Wave equation

Three-layer schemes

1st boundary problem for the wave equation

Let $u(x,t)$ is defined for $x \in [0,1]$ and $t \in [0,T]$.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Boundary conditions

$$u(0,t) = \mu_1(t) \qquad\qquad u(1,t) = \mu_2(t)$$

Initial conditions

$$u(x,t=0) = u_0(x) \qquad\qquad u_t(x,t=0) = \widetilde{u}_0(x)$$

The problem is *correct*:
the solution is unique; the dependence on initial and boundary conditions is continuous.

Numerical solutions: grids and grid functions

Consider a 2D grid $\omega_h \times \omega_\tau$:

$$\omega_\tau = \{t_n = n\,\tau\,, \quad n = 0, \ldots, K\,; \quad K\,\tau = T\}$$
$$\omega_h = \{x_j = j\,h\,, \quad j = 0, \ldots, N\,; \quad N\,h = 1\}$$

Define a *grid function* $y_j^n \equiv y(x_j, t_n)$.

The five-point stencil

An FDE scheme has at least three layers. Using the five-point stencil

Eq.(1) generates at (x_j, t_n),

$$\frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\tau^2} = \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{h^2} \tag{2}$$

At the *internal nodes*,
$n = 1, \ldots, K-1$ and $j = 1, \ldots, N-1$
the scheme approximates Eq. (1) with $O(\tau^2 + h^2)$.

The five-point stencil

$$\frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\tau^2} = \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{h^2}$$

The boundary conditions:

$$y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1}), \qquad n = 0, \dots, K$$

The five-point stencil

$$\frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\tau^2} = \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{h^2}$$

The scheme is *explicit*: on the layer $n+1$

$$y_j^{n+1} = -y_j^{n-1} + 2y_j^n + \gamma^2 \left(y_{j+1}^n - 2y_j^n + y_{j-1}^n \right)$$

$$y_0^{n+1} = \mu_1(t_{n+1}), \quad y_N^{n+1} = \mu_2(t_{n+1})$$

Here $\gamma = \tau/h$

Initial conditions

Need the initial conditions for $n = 0$ and 1.

$n = 0$

$$u(x, t = 0) = u_0(x) \quad \Longrightarrow \quad y_j^0 = u_0(x_j)$$

$n = 1$

$$u_t(x, t = 0) = \widetilde{u}_0(x) \quad \Longrightarrow \quad \frac{y_j^1 - y_j^0}{\tau} = \widetilde{u}_0(x_j)$$

This approximates the initial conditions with $O(\tau)$ only.

Initial conditions: $O(\tau^2)$

Note that

$$\frac{u(x,\tau)-u(x,0)}{\tau} = u_t(x,0) + \frac{\tau}{2}u_{tt}(x,0) + O(\tau^2)$$

Using Eq. (1),

$$u_{tt}(x,0) = u_{xx}(x,0) = u_0''(x)$$

Therefore,

$$\frac{y_j^1 - y_j^0}{\tau} = \widetilde{u}_0(x_j) + \frac{\tau}{2}\mathcal{D}_{xx}u_0(x_j)$$

approximates the initial condition with $O(\tau^2 + h^2)$.

von Neumann stability analysis

Consider the homogenous equation, look for particular solutions of the form

$$y_j^n(\varphi) = q^n e^{i\varphi jh}$$

where $\varphi \in \mathbb{R}$ and $q \in \mathbb{R}$ is unknown.

Require that $|q| \leqslant 1$ for all φ

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von Neumann stability analysis

For Eq. (1),

$$q^2 - 2\left(1 - 2\gamma^2 \sin^2 \varphi h/2\right) q + 1 = 0$$

The roots satisfy $q_1 q_2 = 1$. Two possibilities:

- $q_{1,2} \in \mathbb{R}$. For some φ, $|q_1| > 1$.
- The roots are complex conjugate. Then $|q_1| = |q_2| = 1$.

The Courant condition

This way, stability requires that

$$\left(1 - 2\gamma^2 \sin^2 \varphi h/2\right)^2 - 1 < 0$$

For $\varphi \in \mathbb{R}$,

$$\gamma^2 \leqslant 1 \qquad \Longleftrightarrow \qquad \tau \leqslant h$$

which is known as the *Courant condition*.

Notice the difference to the parabolic equation, where $\tau \leqslant h^2/2$.