Stability of linear multistep methods

Consider a telling example:

$$\begin{cases} \dot{u} = u - \sin t + \cos t \\ u(0) = 0 \end{cases}$$

Solution(can check):

$$u(t) = \sin t$$

Tweak the initial condition:

$$u(0) = \varepsilon, \qquad |\varepsilon| \ll 1$$

Then,

$$u(t) = \sin t + \varepsilon\, e^t$$

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Initial conditions are infinitesimally close, but solutions are very different for $t \geqslant 1/\ln \varepsilon$.

Lyapunov stability

Let $u(t)$ and $w(t)$ are solutions of

$$\dot{u} = f(t, u)$$

with initial conditions $u(0) = u_0$ and $w(0) = w_0$.

Lyapunov stability on $[0, T]$:

$$\max_{t \in [0,T]} |u(t) - w(t)| \leqslant K |u_0 - w_0|$$

(Here K is T-independent.)

Want a numerical scheme to be stable if the IVP is Lyapunov stable.

Stability of LMM

Consider (again) the model equation:

$$\dot{u} = \lambda u \qquad\qquad u(0) = u_0$$

Lyapunov stability: $\lambda \leqslant 0$.

A *linear s-step method*

$$\frac{a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s}}{\tau} = b_0 f_n + b_1 f_{n-1} + \cdots + b_s f_{n-s}$$

with $f_n = \lambda y_n$.

Stability of LMM

A linear s-point scheme for the model equation becomes

$$\gamma_0 y_n + \gamma_1 y_{n-1} + \cdots + \gamma_s y_{n-s} = 0\,.$$

Here

$$\gamma_p = a_p - z b_p\,, \qquad\qquad p = 0, \ldots, s$$

and

$$z \equiv \lambda\tau\,.$$

Perturb the initial condition. The error at time step n satisfies the same equation.

Stability of LMM

Have a linear homogenous recurrence relation

$$\gamma_0 y_n + \gamma_1 y_{n-1} + \cdots + \gamma_s y_{n-s} = 0\,.$$

Look for solutions $y_n = q^n$. The characteristic polynomial (C.P.):

$$\gamma_0 q^s + \gamma_1 q^{s-1} + \cdots + \gamma_{s-1} q + \gamma_s = 0.$$

Note that the roots of C.P. depend on z.

Absolutely stable schemes

The scheme *absolutely stable* for a given z if all roots of the C.P. satisfy the *root condition*:

- $|q| \leqslant 1$
- there are no multiple roots with $|q| = 1$

The *region of absolute stability*: The locus $\mathcal{D} \subset \mathbb{C}$ of the complex plane of z where the scheme is absolutely stable.

A-stability

The model equation

$$\dot{u} = \lambda u$$

is Lyapunov stable for $\lambda \leqslant 0$.

Since $\tau > 0$

$$z = \lambda\tau \leqslant 0\,.$$

A scheme is *A-stable* if its region of stability $\mathcal{D}$ includes the half-plane $\mathrm{Re}\, z < 0$.

Can also define $A(\alpha)$-stable if $\mathcal{D}$ contains the α-angle around the negaive real axis.

Examples: Euler scheme

$$y_{n+1} = (1 + \tau\lambda)y_n$$

The C.E. is

$$q - (1 + z) = 0$$

The stability region $|q| \leqslant 1$ is

$$\mathcal{D} = \{z \in \mathbb{C} : |z + 1| \leqslant 1\}$$

Examples: Implicit Euler scheme

$$y_{n+1} - y_n = \tau \lambda y_{n+1}$$

The C.E. is

$$(1 - z)q - 1 = 0 \qquad \Rightarrow \qquad q = \frac{1}{1 - z}$$

The stability region $|q| \leqslant 1$ is

$$\mathcal{D} = \{z \in \mathbb{C} : |z - 1| > 1\}$$

Examples: symmetrized Euler scheme

$$\frac{y_{n+1} - y_n}{\tau} = \frac{\lambda}{2}\left(y_{n+1} + y_n\right)$$

The C.E. is

$$\left(1 - \frac{z}{2}\right) q - \left(1 + \frac{z}{2}\right) = 0 \qquad \Rightarrow \qquad q = \frac{2+z}{2-z}$$

The stability region $|q| \leqslant 1$ is

$$\mathcal{D} = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$

Can an explicit scheme be A-stable?

$$\frac{a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s}}{\tau} = b_0 f_n + b_1 f_{n-1} + \cdots + b_s f_{n-s}$$

Consider an explicit scheme:

$$b_0 = 0 \qquad a_0 \neq 0$$

Let

$$b_0 = b_1 = \cdots = b_{p-1} = 0 \qquad \text{and} \qquad b_p \neq 0$$

(at least one $b_j \neq 0$ since $\sum_{j=0}^{s} b_j = 1$)

Can an explicit scheme be A-stable?

Apply the linear scheme to the model equation.

Rewrite the C.E.

$$z = \frac{a_0 q^s + a_1 q^{n-1} + \cdots + a_s}{b_0 q^s + b_1 q^{n-1} + \cdots + b_s}$$

For $|q| \gg 1$

$$z(q) = \frac{a_0}{b_p} q^p + \ldots$$

$\Rightarrow$ For $|z| \gg 1$ (incl $\mathrm{Re}\, z < 0$) there is a root of C.P. with $|q| > 1$.

Dahlquist Second Barrier

If an s-step scheme is A-stable, then

- it must be implicit.
- The order of the scheme is at most $s = 2$.