Outline

Para- and diamagnetism of metals

- Degenerate Fermi gas (*eg*: conduction electrons)
- Pauli paramagnetism of a non-interacting FG
- Quantization of a particle motion in the magnetic field
- Landau diamagnetism of a non-interacting FG
- de Haas–Van Alphen effect

Quantum states of a free particle

Condsider a single particle in a cube $L \times L \times L$ with periodic boundary conditions (PBC).

$$\widehat{H} = \frac{\widehat{\mathbf{p}}^2}{2m}$$

$\left[\widehat{H}, \widehat{\mathbf{p}}\right] = 0 \quad \Rightarrow \quad$ momentum $\mathbf{p}$ is a good quantum number.

$$\mathbf{p} = \hbar \mathbf{k} \qquad \varepsilon_{\mathbf{k}} = \frac{\hbar^2}{2m}\mathbf{k}^2 \qquad \psi_k(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

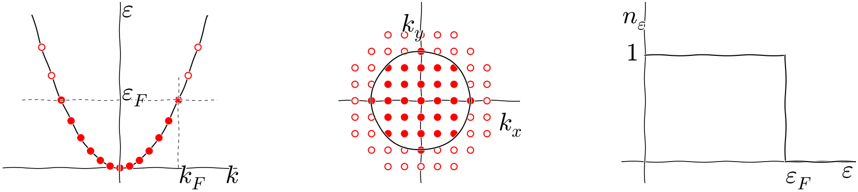
Periodic boundary conditions mean that

$$k_x, k_y, k_z = 0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \ldots$$

NB: in solids, $\mathbf{k} \in$ Brillouin zone

Non-interacting Fermi gas, $T = 0$

Pauli exclusion principle: one particle per state. $\Rightarrow$ For N particles, all the states with $\varepsilon < \varepsilon_F$ are filled, the states with $\varepsilon > \varepsilon_F$ are empty.



Fermi energy ε_F
Fermi momentum k_F
Fermi surface (in $\mathbf{k}$ space) $\varepsilon(\mathbf{k}) = \varepsilon_F$

Non-interacting Fermi gas, $T = 0$

What is the relation between ε_F and N?

In the limit of $L \to \infty$

$$N = (2s+1)\int \frac{\mathrm{d}\mathbf{k}}{(2\pi/L)^d}\Theta(k_F - |\mathbf{k}|) \qquad (d=3, s=1/2)$$

$$= 2\int_0^{k_F} \frac{4\pi k^2 \mathrm{d}k}{8\pi^3/V} = V\frac{k_F^3}{3\pi^2} \qquad (\text{volume } V = L^3)$$

$$k_F = \left(3\pi^2 N/V\right)^{1/3} \qquad \varepsilon_F = \frac{\hbar^2}{2m}k_F^2 = \frac{\hbar^2}{2m}\left(3\pi^2 N/V\right)^{2/3}$$

Non-interacting Fermi gas, $T = 0$, cont'd

What is the energy of a Fermi gas at $T = 0$ and $B = 0$?

$$\frac{E}{N} = \frac{(2s+1)\int\limits_{\mathrm{FS}} \frac{\mathrm{d}\mathbf{k}}{(2\pi/L)^3}\varepsilon_k}{(2s+1)\int\limits_{\mathrm{FS}} \frac{\mathrm{d}\mathbf{k}}{(2\pi/L)^3}} = \frac{(2s+1)\int\limits_0^{k_F} \frac{4\pi k^2 \mathrm{d}k}{8\pi^3/V}\frac{\hbar^2 k^2}{2m}}{(2s+1)\int\limits_0^{k_F} \frac{4\pi k^2 \mathrm{d}k}{8\pi^3/V}}$$

$$= \frac{\hbar^2}{2m}\frac{\int\limits_0^{k_F} k^4 \mathrm{d}k}{\int\limits_0^{k_F} k^2 \mathrm{d}k} = \frac{\hbar^2}{2m}\frac{k_F^5/5}{k_F^3/3} = \frac{3}{5}\frac{\hbar^2}{2m}k_F^2 = \frac{3}{5}\varepsilon_F$$

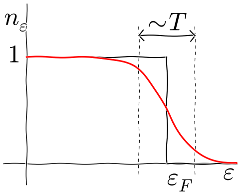
Non-interacting Fermi gas, finite T

Fermi temperature $k_B T_F = \varepsilon_F$

$T \gg T_F$ Boltzmann gas (classical)

$T \sim T_F$ degenerate gas (quantum)

In metals typically $T_F \sim 1 - 10\,\mathrm{eV} \sim 10^4\,\mathrm{K}$



$$n_\varepsilon = \frac{1}{\exp\left(\frac{\varepsilon-\mu}{k_B T}\right) + 1}$$

Chemical potential $\mu = \varepsilon_F$ at $T = 0$.
$(\mu/T \to -\infty$ as $T \to \infty)$

Qualitatively: out of N electrons, only $N(T/T_F)$ are "active", $\Rightarrow$
paramagnetic susceptibility $\chi \propto \frac{1}{T}\left(N\frac{T}{T_F}\right)$

Intermezzo: Density of states (DOS)

Density of states (DOS) = # of states per unit energy interval. Depends on (i) the dimension of space, d, and (ii) the dispersion relation, $\varepsilon_{\mathbf{k}}$

$$V\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\,(\dots) = \int \mathrm{d}\varepsilon\,\mathcal{D}(\varepsilon)\,(\dots)$$

For example: $d = 3$, $\varepsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m$

$$d\mathbf{k} = 4\pi k^2 dk\ , \quad k = \left(2m\varepsilon/\hbar^2\right)^{1/2} \Rightarrow$$

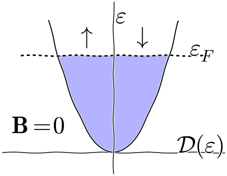
$$\mathcal{D}(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$$

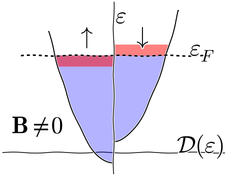
Note that in $d = 3$

$$\mathcal{D}(\varepsilon_F) = \frac{3}{2}\frac{N}{\varepsilon_F}$$

Non-interacting FG: paramagnetism Pauli

Weak fields, degenerate gas: $k_B T \ll \mu_B B \ll \varepsilon_F$.



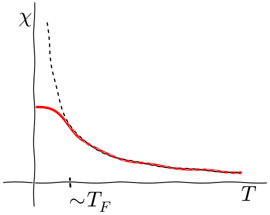


$$N_\uparrow = \frac{1}{2}\int_{-\mu_B B}^{\varepsilon_F} \mathrm{d}\varepsilon\, \mathcal{D}(\varepsilon + \mu_B B) \approx \frac{1}{2}\int_0^{\varepsilon_F} \mathrm{d}\varepsilon\, \mathcal{D}(\varepsilon) + \frac{1}{2}\mu_B B\, \mathcal{D}(\varepsilon_F)$$

$$N_\downarrow = \frac{1}{2}\int_{\mu_B B}^{\varepsilon_F} \mathrm{d}\varepsilon\, \mathcal{D}(\varepsilon - \mu_B B) \approx \frac{1}{2}\int_0^{\varepsilon_F} \mathrm{d}\varepsilon\, \mathcal{D}(\varepsilon) - \frac{1}{2}\mu_B B\, \mathcal{D}(\varepsilon_F)$$

$$\chi = \frac{M}{B} = \frac{\mu_B\,(N_\uparrow - N_\downarrow)}{B} = \mu_B^2 \mathcal{D}(\varepsilon_F) = \mu_B^2 \frac{3}{2}\frac{N}{\varepsilon_F} > 0$$

Non-interacting FG: paramagnetism Pauli

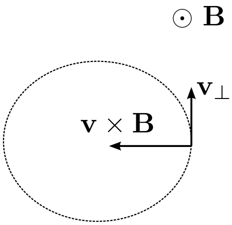


$$T \gg T_F \qquad \chi = \frac{N\mu_B^2}{k_B T} \qquad \text{Curie law}$$

$$T < T_F \qquad \chi = \frac{3}{2}\frac{N\mu_B^2}{\varepsilon_F} \qquad \text{Paramagnetism Pauli}$$

But... This is not the end of the story! Diamagnetic contribution is comparable (next step).

Classical particle in a field: cyclotron orbits



$$m\dot{\mathbf{v}} = \frac{e}{c}\mathbf{v}\times\mathbf{B}$$

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \qquad \mathbf{v}_\parallel \times \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{v}_\parallel = \text{const}$$

$$m\dot{\mathbf{v}}_\perp = \frac{e}{c}\mathbf{v}_\perp\times\mathbf{B} \quad \Rightarrow \quad |\mathbf{v}_\perp| = \text{const}$$

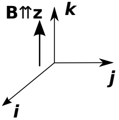
The motion is circular around the field lines.

$$\left.\begin{aligned} \text{acceleration} &= \omega^2 r \\ \text{velocity} &= \omega r \end{aligned}\right\} \Rightarrow m\,\omega^2 r = \frac{e}{c}\,\omega r\, B$$

$$\text{cyclotron frequency } \omega_c = \frac{eB}{mc} = 2\times(\text{Larmor frequency})$$

cf: Wilson chambers, cosmic rays, Hall effect etc etc

Intermezzo: Vector potential of a uniform magnetic field



$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{B} = B\mathbf{k}$$

Coulomb gauge $\quad \mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & H \\ x & y & z \end{vmatrix} = -\frac{1}{2}\mathbf{i}By + \frac{1}{2}\mathbf{j}Bx$

Indeed, $\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -By/2 & Bx/2 & 0 \end{vmatrix} = \mathbf{k}\left(\frac{B}{2} + \frac{B}{2}\right) = \mathbf{k}B$

Landau gauge $\quad \mathbf{A} = -\mathbf{i}By \qquad \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -By & 0 & 0 \end{vmatrix} = \mathbf{k}B$

Quantum particle in magnetic field

For a particle in a magnetic field, the *generalized momentum* $\mathbf{p} = m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{r})$.

$$\text{Kinetic energy } = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2$$

The Hamiltonian

$$\widehat{H} = \frac{1}{2m}\left(\widehat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + \widehat{\mu}\cdot\mathbf{B}$$

For a spinless particle in a uniform field in z direction, in the Landau gauge

$$\widehat{H} = \frac{1}{2m}\left(\widehat{p}_x + \frac{eB}{c}y\right)^2 + \frac{\widehat{p}_y^2 + \widehat{p}_z^2}{2m}$$

Quantum particle in magnetic field II

$$\widehat{H} = \frac{1}{2m}\left(\widehat{p}_x + \frac{eB}{c}y\right)^2 + \frac{\widehat{p}_y^2 + \widehat{p}_z^2}{2m}$$

$$\left[\widehat{H}, \widehat{p}_x\right] = \left[\widehat{H}, \widehat{p}_z\right] = 0$$

$$\Rightarrow \psi = e^{i(p_x x + p_z z)/\hbar}\phi(y) \qquad\qquad p_x, p_z \in (-\infty, \infty)$$

The Schrödinger equation $\widehat{H}\psi = E\psi$ is then

$$\phi'' + \frac{2m}{\hbar^2}\left[\varepsilon - \frac{m}{2}\omega_c^2(y - y_0)^2\right]\phi = 0$$

where $\varepsilon = E - p_z^2/2m$ and $y_0 = -cp_x/eB$.

...which is nothing but a harmonic oscillator.

Quantum particle in magnetic field III

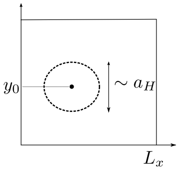
Landau levels: A quantum counterpart of the cyclotron orbits.

energy spectrum $E = \hbar\omega_c \left(n + \frac{1}{2} \right) + \frac{p_z^2}{2m}$, $\qquad n = 0, 1, \ldots$

magnetic length $a_H = \sqrt{\hbar/m\omega_c}$

The motion along the field is not quantized.

Wave functions:
$\psi_{n,p_z,p_x}(\mathbf{r}) = e^{i(p_x x + p_z z)/\hbar} \times (\text{oscillator w.f.})(y - y_0)$



Energy levels do not depend on $p_x \Rightarrow$ degeneracy.

$0 < y_0 < L_y$ and $-p_x = \frac{eB}{c} y_0 \quad \Rightarrow \quad -p_x \in \left(0, \frac{eB}{c} L_y\right)$

$$\mathcal{N} = \frac{L_x}{2\pi\hbar} \frac{eBL_y}{c} \propto L_x L_y$$

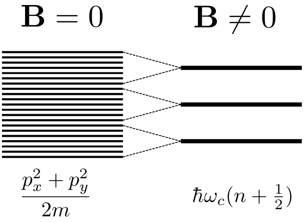
Landau levels degeneracy: A closer look

At a given p_z

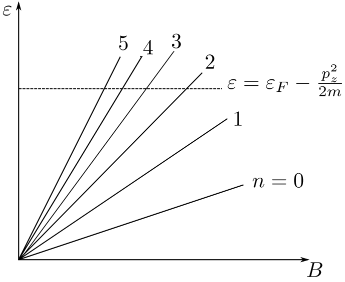
$$\mathcal{N} = \frac{L_x L_y}{2\pi\hbar}\frac{eB}{c} = \frac{L_x L_y}{2\pi\hbar} m\omega_c \qquad \text{(since } \omega_c = \frac{eB}{mc}\text{)}$$

The level spacing is $\hbar\omega_c \ \Rightarrow$ $\qquad \text{DOS} = \mathcal{N}/\hbar\omega_c = \frac{L_x L_y}{2\pi\hbar^2} m$,

which is exactly the DOS for the 2D motion in the x-y plane!



Energy levels for the transverse motion merge into Landau levels.



At a given p_z, the LL with $n > n_{\max}$ are empty:

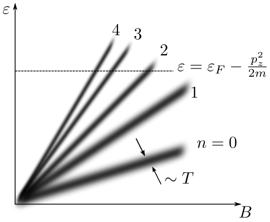
$$\hbar\omega_c \left(n_{\max} + \frac{1}{2}\right) = \varepsilon_F - \frac{p_z^2}{2m}$$

$$\omega_c \propto B \quad \Rightarrow \quad n_{\max} \propto \frac{1}{B}$$

The degeneracy of LL,

$$\mathcal{N} = \nu B \qquad\qquad (\text{where } \nu = \frac{L_x L_y}{2\pi\hbar}\frac{e}{c})$$

Upon increasing B, once a LL with $n = n_{\max}$ crosses $\varepsilon_F - p_z^2/2m$, $n_{\max} \longrightarrow n_{\max} - 1$.



Recall that $\varepsilon_F \gg \mu_B B, k_B T$.
Nevertheless, $\mu_B B \gtrsim k_B T$

WEAK FIELDS, $\mu_B B \ll k_B T$: Landau diamagnetism,

$\chi_{\text{Landau}} = -\frac{1}{3}\chi_{\text{Pauli}}$, so that

$$\chi = \chi_{\text{Landau}} + \chi_{\text{Pauli}} = \left(1 - \frac{1}{3}\right)\mu_B^2 \frac{3}{2}\frac{N}{\varepsilon_F} = \mu_B^2 \frac{N}{\varepsilon_F}$$

STRONG FIELDS, $\mu_B B \gg K_B T$: individual LLs are resolved.

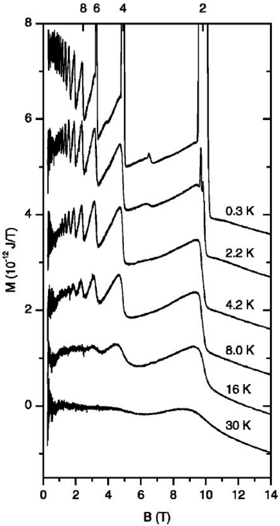
de Haas – van Alphen effect: *Magnetisation* oscillates in $1/B$

Shubnikov – de Haas effect: *Resistance* oscillates in $1/B$

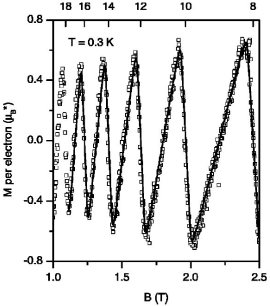
A couple of remarks

- We looked at the non-interacting Fermi gas only. In real metals, we need to take into account
  - Diamagnetism of the ionic cores
  - Band effects (Fermi surface is not spherical)
  - Electron-electron interactions

  Overall, the combined effect can be comparable to what we've found.
- To observe the de Haas – van Alphen effect, one needs
  - Pure samples
  - Magnetic fields of several teslas — need very high field stability
  - Temperatures below 20–30 Kelvin
- 2D samples (*eg* GaAs quantum wells or graphene), in still stronger magnetic field: *quantum Hall effect*.
  In graphene, observed even at room temperature. Possibly: redefinition of the base SI units.



2D electron gas (AlGaAs/GaAs heterostructure). Exp data from M.P. Schwarz *et al.* Phys. Rev. B **65**, 245315 (2002)



Still stronger fields: Quantum Hall Effect

Fast-forward to room temperature, weak fields

Nothing quantum in this regime (yet)

Edwin Hall, 1879:

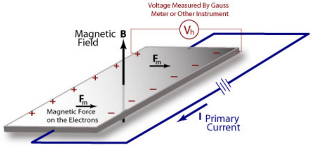


image (c)

ndt-ed.org

Hall resistance:

$$\rho_H = \frac{V_y}{I_x} \propto B$$

typically discussed for semiconductors, not metals.

Typically use *conductivity*, $\mathbf{j} = \sigma \mathbf{E}$.
With $\mathbf{B} \neq 0$,

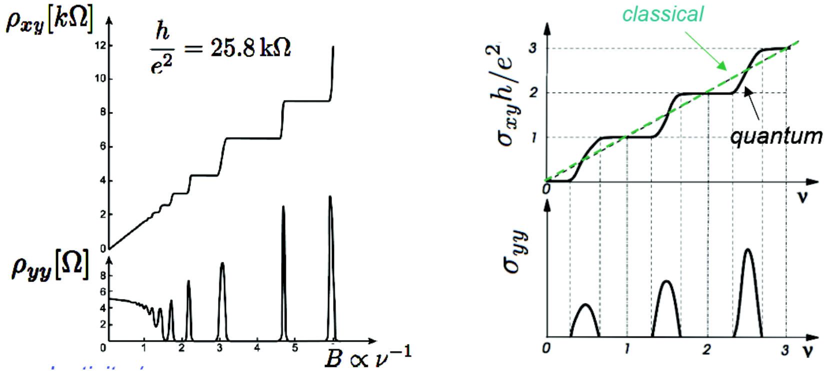
$$j_\alpha = \sigma_{\alpha\beta} E_\beta$$

need to distinguish between σ_{xx} and σ_{xy}.
(details: *e.g.*, Kittel, Chapter 6, Problem 7)

Still stronger fields: Quantum Hall Effect

Low temperature, *very* strong fields

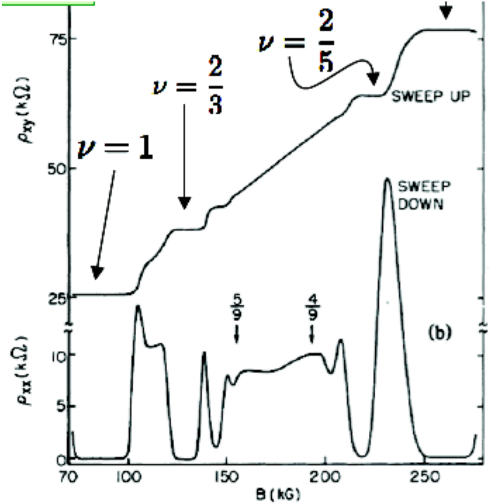
Also confine carriers to two dimensions (jargon: *2DEG*)



von Klitzing, Dorda, and Pepper, 1980

image from M. Sigrist, ETH Zürich; `http://www.itp.phys.ethz.ch/education/fs11/sst/QHE.pdf`

Fractional QHE



Störmer, Tsui, and Gossard, 1982

Still stronger fields: Quantum Hall Effect

IQHE

Qualitatively: LL degeneracy $\propto B$. Carrier density is fixed.
Once a LL is completely full, there's a plateau in ρ_{xy}.
An alternative picture: each electron has an attached "flux tube".
Electron carry with them integer number of magnetic flux quanta.

FQHE

Plateaus at a *fractional* occupation of a LL.
Electrons carry a fraction of the flux quantum.