Multidimensional problems

Example: 2D diffusion equaton

(2+1)D diffusion equation

Let $u(x,t)$ is defined for $x=(x_1,x_2)\in G=[0,a_1]\times[0,a_2]$ and $t\in[0,T]$.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2} \tag{1}$$

Initial and boundary conditions

$$u(x,t)=\mu(x,t) \qquad \text{for } x\in\Gamma\,, \quad t\in[0,T]$$
$$u(x,0)=u_0(x) \qquad \text{for } x\in G+\Gamma$$

Here $\Gamma=\partial G$.

Grids and grid functions

Define the grids:

$$\omega_\tau = \{t_n = n\,\tau\,, \quad n = 0, \dots, K\,; \quad K\,\tau = T\}$$

and

$$\begin{aligned}\Omega_h = \{x_{ij} = (x_1^i, x_2^j)\ : \quad & x_1^i = h_1 i, x_2^j = h_2 j \\ & i = 0, 1, \cdots, N_1\,, \quad j = 0, 1, \cdots, N_2 \\ & N_\alpha h_\alpha = a_\alpha\,, \alpha = 1, 2\}\end{aligned}$$

Call the ***internal nodes*** of Ω_h by ω_h and ***boundary nodes*** by γ_h.

Define a ***grid function*** $y_{ij}^n \equiv y(x_{ij}, t_n)$.

The explicit scheme

$$\frac{y_{ij}^{n+1} - y_{ij}^{n}}{\tau} = \Lambda y_{ij}^{n}\,, \qquad x_{ij} \in \omega_h\,, t_n \in \omega_\tau$$

$$y_{ij}^{n+1} = \mu(x_{ij}, t_{n+1})\,, \qquad x_{ij} \in \gamma_h\,, t_n \in \omega_\tau$$

$$y_{ij}^{0} = u_0(x_{ij})\,, \qquad x \in \Omega_h\,, n = 0$$

Here

$$\Lambda y_{ij}^{n} = (\Lambda_1 + \Lambda_2) y_{ij}^{n}$$

$$\Lambda_1 y_{ij}^{n} = \mathcal{D}_{x_1}^{2} y_{ij}^{n}$$
$$\Lambda_2 y_{ij}^{n} = \mathcal{D}_{x_2}^{2} y_{ij}^{n}$$

The explicit scheme

The solution is constructed layer by layer, starting from for $n = 0$:

$$y_{ij}^{n+1} = y_{ij}^{n} + \tau \Lambda y_{ij}^{n}\,, \qquad x_{ij} \in \omega_h\,, \qquad n = 1, \cdots, K$$

Stability requires that

$$\tau \left(\frac{1}{h_1^2} + \frac{1}{h_2^2} \right) \leqslant \frac{1}{2}$$

This is too stringent to be practical.

The fully implicit scheme

$$\frac{y_{ij}^{n+1} - y_{ij}^{n}}{\tau} = \Lambda y_{ij}^{n+1}\,, \qquad x_{ij} \in \omega_h\,, t_n \in \omega_\tau$$

$$y_{ij}^{n+1} = \mu(x_{ij}, t_{n+1})\,, \qquad x_{ij} \in \gamma_h\,, t_n \in \omega_\tau$$

$$y_{ij}^{0} = u_0(x_{ij})\,, \qquad x \in \Omega_h\,, n = 0$$

The scheme is unconditionally stable.

The fully implicit scheme

Need to solve ***on each layer*** a linear system of the size $O(1/h^2)$

$$y_{ij}^{n+1} - \tau \Lambda y_{ij}^{n+1} = y_{ij}^{n}\,, \qquad x_{ij} \in \omega_h\,, t_n \in \omega_\tau$$

$$y_{ij}^{n+1} = \mu(x_{ij}, t_{n+1})\,, \qquad x_{ij} \in \gamma_h\,, t_n \in \omega_\tau$$

The computational complexity is not manageable.

Operator splitting

Alternating directions implicit scheme (ADI)

Peaceman-Rachford scheme

Продольно-поперечная схема

Alternating directions implicit method

The main idea: convert a $d > 1$ problem into a sequence of 1D problems.

Make the transition from layer n to layer $n+1$ in two steps, via layer $n + 1/2$.

Peaceman-Rachford ADI scheme

At internal nodes, $x_{ij} \in \omega_h \,, t_n \in \omega_\tau$:

$$\frac{y_{ij}^{n+1/2} - y_{ij}^{n}}{\tau/2} = \Lambda_1 y_{ij}^{n+1/2} + \Lambda_2 y_{ij}^{n} \tag{2}$$

implicit over i, explicit over j

$$\frac{y_{ij}^{n+1} - y_{ij}^{n+1/2}}{\tau/2} = \Lambda_1 y_{ij}^{n+1/2} + \Lambda_2 y_{ij}^{n+1} \tag{3}$$

explicit over i, implicit over j

Peaceman-Rachford ADI scheme

Questions

- How to organize computations
- Approximation: does the FDE approximate the PDE
- Stability

Peaceman-Rachford ADI scheme

Write out the 1st ADI step, Eq. (2),

$$\frac{w_{ij} - y_{ij}^n}{\tau/2} = \frac{w_{i-1,j} - 2w_{ij} + w_{i+1,j}}{h_1^2} + \Lambda_2 y_{ij}^n$$
$$i = 1, 2, \cdots, N_1 - 1$$

Here $w_{ij} \equiv y_{ij}^{n+1/2}$ for brevity.

At a fixed j, this is a triagiagonal system of equations w.r.t. i.

NB: need extra boundary conditions for $y_{0,j}^{n+1/2}$ and $y_{N_1,j}^{n+1/2}$

Peaceman-Rachford ADI scheme

Write out the 2st ADI step, Eq. (3),

$$\frac{w_{ij} - y_{ij}^{n+1/2}}{\tau/2} = \Lambda_1 y_{ij}^{n+1/2} + \frac{w_{i,j-1} - 2w_{ij} + w_{i,j+1}}{h_2^2}$$

$$j = 1, 2, \cdots, N_1 - 1$$

$$w_{i,0} = \mu(x_{i,0}, t_{n+1}), \qquad w_{i,N_2} = \mu(x_{i,N_2}, t_{n+1})$$

Here $w_{ij} \equiv y_{ij}^{n+1}$ for brevity.

At a fixed i, this is a triagiagonal system of equations w.r.t. j.

Computational complexity of the ADI scheme

For the transition $n \longrightarrow n+1$:

- The 1st ADI step is N_2 tridiagonal order-N_1 solves. The complexity is $O(N_1 N_2)$
- The 2nd ADI step is N_1 triadiagonal order-N_2 solves. The complexity is also $O(N_1 N_2)$

For comparison, FFT approach is $O(N_1 N_2 \ln N_2)$ per layer.
Direct Gauss solve is $O(N_1^3 N_2^3)$

Boundary conditions for $n + 1/2$

Express $y^{n+1/2}$ from the ADI equations for $x_{ij} \in \omega_h$, extend to γ_h.

Subtract Eq.(2) from (3) (drop ij subscripts for brevity)

$$\frac{y^{n+1} - 2y^{n+1/2} + y^n}{\tau/2} = \Lambda_2(y^{n+1} - y^n)$$

therefore

$$y^{n+1/2} = \frac{1}{2}\left(y^{n+1} + y^n\right) - \frac{\tau}{4}\Lambda_2\left(y^{n+1} - y^n\right)$$

Boundary conditions for $n + 1/2$

$$y^{n+1/2} = \frac{1}{2}\left(y^{n+1} + y^{n}\right) - \frac{\tau}{4}\Lambda_2\left(y^{n+1} - y^{n}\right)$$

This is valid for $x_{ij} \in \omega_h$. Extend to $x_{ij} \in \gamma_h$, use the fact that

$$y_{0,j}^{n+1} = \mu(x_{0,j}, t_{n+1})\,, \qquad y_{0,j}^{n} = \mu(x_{0,j}, t_{n})$$

and likewise for $i = N_1$.

Approximation properties of the ADI scheme

Approximation properties of the ADI method

Define the ***deviations***, z_{ij}^n, via

$$y_{ij}^n = u_{ij}^n + z_{ij}^n\,, \qquad y_{ij}^{n+1/2} = u_{ij}^{n+1/2} + z_{ij}^{n+1/2}$$

Here $u_{ij}^n \equiv u(x_{ij}, t_n)$ is the solution of the PDE (1).

Substitute into the FDE. The errors, z_{ij}^n, satisfy the FDE equations with the ***residuals***

$$\psi^n = -\frac{u^{n+1/2} - u^n}{\tau/2} + \Lambda_1 u^{n+1/2} + \Lambda_2 u^n$$

$$\phi^n = -\frac{u^{n+1} - u^{n+1/2}}{\tau/2} + \Lambda_1 u^{n+1/2} + \Lambda_2 u^{n+1}$$

Approximation properties of the ADI method

Taylor-expand the residuals, $u_{ij}^{n+1} \equiv u(x_{ij}, t_n + \tau)$

$$\psi^n = -\frac{\tau}{4} u_{tt} + \frac{\tau}{2} L_1 u_t + O(\tau^2 + h^2)$$

$$\phi^n = -\frac{3\tau}{4} u_{tt} + \frac{\tau}{2} L_1 u_t + \tau L_2 u_t + O(\tau^2 + h^2)$$

This way, ϕ^n and ϕ^n are $O(\tau + h^2)$ but

$$\psi^n + \phi^n = O(\tau^2 + h^2)$$

суммарная аппроксимация

Alternative ADI splittings

$$\frac{y_{ij}^{n+1/2} - y_{ij}^{n}}{\tau} = \Lambda_1 y_{ij}^{n+1/2}\,, \qquad x_{ij} \in \omega_h\,, t_n \in \omega_\tau$$
$$\frac{y_{ij}^{n+1} - y_{ij}^{n+1/2}}{\tau} = \Lambda_2 y_{ij}^{n+1}\,, \qquad x_{ij} \in \omega_h\,, t_n \in \omega_\tau$$

Each of the equations individually is $O(1)$ but taken together the scheme is

$$O(\tau + h^2)$$

Stability of the ADI scheme

The factorized form of the ADI scheme

Excluding $y^{n+1/2}$ from the ADI equations, (2) and (3), we get

$$\mathcal{D}_t y^n = \frac{1}{2}\Lambda\left(y^{n+1} + y^n\right) - \frac{\tau^2}{4}\Lambda_1\Lambda_2\mathcal{D}_t y^n$$

Here $\mathcal{D}_t y^n \equiv (y^{n+1} - y^n)/\tau$

Identically, in the ***factorized form***,

$$\left(1 - \frac{\tau}{2}\Lambda_1\right)\left(1 - \frac{\tau}{2}\Lambda_2\right)\mathcal{D}_t y^n = \Lambda y^n$$

von Neumann stability analysis

Take a particular solution of the form

$$y_{ij}^n = q^n e^{i\alpha x_1^i} e^{i\beta x_2^j}$$

Note that

$$\mathcal{D}_t y_{ij}^n = \frac{q-1}{\tau} y_{ij}^n$$

$$\Lambda_1 y_{ij}^n = \frac{-4}{h_1^2} \sin^2 \alpha h_1/2 \, y_{ij}^n$$

$$\Lambda_2 y_{ij}^n = \frac{-4}{h_2^2} \sin^2 \beta h_2/2 \, y_{ij}^n$$

von Neumann stability analysis

Collect all terms and rearrange to

$$q = \frac{1-\xi}{1+\xi}\frac{1-\eta}{1+\eta}$$

where

$$\xi = \frac{2\tau}{h_1^2}\sin^2 \alpha h_1/2\,, \qquad \eta = \frac{2\tau}{h_2^2}\sin^2 \beta h_2/2$$

Since $\xi, \eta > 0$,

$$|q| \leqslant 1\,, \quad \text{for all} \quad \alpha, \beta$$

Hence the ADI scheme is unconditionally stable.