PSTAT 160B Homework 3

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Exercise 1

Use the Chapman-Kolmogorov equations to find the set $A \subseteq \mathbb{R}^2$ and function f (in terms of the transition probability p of B) such that

$$P(B_1 < 0, B_2 > 0) = \iint_A f(x,y)\,dx\,dy$$

Use numerical integration (in R or Python) to calculate this probability.
Chapman-Kolmogrov Equation (From Lecture):

$$p_t(x,A) = \iint_A p_s(x,z)p_{t-s}(z,y)\,dy\,dz$$

For our scenario, we have...

$$P(B_1 < 0, B_2 > 0) = \int_{-\infty}^{0}\int_{0}^{\infty} p_1(0,x)p_{2-1}(x,y)\,dx\,dy$$

Since we are given...

$$p_h(x,y) = \frac{1}{\sqrt{2h\pi}}e^{-(\frac{x-y}{\sqrt{2h}})^2}$$

we can write our $p_1(0,x)$ and $p_{2-1}(x,y)$ as...

$$p_1(0,x) = \frac{1}{\sqrt{2(1)\pi}}e^{-(\frac{0-x}{\sqrt{2(1)}})^2}$$

=

$$p_1(0,x) = \frac{1}{\sqrt{2\pi}}e^{-(\frac{-x}{\sqrt{2}})^2}$$

$$p_{2-1}(x,y) = \frac{1}{\sqrt{2(2-1)\pi}}e^{-(\frac{x-y}{\sqrt{2(2-1)}})^2}$$

=

$$p_1(x,y) = \frac{1}{\sqrt{2\pi}}e^{-(\frac{x-y}{\sqrt{2}})^2}$$

Therefore,

$$f(x,y) = p_1(0,x) * p_1(x,y)$$
$$= \frac{1}{2\pi} e^{\frac{-2x^2+2xy-y^2}{2}}$$

Now, we will use numerical integration in R to calculate the probability.

$$P(B_1 < 0, B_2 > 0) = \int_{-\infty}^{0} \int_{0}^{\infty} \frac{1}{2\pi} e^{\frac{-2x^2+2xy-y^2}{2}} dx\ dy$$

```
f = function(x,y) { (1/(2*pi))*exp((-2*x^2 + 2*x*y - y^2)/(2)) }
g = function(y) { integrate(function(x) { f(x,y) },0,Inf)$value }
integrate(Vectorize(g), -Inf, 0)
> 0.125 with absolute error < 2.5e-06
```

Thus it is shown that

$$0.125 = \frac{1}{8} = P(B_1 < 0, B_2 > 0)$$

Exercise 2

Write down the covariance matrix Σ for the vector $X = (B_1, B_2)$. Compute the determinant and inverse of Σ to write down the density f of X and confirm the expression for f found in the previous exercise.

$$\Sigma = \begin{pmatrix} \mathrm{Cov}(B_1, B_1) & \mathrm{Cov}(B_1, B_2) \\ \mathrm{Cov}(B_2, B_1) & \mathrm{Cov}(B_2, B_2) \end{pmatrix}$$

In general,

$$\mathrm{Cov}(B_s, B_t) = E(B_s B_t) - E(B_s)E(B_t) = E(B_s B_t)$$

For $s < t$, let's write $B_t = (B_t - B_s) + B_s$. So,

$$E(B_s B_t) = E(B_s(B_t - B_s + B_s))$$

$$= E(B_s(B_t - B_s)) + E(B_s^2)$$

$$= E(B_s)E(B_t - B_s)) + E(B_s^2)$$

$$= 0 + E(B_s^2)$$

By Definition,

$$E(B_s^2) = \mathrm{Var}(B_s) + [E(B_s)]^2$$

$$s + 0 = s$$

Thus,

$$\text{Cov}(B_s, B_t) = s$$

By symmetry, for $t < s$,

$$\text{Cov}(B_s, B_t) = t$$

So,

$$\text{Cov}(B_s, B_t) = \min\{s, t\}$$

$$\implies \text{Cov}(B_1, B_1) = \text{Var}(B_1) = 1$$

$$\implies \text{Cov}(B_1, B_2) = 1$$

$$\implies \text{Cov}(B_2, B_1) = 1$$

$$\implies \text{Cov}(B_2, B_2) = \text{Var}(B_2) = 2$$

$$\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\det(\Sigma) = 1(2) - 1(1) = 1$$

$$\Sigma^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$f(x_1, ..., x_n) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \cdot \exp(-\frac{1}{2}(x - \mu)^T) \cdot \Sigma^{-1}(x - \mu))$$

$$\implies f(x_1, x_2) = \frac{1}{\sqrt{(2\pi)^2 \det(\Sigma)}} \cdot \exp(-\frac{1}{2}(x - \mu)^T) \cdot \Sigma^{-1}(x - \mu))$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies (x - \mu) = \begin{pmatrix} x_1 - 0 \\ x_2 - 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\implies (x - \mu)^T = [x_1, x_2]$$

$$\implies \exp(-\frac{1}{2}(x - \mu)^T) \cdot \Sigma^{-1}(x - \mu)) = \exp(-\frac{1}{2}[x_1, x_2] \cdot \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix})$$

$$= \exp(-\frac{1}{2}[2x_1 - x_2, -x_1 + x_2] \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix})$$

$$= \exp(-\frac{1}{2}(2x_1^2 - x_1x_2 - x_1x_2 + x_2^2))$$

$$= \exp(-\frac{1}{2}(2x_1^2 - 2x_1x_2 - x_2^2))$$

$$f(x_1, x_2) = \frac{1}{\sqrt{4\pi^2 \cdot 1}} \cdot \exp(-\frac{1}{2}(2x_1^2 - 2x_1x_2 - x_2^2))$$

$$= \frac{1}{2\pi} \cdot \exp(-\frac{1}{2}(2x_1^2 - 2x_1x_2 - x_2^2))$$

In comparison to to 1, this is the same expression, where $x_1 = x$ and $x_2 = y$

$$\frac{1}{2\pi} e^{\frac{-2x^2+2xy-y^2}{2}} = \frac{1}{2\pi} \cdot \exp(\frac{-2x_1^2 + 2x_1x_2 - x_2^2}{2})$$

Exercise 3

Show that $Z_s = B_s - B_t(\frac{s}{t})$ and B_t are uncorrelated for $s \leq t$ (i.e., $\mathrm{Cov}(Z_s, B_t) = 0$). Use this to compute the conditional density.

$$f_{B_s|B_t}(x|y) = \frac{\partial}{\partial x} P(B_s \leq x | B_t = y)$$

Express (as an integral) the probability B was in a set $A \subseteq \mathbb{R}$ at time $s \geq 0$ given that at a future time $t > s$, the process is at location $y \in \mathbb{R}$. Verify the following Bayes' formulas with the f from the previous exercise.

$$f_{B_2|B_1}(y|x) = \frac{f(x,y)}{p_1(0,x)}$$

$$f_{B_1|B_2}(x|y) = \frac{f(x,y)}{p_2(0,y)}$$

So,

$$\mathrm{Cov}(Z_s, B_t) = E(Z_sB_t) - E(Z_s)E(B_t) = E(Z_sB_t)$$

$$= E((B_s - B_t(\frac{s}{t}))B_t)$$

$$= E(B_sB_t - B_t^2(\frac{s}{t}))$$

$$= E(B_sB_t) - E(B_t^2(\frac{s}{t}))$$

By Exercise 2, $E(B_sB_t) = \mathrm{Cov}(B_s, B_t) = \min\{s, t\} = s$, since it is given that $s \leq t$. So,

$$= E(B_sB_t) - E(B_t^2(\frac{s}{t})) = s - (\frac{s}{t})E(B_t^2)$$

By Definition,

$$E(B_t^2) = \mathrm{Var}(B_t) + [E(B_t)]^2$$

$$\implies s - (\frac{s}{t})E(B_t^2) = s - (\frac{s}{t})(\mathrm{Var}(B_t) + [E(B_t)]^2)$$

$$= s - (\frac{s}{t})(t + 0)$$

$$= s - (\frac{s}{t})t$$

$$= s - s = 0$$

Thus it is shown that $\mathrm{Cov}(Z_s, B_t) = 0$.

Verifying the following first Bayes' formula with the f from previous exercise:

$$f_{B_2|B_1}(y|x) = \frac{f(x,y)}{p_1(0,x)}$$

$$f_{B_2|B_1}(y|x) = \frac{\partial}{\partial y}P(B_2 \leq y|B_1 = x) = \frac{\partial}{\partial y}P(B_{1+1} \leq y|B_1 = x)$$

$$= \frac{1}{\sqrt{2\pi(1)}} \cdot e^{-(\frac{x-y}{\sqrt{2(1)}})^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-(\frac{x-y}{\sqrt{2}})^2}$$

$$f_{B_2|B_1}(y|x) = \frac{f(x,y)}{p_1(0,x)} = \frac{\frac{1}{2\pi} \cdot e^{\frac{-2x^2+2xy-y^2}{2}}}{\frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}}}$$

$$= \frac{\sqrt{2\pi}}{2\pi} \cdot e^{\frac{-x^2+2xy-y^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-(\frac{x-y}{\sqrt{2}})^2}$$

Verifying the following second Bayes' formula with the f from the previous exercise: where s ¡ t

$$Z_s = B_s - B_t(\frac{s}{t})$$

$$f_{B_s|B_t}(x|y) = \frac{d}{dx}P(B_s \leq x|B_t = y)$$

$$\frac{d}{dx}P(B_s \leq x | B_t = y)$$

$$= P(Z_s \leq x - \frac{s}{t}B_t | B_t = y)$$

Because Z_s and B_t are normal with covariance = 0, they are independent. Additionally, Z_s can be viewed as a linear combination of two different segments shown as follows. Because both segments are normally distributed, we know a combination of two segments yields a new normal =lly distributed

$$= P(Z_s \leq x - \frac{s * y}{t})$$

$$Z_s = B_s - B_t \cdot \frac{s}{t}$$

$$B_s - \frac{s}{t}(B_t - B_s + B_s)$$

$$B_s - \frac{s}{t}B_s - \frac{s}{t}(B_t - B_s)$$

$$B_s(1 - \frac{s}{t}) - \frac{s}{t}(B_t - B_s)$$

$$(B_s - B_0)(1 - \frac{s}{t}) - \frac{s}{t}(B_t - B_s)$$

$$x = (B_s - B_0) \sim N(0, s)$$

$$y = (B_t - B_s) \sim N(0, t - s)$$

and

$$E[X \cdot C] = C \cdot E[x]$$

and

$$var[X \cdot C] = C^2 \cdot var[x]$$

$$Z_s = (B_s - B_0)(1 - \frac{2}{t}) - (B_t - B_s)\frac{s}{t}$$

$$x = B_s - B_0$$

$$y = B_t - B_s$$

$$E[Z_s] = E[(1 - \frac{s}{t})x] - E[\frac{s}{t}y]$$

$$(1 - \frac{s}{t})E[x] - \frac{s}{t}E[y]$$

$$0 - 0 = 0$$

$$var[Z_s] = var((1 - \frac{s}{t})x) + var(\frac{s}{t}y)$$

$$(1 - \frac{s}{t})^2 var(x) + (\frac{s}{t})^2 var(y)$$

$$(1 - \frac{s}{t})^2 \cdot s + (\frac{s}{t})^2 (t - s)$$

$$(1 - \frac{s}{t})^2 = 1 - \frac{2s}{t} + \frac{s^2}{t^2}$$

$$(1-\frac{s}{t})^2 \cdot s = s - \frac{2s^2}{t} + \frac{s^3}{t}$$

$$= s - \frac{2s^2}{t} + \frac{s^3}{t^2} + \frac{s^2 \cdot t}{t^2} - \frac{s^3}{t^2}$$

$$s - \frac{2s^2}{t} + \frac{s^2}{t}$$

$$= s - \frac{s^2}{t}$$

$$var[Z_s] = s - \frac{s^2}{t}$$

$$= s(1-\frac{s}{t})$$

So,

$$Z_s \sim N(0, S(1-\frac{s}{t}))$$

$$\frac{d}{dx}P(Z_s \leq x - \frac{s \cdot y}{t}$$

So we have,

$$\frac{1}{\sqrt{2}\pi h} \cdot exp(^{\frac{-(x-\frac{sy}{t})^2}{2 \cdot h}})$$

$$h = s(1-\frac{s}{t})$$

$$= \frac{1}{\sqrt{2\pi \cdot s(1-\frac{s}{t})}} \cdot exp(\frac{-(x^2 - \frac{2xy}{t} + \frac{s^2y^2}{t^2})}{2 \cdot s(1-\frac{s}{t})})$$

When s = 1, t= 2, then $S(1-\frac{s}{t} = \frac{1}{2})$

$$= \frac{1}{\sqrt{\pi}} \cdot exp(\frac{-(x^2 - \frac{2xy}{2} + \frac{y^2}{4})}{1})$$

$$= \frac{1}{\sqrt{\pi}} \cdot exp(-\frac{4x^2 - 4xy + y^2}{4})$$

$$4x^2 - 4xy + y^2$$

$$(2x-y)^2$$

$$\frac{1}{\sqrt{\pi}} \cdot exp(-\frac{(2x-y)^2}{4})$$

$$(2x-y)^2 \iff (y-2x)^2$$

$$f_{B_1|B_2}(x|y) = \frac{f(x,y)}{p_2(0,y)} = \frac{\frac{1}{2\pi}e^{\frac{-2x^2+2xy-y^2}{2}}}{\frac{1}{\sqrt{2(2)\pi}}e^{-(\frac{0-y}{\sqrt{(2(2))}})^2}}$$

$$= \frac{\frac{1}{2\pi}e^{\frac{-2x^2+2xy-y^2}{2}}}{\frac{1}{\sqrt{4\pi}}e^{\frac{-y^2}{4}}}$$

$$= \frac{\frac{1}{2\pi}e^{\frac{-4x^2+4xy-2y^2}{4}}}{\frac{1}{\sqrt{4\pi}}e^{\frac{-y^2}{4}}}$$

$$= \frac{\sqrt{4\pi}}{2\pi}e^{(-x^2+xy-\frac{y^2}{4})}$$

$$= \frac{1}{\sqrt{\pi}}e^{(-x^2+xy-\frac{y^2}{4})}$$

Exercise 4

Consider two standard and independent Brownian motions B and W. Set $W_t^x = x + W_t$ for $x \in (0, \infty)$ and $X_t = W_t^x - B_t$ for $t \geq 0$. Show that $P(W_1^x \leq B_1) = P(B_1 \leq \frac{-x}{\sqrt{2}})$. Prove that X is a (nonstandard) Brownian motion and find the volatility and starting position of X. Let $T_{collide} = \min\{t > 0 : W_t^x = B_t\}$, first time W^x and B collide into one another. Find the density of the time $T_{collide}$ and calculate $P(T_{collide} \leq 1)$.

$$P(W_1^x \leq B_1) = P(x + W_1 - B_1 \leq 0) = P(W_1 - B_1 \leq -x)$$

$$E(W_1 - B_1) = E(W_1) - E(B_1) = 0 - 0 = 0$$

$$\text{Var}(W_1 - B_1) = \text{Var}(W_1) + \text{Var}(B_1) - 2\text{Cov}(W_1, B_1)$$

$$= 1 + 1 - 2(0) = 2$$

Since it is given that B and W are independent Brownian motions, $\text{Cov}(W_1, B_1) = 0$
So,

$$W_1 - B_1 \sim N(0, 2)$$

$$\implies P(W_1 - B_1 \leq -x) = P(\frac{W_1 - B_1}{\sqrt{2}} \leq -\frac{x}{\sqrt{2}})$$

$$\text{Let } Z = \frac{W_1 - B_1}{\sqrt{2}}$$

$$E(Z) = E(\frac{W_1 - B_1}{\sqrt{2}}) = \frac{E(W_1 - B_1)}{E(\sqrt{2})}$$

$$E(Z) = \frac{E(W_1) - E(B_1)}{E(\sqrt{2})} = \frac{0}{\sqrt{(2)}} = 0$$

$$\text{Var}(Z) = (\frac{1}{\sqrt{2}})^2 \cdot (\text{Var}(W_1) + \text{Var}(B_1)) = \frac{1}{2} \cdot (1 + 1) = 1$$

Thus,

$$Z \sim N(0,1)$$

$$\implies P(\frac{W_1 - B_1}{\sqrt{2}} \leq -\frac{x}{\sqrt{2}}) = P(Z \leq -\frac{x}{\sqrt{2}})$$

$Z \sim N(0,1)$ and $B_1 \sim N(0,1)$,

$$\implies P(Z \leq -\frac{x}{\sqrt{2}}) = P(B_1 \leq -\frac{x}{\sqrt{2}})$$

Thus it is shown that

$$P(W_1^x \leq B_1) = P(B_1 \leq \frac{-x}{\sqrt{2}})$$

Now, we will prove that X is a (nonstandard) Brownian motion.
Let $0 \leq s < t$,

$$X_t - X_s = W_t^x - B_t - W_s^x + B_s$$

$$= x + W_t - B_t - x - W_s + B_s$$

$$= (W_t - W_s) - (B_t - B_s)$$

Where $(W_t - W_s)$ and $(B_t - B_s)$ are iid normally distributed stationary increments each with mean 0 and variance $t - s$.

$$E(X_t - X_s) = E(W_t - W_s) - E(B_t - B_s) = 0 - 0 = 0$$

$$\text{Var}(X_t - X_s) = \text{Var}(W_t - W_s) + \text{Var}(B_t - B_s) - 2\text{Cov}((W_t - W_s), (B_t - B_s))$$

$$= (t - s) + (t - s) - 2(0) = 2(t - s)$$

$\text{Cov}((W_t - W_s), (B_t - B_s)) = 0$ since it is given that B and W are independent Brownian motions.
So,

$$X_t - X_s \sim N(0, 2(t - s))$$

$$X_0 = W_0^x - B_0 = x$$

Thus X is a non standard Brownian Motion with a starting position of X is x, and a volatility of X is $\sqrt{2}$.

$T_{collide} = \min\{t > 0 : W_t^x = B_t\}$
$X_t \sim N(0, 2t)$
$T_{collide} = \min\{t > 0 : X_t = x\}$

$$f_{T_{collide}}(t) = \frac{x}{\sqrt{4\pi t^3}} e^{-\frac{x^2}{4t}}$$

$$P(T_{collide} \leq 1) = \int_0^1 \frac{x}{\sqrt{4\pi t^3}} e^{-\frac{x^2}{4t}}\ dt$$

$$= x \cdot e^{-x^2} \int_0^1 \frac{1}{\sqrt{4\pi t^3}} e^{-\frac{1}{4t}}\ dt$$

$$= 0.4795 e^{-x^2} \cdot x$$

```
f = function(x) { 0.4795*exp(-x^2)*x }

x <- seq(0.01, 2, length.out = 100)

prob <- sapply(x, function(x) integrate(function(x), 0, 2)$value)
```

Exercise 5

In 1901 Bachalier proposed the Brownian motion $W = \sigma B$ to model a stock price. Since prices are nonnegative, let $X = (W)_+$ as the price where $(x)_+ = \max(x, 0)$. The price at T of an option with strike K is

$$C(K) = E((X_T - K)_+); \quad K > 0$$

i.e., the expected payoff. Let $\psi(x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{x^2}{2\sigma^2 T}}$ and $\Psi(x) = \int_{-\infty}^{x} \psi(z)\ dz$.

1.

Show that

$$C(K) = \int_K^\infty x\psi(x)\ dx - K \int_K^\infty \psi(x)\ dx$$

$$C(K) = K\Psi(K) + \sigma^2 T \psi(K) - K$$

So,

$$B \sim N(0, \sigma^2)$$

$$X = (W)_+ = (\sigma B)_+ \sim N(0, \sigma^2 T)$$

Therefore the pdf of X is $\psi(x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot e^{-\frac{x^2}{2\sigma^2 T}}$

$$C(K) = E[(X_T - K)_+] = \int_{-\infty}^{\infty} (x - K)\psi(x) \mathbf{1}_{\{x>K\}}\ dx$$

$$= \int_K^\infty (x - K)\psi(x)\ dx$$

$$= \int_K^\infty x\psi(x)\ dx\ -\ K\int_K^\infty \psi(x)\ dx$$

Thus it is shown that

$$\implies C(K) = \int_K^\infty x\psi(x)\ dx\ -\ K\int_K^\infty \psi(x)\ dx$$

Now to prove the second part,

$$= \int_K^\infty x \cdot \frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot e^{\frac{-x^2}{2\sigma^2 T}}\ dx\ -\ K(1 - \int_{-\infty}^K \psi(x)\ dx)$$

Since $\Psi(x) = \int_{-\infty}^x \psi(z)dz$,

$$= \frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot \int_K^\infty x \cdot e^{-\frac{x^2}{2\sigma^2 T}}\ dx - K(1 - \Psi(K))$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot \int_K^\infty x \cdot e^{-\frac{x^2}{2\sigma^2 T}}\ dx + K\Psi(K) - K$$

To solve $\int_K^\infty x \cdot e^{-\frac{x^2}{2\sigma^2 T}}\ dx$, we will use u-substitution. Let

$$u = x^2$$

$$\implies du = 2x\ dx,\ dx = \frac{du}{2x}$$

$$x = k,\ \implies\ u = k^2$$

$$\int_K^\infty x \cdot e^{-\frac{x^2}{2\sigma^2 T}}\ dx = \int_{K^2}^\infty \frac{x}{2x} \cdot e^{-\frac{u}{2\sigma^2 T}}\ du$$

$$= \frac{1}{2}\int_{K^2}^\infty e^{-\frac{u}{2\sigma^2 T}}\ du$$

$$= \frac{1}{2}(-2\sigma^2 T \cdot e^{-\frac{u}{2\sigma^2 T}}\,|_{K^2}^\infty)$$

$$= -\sigma^2 T(e^{-\frac{u}{2\sigma^2 T}}\,|_{K^2}^\infty)$$

$$= -\sigma^2 T[(e^{-\frac{\infty}{2\sigma^2 T}}) - (e^{-\frac{K^2}{2\sigma^2 T}})]$$

$$= -\sigma^2 T[0 - (e^{-\frac{K^2}{2\sigma^2 T}})]$$

$$= -\sigma^2 T(-e^{-\frac{K^2}{2\sigma^2 T}})$$

$$= \sigma^2 T e^{-\frac{K^2}{2\sigma^2 T}}$$

$$\frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot \int_K^{\infty} x \cdot e^{-\frac{x^2}{2\sigma^2 T}} \ dx + K\Psi(K) - K = \frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot \sigma^2 T e^{-\frac{K^2}{2\sigma^2 T}} + K\Psi(K) - K$$

$$= \sigma^2 T(\frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot e^{-\frac{K^2}{2\sigma^2 T}}) + K\Psi(K) - K$$

Since $\frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot e^{-\frac{K^2}{2\sigma^2 T}} = \psi(K)$

$$= \sigma^2 T\psi(K) + K\Psi(K) - K$$

Thus it is shown that,

$$C(K) = K\Psi(K) + \sigma^2 T\Psi(K) - K$$

Extra Credit: Compute option price when $K \leq 0$

2.

Evaluate the first two derivatives with respect to K to show

$$\frac{dC(K)}{dK} = C^{(1)}(K) = -\int_K^{\infty} \psi(x) \ dx$$

$$\frac{d^2C(K)}{dK^2} = C^{(2)}(K) = \psi(K)$$

and deduce that $C^{(k)}(K) = \psi^{(k-2)}(K)$ for all integers $k \geq 2$.

Firstly,

$$C(K) = K\Psi(K) + \sigma^2 T\psi(K) - K$$

$$= K(\Psi(K) - 1) + \sigma^2 T\psi(K)$$

$$= K(\int_{-\infty}^{K} \psi(x) \ dx - 1) + \sigma^2 T\psi(K)$$

$$= K(-\int_K^{\infty} \psi(x)dx) + \sigma^2 T\psi(K)$$

$$fg = fg' + f'g$$

$$\frac{dC(K)}{dK} = C^{(1)}(K) = \frac{d}{dK}[K(-\int_K^{\infty} \psi(x)dx) + \sigma^2 T\psi(K)]$$

$$= -\int_K^\infty \psi(x)dx - K\frac{d}{dK}\int_K^\infty \psi(x)\ dx + \sigma^2 T\frac{d}{dK}\psi(K) \tag{1}$$

By Leibniz's Rule,

$$f'(x) = \frac{d}{dx}\int_{u(x)}^{v(x)} g(t)dt \qquad G'(t) = g(t)$$

$$= \frac{d}{dx}G(v(x)) - G(u(x))$$

$$= G'(v(x))v'(x) - G'(u(x))u'(x)$$

So,

$$\frac{d}{dK}\int_K^\infty \psi(x)dx = \psi(\infty)\cdot\frac{d}{dK}(\infty) - \psi(K)\frac{d}{dK}(K)$$

$$= -\psi(K)$$

(1) becomes:

$$= -\int_K^\infty \psi(x)dx + K\psi(K) + \sigma^2 T\frac{d}{dK}\psi(K) \tag{2}$$

$$\frac{d}{dK}\psi(K) = \frac{d}{dK}[\frac{1}{\sqrt{2\pi\sigma^2 T}}\cdot e^{\frac{-K^2}{2\sigma^2 T}}]$$

(2) becomes:

$$= -\int_K^\infty \psi(x)dx + K\psi(K) + \sigma^2 T\cdot\frac{K}{\sigma^2 T}(-\psi(K))$$

$$= -\int_K^\infty \psi(x)dx$$

Thus it is shown that

$$\frac{dC(K)}{dK} = C^{(1)}(K) = -\int_K^\infty \psi(x)dx$$

Secondly,

$$\frac{d^2C(K)}{dK^2} = \frac{d}{dK}(C^{(1)}(K)) = \frac{d}{dK}[-\int_K^\infty \psi(x)dx] = -[\frac{d}{dK}\int_K^\infty \psi(x)dx]$$

By Leibniz's rule,

$$\frac{d}{dK}\int_K^\infty \psi(x)dx = \psi(\infty)\cdot\frac{d}{dK}(\infty) - \psi(K)\frac{d}{dK}(K)$$

$$= -\psi(K)$$

$$-[\frac{d}{dK}\int_K^\infty \psi(x)dx] = -(-\psi(K)) = \psi(K)$$

Thus it is shown that
$$\frac{d^2C(K)}{dK^2} = C^{(2)}(K) = \psi(K)$$

Claim: $P(K) = C^K(K) = \psi^{K-2}(K)$ for all integers $K \geq 2$.

Base case:
$$P(2) : C^{(2)} = \psi^{(2-2)}(K) = \psi^{(0)}(K) = \psi(K)$$
So, P(2) is true as above shows.

We then need to show if P(K) is true, where K $\geq$2, P(K+1) is also true.
WTS:
$$C^{(K+1)}(K) = \psi^{(K+!-2)}(K) = [\psi^{(K-1)}(K)]$$

$$C^{(K+1)}(K) = \frac{d}{dK}[c^{(K)}(K)] = \frac{d}{dK}[\psi^{(K-2)}(K)] = \psi^{(K-2+1)}(K)$$
$$= \psi^{(K-1)}\checkmark$$
So, $P(K) \implies P(K+1)$ and thus $C^K(K) = \psi^{(K-2)}(K),\ \forall K \geq 2$

3.

By Taylor's theorem conclude that $C(K)$ has the expansion
$$C(K) = \sum_{k\in\mathbb{N}} c_k K^k$$
where $c_0 = \int_0^\infty x\psi(x)\ dx$, $c_1 = -\int_0^\infty \psi(x)\ dx$ and $c_k = \frac{1}{k!}\psi^{(k-2)}(0)$.

By Taylor's Theorem, we have:

$$C(K) = C(0) + C'(0)K + \frac{C''(0)}{2!}K^2 + \frac{C'''(0)}{3!}K^3 + \frac{C^{(4)}(0)}{4!}K^4 + [...] + \frac{C^{(n)}(0)}{n!}K^n + R$$
Where R is the remainder but is negligible since it is an estimation for (K)
$$C(K) = \int_K^\infty x\psi(x)dx - K\int_K^\infty \psi(x)dx$$
$$C^K(K) = \psi^{(K-2)}(K) \qquad \forall K \geq 2$$
$$\implies C(0) = \int_0^\infty x\psi(x)dx$$
$$C^{(1)}(K) = -\int_K^\infty \psi(x)dx \implies C'(0) = -\int_0^\infty \psi(x)dx$$

$$C^{(2)}(K) = \psi(K) \implies c^{(2)}(0) = \psi(0) = \psi^{(0)}(0)$$

$$C^{(3)}(K) = \psi^{(1)}(K) \implies C^{(3)}(0) = \psi^{(1)}(0)$$

$$C^{(4)}(K) = \psi^{(2)}(K) \implies C^{(4)}(0) = \psi^{(2)}(0)$$

$$C^{(n)}(K) = \psi^{(n-2)}(K) \implies C^{(n)}(0) = \psi^{(n-2)}(0)$$

$$C(K) = \int_0^\infty x\psi(x)dx + (-\int_0^\infty \psi(x)dxK) + \frac{\psi(0)}{2!}K^2 + \frac{\psi^{(1)}(0)}{3!}K^3 + \frac{\psi^{(2)}(0)}{4!}K^4 + [...] + \frac{\psi^{(n-2)}(0)}{n!}K^n$$

$$= \int_0^\infty x\psi(x)dx \cdot K^0 + (-\int_0^\infty \psi(x)dxK^1) + \frac{1}{2!}\psi^{(0)}K^2 + \frac{1}{3!}\psi^{(1)}(0)K^3 + \frac{1}{4}\psi^{(2)}(0)K^4 + [...] + \frac{1}{n!}\psi^{(n-2)}(0)K^n$$

Now, let

$$c_0 = \int_0^\infty x\psi(x)dx, \qquad c_1 = -\int_0^\infty \psi(x)dx, \qquad c_K = \frac{1}{k!}\psi^{(k-2)}(0)$$

$$= c_0K^0 + c_1K + c_2K^2 + c_3K^3 + c_4K^4 + [...] + c_nK^n$$

$$= \sum_{n\in\mathbb{N}} c_nK^n = \sum_{k\in\mathbb{N}} c_kK^k$$

Thus it is shown that

$$C(K) = \sum_{k\in\mathbb{N}} c_kK^k$$

4.

Setting $A = \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}$, calculate the leading terms to show that

$$C(K) = A - \frac{K}{2} + \frac{K^2}{4\pi A} - \frac{K^4}{96\pi^2 A^3} + O(\frac{K^6}{A^5})$$

$$\psi(x) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \cdot e^{\frac{-x^2}{2\sigma^2 T}}$$

$$\longrightarrow \psi(0) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \tag{1}$$

$$\psi'(x) = \frac{x}{\sigma^2 T}(-\psi(x))$$

$$\longrightarrow \psi'(0) = 0 \tag{2}$$

$$\psi''(x) = \frac{1}{\sigma^2 T}(-\psi(x)) - \frac{x}{\sigma^2 T}\psi'(x) = \frac{-1}{\sigma^2 T}(\psi(x) + x\psi'(x)$$

$$\longrightarrow \psi^{(2)}(0) = \frac{1}{\sigma^2 T\sqrt{2\pi\sigma^2 T}} \tag{3}$$

$$\psi^{(3)}(x) = \frac{1}{\sigma^2 T}(-\psi'(x))) - [2x\psi(x) + x^2\psi'(x))]$$

$$\longrightarrow \psi^{(3)}(0) = 0 \tag{4}$$

$$\psi^{(4)}(x) = \frac{1}{\sigma^4 T^2}(\psi(x) + x\psi'(x)) + 2(\psi(x) + x\psi'(x)) - \frac{1}{\sigma^2 T}(3x^2\psi(x) + x^3\psi'(x))$$

$$\longrightarrow \psi^{(4)}(0) = \psi(0) * (\frac{1}{\sigma^4 T^2} + 2) = \frac{1}{\sqrt{2\pi\sigma^2 T}}(\frac{1}{\sigma^4 T^2} + 2) \tag{5}$$

$$C(K) = \int_K^\infty x\psi(x)dx - K\int_K^\infty \psi(x)dx$$
$$= K\Psi(K) + \sigma^2 T\psi(K) - K$$

By Taylor Theorem, we have

$$C(K) = \sum_{K\in\mathbb{N}} C_K K^K = \sum_{n\in\mathbb{N}} C_n K^n$$

$$C(K) = \int_0^\infty x\psi(x)dx - \int_0^\infty x\psi(x)dxK + \frac{1}{2!}\psi(0)K^2 + \frac{1}{3!}\psi^{(1)}(0)K^3 + \frac{1}{4!}^{(2)}(0)K^4 + \frac{1}{5!}\psi^{(3)}(0)K^5 + \frac{1}{6!}\psi^{(4)}(0)K^6 + [...]$$
$+\frac{1}{n!}\psi^{(n-2)}(O)K^n$

$$= \int_0^\infty x\psi(x)dx - \int_0^\infty x\psi(x)dxK + \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi\sigma^2 T}}K^2 - \frac{1}{24}\frac{1}{\sigma^2 T\sqrt{2\pi\sigma^2 T}} - \frac{1}{720}\frac{1}{\sqrt{2\pi\sigma^2 T}}\cdot(\frac{1}{\sigma^4 T^2} + 2)K^6 + [...]$$

$$= A - \frac{1}{2}K + \frac{1}{4\pi A}K^2 + \frac{1}{96\pi^2 A^3}K^4 + O(\frac{K^6}{A^5})\checkmark$$

From (1), Since:

$$\int_K^\infty x\psi(x)dx = \sigma^2 T\psi(K)$$

as shown above,

$$\int_0^\infty x\psi(x)dx = \sigma^2\psi(K=0)$$

$$\int_0^\infty x\psi(x)dx = \sigma^2 T\cdot\psi(K=0) = \frac{\sigma^2 T}{\sqrt{2\pi\sigma^2 T}}\cdot e^{\frac{-0^2}{2\sigma^2 T}}$$

$$= \frac{\sigma^2 T}{\sqrt{2\pi\sigma^2 T}} = \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} = A$$

From (2),

$$\int_0^\infty x\psi(x)dx = \frac{K}{\sqrt{2\pi\sigma^2 T}} \int_0^\infty e^{\frac{-x^2}{2\sigma^2 T}} dx$$

$$\int_0^\infty e^{\frac{-x^2}{2\sigma^2 T}} dx \implies u = x^2 \implies \sqrt{u} = x$$

so,

$$du = 2xdx \implies \frac{1}{2x} du = dx$$

$$\frac{1}{2\sqrt{u}} du = \frac{\sqrt{u}}{2u} du = dx$$

$$\int_0^\infty \frac{\sqrt{u}}{2u} e^{\frac{-u}{2\sigma^2 T}} du$$

$$= \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{\frac{1}{\sigma^2 T}}} = \sqrt{\frac{\pi}{2}\sigma^2 T}$$

Finally we have,

$$= \frac{K}{\sqrt{2\pi\sigma^2 T}} \cdot \sqrt{\frac{\pi}{2}\sigma^2 T} = \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi}} K = \frac{1}{2} K$$

From 3,

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2\pi\sigma^2 T}} K^2 = \frac{1}{\sqrt{8\sigma^2 T}} K^2 = \sqrt{\frac{2\pi}{16\pi\sigma^2 T}} K^2$$

$$= \frac{1}{\frac{4\pi\sqrt{\sigma^2 T}}{\sqrt{2\pi}}} K^2 = \frac{K^2}{4\pi \cdot \frac{\sigma\sqrt{T}}{\sqrt{2\pi}}} = \frac{K^2}{4\pi A}$$

From 4,

$$\frac{1}{24} \cdot \frac{1}{\sigma^2 T\sqrt{2\pi\sigma^2 T}} K^4$$

$$= \frac{1}{24 \cdot \sigma^2 T \cdot \sqrt{2\pi\sigma^2 T}} K^4$$

$$= \frac{1}{\sqrt{576 \cdot 2 \cdot \sigma^4 \cdot T^2 \cdot \pi\sigma^2 T}} K^4 = \frac{1}{1152\pi\sigma^6 T^3} K^4$$

$$= \frac{1}{\sqrt{\frac{2304\pi^2\sigma^4 T^2 \sigma^2 T}{2\pi}}} K^4$$

$$= \frac{1}{48\pi\sigma^2 T \frac{\sqrt{\sigma^2 T}}{2\pi}} K^4 = \frac{1}{96\pi^2 \frac{\sigma^2 T}{2\pi} (\sqrt{\frac{\sigma^2 T}{2\pi}})^3} K^4$$

$$\implies \frac{1}{96\pi^2 A^3} K^4$$

From 5,

$$\begin{aligned}
\psi^{(4)}(O) &= \frac{1}{\sqrt{\pi\sigma^2 T}} \cdot (\frac{1}{\sigma^4 T^2} + 2) \\
&= \frac{\sqrt{2pi}}{2\pi\sqrt{\sigma^2 T}} \cdot (\frac{1}{\sigma^4 T^2} + 2) \\
&= \frac{1}{2\pi A} \cdot (\frac{(\sqrt{2\pi})^4}{4\pi^2(\sqrt{\sigma^2 T})^4} + 2) \\
&= \frac{1}{2\pi A} \cdot \frac{1}{4\pi^2 A^4} + \frac{1}{\pi A} = \frac{1}{8\pi^3 A^5} + \frac{1}{\pi A} \\
\frac{1}{6!}\psi^{(4)}(0) \cdot K^6 &= \frac{K^6}{6!}(\frac{1}{8\pi^3 A^5} + \frac{1}{\pi A}) \\
&= \frac{K^6}{6!}\frac{1}{8\pi^3 A^5}(1 + \frac{8\pi^3 A^5}{\pi A}) \\
&= \frac{K^6}{A^5}\frac{(1 + \frac{8\pi^3 A^5}{\pi A})}{6!8\pi^3} = O(\frac{K^6}{A^5})
\end{aligned}$$

Exercise 6

Let $X_t = bt + B_t$ for a constant $b \in \mathbb{R}$. X is called a Brownian motion with drift b. Compute the transition density for X by evaluating,

$$p_h(x, y) = \frac{\partial}{\partial y} P(X_{t+h} \leq y | X_t = x)$$

Let $\{Z_j\}_{j \geq 1}$ be an i.i.d. sequence of {-1,+1}-valued random variables with $p = P(Z_1 = 1)$, and consider the biased random walk $S_k = \sum_{j=1}^{k} Z_j$ at step k. Scale this walk by $\triangle x = \sqrt{\triangle t}$ and interpolate it to write down a continuous process $X_\triangle(t)$ for times $t \geq 0$. Set $p = \frac{1}{2}(1 + b\sqrt{\triangle t}$ and taking $\triangle t \downarrow 0$, use the Lindenberg CLT to show that the increment of $X_\triangle$ have

$$\lim_{\triangle t \downarrow 0} P(X_\triangle(t) - X_\triangle(s) \leq z) = \int_{-\infty}^{z} p_{t-s}(0, y)\ dy \quad (s < t)$$

Exercise 7

We know that B is a process on state space $\mathbb{R}$ which has continuous paths starting in zero and satisfies the following two properties:

$$(1) \quad \text{B is a mean-zero Guassian process}$$

$$(1) \quad \text{B has the auto-covariance function } \operatorname{cov}(B_t, B_s) = t \wedge s$$

Show that the B satisfying (1) and (2) must also have stationary and independent increments with any $B_t - B_s$ having the distribution $N(0, t - s)$.

Since it is given B is a mean-zero Gaussian process, for s,t ≥ 0, $B_t - B_s$ is normally distributed with mean $E(B_t - B_s) = E(B_t) - E(B_s) = 0$ and auto-covariance function $\text{cov}(B_t, B_s) = t \wedge s$

For $0 \leq s < t$,

$$\text{Var}(B_t - B_s) = \text{Var}(B_t) + \text{Var}(B_s) - 2\text{Cov}(B_t, B_s)$$

where by definition,

$$\text{Var}(B_t) = cov(B_t, B_t) = t$$

$$\text{Var}(B_t) + \text{Var}(B_s) - 2\text{Cov}(B_t, B_s) = (t) + s - 2s = t - s.$$

Thus, it is show that the distribution of $B_t - B_s$ only depends on the length of the segment t - s, regardless of what the initial starting point s was, which implies stationary increments.

For some $0 \leq q < r \leq s < t$,

$$E((B_r - B_q)(B_t - B_s)) = E(B_r B_t) - E(B_r B_s) - E(B_q B_t) + E(B_q B_s)$$

$$= \text{Cov}(B_r, B_t) - \text{Cov}(B_r, B_s) - \text{Cov}(B_q, B_t) + \text{Cov}(B_q, B_s)$$

$$= r - r - q + q = 0$$

Thus, segments are uncorrelated. Since the segments $B_r - B_q$ and $B_t - B_s$ are normally distributed and have a covariance of 0, this implies that they are independent.

Exercise 8

Prove that for any fixed time $\tau \in (0, \infty)$, we have

$$P(B_\tau < 0, B_{2\tau} > 0) = \frac{1}{8}$$

and

$$P(B_{2\tau} > 0 | B_\tau < 0) = \frac{1}{4}$$

Hint. Use the Chapman-Kolmogorov equations and integration by parts.

Extra Credit. Prove the above holds for any random time $\tau \geq 0$ that is independent of $\mathbf{B}$. Construct τ depending on $\mathbf{B}$ for which this fails.

$$P(B_\tau < 0, B_{2\tau} > 0) = P(B_\tau < 0, B_{2\tau} > 0 | B_\tau < 0) * P(B_\tau < 0)$$

$$= P(B_{2\tau} > 0 | B_\tau < 0) * P(B_\tau < 0)$$

$$= \int_{-\infty}^{0} P(B_{2\tau} - B_\tau > -x) * p_\tau(0, x)\ dx$$

$$= \int_{-\infty}^{0} (1 - P(B_{2\tau} - B_\tau \leq -x)) * p_\tau(0, x)\ dx$$

By symmetry of the normal distribution, this is equivalent to

$$\int_{-\infty}^{0} P(B_{2\tau} - B_\tau \le x) * p_\tau(0,x)\ dx$$
$$= \int_{-\infty}^{0} P(B_\tau \le x) * p_\tau(0,x)\ dx$$

Let $p_\tau(0,x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}} = f(x)$
Let $P(B_\tau \le x) = F(x)$, the CDF of the normal distribution with density f, as defined above.

$$\implies \int_{-\infty}^{0} P(B_\tau \le x) * p_\tau(0,x)\ dx = \int_{-\infty}^{0} F(x)f(x)\ dx$$

Lemma:

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

Let $u = F(x)$, and let $v = F(x)$ So, $du = f(x)\ dx,\ dv = f(x)dx$

$$\int_{-\infty}^{0} F(x)f(x)\ dx = (F(x) * F(x))\Big|_{-\infty}^{0} - \int_{-\infty}^{0} F(x)f(x)\ dx$$
$$\implies 2\int_{-\infty}^{0} F(x)f(x)\ dx = F(x)^2\Big|_{-\infty}^{0}$$
$$\implies \int_{-\infty}^{0} F(x)f(x)\ dx = \frac{F(x)^2\Big|_{-\infty}^{0}}{2}$$
$$= \frac{F(0)^2}{2} = \frac{(\frac{1}{2})^2}{2} = \frac{\frac{1}{4}}{2} = \frac{1}{8}$$

Thus it is shown that for any time T

$$P(B_\tau < 0, B_{2\tau} > 0) = \frac{1}{8}$$

Now, we prove

$$P(B_{2\tau} > 0 | B_\tau < 0) = \frac{1}{4}$$

$$P(B_{2\tau} > 0 | B_\tau < 0) = \frac{P(B_\tau < 0, B_{2\tau} > 0)}{P(B_\tau < 0)}$$

By previous part, we know $P(B_\tau < 0, B_{2\tau} > 0) = \frac{1}{8}$. So,

$$\frac{P(B_\tau < 0, B_{2\tau} > 0)}{P(B_\tau < 0)} = \frac{\frac{1}{8}}{P(B_\tau < 0)}$$

Since $B_\tau \sim N(0,\tau)$, $P(B_\tau < 0) = \frac{1}{2}$ by symmetry of the normal distribution about mean 0.

$$\implies \frac{\frac{1}{8}}{P(B_\tau < 0)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

Thus it is shown that for any time T

$$P(B_{2\tau} > 0 | B_\tau < 0) = \frac{1}{4}$$

Exercise 9

Let X and Y be two standard independent Brownian motions. Suppose we model the position of a molecule moving in a plane by (X,Y).

Let $R_t = \sqrt{X_t^2 + Y_t^2}$ be its distance from the origin at time t.

1.) The location of the molecule is described by $(R, \Theta) \in R_+$ x $[0, 2\pi]$ where $X = Rcos(\Theta)$ and $Y = Rsin(\Theta)$. Show that R_t and Θ_t are independent and find their cumulative distribution functions and probability densities (for example, $P(R_t \leq r) = F_{R_t}(r) = \int_0^r f_{R_t}(\rho)d\rho$)

$$P(R_t, \Theta_t) = \int_0^{2\pi}\int_0^r \sqrt{(Rcos(\Theta))^2 + (Rsin(\Theta))^2}RdRd\Theta$$

$$= \int_0^{2\pi}\int_0^r \sqrt{R^2(cos^2(\Theta) + sin^2(\Theta))}RdRd\Theta$$

$$= \int_0^{2\pi}\int_0^r R^2\sqrt{1}dRd\Theta$$

$$= \int_0^{2\pi} 1d\Theta \int_0^r R^2 dR$$

$$Because F(R,\theta) \xrightarrow{independence} F(R) * F(\theta)$$

$$\int_0^{2\pi} 1d\Theta = 2\pi, \text{so scaled is} \frac{1}{2\pi}$$

$$\frac{1}{2\pi}\int_0^{2\pi} 1d\theta \checkmark \text{so } \theta \text{ is constant and uniform on} (0, 2\pi)$$

$$\int_0^r R^2 dR =$$

$$\text{so PDF of P is } C * P^2 \text{where C is a constant such that}$$

$$C * \int R^2 dR = 1$$

2.) Argue that the point $(\hat{X}_t, \hat{Y}_t)$ with $\hat{X} = \frac{X}{R}$ and $\hat{Y} = \frac{Y}{R}$ is uniformly distributed on the unit circle $\{(x, y)|x^2 + y^2 = 1\ \}$. Determine the probability density of $\hat{Y}_t$, graph it, and find its mean and variance.

$$(\hat{X}_t, \hat{Y}_t)$$

When $\hat{X}_t = \frac{x}{R}$ and $\hat{Y}_t = \frac{y}{R}$

$$\hat{X}_t = \frac{Rcos(\theta)}{R} and \hat{Y}_t = \frac{Rsin(\theta}{R}$$

$$\hat{X}_t = cos(\theta) and \hat{Y}_t = sin(\theta)$$

Because θ is uniform on $[0, 2\pi]\hat{X}_t + \hat{Y}_t = cos(\theta)^2 + sin(\theta)^2 = 1$.

because θ is uniform,

$cos(\theta)$ and $sin(\theta)$ will always be on the unit circle and be uniformly distributed as θ is uniform. Because θ is uniform, the distributions of Cos(θ) and Sin(θ) will be uniform on the unit circle.

$$P * \hat{Y}_t \leq a)$$

$$P(Y_t^2 \leq a \cdot R_t)$$

$$P(Y_t^2 \leq a^2 \cdot R_t^2)$$

$$P(Y_t^2 \leq a^2 \cdot (X_t^2 + Y_t^2))$$

$$P(Y_t^2 - a^2(x_t^2 + y_y^2) \leq 0)$$

$$P(Y_t^2 - a^2 Y_t^2 - a^2 X_t^2 \leq 0)$$

$$P(Y_T^2(1 - a^2) - a^2 X_t^2 \leq 0)$$

Or,

$$P(Y_t \leq a \cdot R_t)$$

$$Y_t = R_t sin(\theta_t)$$

$$P(R_t sin(\theta) \leq aR_t)$$

$$P(sin(\theta) \leq a)$$

where $\theta \in [0, 2\pi] unif$

$$P(arcsin(sin(\theta)) \leq arcsin(a))$$

$$P(\theta \leq arcsin(a))$$

$$P(Q \leq arcsin(a))$$

As θ goes from 0 to $\frac{pi}{2}$, arcsin goes from [-1,1]

$$= \frac{1}{2pi} \int_0^{arcsin(a)} 1 d\theta$$

$$= \frac{arcsin(a)}{2\pi}$$

When arcsin(a) $= \int_0^a \frac{dz}{(1-z)^2}$

$$= \frac{\int_0^1 \frac{dz}{(1-z)^2}}{2\pi}$$

$\hat{Y}_t$ has mean 0 and variance $\frac{1}{R}$

$$E[\hat{Y}_t] = E[\frac{Y}{R_t}] = \frac{0}{E[R_t]} = 0$$