Projection and variational methods

Consider a functional

$$S[u] = \int_a^b L(x, u, \dot{u})\, dx$$

Here $\dot{u} = \dfrac{du}{dx}$

$S[u]$ is defined for $u(x) \in U$: the set of

$$u(x) \in C^1[a, b]\,, \qquad u(a) = u_a, \quad u(b) = u_b$$

can generalize for piecewise-smooth $u(x)$.

Convert a BVP to a problem of minimizing a functional $S[u]$.

Euler-Lagrange equations

Let $u(x)$ is a stationary point of $S[u]$.

Then, $u(x)$ satisfies the Euler-Lagrange equation:

$$-\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{u}}\right) + \frac{\partial L}{\partial u} = 0 \tag{1}$$

i.e. the stationary point of $S[u] \Longleftrightarrow$ BVP of (1).

Example: a quadratic functional

Consider

$$S[u] = \frac{1}{2}\int_a^b \left(p(x)(\dot{u})^2 + q(x)u^2 \right) dx - \int_a^b f(x)u\,dx \qquad (2)$$

with $p(x),\ q(x),\ f(x) \in C^1[a,b]$, and $p(x) \geqslant p_0 > 0$, $q(x) \geqslant 0$ for $x \in [a,b]$

Example: a quadratic functional

$$\begin{cases} \dfrac{\partial L}{\partial \dot{u}} = p(x)\dot{u} \\ \dfrac{\partial L}{\partial u} = q(x)u - f(x) \end{cases}$$

The Euler-Lagrange equation becomes

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = f(x)\ .$$

Ritz method

Need to minimize $S[u]$ for $u(x) \in U$.

Look for a minimum over a set $U_N \subset U$, spanned by the ***basis*** functions $\varphi_k(x)$:

$$u_N(x) = \sum_{k=0}^{N} \beta_k \varphi_k(x) \,,$$

with boundary conditions $u_N(a) = u_a$ and $u_N(b) = u_b$.

Assuming linear independence of $\{\varphi_k(x)\}$,

$$\min_{u_N} S[u_N] \qquad \text{is a minimization w.r.t. } \{\beta_0, \cdots, \beta_N\}$$

Ritz method

For convenience, take

$$\begin{cases} \varphi_0(a) = 1 & & \varphi_N(b) = 1 & \\ \varphi_k(a) = 0 & k > 0 & \varphi_k(b) = 0 & k < N \end{cases}$$

Then $\quad \beta_0 = u_a, \quad \beta_N = u_b$

Need to minimize a function of $N-1$ variables, $\beta_1, \cdots, \beta_{N-1}$,

$$S(\vec{\beta}) \equiv S\left[\sum_{k=0}^{N} \beta_k \varphi_k(x)\right]$$

$$0 = \frac{\partial S(\vec{\beta})}{\partial \beta_l}, \qquad l = 1, \cdots, N-1$$

Example: a quadratic functional

Consider the quadratic functional (2):

$$
\begin{aligned}
S(\vec{\beta}) &\equiv S\left[\sum_{k=0}^{N} \beta_k \varphi_k(x)\right] \\
&\equiv \frac{1}{2}\int_a^b dx \left[p(x)\left(\sum_{k=0}^{N} \beta_k \dot{\varphi}_k\right)^2 + q(x)\left(\sum_{k=0}^{N} \beta_k \varphi_k\right)^2\right] \\
&- \int_a^b dx\, f(x) \sum_{k=0}^{N} \beta_k \varphi_k
\end{aligned}
$$

Example: a quadratic functional

Consider the quadratic functional (2):

$$\frac{\partial}{\partial \beta_l} S(\vec{\beta}) = \int_a^b dx \left[p(x) \left(\sum_{k=0}^{N} \beta_k \dot{\varphi}_k \right) \dot{\varphi}_l + q(x) \big(\sum_{k=0}^{N} \beta_k \varphi_k \big) \varphi_l \right]$$

$$- \int_a^b dx \, f(x) \varphi_l(x)$$

Example: a quadratic functional

The stationary point of $S(\vec{\beta})$ is the solution of the algebraic system:

$$\sum_{k=0}^{N} A_{kl}\beta_k = b_l \qquad l = 1, \cdots, N-1$$

$$\beta_0 = u_a, \quad \beta_N = u_b$$

with

$$A_{kl} = \int_a^b dx \left[p(x)\dot{\varphi}_k\dot{\varphi}_l + q(x)\varphi_k\varphi_l \right]$$

$$b_l = \int_a^b dx f(x)\varphi_l$$

Ritz method

- For a quadraric functional, $\mathbf{A}$ is symmetric and positive definite.

- In general, $\mathbf{A}$ is dense. Unless the basis functions are localized.

- Ritz method works iff $S[u]$ exists s.t. the original BVP is the Euler equation for $S[u]$.

Galerkin method

Limitations of the variational approach

Ritz method works iff $S[u]$ exists s.t. the original BVP is the Euler equation for $S[u]$.

This is not always the case. E.g.

$$-\frac{d}{dx}\left(p(x)\dot{u}\right) + v(x)\dot{u} + q(x)u = f$$

There is no $S[u]$ which has this as its Euler equation for $v(x) \neq 0$.

Galerkin's approach: projection formulation

Given a BVP

$$Q[u] = f(x)\,, \qquad u(a) = u_a\,, \quad u(b) = u_b$$

The integral identity

Multiply by some ***trial function*** $\phi(x)$ and integrate:

$$\int_a^b Q[u]\phi(x)\,dx = \int_a^b f(x)\phi(x)\,dx$$

Here the trial function is piecewise differentiable and $\phi(a) = \phi(b) = 0$. Let Φ is the set of all such functions.

Galerkin's approach

The main lemma of the variational calculus

1. If $u(x)$ satisfies $Q[u] = f$, it satisfies the integral identity.
2. Conversely, if the integral identity holds $\forall \phi(x) \in \Phi$, then $u(x)$ satisfies $Q[u] = f$

Instead of of solving the BVP, look for $u(x)$ which satifies the boundary conditions, and satisfies the integral identity for any $\phi(x) \in \Phi$.

Galerkin's approach

Look for an approximate solution in the form

$$u_N(x) = \sum_{k=0}^{N} \beta_k \varphi_k(x)\,,$$

Take as an (approximate) solution the set of $\beta_0, \cdots, \beta_N$, s.t. $u_N(x)$ satisfies the integral identity for

$$\phi(x) = \varphi_k(x)\,, \qquad k = 1, \cdots, N-1$$

Example: For the heat conduction equation, this coincides with the Ritz method. However, it generalizes.