sets

Axiom of completeness

Every non-empty set of real numbers that is bounded from above has a least upper bound

supremum

$s \in \mathbb{R}$ upper bound for A

$s = \sup A$ iff every $\varepsilon > 0$

$\exists\, a \in A$ such that $s - \varepsilon < a$

Nested interval property

Sequence of non-empty, bounded closed intervals where each interval contains the next has a non-empty intersection

functions $f : A \to B$

(i) one to one

$a_1 \neq a_2 \quad f(a_1) \neq f(a_2)$

(ii) onto

for any $b \in B$

$\exists\, a \in A$ such that

$f(a) = b$

countably infinite

A can be indexed by $\mathbb{N}$

Triangle inequality

$a, b, c \in \mathbb{R}$

$|a-b| = |(a-c)+(c+b)| \leq |a-c| + |c+b|$

Examples

$I_n = [0, \frac{1}{n}] \quad n \in \mathbb{N}$

$A = \{x \in \mathbb{Q} : x^2 < 2\}$

$\mathbb{R}$ is the closure of $\mathbb{Q}$

$a_n = (-1)^n$

convergence

topological too

1. a_n converges to a if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$ $|a_n - a| < \varepsilon$

2. a_n is bounded if there exists $M > 0$ such that for all $n \in \mathbb{N}$ $|a_n| \leq M$

3. Every convergent sequence is bounded

Algebraic limit theorems

(i) $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$

(ii) $\lim_{n\to\infty} (a_n + b_n) = a + b$

(iii) $\lim_{n\to\infty} a_n b_n = ab$

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b} \quad b \neq 0$

Basic Structure of Proof

1. "Fix an arbitrary $\varepsilon > 0$"
2. State the condition $\mathbb{N}$ will need to satisfy
3. show that if $n > N$ for this N

$|a_n - a| < \varepsilon$

If a sequence is monotone and bounded, it converges

If b_n is a sequence $\sum_{n=1}^{\infty} b_m$ converges to B if the sequence of partial sums $S_n = \sum_{m=1}^{n} b_m$ converges (to B)

Cauchy Condensation

If b_n

1. decreasing
2. $b_n \geq 0$ for all $n \in \mathbb{N}$

$\sum_{n=1}^{\infty} b_n$ converges iff

$\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges

a) $\underbrace{\sum_{n=1}^{\infty} 2^n b_{2^n}}_{\text{converges}} \to \underbrace{\sum_{n=1}^{\infty} b_n}_{\text{converges}}$

b) $\neg \sum_{n=1}^{\infty} b_n \to \neg \sum_{n=1}^{\infty} 2^n b_{2^n}$

Bolzano-Weierstrass

If a_n converges to a and a_{n_k} is a subsequence of a_n then a_{n_k} converges to a

Good for $a_n = (-1)^n$ divergence

Every bounded sequence a_n contains a convergent subsequence

Cauchy criterion

A sequence a_n is Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n, m \geq N$

$|a_n - a_m| < \varepsilon$

$\to$ Cauchy in $\mathbb{R}$ iff it converges

A series is Cauchy iff $\sum_{i=1}^{\infty} a_i$ for all $\varepsilon > 0$, there exists $N \in \mathbb{N} \quad n > m \geq N$

$|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon$

corollary: if $\sum a_i$ converges $a_n \to 0$

comparison $k \in \mathbb{N} \quad 0 \leq a_k \leq b_k$

(i) if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges

(ii) if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges

Absolute Convergence

$\sum_{n=1}^{\infty} |a_n|$ and $\sum_{i=1}^{\infty} a_n$ converge

$\to$ a rearrangement of a series only adds to same value if convergence is absolute

Conditional convergence

$\sum_{n=1}^{\infty} |a_n|$ diverges $\sum_{i=1}^{\infty} a_n$ converges

Alternating series test

If a_n

(i) $a_n \geq 0$ for all $n \in \mathbb{N}$

(ii) $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$

(iii) $a_n \to 0$

then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

Topology

A set $O \in \mathbb{R}$ is open if for all points $a \in O$, there exists $\varepsilon > 0$ for which $V_\varepsilon(a) \subseteq O$

→ there exists $a \in A$ such that for all $\varepsilon > 0$, $V_\varepsilon(a) \not\subseteq A$

(i) an arbitrary union of open sets is open

(ii) intersection of finitely many open sets is open

Basic structure

1. Λ be any set $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets
2. $O = \bigcup_{\lambda \in \Lambda} O_\lambda$
3. $a \in O$ so $\lambda \in \Lambda$ $a \in O_\lambda$
4. $V_\varepsilon(a) \subseteq O_\lambda \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$

Limit point

- if for all $\varepsilon > 0$ $V_\varepsilon(x) \cap A$ contains at least one point other than x
- iff a_n contained in A with $a_n \to x$ and $a_n \neq x$ for all $n \in \mathbb{N}$

isolated point

a point in A that is not a limit point

use $V_\varepsilon(x)$ with $\varepsilon > 0$ to prove

closed set

Set w/ all its limit points

(i) a set O is open iff O^c is closed

(ii) Arbitrary intersection of closed sets is closed

(iii) Finite union of closed sets is closed

Compact set ⇒ Use BW

A set $K \subseteq \mathbb{R}$ is $\boxed{\text{sequentially}}$ compact if every sequence in K has a subsequence converging to a limit that is also in K

A set $K \subseteq \mathbb{R}$ is compact iff it is closed and bounded

→ bounded set is $A \subseteq \mathbb{R}$ $M > 0$ $|a| < M$ $a \in A$

(i) finite unions of closed bounded intervals

(ii) arbitrary intersection of compact sets

Open cover compactness

For $A \subseteq \mathbb{R}$ an open cover of A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ with $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$

→ finite sub cover $\{O_\lambda : i = 1 \text{ to } N\}$

Heine-Borel $K \subseteq \mathbb{R}$

(i) K is compact

(ii) K is closed and bounded

(iii) every open cover of K has a finite subcover

Perfect Set

closed and contains no isolated points

(i) non-empty perfect set is uncountable

$\mathbb{R}$, Cantor

Structures of Common Proofs

I. Proof by Double Inclusion

Show $X = Y$ by proving $X \subseteq Y$ and $Y \subseteq X$.

A. Proving $X \subseteq Y$:

1. Assume $x \in X$.
2. Use definition of X (e.g., $x \notin A \cup B$).
3. Deduce $x \in A^c$ and $x \in B^c$.
4. Conclude $x \in Y$.

B. Proving $Y \subseteq X$:

1. Assume $x \in Y$.
2. Use definitions to show $x \notin A, B$.
3. Conclude $x \in X$.

II. Proof by Contradiction

1. Assume $\neg P$ (e.g., $\sqrt{p}$ is rational).
2. Derive divisibility ($p|a^2 \Rightarrow p|a$).
3. Substitute and recurse to get $p|b$.
4. Contradiction $\Rightarrow P$ holds.

III. Proof by Induction

1. **Base:** Verify $P(1)$.
2. **IH:** Assume $P(n)$ true.
3. **Step:** Show $P(n) \Rightarrow P(n+1)$ using IH.

IV. Epsilon–N Proof

To show $x_n \to x$:

1. Fix $\epsilon > 0$.
2. Bound $|x_n - x|$ (e.g., via $\frac{|a_n - a|}{|\sqrt{a_n} + \sqrt{a}|}$).
3. Find N so $|x_n - x| < \epsilon$ for $n \geq N$.

V. Infimum/Supremum Proofs

1. Show $L = \sup B$ is a lower bound of A.
2. Show L is the *greatest* lower bound.

VI. Proof of Equivalence ($\iff$)

1. **Forward:** $a_n \to a \Rightarrow \liminf = \limsup = a$.
2. **Backward:** If $\liminf = \limsup = a$, use ϵ–N definition to show $a_n \to a$.