METRIC & NEIGHBORHOODS

Metric: Standard metric on $\mathbb{R}$: $d(x,y) = |x-y|$
Open Ball: Radius $r > 0$ around x: $\{y : |y-x| < r\}$
Neighborhood: Set containing some open ball around x

OPEN & CLOSED SETS

Open: O open if $\forall x \in O$, $\exists r > 0$ s.t. $(x-r, x+r) \subseteq O$
Closed: F closed if $\mathbb{R} \setminus F$ is open, or F contains all limit points
Interior: $\text{int}(A)$ = largest open subset of A
Closure: $\overline{A}$ = smallest closed set containing $A = A$ + all limit points
Boundary: $\partial A = \overline{A} \setminus \text{int}(A)$

LIMIT POINTS & DENSE SETS

Limit Point: x is limit point of A if every neighborhood of x contains point of A different from x
Isolated Point: $x \in A$ isolated if $\exists r > 0$ s.t. $(x-r, x+r) \cap A = \{x\}$
Dense: D dense in $\mathbb{R}$ if every nonempty open interval contains point of D, i.e., $\overline{D} = \mathbb{R}$

COMPACT SETS

Def (Open Cover): K compact if every open cover has finite subcover
Heine-Borel: $K \subseteq \mathbb{R}$ compact $\iff$ K closed and bounded
Sequential: K compact $\iff$ every sequence in K has subsequence converging to point in K
Thms:

- Nested: $K_1 \supseteq K_2 \supseteq \cdots \Rightarrow \bigcap K_n \neq \emptyset$
- Continuous image of compact is compact
- Continuous on compact $\Rightarrow$ bounded, attains bounds, uniformly continuous

Ex: $[a,b]$, finite sets **Non-ex:** (a,b), $[0,\infty)$, $\mathbb{Q} \cap [0,1]$

PERFECT SETS

Def: P perfect if P closed and every point is limit point (no isolated points)
Thms: Cantor set is perfect and uncountable; every perfect set is uncountable
Ex: $\mathbb{R}$, $[a,b]$, Cantor set **Non-ex:** $\mathbb{Q}$, $\{1/n\} \cup \{0\}$

CONNECTED SETS

Def: E connected if cannot write $E = A \cup B$ where A, B nonempty separated ($\overline{A} \cap B = A \cap \overline{B} = \emptyset$)
Def (Top): No disjoint open U, V with $E \subseteq U \cup V$, $E \cap U \neq \emptyset$, $E \cap V \neq \emptyset$
Characterization: $E \subseteq \mathbb{R}$ connected $\iff$ E is interval
Thm: Continuous image of connected is connected
Ex: All intervals **Non-ex:** $[0,1] \cup [2,3]$, $\mathbb{Q}$, $\mathbb{Z}$

FUNCTIONAL LIMITS

Def (ε-δ): $\lim_{x\to c} f(x) = L$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$
Top View: For every neighborhood V of L, $\exists$ neighborhood U of a s.t. $\forall x \in U \setminus \{a\}$, $f(x) \in V$
Sequential: $\lim_{x\to c} f(x) = L \iff f(x_n) \to L$ for every $x_n \to c$ with $x_n \neq c$

CONTINUITY

Def (ε-δ): f continuous at c if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$
Sequential: f continuous at $c \iff x_n \to c \Rightarrow f(x_n) \to f(c)$
Topological: f continuous $\iff f^{-1}(O)$ open for every open $O \iff f^{-1}(F)$ closed for every closed F
Thms: Composition, $|f|$, max/min of continuous is continuous
Ex: Polynomials, $\sin(x)$, e^x, $|x|$ **Non-ex:** $1/x$ at $x = 0$, $\lfloor x \rfloor$

UNIFORM CONTINUITY

Def: f uniformly continuous on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for all $x, y \in A$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
Lipschitz: $|f(x) - f(y)| \leq M|x-y|$ for all $x, y \in A$ (constant $M > 0$)
Thms:

- Lipschitz $\Rightarrow$ uniformly continuous
- f continuous on compact $K \Rightarrow f$ uniformly continuous on K

Ex: x, $\sin(x)$, $\sqrt{x}$ on $[0,\infty)$ **Non-ex:** x^2 on $\mathbb{R}$, $1/x$ on $(0,1)$

PREIMAGE PROPERTIES

$f^{-1}(B) = \{x \in A : f(x) \in B\}$

- $f^{-1}(\bigcup B_i) = \bigcup f^{-1}(B_i)$
- $f^{-1}(\bigcap B_i) = \bigcap f^{-1}(B_i)$
- $f^{-1}(B^c) = (f^{-1}(B))^c$

.

EXTREME VALUE THEOREM

Thm (EVT): If f continuous on $[a,b]$, then f attains max and min. $\exists c, d \in [a,b]$ s.t. $f(c) \geq f(x)$ and $f(d) \leq f(x)$ $\forall x \in [a,b]$

INTERMEDIATE VALUE THEOREM

Thm (IVT): If f continuous on $[a,b]$ and $f(a) < L < f(b)$, then $\exists c \in (a,b)$ s.t. $f(c) = L$
Equiv: Image of interval under continuous function is interval
Ex: $x^5 + x - 1 = 0$ has solution in $(0,1)$

DISCONTINUITY

Types:

1. **Removable:** $\lim_{x\to c} f(x)$ exists but $\neq f(c)$ or $f(c)$ undefined
2. **Jump:** One-sided limits exist but unequal
3. **Essential:** At least one one-sided limit doesn't exist

Ex: Removable: $\frac{x^2-1}{x-1}$ at $x = 1$; Jump: $\lfloor x \rfloor$; Essential: $\sin(1/x)$ at $x = 0$

MONOTONE FUNCTIONS

Thm: If f monotone on (a,b), discontinuities are at most countable and are jump discontinuities
Thm: Monotone + surjective on interval $\Rightarrow$ continuous

DERIVATIVES

Def: $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$
Thm: Differentiable at $c \Rightarrow$ continuous at c
Rules: $(f \pm g)' = f' \pm g'$, $(fg)' = f'g + fg'$, $(f/g)' = \frac{f'g - fg'}{g^2}$
Chain: $(f \circ g)'(x) = f'(g(x))g'(x)$

ROLLE'S THEOREM

Thm: f continuous on $[a,b]$, differentiable on (a,b), $f(a) = f(b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = 0$

MEAN VALUE THEOREM

Thm (MVT): f continuous on $[a,b]$, differentiable on $(a,b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$
Consequences:

- $f' = 0$ on interval $\Rightarrow f$ constant
- $f' > 0 \Rightarrow f$ strictly increasing; $f' < 0 \Rightarrow$ strictly decreasing
- $|f'| \leq M \Rightarrow f$ Lipschitz with constant M

.

DARBOUX'S THEOREM

Thm: f differentiable on $[a,b]$ and $f'(a) < \lambda < f'(b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = \lambda$
IVP for Derivatives: Derivatives have intermediate value property even if not continuous! No jump discontinuities possible.

L'HÔPITAL'S RULE

Thm: If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ (or $\pm\infty$), $g'(x) \neq 0$ near c, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$
Forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$ (rewrite others)

CAUCHY FUNCTIONAL EQUATION

Thm: If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f continuous at one point, then $\exists a \in \mathbb{R}$ s.t. $f(x) = ax$ for all x
Proof Sketch: Show $f(nx) = nf(x)$ by induction, get $f(0) = 0$, $f(-x) = -f(x)$, so $f(n) = nf(1)$ for $n \in \mathbb{Z}$. Then $f(m/n) = \frac{m}{n} f(1)$. Use continuity for irrationals.

DISCONTINUITY SETS OF DERIVATIVES

Thm: If f differentiable on (a,b), set of discontinuities of f' has measure zero (meager)
Dense Continuity: Set of points where f' is continuous is dense
Key: f' *cannot* be nowhere continuous!

BAIRE'S THEOREM

Thm: $\mathbb{R}$ (complete metric space) is not countable union of nowhere dense sets
Equiv: Countable intersection of dense open sets is dense

SEQUENCES OF FUNCTIONS

Pointwise: $f_n \to f$ pointwise if $\forall x$, $\forall \varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$
Uniform: $f_n \to f$ uniformly if $\sup_{x \in A} |f_n(x) - f(x)| \to 0$ as $n \to \infty$
Key Difference: Uniform: same N works for all x; Pointwise: N depends on x
Ex: $f_n(x) = x^n$ on $[0,1]$ converges pointwise to $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$ but not uniformly

UNIFORM CONVERGENCE PROPERTIES

Thms:

- $f_n \to f$ uniformly, f_n continuous $\Rightarrow f$ continuous
- Pointwise limit of continuous may be discontinuous
- $f_n \to f$ uniformly on $[a,b] \Rightarrow \int_a^b f_n \to \int_a^b f$
- $f_n \to f$ pointwise, $f_n' \to g$ uniformly $\Rightarrow f$ differentiable, $f' = g$

.

DIFFERENTIABLE LIMIT THEOREM

Thm: f_n differentiable, $f_n \to f$ pointwise, $f_n' \to g$ uniformly $\Rightarrow f$ differentiable and $f' = g$
Critical: Need uniform convergence of *derivatives*!

SERIES OF FUNCTIONS

Pointwise: $\sum f_n$ converges pointwise if $S_n = \sum_{i=1}^n f_i$ converges pointwise
Uniform: $\sum f_n$ converges uniformly if S_n converges uniformly
Weierstrass M-Test: If $|f_n(x)| \leq M_n$ $\forall x$ and $\sum M_n$ converges, then $\sum f_n$ converges uniformly

POWER SERIES

Form: $\sum_{n=0}^\infty a_n (x-c)^n$
Radius: $R = \frac{1}{\limsup |a_n|^{1/n}}$
Convergence: Absolutely for $|x-c| < R$; diverges for $|x-c| > R$
Uniform: Converges uniformly on $[c-r, c+r]$ for any $r < R$

ABEL'S THEOREM

Thm: If $\sum a_n x^n$ has radius $R > 0$ and $\sum a_n R^n$ converges, then $\lim_{x \to R^-} \sum a_n x^n = \sum a_n R^n$

DIFFERENTIATION OF POWER SERIES

Thm: $f(x) = \sum a_n (x-c)^n$ with radius $R > 0 \Rightarrow f$ infinitely differentiable on $(c-R, c+R)$
$f'(x) = \sum_{n=1}^\infty n a_n (x-c)^{n-1}$ (same radius R)
Corollary: $a_n = \frac{f^{(n)}(c)}{n!}$

TAYLOR SERIES

Def: $T(x) = \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(x-c)^n$
Taylor's Thm: $f(x) = T_n(x) + R_n(x)$ where $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$ for some ξ between c and x
Convergence: $f(x) = T(x) \iff R_n(x) \to 0$
Common Series:

- $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$, $R = \infty$
- $\sin(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $R = \infty$
- $\cos(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!}$, $R = \infty$
- $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$, $R = 1$

.

INFINITELY DIFFERENTIABLE

Def: $f : \mathbb{R} \to \mathbb{R}$ is C^∞ (smooth) if derivatives of all orders exist at every point

INTERMEDIATE VALUE PROPERTY (IVP)

Def: f has IVP on interval I if $\forall a, b \in I$ with $a < b$ and all L between $f(a)$ and $f(b)$, $\exists c \in (a,b)$ with $f(c) = L$
Note: Continuous on interval $\Rightarrow$ IVP, but IVP $\not\Rightarrow$ continuous
Ex (IVP, not cont): $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

CLOSED SETS & SEQUENCES

Thm: $F \subseteq \mathbb{R}$ closed $\iff$ whenever (x_n) in F converges to $x \in \mathbb{R}$, we have $x \in F$
Use: Show set closed by verifying it contains all limits of its sequences

PATHOLOGICAL EXAMPLES

All derivatives zero at point, $f \neq 0$:

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is C^∞, all derivatives at 0 are 0, but f not identically zero
Differentiable only at one point:

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is differentiable at $x = 0$ only

IVP but discontinuous:

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Continuous image of closed not closed:

$$f(x) = \frac{x}{1+|x|}$$

$\mathbb{R}$ is closed but $f(\mathbb{R}) = (-1,1)$ is open
Pointwise limit discontinuous: $f_n(x) = x^n$ on $[0,1]$, limit is $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$

KEY THEOREMS SUMMARY

- **Heine-Borel:** Compact $\iff$ closed & bounded (in $\mathbb{R}$)
- **EVT:** Continuous on compact $\Rightarrow$ attains max/min
- **IVT:** Continuous on interval $\Rightarrow$ image is interval
- **Uniform Continuity:** Continuous on compact $\Rightarrow$ uniformly continuous
- **MVT:** $\exists c$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$
- **Darboux:** f' has IVP (even if discontinuous)
- **Uniform Limit:** $f_n \to f$ uniformly, f_n cont $\Rightarrow f$ cont