Systems of linear algebraic equations

Systems of linear algebraic equations

Have a system of m equations for m unknowns, $x_1, x_2, \ldots, x_m$

$$\begin{cases} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1m}x_m = \ b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2m}x_m = \ b_2 \\ \qquad\qquad\qquad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m = b_m \end{cases}$$

Systems of linear algebraic equations

Have a system of m equations for m unknowns, $x_1, x_2, \ldots, x_m$

$$\begin{cases} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1m}x_m = \ b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2m}x_m = \ b_2 \\ \qquad\qquad\qquad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m = b_m \end{cases}$$

Or, equivalently

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ & & \cdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$$

Cramer's rule

Given

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

the formal solution is ($k = 1, \cdots, m$)

$$x_k = \frac{\det \mathbf{A}_{(k)}}{\det \mathbf{A}}$$

Cramer's rule: the computational complexity of

Compute the determinants using a cofactor expansion.

Let C_m is the number of operations for an $m \times m$ matrix.

$$C_m = mC_{m-1}$$

$\Rightarrow C_m = O(m!)$

Cramer's rule: the computational complexity of

m	$m!$	time	
10	3.6×10^6	$\approx 3.6 \times 10^{-3}$ sec	
20	2.4×10^{18}	$\approx 2.4 \times 10^9$ sec	≈ 760 yrs
50	3×10^{64}	$\approx 3 \times 10^{53}$ sec	$\approx 10^{46}$ yrs

Need a better method.

Systems of linear algebraic equations

Gaussian elimination

Gaussian elimination

- Forward sweep
- Back substitution

Gaussian elimination

Forward sweep:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ & & \cdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}$$

Gaussian elimination

Forward sweep:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ & & \cdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix} \gamma_{21} = a_{21}/a_{11}$$

$$\begin{aligned} a_{11}x_1 + \quad & a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ (a_{21} - \gamma_{21}a_{11})x_1 + \quad & (a_{22} - \gamma_{21}a_{12})x_2 + \cdots &= b_2 - \gamma_{21}b_1 \end{aligned}$$

Gaussian elimination

Forward sweep:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ & & \cdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix} \begin{matrix} \gamma_{21} = a_{21}/a_{11} \\ \gamma_{31} = a_{31}/a_{11} \\ \\ \\ \end{matrix}$$

$$\begin{aligned} a_{11}x_1 + \quad & a_{12}x_2 + \cdots + a_{1m}x_m && = b_1 \\ (a_{31} - \gamma_{31}a_{11})x_1 + \quad & (a_{32} - \gamma_{31}a_{12})x_2 + \cdots && = b_3 - \gamma_{31}b_1 \end{aligned}$$

Gaussian elimination

After $m-1$ steps, we arrive at

$$\mathbf{A}^{(1)}\mathbf{x} = \mathbf{b}^{(1)}$$

with

$$\mathbf{A}^{(1)} = \begin{pmatrix} \times & \times & \times & \cdots & \times \\ 0 & \times & \times & \cdots & \times \\ 0 & \times & \times & \cdots & \times \\ & & \cdots & & \\ 0 & \times & \times & \cdots & \times \end{pmatrix}$$

Gaussian elimination

After $m-2$ steps, we arrive at

$$\mathbf{A}^{(2)}\mathbf{x} = \mathbf{b}^{(2)}$$

with

$$\mathbf{A}^{(2)} = \begin{pmatrix} \times & \times & \times & \cdots & \times \\ 0 & \times & \times & \cdots & \times \\ 0 & 0 & \times & \cdots & \times \\ & & \cdots & & \\ 0 & 0 & \times & \cdots & \times \end{pmatrix}$$

Gaussian elimination

Having annihilated $m-1$ columns, we arrive at

$$\mathbf{A}^{(m-1)}\mathbf{x} = \mathbf{b}^{(m-1)}$$

with an *upper triangular* matrix

$$\mathbf{A}^{(m-1)} = \begin{pmatrix} \times & \times & \times & \cdots & \times & \times \\ 0 & \times & \times & \cdots & \times & \times \\ 0 & 0 & \times & \cdots & \times & \times \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \times & \times \\ 0 & 0 & 0 & \cdots & 0 & \times \end{pmatrix}$$

Gaussian elimination: back substitution

$$\mathbf{A}^{(m-1)}\mathbf{x} = \mathbf{b}^{(m-1)}$$

$$\begin{pmatrix} \times & \times & \times & \cdots & \times & \times \\ 0 & \times & \times & \cdots & \times & \times \\ 0 & 0 & \times & \cdots & \times & \times \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \times & \times \\ 0 & 0 & 0 & \cdots & 0 & \times \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_{m-1} \\ x_m \end{pmatrix} = \begin{pmatrix} \times \\ \times \\ \times \\ \cdots \\ \times \\ \times \end{pmatrix}$$

Gaussian elimination: computational complexity

Forward sweep

1st row: multiply $m-1$ elements, a_{1k}, $k = 2, \ldots, m$
by $\gamma_{21} \Rightarrow m-1$ multiplies.

2nd row: $m-1$ multiplies

...

Gaussian elimination: computational complexity

Forward sweep

1st row: multiply $m-1$ elements, a_{1k}, $k = 2, \ldots, m$
by $\gamma_{21} \Rightarrow m-1$ multiplies.

2nd row: $m-1$ multiplies

...

Total for the first column: $(m-1)^2$ operations.

Gaussian elimination: computational complexity

Forward sweep

1st row: multiply $m-1$ elements, a_{1k}, $k=2,\ldots,m$ by $\gamma_{21} \Rightarrow m-1$ multiplies.

2nd row: $m-1$ multiplies

...

Total for the first column: $(m-1)^2$ operations.
Total for the 2nd column: $(m-2)^2$ operations.

Gaussian elimination: computational complexity

Forward sweep

1st row: multiply $m-1$ elements, a_{1k}, $k = 2, \ldots, m$ by $\gamma_{21} \Rightarrow m-1$ multiplies.

2nd row: $m-1$ multiplies

...

Total for the first column: $(m-1)^2$ operations.
Total for the 2nd column: $(m-2)^2$ operations.

Total for the whole forward sweep:

$$(m-1)^2 + (m-2)^2 + \cdots \sim O(m^3)$$

Gaussian elimination: computational complexity

Back substitution

m-th row: 1 multiplication

$(m-1)$-th row: 2 multiplications

...

Gaussian elimination: computational complexity

Back substitution

m-th row: 1 multiplication

$(m-1)$-th row: 2 multiplications

...

Total is $\sim O(m^2)$

Compared to the forward sweep, back substitution is cheap.

When is the Gaussian elimination applicable?

$\mathbf{A} \longrightarrow \mathbf{A}^{(1)}$:

$$\begin{matrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} \qquad\qquad a_{11} \neq 0$$

When is the Gaussian elimination applicable?

$\mathbf{A} \longrightarrow \mathbf{A}^{(1)}$:

$$\begin{matrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} \qquad\qquad a_{11} \neq 0$$

$\mathbf{A}^{(1)} \longrightarrow \mathbf{A}^{(2)}$:

$$\begin{matrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22}^{(1)} & \cdots \\ \vdots & \vdots & \ddots \end{matrix} \qquad\qquad \begin{aligned} a_{22}^{(1)} &= a_{22} - \gamma_{21} a_{12} \\ &= a_{22} - \frac{a_{21}}{a_{11}} a_{12} \neq 0 \end{aligned}$$

When is the Gaussian elimination applicable?

Need all leading minors of the matrix $\mathbf{A}$ to be non-zero

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3m} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4m} \\ & & \cdots & & & \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & a_{mm} \end{pmatrix}$$