Phys 481 : Advanced Magnetism

Lecturer: Evgeni Burovski
(B75 Physics building)

Lectures:

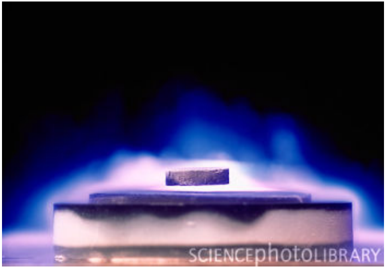
Monday	10am–11am, Charles Carter A16
	4pm–5pm, George Fox LT4
Tuesday	2pm–3pm, George Fox A18
Friday	10am–11am, BNSR 23 (week 1)

Seminars:

Thursday	10am–11am, BNSR 23 (weeks 2–5)

Office hours: Open door policy.

Magnetic phenomena in condensed matter systems: things do levitate



Assessment: 80% exam (Summer term), 20% coursework.

Worksheets will be posted to MLE on Mondays weeks 1–4, you have a week to complete a worksheet. For example, worksheet 1 will be posted to MLE on Monday week one, and should be handed in by 10am Monday week 2 in the relevant INBOX in the physics foyer. Work handed in after that and 10am Thursday is subject the mark reduction equivalent to one full letter grade. Work handed in later than this does not count towards your continuous assessment.

Worksheets will have assessed and "practice" parts. The latter need not be submitted, we will be discuss them in seminars.

NB: *No units, no marks.*

Lecture notes: will be on Moodle.

There is no single textbook for the course.

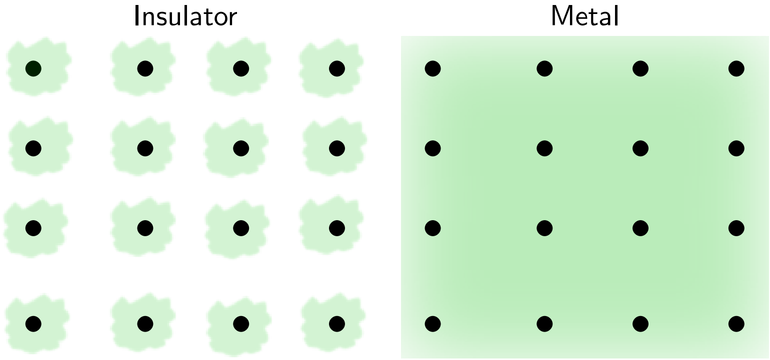
The most useful book: C Kittel, *Introduction to the solid state physics*
Additional material:
C Kittel, *Quantum theory of solids*
LD Landau and EM Lifshitz, *Statistical Physics I*
and other reading material, as necessary.

Please revise:

your E&M course [E Purcell, *Electricity and Magnetism* or Phys 221 lecture notes] and
your solid state course [Phys 313].

Magnetic responce of solids

Solid = nuclei + electrons

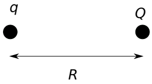


Need to revise the E&M first.

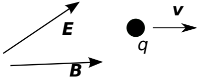
Outline

- Revision of the E&M
- Magnetic field in media
- Diamagnetism and paramagnetism of insulators: a microscopic theory

Systems of units: SI vs CGS (Gaussian)



$$F = \frac{1}{4\pi\epsilon_0}\frac{q\,Q}{R^2}\ \text{(SI)} \qquad\qquad F = \frac{q\,Q}{R^2}\ \text{(CGS)}$$



$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \text{(SI)}$$
$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \quad \text{(CGS)}$$

SI system introduces *spurious* permittivities of "free space", ϵ_0 and μ_0.

CGS (Gaussian) system only uses a physical quantity, the speed of light c.

Discussion and conversion dictionaries: R. Littlejohn, Phys 221A UC Berkeley lecture notes (2007). Also available on MLE.

MAXWELL'S EQUATIONS *in vacuo*

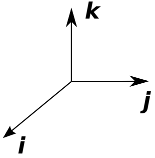
SI	CGS (Gaussian)
$\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{D} = 4\pi\rho$
$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$	$\nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = 0$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = \mathbf{j}$	$\nabla \times \mathbf{H} - \frac{1}{c}\partial_t \mathbf{D} = \frac{4\pi}{c}\mathbf{j}$

Electric and magnetic inductions:

$\mathbf{D} = \epsilon_0 \mathbf{E}$	$\mathbf{D} = \mathbf{E}$
$\mathbf{B} = \mu_0 \mathbf{H}$	$\mathbf{B} = \mathbf{H}$

$$\epsilon_0\, \mu_0 = 1/c^2$$

Intermezzo: Vector calculus



$$
\begin{aligned}
\nabla &= \mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z \\
\nabla \cdot \mathbf{E} &\equiv \operatorname{div} \mathbf{E} \equiv \partial_x E_x + \partial_y E_y + \partial_z E_z \\
\nabla \times \mathbf{B} &\equiv \operatorname{rot} \mathbf{B} \equiv \operatorname{curl} \mathbf{B} \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix}
\end{aligned}
$$

$$\text{Gauss theorem:} \quad \oint_{\partial\Omega} \mathbf{K} \cdot \mathrm{d}\mathbf{S} = \int_{\Omega} \nabla \cdot \mathbf{K} \, \mathrm{d}\Omega$$

“The flux across a surface equals the volume integral of the grad”

$$\text{Stokes theorem:} \quad \int_{\mathcal{S}} (\nabla \times \mathbf{K}) \cdot \mathrm{d}\mathbf{S} = \oint_{\mathcal{C}} \mathbf{K} \cdot \mathrm{d}\mathbf{l}$$

“The flux of a curl equals the circulation along the boundary”

ELECTROSTATICS *in vacuo* (CGS)

$$\nabla \times \mathbf{E} = 0 \qquad\qquad \nabla \cdot \mathbf{E} = 4\pi\rho$$

Recall: $\mathrm{curl}\,(\mathrm{grad}\,\varphi) = 0 \qquad \forall$ "reasonable" $\varphi(\mathbf{r})$

Scalar potential $\varphi(\mathbf{r})$: $\mathbf{E} = -\nabla\varphi$

Gauss's theorem: $\nabla \cdot \mathbf{E} = -\nabla^2\varphi = 4\pi\rho$

For a system of charges

$$\rho(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i) \qquad\qquad \varphi(\mathbf{r}) = \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}$$

Magnetostatics *in vacuo* (CGS)

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} \qquad\qquad \nabla \cdot \mathbf{H} = 0$$

Vector potential $\mathbf{A}(\mathbf{r})$:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad\qquad \nabla \cdot \mathbf{A} = 0 \qquad \text{(Coulomb gauge)}$$

Using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

$$\nabla \times \mathbf{H} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{j}$$

For a system of moving charges $\qquad \mathbf{A}(\mathbf{r}) = \dfrac{1}{c}\displaystyle\sum_i \frac{q_i\mathbf{v}_i}{|\mathbf{r} - \mathbf{r}_i|}$

Intermezzo: gauge invariance

In general: $\begin{cases} \mathbf{E} = -\nabla\varphi - \partial_t \mathbf{A} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$

The fields are invariant under a *gauge transformation*

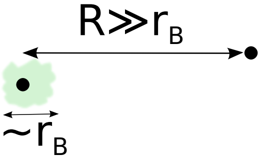
$$\varphi \longrightarrow \varphi - \partial_t g$$
$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla g$$

where $g(\mathbf{r}, t)$ is *any* function.

In quantum mechanics, the wave function acquires a phase under a gauge transformation:

$$\Psi(\mathbf{r}, t) \longrightarrow \Psi(\mathbf{r}, t) \exp\left(\frac{ie}{\hbar c} g(\mathbf{r}, t)\right), \qquad \text{so that} \qquad |\Psi(\mathbf{r}, t)|^2 = \mathrm{inv}$$

ATOM IN A MAGNETIC FIELD (CGS)



Only observe the far fields:

$$\begin{aligned}\frac{1}{|\mathbf{R}-\mathbf{r}|} &= \frac{1}{\sqrt{R^2 - 2\mathbf{R}\cdot\mathbf{r} + r^2}} \\ &= \frac{1}{R} + \frac{\mathbf{R}\cdot\mathbf{r}}{R^3} + \ldots \quad \text{for} \quad R \gg r\end{aligned}$$

Microscopic fields fluctuate wildly. Need to average:

OVER THE ORBITAL PERIOD:

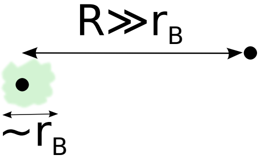
$$\langle(\cdots)\rangle = \int_0^\tau \frac{\mathrm{d}t}{\tau}(\cdots)$$

OVER THE QUANTUM STATE:

$$\langle(\cdots)\rangle = \int \mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_n \Psi^*(\mathbf{r}_1 \ldots \mathbf{r}_n)\,(\cdots)\,\Psi(\mathbf{r}_1 \ldots \mathbf{r}_n)$$

$$\mathbf{A}(\mathbf{R}) = \frac{1}{c}\left\langle \sum_i \frac{q_i \mathbf{v}_i}{|\mathbf{R}-\mathbf{r}_i|} \right\rangle$$

Magnetic dipole (CGS)



$$\mathbf{A}(\mathbf{R}) = \frac{1}{c}\left\langle \sum_i \frac{q_i \mathbf{v}_i}{|\mathbf{R}-\mathbf{r}_i|} \right\rangle$$

$$= \frac{1}{c}\left\langle \sum_i \left(\frac{-e\mathbf{v}_i}{R} + \frac{-e\mathbf{v}_i(\mathbf{R}\cdot\mathbf{r}_i)}{R^3} + \cdots \right) \right\rangle$$

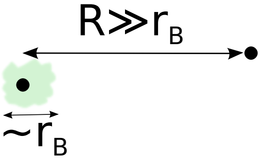
$$\langle \mathbf{v}_i \rangle \equiv \int_0^\tau \frac{\mathrm{d}t}{\tau}\frac{d\mathbf{r}}{dt} = \frac{\mathbf{r}(\tau)-\mathbf{r}(0)}{\tau} = 0$$

Using $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$

$$\mathbf{v}(\mathbf{r}\cdot\mathbf{R}) = \underbrace{\frac{1}{2}\mathbf{v}(\mathbf{r}\cdot\mathbf{R}) + \frac{1}{2}\Big(\mathbf{r}(\mathbf{v}\cdot\mathbf{R})}_{=\frac{1}{2}\frac{d}{dt}\left[\mathbf{r}(\mathbf{r}\cdot\mathbf{R})\right]} + (\mathbf{r}\times\mathbf{v})\times\mathbf{R}\Big)$$

$$= \frac{1}{2}\frac{d}{dt}\left[\mathbf{r}(\mathbf{r}\cdot\mathbf{R})\right] \quad \Rightarrow \text{yields zero when averaged}$$

Magnetic dipole (CGS)



$$\mathbf{A}(\mathbf{R}) = \frac{-e}{2c}\sum_i \langle \mathbf{r}_i \times \mathbf{v}_i \rangle \times \frac{\mathbf{R}}{R^3}$$

General case: $\mathbf{A}(\mathbf{R}) = \dfrac{\mathbf{m} \times \mathbf{R}}{R^3}$

Magnetic dipole moment $\quad \mathbf{m} = \dfrac{1}{2c}\sum_i \langle \mathbf{r}_i \times (-e\mathbf{v}_i) \rangle = \dfrac{1}{2c}\displaystyle\int \mathrm{d}\Omega\,[\mathbf{r} \times \mathbf{j}]$

In SI system: $\mathbf{A}(\mathbf{R}) = \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m} \times \mathbf{R}}{R^3}$ and $\mathbf{m} = \dfrac{1}{2}\displaystyle\int \mathrm{d}\Omega\,[\mathbf{r} \times \mathbf{j}]$

Magnetic dipole (CGS)

Angular momentum $$\mathbf{L} = \sum_i \langle \mathbf{r}_i \times \mathbf{p}_i \rangle = m_e \sum_i \langle \mathbf{r}_i \times \mathbf{v}_i \rangle$$

$$\Longrightarrow \mathbf{m} = \frac{-e}{2m_e c}\mathbf{L}$$

Magnetic moments are quantized in units of $\hbar$, hence a natural measure for magnetic moments is a

Bohr magneton $$\mu_B = \frac{e\hbar}{2m_e c}$$

In SI system there is no c in the denominator.

Numeric value: $$\mu_B \approx 9.27 \times 10^{-21} \text{ erg/Gauss} \qquad \text{(CGS)}$$

$$\approx 9.27 \times 10^{-24} \text{ Joule/Tesla} \qquad \text{(SI)}$$

Magnetic field of a magnetic dipole (CGS)

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left(\frac{\mathbf{m} \times \mathbf{R}}{R^3} \right) = \mathbf{m} \left(\nabla \cdot \frac{\mathbf{R}}{R^3} \right) - (\mathbf{m} \cdot \nabla) \frac{\mathbf{R}}{R^3}$$

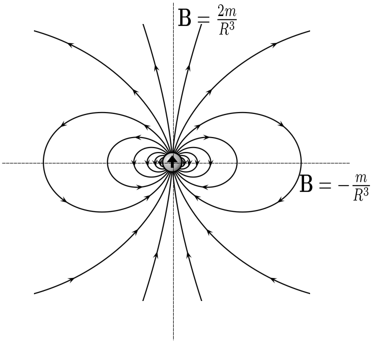
- $\frac{\mathbf{R}}{R^3} = -\nabla \frac{1}{R} \quad \Rightarrow \quad \nabla \frac{\mathbf{R}}{R^3} = -\nabla^2 \frac{1}{R} = 4\pi\delta(\mathbf{R}) = 0$ for $R \neq 0$
- $\nabla \times \frac{\mathbf{R}}{R^3} = \nabla \times \left(-\nabla \frac{1}{R} \right) = 0$
- $0 = \mathbf{m} \times \left(\nabla \times \frac{\mathbf{R}}{R^3} \right) = \nabla \left(\mathbf{m} \cdot \frac{\mathbf{R}}{R^3} \right) - (\mathbf{m} \cdot \nabla) \frac{\mathbf{R}}{R^3}$

Finally,

$$\mathbf{B} = -\nabla \left(\mathbf{m} \cdot \frac{\mathbf{R}}{R^3} \right) = -\frac{\nabla (\mathbf{m} \cdot \mathbf{R})}{R^3} - (\mathbf{m} \cdot \mathbf{R}) \nabla \frac{1}{R^3}$$

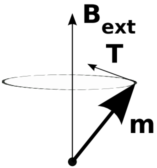
$$= -\frac{\mathbf{m}}{R^3} + 3 (\mathbf{m} \cdot \mathbf{R}) \frac{\mathbf{R}}{R^5}$$

Magnetic field of a magnetic dipole (CGS)



$$\mathbf{B} = -\frac{\mathbf{m}}{R^3} + 3\,(\mathbf{m}\cdot\mathbf{R})\,\frac{\mathbf{R}}{R^5}$$

In an external field



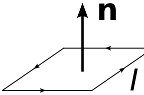
Torque: $\mathbf{T} = \mathbf{m} \times \mathbf{B}_{ext}$

Energy: $U = -\mathbf{m}\cdot\mathbf{B}_{ext}$

Magnetic moment of a current-carrying loop

$$\mathbf{m} = \frac{1}{2c}\int \mathrm{d}\Omega\,[\mathbf{r}\times\mathbf{j}]$$

Far field *only*!

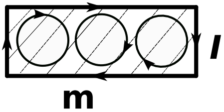


$$\mathbf{m} = \frac{1}{c} IS\mathbf{n}$$

- S is the area of the loop
- *irrespective of its shape*.
- $\mathbf{n}$ is the normal vector to the loop.

Other derivation: E Purcell, Chapter 11.3

Magnetized medium: Amperian currents



$$\mathbf{m} = \int \mathrm{d}\Omega\,\mathbf{M}(\mathbf{r})$$

M is density of the magnetic moments, a.k.a. *magnetization of a medium*

Magnetic field in a medium

Microscopic fields fluctuate wildly. Need to average:

Over the orbital period:

$$\langle(\cdots)\rangle = \int_0^\tau \frac{\mathrm{d}t}{\tau}(\cdots)$$

Over the thermal distribution:

$$\langle(\cdots)\rangle = \frac{\int \mathrm{d}\Omega\, e^{-\beta U(\Omega)}(\dots)}{\int \mathrm{d}\Omega\, e^{-\beta U(\Omega)}}$$

(in thermodynamic equilibrium with temperature T)

Or,

Over the quantum state:

$$\langle(\cdots)\rangle = \int \mathrm{d}\mathbf{r}_1 \dots \mathrm{d}\mathbf{r}_n \Psi^*(\mathbf{r}_1 \dots \mathbf{r}_n)\,(\cdots)\,\Psi(\mathbf{r}_1 \dots \mathbf{r}_n)$$

Over the thermal distribution:

$$\langle(\cdots)\rangle = Z^{-1}\,\mathrm{tr}\left[e^{-\beta \widehat{H}}(\dots)\right]$$

Magnetic field in a medium

Bulk density of the “bound” currents (equivalent to the Amperian currents)

$$\mathbf{j}_b = c\nabla \times \mathbf{M}_{\rm micro}$$

Both “free” currents $\mathbf{j}$ and bound currents:

$$\begin{aligned}\nabla \times \mathbf{B}_{\rm micro} &= \frac{4\pi}{c}\left(\mathbf{j} + \mathbf{j}_b\right) \\ &= \frac{4\pi}{c}\left(\mathbf{j} + c\nabla \times \mathbf{M}_{\rm micro}\right)\end{aligned}$$

Averaging: $\mathbf{B} \equiv \langle \mathbf{B}_{\rm micro} \rangle$, $\mathbf{M} \equiv \langle \mathbf{M}_{\rm micro} \rangle$

CGS: $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, $\qquad \nabla \times \mathbf{H} = \dfrac{4\pi}{c}\mathbf{j}$

SI system: $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, $\qquad \nabla \times \mathbf{H} = \mathbf{j}$

Magnetic field in a medium

For linear, isotropic, homogenous media

$$\mathbf{M} = \chi \mathbf{H}$$

where χ is *magnetic susceptibility*.
Magnetic induction

$$\mathbf{B} = (1 + 4\pi\chi)\,\mathbf{H}$$

magnetic permeability $\mu = 1 + 4\pi\chi$

In the SI system: $\mathbf{B} = \mu_0 \mu \mathbf{H}$ and $\mu = 1 + \chi$

If the medium is

- anisotropic: $M_\alpha = \chi_{\alpha\beta} H_\beta$
- non-uniform: $\chi = \chi(\mathbf{r})$
- non-linear: $\mathbf{M} \neq \chi \mathbf{H}$

Magnetic field in a medium: confusion points

4π strikes back

$$\chi^{\mathrm{SI}} = 4\pi\chi^{\mathrm{CGS(Gaussian)}}$$

(older books/tables typically use CGS)

Units

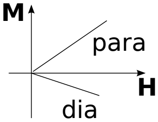
χ is dimensionless. Often referred to as a *volume susceptibility*.

But. Also *molar susceptibility*. Measured in meters cubed per mole. (older books/tables: $\mathrm{cm}^3/\mathrm{mol}$)
Also *mass susceptibility*, m^3/kg.

A zoo of solid state magnetism

Paramagnetic $\chi > 0$

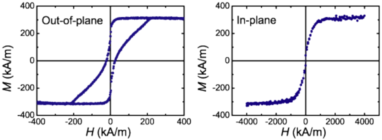
Diamagnetc $\chi < 0$



Ferromagnetic

$\mathbf{M}(\mathbf{H} = 0) \neq 0.$

Often anisotropic.



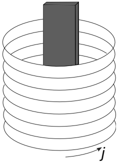
(GdTbFe film;
image courtesy Univ Amsterdam)

antiferromagnetic

ferrimagnetic

...and many more

Energetics of the magnetic media



Total energy $=$ energy stored in the field
$+$energy of polarized magnetic moments

$$E - E^{(0)} = \int d\Omega \left(\frac{\mathbf{B}\cdot\mathbf{H}}{8\pi} - \frac{H^2}{8\pi} \right) - \left\langle \sum_i \mathbf{m}_i \cdot \mathbf{B} \right\rangle$$
$$= \int d\Omega \left(\frac{(\mu-1)H^2}{8\pi} - \mathbf{M}\cdot\mathbf{B} \right)$$
$$= \int d\Omega \left(\frac{4\pi\chi H^2}{8\pi} - \chi H^2 \right) = -\int d\Omega \frac{\chi H^2}{2}$$

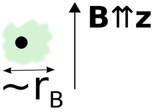
Paramagnets ($\chi > 0$) are attracted into a solenoid. Diamagnets ($\chi < 0$) are repelled out.

Diamagnetism and paramagnetism: microscopic treatment(s)

Magnetic moment of a single atom:

- the change of the orbital motion induced by the magnetic field (diamagnetic)
- electrons' spin (paramagnetic)
- angular momentum of the orbital motion around the nucleus (paramagnetic)
- *nuclear paramagnetism* due to the magnetic moments of the nuclei (some 10^{-3} smaller)

Langevin diamagnetism I (classical treatment)



Larmor theorem: in a weak **B**-field the motion of an electron is a superposition of the motion in zero field *and* precession around the nucleus with angular frequency

$$\omega_L = \frac{eB}{2m_e c} \quad \text{(CGS)} \qquad\qquad \omega_L = \frac{eB}{2m_e} \quad \text{(SI)}$$

"Weak field" means $\omega_L \ll$ other frequencies

Atom $\longleftrightarrow$ a current loop with $m = \frac{1}{c} IS$

The current: $I = (\textit{charge})(\#\text{ of revolutions per second}) =$

$$(-Ze)\left(\frac{1}{2\pi}\omega_L\right) = -Ze\frac{1}{2\pi}\frac{eB}{2m_e c}$$

The area: $S = \pi\left\langle x^2 + y^2\right\rangle = \pi\frac{2}{3}\left\langle r^2\right\rangle$

(Assuming the spherical symmetry at $\mathbf{B} = 0$: $\left\langle x^2\right\rangle = \left\langle y^2\right\rangle = \left\langle z^2\right\rangle$)

Langevin diamagnetism II

Putting this all together: for n atoms per unit volume

$$\chi = \frac{nm}{B} = -n\frac{Ze^2}{6m_e c^2}\left\langle r^2 \right\rangle \qquad \text{(CGS)}$$

$$= \mu_0 \frac{nm}{B} = -\mu_0 n \frac{Ze^2}{6m_e}\left\langle r^2 \right\rangle \qquad \text{(SI)}$$

- Physics is simple. Recall the Lenz law: (diamagnetic) currents try to shield a sample from the applied field.
- No temperature dependence.
- Quantum mechanics enters through $\left\langle r^2 \right\rangle$.
- Full QM treatment: The result is exactly the same. (See, *e.g.* Landau&Lifshitz, vol III, Sec 113.)
- There are additional effects (for which we don't have time, see above or Kittel's *ISSP*.)

Paramagnetism

Paramagnetism: intrinsic magnetic moments tend to align along the applied field.

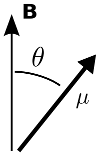
Usually found in

- Atoms, molecules, lattice defects with odd number of electrons
- Ions with partially filled shells
- Metals

Start with a single localized magnetic moment.

Paramagnetism (classical magnetic moment)

Paramagnetism: intrinsic magnetic moments tend to align along the applied field.



$$\mu_x = \mu \cos\phi \sin\theta$$
$$\mu_y = \mu \sin\phi \sin\theta$$
$$\mu_z = \mu \cos\theta$$

Energy: $U = -\boldsymbol{\mu}\cdot\mathbf{B} = -\mu B\cos\theta$
At finite temperature T: balance energy vs entropy.
In other words, find the *thermal average*

$$\langle\boldsymbol{\mu}\rangle = \frac{\int e^{-U/k_BT}\boldsymbol{\mu}\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\int e^{-U/k_BT}\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}$$

Paramagnetism (classical magnetic moment)

In-plane magnetization is zero:

$$\langle \mu_x \rangle = \langle \mu_y \rangle = 0$$

because $\int_0^{2\pi} \sin\phi \, \mathrm{d}\phi = \int_0^{2\pi} \cos\phi \, \mathrm{d}\phi = 0$.
But $\langle \mu_z \rangle \neq 0$.

Changing the integration variables to $\xi = \cos\theta$,

$$\langle \mu_z \rangle = \frac{\int\limits_{-1}^{1} e^{\mu B/k_B T \xi} (\mu\xi) \, \mathrm{d}\xi}{\int\limits_{-1}^{1} e^{\mu B/k_B T \xi} \, \mathrm{d}\xi}.$$

Actually, we don't even need to do the full integral: susceptibility is defined at $B \to 0$.

Paramagnetism (classical magnetic moment)

For $\mu B/k_B T \ll 1$, expand the exponentials: $e^x \approx 1 + x$ for $x \ll 1$, and obtain

$$\langle \mu_z \rangle = \frac{\mu^2 B}{3k_B T}$$

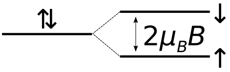
For n magnetic moments per unit volume,

$$M = n\langle \mu_z \rangle = n\frac{\mu^2}{3k_B T} \times B,$$

and

$$\chi = \frac{n\mu^2}{3k_B T} \qquad \text{(Curie law)}$$

Paramagnetism II: Zeeman splitting, $S = 1/2$



$$U = -\boldsymbol{\mu} \cdot \mathbf{B} = s_z g \mu_B B = \pm \mu_B B$$

where for an electron $g = 2$ and $s_z = \pm\frac{1}{2}$.

In thermodynamic equilibrium with temperature T

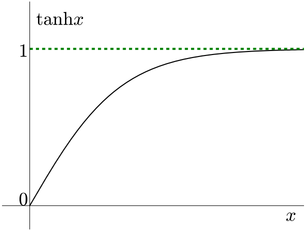
$$w_i \propto e^{-\varepsilon_i / k_B T}$$

$$\Rightarrow \quad \frac{n_{\uparrow,\downarrow}}{n} \propto e^{\pm x} \qquad \text{with} \quad x = \frac{\mu_B B}{k_B T}$$

For n spins per unit volume

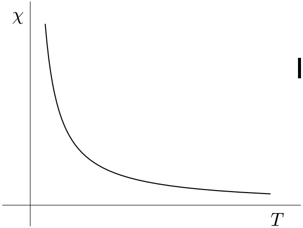
$$\begin{aligned} M &= \mu_B \left(n_\uparrow - n_\downarrow \right) \\ &= n \mu_B \frac{e^x - e^{-x}}{e^x + e^{-x}} = n \mu_B \tanh x \end{aligned}$$

Paramagnetism II: Zeeman splitting, $S = 1/2$



$$\tanh(x \to \infty) \to 1$$

$$\tanh(x \ll 1) \approx x - \frac{x^3}{3} + \dots$$



In weak fields / at high T-s

$$\chi = \frac{M}{B} \approx \frac{n\mu_B^2}{k_B T} \qquad \text{(Curie law)}$$

Paramagnetism III: $J \neq 1/2$

For an atom with a total angular momentum J

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} = m_J g \mu_B B \ , \qquad m_J = J, J-1, \ldots, -J$$

Magnetization $M = ngJ\mu_B B_J(x)$ with $x = gJ\mu_B B / k_B T$ and

Brillouin function: $B_J(x) = \frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} x\right) - \frac{1}{2J} \coth \frac{x}{2J}$

$$\coth(x \ll 1) \approx \frac{1}{x} + \frac{x}{3} + \ldots \quad \Rightarrow \quad \chi = \frac{M}{B} \approx \frac{n\mu_B^2 g^2 J(J+1)}{3k_B T}$$

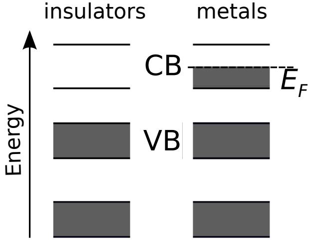
What is g? $\mathbf{J} = \mathbf{L} + \mathbf{S}$

Landé factor $g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$

Paramagnetism of metals

Expect $\chi \propto 1/T$

In reality most non-ferromagnetic metals have T-independent susceptibility.



In metals, the Fermi level lies *in* the conduction band, $\Rightarrow$ there are *free carriers*.

To a good approximation, these are just electrons with an effective mass $m_* \neq m_e$.

Next step: itinerant particles in magnetic field.