Real Analysis Cheat Sheet

1. Basic Topology of $\mathbb{R}$

Open and Closed Sets

- **Open Set:** A set $O \subseteq \mathbb{R}$ is open if for every point $a \in O$, there exists an $\epsilon > 0$ such that the ϵ-neighborhood $V_\epsilon(a) = (a-\epsilon, a+\epsilon)$ is entirely contained in O.
- **Closed Set:** A set $F \subseteq \mathbb{R}$ is closed if it contains all of its limit points. Equivalently, F is closed if and only if its complement F^c is an open set.
- **Closure ($\overline{A}$):** The smallest closed set containing A. $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

Compact Sets

A set $K \subseteq \mathbb{R}$ is **Compact** if it satisfies any of the following equivalent conditions:

1. **Heine-Borel Theorem:** K is both closed and bounded.
2. **Sequential Compactness:** Every sequence contained in K has a subsequence that converges to a limit that is also in K.
3. **Open Cover Definition:** Every open cover of K (a collection of open sets whose union contains K) admits a **finite subcover** (a finite selection of those sets that still covers K).

Advanced Set Properties

- **Perfect Set:** A set that is closed and has no isolated points (every point is a limit point). *Example:* The Cantor Set.
- **Dense Set:** A set A is dense in $\mathbb{R}$ if $\overline{A} = \mathbb{R}$. Equivalently, A intersects every non-empty open set.
- **Nowhere Dense:** A set A is nowhere dense if the interior of its closure is empty ($(\overline{A})^\circ = \emptyset$). *Example:* $\mathbb{Z}$ or the Cantor Set.
- **Baire Category Theorem:** The real numbers $\mathbb{R}$ cannot be written as a countable union of nowhere dense sets (closed sets with empty interior).
- **F_σ Set:** A set that is a countable union of closed sets. *Example:* $\mathbb{Q}$ (union of singletons), or the set of discontinuities of any function.
- **G_δ Set:** A set that is a countable intersection of open sets. *Example:* The Irrationals $\mathbb{I}$.

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2. Functional Limits

Limit Definitions

Let c be a limit point of the domain A.

- **ϵ-δ Definition:** We say $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x-c| < \delta \implies |f(x) - L| < \epsilon$$

- **Sequential Criterion:** $\lim_{x\to c} f(x) = L$ if and only if for every sequence (x_n) in A such that $x_n \to c$ and $x_n \neq c$, the sequence $f(x_n) \to L$.
- **Divergence Criterion:** If you can find two sequences converging to c whose images converge to different values, the limit does not exist.
- **Infinite Limits:** $\lim_{x\to c} f(x) = \infty$ if for every $M > 0$, there exists $\delta > 0$ such that $0 < |x-c| < \delta \implies f(x) \geq M$.

3. Continuity

Definitions of Continuity

A function f is **continuous at a point** c if:

- **Standard Definition:** For every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
- **Limit Definition:** $\lim_{x\to c} f(x) = f(c)$.
- **Sequential Definition:** For every sequence $x_n \to c$, $f(x_n) \to f(c)$.

Global Continuity Properties

A function is **Continuous on** $\mathbb{R}$ if and only if:

- **Topological Characterization:** The inverse image $f^{-1}(O)$ of every open set O is open. (Equivalently, $f^{-1}(F)$ is closed for every closed set F).
- **Preservation of Compactness:** If K is compact, then the image $f(K)$ is compact (closed and bounded).
- **Preservation of Connectedness:** If E is connected (an interval), then $f(E)$ is connected (an interval).

Important Continuity Theorems

- **Extreme Value Theorem (EVT):** A continuous function on a compact set $[a, b]$ attains a maximum and a minimum value.
- **Intermediate Value Theorem (IVT):** If f is continuous on $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$.
- **Continuous Inverses:** If $f : [a, b] \to \mathbb{R}$ is continuous and one-to-one (injective), then its inverse f^{-1} is continuous on its domain $f([a, b])$.
- **Monotone Functions:**
  - A monotone function can only have **jump discontinuities**. The set of discontinuities is at most countable.
  - If f is monotone and satisfies the Intermediate Value Property, then f must be continuous.
- **Additive Functions:** If f is continuous and satisfies $f(x + y) = f(x) + f(y)$ for all x, y, then $f(x) = ax$ where $a = f(1)$.

Uniform Continuity

Definition: f is uniformly continuous on a set A if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for **all pairs** $x, y \in A$:

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Key Difference: In uniform continuity, δ depends only on ϵ, not on the specific location x.

Criteria for Uniform Continuity:

- **Compact Domain:** Any continuous function on a compact set (e.g., $[a, b]$) is uniformly continuous.
- **Bounded Derivative:** If f is differentiable on an interval and $|f'(x)| \leq M$ (bounded), then f is Lipschitz continuous ($|f(x)-f(y)| \leq M|x-y|$), which implies uniform continuity.
- **Cauchy Sequences:** A uniformly continuous function maps Cauchy sequences to Cauchy sequences. (Useful for proving non-uniform continuity: find a Cauchy sequence x_n where $f(x_n)$ is not Cauchy).

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4. The Derivative

Definitions

- **The Derivative:** $g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$.
- **Differentiability implies Continuity:** If g is differentiable at c, it must be continuous at c. The converse is false (e.g., $|x|$ at 0).
- **Inverse Function Theorem:** If f is one-to-one, differentiable, and $f'(x) \neq 0$, then:

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x)$$

Core Theorems of Calculus

- **Interior Extremum Theorem:** If f attains a maximum or minimum at an interior point $c \in (a, b)$ and is differentiable there, then $f'(c) = 0$.
- **Rolle's Theorem:** If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, there exists $c \in (a, b)$ such that $f'(c) = 0$.
- **Mean Value Theorem (MVT):** If f is continuous on $[a, b]$ and differentiable on (a, b), there exists $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Darboux's Theorem:** If f is differentiable on $[a, b]$, then the derivative f' satisfies the **Intermediate Value Property**, even if f' is not continuous. This means f' cannot have simple jump discontinuities.
- **L'Hôpital's Rule:** Used for $0/0$ or ∞/∞ limits. If $\lim \frac{f'(x)}{g'(x)} = L$, then $\lim \frac{f(x)}{g(x)} = L$.

Applications of MVT

- **Zero Derivative:** If $f'(x) = 0$ for all x in an interval, then f is constant ($f(x) = k$).
- **Equal Derivatives:** If $f'(x) = g'(x)$ for all x, then $f(x) = g(x) + C$.
- **Monotonicity Test:**
  - If $f'(x) > 0$ for all x, f is strictly increasing.
  - If $f'(x) < 0$ for all x, f is strictly decreasing.
- **Lipschitz Continuity:** If $|f'(x)| \leq M$ for all $x \in (a, b)$ (bounded derivative), then $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in (a, b)$. This implies f is Uniformly Continuous.
- **L'Hôpital's Rule:** Derived using Generalized MVT. Limits of $0/0$ or ∞/∞ can be solved via limits of derivatives.

Darboux's Theorem

If f is differentiable on $[a, b]$, then f' satisfies the **Intermediate Value Property** (even if f' is not continuous).

- If $f'(a) < \lambda < f'(b)$, there exists $c \in (a, b)$ such that $f'(c) = \lambda$.
- **Consequence:** A derivative function cannot have simple jump discontinuities.

Tests for Functions using Derivatives

- **Constant Function:** $f'(x) = 0$ on an interval implies f is constant.
- **Strict Monotonicity:** If $f'(x) \neq 0$ on an interval, f is strictly monotone (either always increasing or always decreasing).
- **Fixed Points:** If $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(x) \neq 1$ everywhere, f has at most one fixed point ($f(x) = x$).

5. Important Counterexamples

Limits & Continuity Examples

- **$\sin(1/x)$ at $x = 0$:** The limit does not exist because the function oscillates between -1 and 1 infinitely often near 0.
- **Dirichlet Function:** $f(x) = 1$ if $x \in \mathbb{Q}$, and 0 if $x \notin \mathbb{Q}$. This function is **discontinuous at every point**.
- **Thomae's Function:** $f(x) = 1/n$ if $x = m/n \in \mathbb{Q}$ (in lowest terms), and 0 if $x \notin \mathbb{Q}$. This function is continuous at every irrational number but discontinuous at every rational number.

Uniform Continuity Pathologies

- **$f(x) = x^2$ on $\mathbb{R}$:** Continuous but **not uniformly continuous**. As x gets larger, you need a smaller δ for the same ϵ. (Sequence test: $x_n = n, y_n = n + 1/n$).
- **$f(x) = 1/x$ on $(0, 1)$:** Continuous but **not uniformly continuous** because the function is unbounded near 0.
- **$f(x) = \sin(1/x)$ on $(0, 1)$:** Bounded and continuous, but **not uniformly continuous** due to infinitely rapid oscillations near 0.

Differentiability Pathologies ($g_n(x) = x^n \sin(1/x)$)

Consider the family of functions defined by $x^n \sin(1/x)$ for $x \neq 0$ and 0 at $x = 0$:

- **$n = 0$ ($\sin(1/x)$):** Discontinuous at 0.
- **$n = 1$ ($x \sin(1/x)$):** Continuous at 0, but **not differentiable** at 0 (slopes oscillate between -1 and 1).
- **$n = 2$ ($x^2 \sin(1/x)$):** Differentiable at 0 with $f'(0) = 0$. However, the derivative function $f'(x)$ is **discontinuous** at 0 (limit of f' does not exist).
- **Weierstrass Function:** A function that is continuous everywhere on $\mathbb{R}$ but differentiable nowhere.