution starting from x_m.

The total computational complexity is $O(m)$.

Tridiagonal systems: Thomas algorithm

The corresponding LU factorization is

$$\mathbf{A} = \begin{pmatrix} \times & \times & & & \\ & \times & \times & & \\ & & \ddots & \ddots & \\ & & & \times & \times \\ & & & & \times \end{pmatrix} \begin{pmatrix} \times & & & & \\ \times & \times & & & \\ & \ddots & \ddots & & \\ & & & \times & \\ & & & \times & \times \end{pmatrix}$$

Tridiagonal systems: Thomas algorithm

The corresponding LU factorization is

$$\mathbf{A} = \begin{pmatrix} \times & \times & & & \\ & \times & \times & & \\ & & \ddots & \ddots & \\ & & & \times & \times \\ & & & & \times \end{pmatrix} \begin{pmatrix} \times & & & & \\ \times & \times & & & \\ & \ddots & \ddots & & \\ & & & \times & \\ & & & \times & \times \end{pmatrix}$$

Stability

The algorithm is stable if $\mathbf{A}$ is *diagonally dominant*:

$$|b_k| \geq |a_k| + |c_k|, \qquad k = 1, \ldots, m$$