Flux-conservative IVPs

Flux-conservative equations

Let $\mathbf{u}(x, t)$ is a vector field, and $\mathbf{F}(\mathbf{u})$ is a known vector function.

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} \tag{1}$$

$\mathbf{F}(\mathbf{u})$ is called a ***conserved flux***.

Flux-conservative formulation of the wave equation

Consider the 1D wave equation

$$u_{tt} = c^2 u_{xx}$$

Identically rewrite as

$$\frac{\partial r}{\partial t} = c\frac{\partial s}{\partial x}$$
$$\frac{\partial s}{\partial t} = c\frac{\partial r}{\partial x}$$

where $r \equiv c\,\partial u/\partial x$ and $s \equiv \partial u/\partial t$

Flux-conservative formulation of the wave equation

Introduce

$$\mathbf{u} = \begin{pmatrix} r \\ s \end{pmatrix}$$

In the vector form, the system is

$$\mathbf{F}(\mathbf{u}) = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} \cdot \mathbf{u}$$

Advection equation

For clarity, consider a scalar version:

$$\frac{\partial u}{\partial t} = -c\,\frac{\partial u}{\partial x}$$

The solution is

$$u(x,t) = f(x - c\,t)$$

with arbitrary $f(\cdot)$. Qualitatively, the solution at $t = 0$ is transported towards the +ve x direction.

FTCS scheme

Numerically, discretize and use the FTCS scheme (forward time centered space)

$$\frac{y_j^{n+1} - y_j^n}{\tau} = -c\frac{y_{j+1}^n - y_{j-1}^n}{2h}$$

Stability analysis of the FTCS scheme

Consider the homogenous equation, look for particular solutions of the form

$$y_j^n(k) = q^n e^{ikjh}$$

where $k \in \mathbb{R}$ and $q \in \mathbb{R}$ is unknown.

Require that the ***amplification factor*** $|q| \leqslant 1$ for all k

Stability analysis of the FTCS scheme

FTCS scheme

$$y_j^{n+1} = y_j^n - \gamma \frac{y_{j+1}^n - y_{j-1}^n}{2}$$

is ***unconditionally unstable***:

$$q = 1 - i\gamma \sin kh$$

Here $\gamma = c\tau/h$

Lax method

Vanilla FTCS scheme

$$\frac{y_j^{n+1} - y_j^n}{\tau} = -c\frac{y_{j+1}^n - y_{j-1}^n}{2h}$$

Lax construction: in the time derivative, replace

$$y_j^n \longrightarrow \frac{1}{2}\left(y_{j+1}^n + y_{j-1}^n\right)$$

This way, the scheme is

$$y_j^{n+1} = \frac{1}{2}\left(y_{j+1}^n + y_{j-1}^n\right) - \gamma\frac{1}{2}\left(y_{j+1}^n - y_{j-1}^n\right)$$

Stability of the Lax scheme

Using

$$y_j^n = q^n e^{ikhj}$$

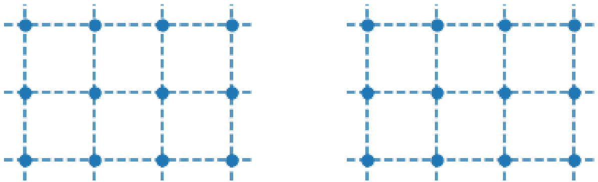
gives

$$q = \cos kh - i\gamma \sin kh$$

I.e., the scheme is stable if $\gamma \leqslant 1$.

A qualitative meaning of the Courant condition

The Courant condition, $c\tau/h \leqslant 1$, compares the grid spacings, τ and h, to the propagation velocity c.



Stability of the Lax scheme: why and how?

Identically rewrite the Lax scheme

$$\frac{y_j^{n+1} - \frac{1}{2}\left(y_{j+1}^n + y_{j-1}^n\right)}{\tau} = -c\frac{y_{j+1}^n - y_{j-1}^n}{2h}$$

as

$$\frac{y_j^{n+1} - y_j^n}{\tau} - \frac{1}{2}\frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{\tau} = -c\frac{y_{j+1}^n - y_{j-1}^n}{2h}$$

Numerical dissipation

This approximates

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + \frac{h^2}{2\tau}\frac{\partial^2 u}{\partial x^2}$$

Numerical dissipation

Note that the amplitude of a harmonic with the wavenumber k is q^n, with

$$q = \cos kh - i\gamma \sin kh$$

Unless $\gamma = 1$, the amplitude decreases with n.

This suppression is spurious. This is an artifact of the Lax scheme.

Numerical dissipation

$$q = \cos kh - i\gamma \sin kh$$

Qualitatively

- We are interested in the long-wavelength harmonics, i.e. $kh \ll 1$.
- In this regime, $q \approx 1$ for both stable and unstable schemes.
- For $\gamma > 1$, harmonics with $kh \sim 1$ grow and swamp the interesting part of the solution
- For $\gamma < 1$, these harmonics are artificially suppressed. The scheme is not accurate for these harmonics, but ***we are not interested in them***.

Recap

- The Lax scheme is stable for $\tau \leqslant h$.
- The approximation is $O(\tau + h^2)$

The time step size is limited by the accuracy, not stability. Can we construct a scheme which is $O(\tau^2 + h^2)$?

Staggered Leapfrog scheme

For the simplified scalar equation,

$$u_t = -cu_x$$

$$\frac{y_j^{n+1} - y_j^{n-1}}{2\tau} = -c\frac{y_{j+1}^n - y_{j-1}^n}{2h}$$

von Neumann stability analysis of the staggered Leapfrog scheme

Using

$$y_j^n = q^n e^{ikhj} \, ,$$

we obtain

$$q^2 - 1 = -2i\gamma \sin kh \cdot q$$

- The scheme is stable for $\gamma = c\tau/h \leqslant 1$
- For $\gamma \leqslant 1$, have $|q| = 1$

Staggered leapfrog

- For a flux-conservative formulation of the wave equation, staggered leapfrog is equivalent to the three-layer five-point stencil scheme.
- Mesh drifting instability: even and odd sublattices are decoupled.

Two-step Lax-Wendroff scheme

Consider a general form with $F = F(u)$

$$\frac{\partial u}{\partial t} = -\frac{\partial F}{\partial x}$$

Transition $n \longrightarrow n+1$ via an auxilliary layer $n+1/2$.

At layer $n+1/2$, the mesh is shifted by by $h/2$.

Auxilliary values $y_{j+1/2}^{n+1/2} \approx u(t_n + \tau/2, x_j + h/2)$.

Two-step Lax-Wendroff scheme

$n \to n+1/2$: a Lax step

$$y_{j+1/2}^{n+1/2} = \frac{1}{2}\left(y_{j+1}^n + y_j^n\right) - \frac{\tau}{h}\left(F_{j+1}^n - F_j^n\right)$$

fluxes at $n+1/2$: compute

$$F_{j+1}^{n+1/2} \equiv F(y_{j+1/2}^{n+1/2})$$

$n+1/2 \to n$: a central difference

$$y_j^{n+1} = y_j^n - \frac{\tau}{h}\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}\right)$$