Sherman-Morrison formula

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$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Now, we “slighly” change $\mathbf{A}$. Can we update $\mathbf{x}$?

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Yes, for an outer product update:

$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)\mathbf{x} = \mathbf{b}$$

where $\mathbf{u}$ and $\mathbf{v}$ are known column vectors.

If $u_j = \delta_{jp}$, this updates the p-th row of $\mathbf{A}$.
If $v_j = \delta_{jp}$, this updates the p-th column of $\mathbf{A}$

Sherman-Morrison formula

$$\left(\mathbf{A} + \mathbf{u}\,\mathbf{v}^T\right)^{-1} = \left(\mathbf{1} + \mathbf{A}^{-1}\mathbf{u}\,\mathbf{v}^T\right)^{-1} \mathbf{A}^{-1}$$

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(expand into a formal power series)

$$= \left(\mathbf{1} - \mathbf{A}^{-1}\mathbf{u}\,\mathbf{v}^T + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T + \cdots\right)\mathbf{A}^{-1}$$

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(define $\lambda = \mathbf{v}^T\mathbf{A}\mathbf{u}$)

$$= \mathbf{A}^{-1}-\left(\mathbf{A}^{-1}\mathbf{u}\right)\left(\mathbf{v}^T\mathbf{A}^{-1}\right)\left(1-\lambda+\lambda^2+\cdots\right)$$

Sherman-Morrison formula

$$
\begin{aligned}
\left(\mathbf{A} + \mathbf{u}\,\mathbf{v}^T\right)^{-1} &= \left(\mathbf{1} + \mathbf{A}^{-1}\mathbf{u}\,\mathbf{v}^T\right)^{-1}\mathbf{A}^{-1} \\
&\qquad \textit{(expand into a formal power series)} \\
&= \left(\mathbf{1} - \mathbf{A}^{-1}\mathbf{u}\,\mathbf{v}^T + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T + \cdots\right)\mathbf{A}^{-1} \\
&\qquad \textit{(define } \lambda = \mathbf{v}^T\mathbf{A}\mathbf{u}\textit{)} \\
&= \mathbf{A}^{-1} - \left(\mathbf{A}^{-1}\mathbf{u}\right)\left(\mathbf{v}^T\mathbf{A}^{-1}\right)\left(1 - \lambda + \lambda^2 + \cdots\right) \\
&\qquad \textit{(sum the geometric series)} \\
&= \mathbf{A}^{-1} - \frac{\left(\mathbf{A}^{-1}\mathbf{u}\right)\left(\mathbf{v}^T\mathbf{A}^{-1}\right)}{1 + \lambda}
\end{aligned}
$$

Sherman-Morrison update

Let $\mathbf{y}$ is the (known) solution of $\mathbf{A}\mathbf{y} = \mathbf{b}$.
We want to find the solution of $\left(\mathbf{A} + \mathbf{u}\,\mathbf{v}^T\right)\mathbf{x} = \mathbf{b}$.

Define z as the solution of $\mathbf{A}\mathbf{z} = \mathbf{u}$.
Note that $\lambda = \mathbf{v}^T\mathbf{z}$.

Then the solution $\mathbf{x}$ of $\left(\mathbf{A} + \mathbf{u}\,\mathbf{v}^T\right)\mathbf{x} = \mathbf{b}$ is

$$\begin{aligned}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} - \frac{\left(\mathbf{A}^{-1}\mathbf{u}\right)\left(\mathbf{v}^T\mathbf{A}^{-1}\right)}{1+\lambda}\mathbf{b} \\ &= \mathbf{y} - \mathbf{z}\frac{(\mathbf{v}^T\mathbf{y})}{1+(\mathbf{v}^T\mathbf{z})}\end{aligned}$$

The complexity of the Sherman-Morrison update is $O(m^2)$.