# PSTAT 160B Homework 4

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## Exercise 1

Consider a voter model on a graph on $V = \{1, 2, ..., n\}$ vertices (see Lecture 15). Each vertex represents a person who has a political affiliation denoted by 0 or 1, and each edge $(i, j)$ represents a friendship between persons $i$ and $j$. At each step we pick a person $j$ (vertex) uniformly at random. Then, we select a friend of $j$ uniformly at random and changes the political affiliation of $j$ (if different) to that of the selected friend.

Suppose the graph on $V$ is complete (each pair of vertices $i \neq j$ are connected by an edge) and $X_k^j$ is the political affiliation of $j$ at time $k$.

### 1.

Write down the state space for the process $X_k = (X_k^1, ..., X_k^n)$ for $k \geq 0$. Show that $X$ a Markov chain. Show that $X$ is not irreducible and that $X$ has two stationary distributions (state what they are).

The state space: $\mathbb{S} = \{0, 1\}^n$

To show that $X$ is a Markov chain, we need to show that the transition probabilities depend only on the current state and not on the past history of the process.At each step, we pick a vertex $j$ uniformly at random, and then pick one of its neighbors $i$ uniformly at random.The probability of changing the political affiliation of $j$ from $X_k^j$ to $X_{k+1}^j$ is 1 if $X_k^i = X_k^j$, and 0 otherwise.Therefore, the transition probabilities depend only on the current state of the process and the selected vertex $j$, and not on the past history of the process.

$X$ is not irreducible since the chain is going to be either in the state with full 0's or full 1's in the long run for every single vertex, and both of these total states are absorbing and there is no transitioning out of it,such that each individual X in the total chain is $(1)^n$ or $(0)^n$.

X has two stationary distributions, the chain will reach the either state $(1)^n$ or $(0)^n$ after a long time.

Next, consider the following modified model. The setup is the same as above except the person $j$ who is picked uniformly at random, now changes (if possible) their affiliation to be the opposite of that of the selected friend.

**2.**

Let $Z_k$ be the number of people (vertices) with affiliation 0 at time $k$. Show that $Z$ is a reversible Markov chain on state space $\mathbf{S} = \{0\} \cup \mathbf{V}$ and with a unique stationary distribution $\pi_Z$ given by

$$\pi_Z(j) = \frac{\binom{n}{j}\binom{n-2}{j-1}}{\binom{2(n-1)}{n-1}}$$

$$Z_k \to \begin{cases} Z_k - 1 & \text{w.p } (\frac{Z_k}{n})(\frac{Z_k-1}{n-1}) \\ Z_k & \text{w.p } (\frac{Z_k}{n})(\frac{n-Z_k}{n-1}) + (\frac{n-Z_k}{n})(\frac{Z_k}{n-1}) \\ Z_k + 1 & \text{w.p } (\frac{n-Z_k}{n})(\frac{n-Z_k-1}{n-1}) \end{cases}$$

$$\frac{Z_k(Z_k-1)}{n(n-1)} + \frac{Z_k(n-Z_k)}{n^2-n)} + \frac{Z_k(n-Z_k)}{n^2-n} + \frac{(n-Z_k)(n-Z_k-1)}{n^2-n}$$

$$= \frac{1}{n(n-1)} * (Z_k^2 - Z_k + Z_k n - Z_k^2 + Z_k n - Z_k^2 + n^2 - Z_k n - n - Z_K n + Z_K^2 + Z_k)$$

$$= \frac{1}{n(n-1)}(n^2 - n)$$

$$\frac{n^2-n}{n^2-n} = 1$$

$\therefore$ it is a valid transition matrix

At the edge cases where the state of the entire system is fully 0's or full 1's you instead can only transition out and never return to the initial state. The edge cases have these transition probabilities

For Edge Cases: if $Z_k = 0$

$$Z_k \implies Z_k + 1 = 1 w.p 1$$

if $Z_k = 1$

$$Z_k \implies Z_k w.p \frac{1}{n} + \frac{n-1}{n} \cdot (\frac{1}{n-1})$$

$$Z_k \implies Z_k + 1 = 2 w.p \frac{n-1}{n} \cdot \frac{n-2}{n-1}$$

if $Z_k = n$

$$Z_k \implies Z_k - 1 = n - 1 w.p 1$$

if $Z_k = n - 1$

$$Z_k \implies Z_k w.p \frac{1}{n} + \frac{n-1}{n} \cdot (\frac{1}{n-1})$$

$$Z_k \implies Z_k - 1 = n - 2(\frac{n-1}{n}) \cdot (\frac{n-2}{n-1})$$

Given the stationary distribution is

$$\pi_z(k) = \frac{\binom{n}{k} * \binom{n-2}{k-1}}{\binom{2(n-1)}{(n-1)}}$$

where

$$\pi_z(0) = \pi_z(n) = 0$$

Checking Local Balance Equations:

$$\pi(y)p(y,x) = \pi(x)p(x,y)$$

where $x$ is $x = Z_k$, and $y$ is $y = Z_k + 1 = x + 1$

$$\pi(y)(\frac{Z_k+1}{n})(\frac{Z_k}{n-1}) = \pi(x)(\frac{n-Z_k}{n})(\frac{n-Z_k-1}{n-1})$$

$$\pi(y)(\frac{Z_k^2+Z_k}{n^2-n}) = \pi(x)(\frac{Z_k^2+Z_k-2Z_kn+n^2-n}{n^2-n})$$

$$\frac{\pi(y)}{\pi(x)} = (\frac{Z_k^2+Z_k-2Z_kn+n^2-n}{n(n-1)})(\frac{n(n-1)}{Z_k^2+Z_k})$$

$$\frac{\pi(y)}{\pi(x)} = \frac{Z_k^2+Z_k-2Z_kn+n^2-n}{Z_k^2+Z_k}$$

so,

$$\frac{\pi(y)}{\pi(x)} = \frac{(n-Z_k)(n-Z_k-1)}{Z_k(Z_k+1)}$$

*equation(1)*

Now that we have an expression for the right hand side, we know need to check that the left hand side is equivalent

$$\pi(y) = \frac{\binom{n}{x+1} * \binom{n-2}{x-1}}{\binom{2(n-1)}{n-1}}$$

$$\pi(x) = \frac{\binom{n}{x} * \binom{n-2}{x-1}}{\binom{2(n-1)}{n-1}}$$

$$= \frac{\binom{n}{x+1} * \binom{n-2}{x-1}}{\binom{n}{x}\binom{n-2}{x-1}}$$

$$= \frac{\frac{\frac{n!}{(x+1)!(n-x-1)!} * \frac{(n-2)!}{(x)!(n-2-x)!}}{(n!)(n-2)!}}{x!(n-x)!(x-1)!(n-2-x+1)!}$$

$$\frac{x!(n-x)! \cdot (x-1)!(n-1-x)!}{(x+1)!(n-x-1)! \cdot (x)! \cdot (n-2-x)!}$$

$$\frac{(x-1)! \cdot (n-x)!}{(x+1)! \cdot (n-x-2)!}$$

$$\frac{(x-1)! \cdot (n-x-2)! \cdot (n-x)(n-x-1)}{(x-1)! \cdot (x+1)(x)(n-x-2)!}$$

$$\frac{\pi(y)}{\pi(x)} = \frac{(n-x)(n-x-1)}{x(x+1)} \implies \frac{(n-Z_k)(n-Z_k-1)}{(Z_k^2+Z_k)}$$

$$x = Z_k$$

So,

$$\frac{(n-Z_k)(n-Z_k-1)}{Z_k(Z_k+1)} = \frac{(n-Z_k)(n-Z_k-1)}{Z_k(Z_k+1)}$$

Equal, thus local balance holds. Stationary is verified, and because it holds, it is also reversible.
**Extra credit.** Derive this stationary distribution from scratch.

### 3.

Use part 2 to determine the stationary distribution.

Stationary of X

For each state $Z_k$ with $\pi_Z(i)$ stationary, there are k 0's
this implies that there are $\binom{n}{i}$ different combinations of vertexes that satisfy $Z_k$= i.
where $Z_k = \sum_j^n x_j$
as such, the distribution of a string of X's for a given $Z_k = i$ will have

$$\frac{\pi_z(Z_k = \sum_j^n X_j)}{\binom{n}{Z_k = \sum_j^n X_j}}$$

or,

$$\frac{\pi_z(\sum_j^n X_j)}{\binom{n}{\sum_j^n X_j}}$$

where each string is uniformly likely.
Our guess will therefore be:

$$\frac{\pi_z(\sum_j^n X_j)}{\binom{n}{\sum_j^n X_j}}$$

Checking with local balance:

$$\pi(x)p(x,y) = \pi(y)p(y,z)$$

where $X$ has $Z_{kx} = a$, our vertex $X_j^{(1)} = 0$

$$Y \text{ has } Z_{ky} = a + 1 = Z_{kx} + 1, x_j^{(1)} = 1, \text{ and n-vertices}$$

$$\pi(x)P(x, y) = \pi(y)P(y, x)$$

$$\pi(a)P(a, a + 1) = \pi(a + 1)P(a + 1, a)$$

$$\frac{\pi_Z(a)}{\binom{n}{a}} P(a, a + 1) = \frac{\pi_Z(a + 1)}{\binom{n}{a+1}} \cdot P(a + 1, a)$$

Consider $(a, a + 1)$ is picking our vertex $X_k^j$ and choosing opposite colored friend

$$P(a, a + 1) = \frac{1}{n} \cdot \frac{n - a - 1}{n - 1}$$

$P(a + 1, a)$ is picking our vertex $x_k^j$ and selecting same color friend.

$$P(a + 1, a) = \frac{1}{n} \cdot \frac{(a + 1) - 1}{n - 1}$$

$$\frac{\pi_Z(a)}{\binom{n}{a}} \cdot \frac{1}{n} \left(\frac{n - a - 1}{n - 1}\right) = \frac{\pi_Z(a + 1)}{\binom{n}{a+1}} \cdot \frac{1}{n} \left(\frac{a}{n - 1}\right)$$

$$\pi_Z(a) = \frac{\binom{n}{a} \cdot \binom{n-2}{a-1}}{\binom{(Z(n-1)}{n-1)}}$$

So,

$$\frac{\pi_Z(a)}{\binom{n}{a}} = \frac{\binom{n-2}{a-1}}{\binom{2(n-1)}{n-1}}$$

So,

$$= \frac{\binom{n-2}{a-1}}{\binom{2(n-1)}{n-1}} \cdot \left(\frac{n - a - 1}{n - 1}\right) = \frac{\binom{n-2}{a-1}}{\binom{2(n-1)}{n-1}} \cdot \frac{a}{n - 1}$$

$$\binom{n - 2}{a - 1}(n - a - 1) = \binom{n - 1}{a} \cdot (a)$$

$$\frac{(n - 2)!}{(a - 1)!(n - 2 - a + 1)!}(n - a - 1) = \frac{(n - 2)!}{a!(n - 2 - a)!(a)}$$

$$\frac{1}{(a - 1)!(n - 2 - a)!(n - 1 - a)} \cdot (n - a - 1) = \frac{1}{(a - 1)! \cdot a(n - 2 - a)!}(a))$$

$$\frac{1}{(n - 1 - a)} \cdot (n - a - 1) = \frac{1}{a} \cdot a$$

$$1 = 1$$

Thus local balance holds and it is a valid stationary measure.

Now, we need to normalize the stationary measure. $\pi_z(x)$ is already given to be a stationary distribution of $Z_k$. So,

$$\sum_{\sum X_j=1}^{n-1} \frac{\pi_z(\sum X_j)}{\binom{n}{\sum X_j}} = \frac{1}{\sum_{\sum X_j=1}^{n-1} \binom{n}{\sum X_j}}$$

Where,

$$\sum_{k=1}^{n-1} \binom{n}{k} = \sum_{k=1}^{n} \binom{n}{k} - \binom{n}{n}$$

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$$

So,

$$\sum_{k=1}^{n-1} \binom{n}{k} = 2^n - 2$$

Thus, our normalized $\pi$ is:

$$(2^n - 2) \cdot \frac{\pi_z(\sum_j^n X_j)}{\binom{n}{\sum_j^n X_j}}$$

Since local balance held for our stationary measure, local balance holds trivially for our normalized stationary distribution.

## Exercise 2

In finance it is common to model a firm bankruptcy as the first time that its value drops to zero. Suppose that the value of a firm at time $t$ is $V_t$ for a standard Brownian motion $V$ started in $V_0 = v > 0$. Define $T_0 = min\{t > 0 : V_t = 0\}$, the time of the (first) bankruptcy of the firm.

### 1.

Find the density of the time $T_0$ of bankruptcy and show $E(T_0) = \infty$.

By the Reflection Principle, $T_0 = min\{t > 0 : V_t = 0\}$ is equivalent to $T_v = min\{t > 0 : B_t = v\}$. For a standard Brownian motion, let $T_v$ be the first time the process hits level $v$.

$$\begin{aligned} P(T_v < t) &= 2P(B_t > v) \\ &= 2\int_v^{\infty} \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} dx \end{aligned}$$

For any $v$,

$$= 2\int_{\frac{v}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

Differentiating with respect to $t$ gives the probability density function of the first hitting time. The density function of $T_v$ is given by:

$$f_{T_v}(t) = \frac{v}{\sqrt{2\pi t^3}} \cdot e^{\frac{-v^2}{2t}}$$

which is equivalent to density of the time $T_0$ of bankruptcy.

$$E(T_v) = \int_0^{\infty} \frac{tv}{\sqrt{2\pi t^3}} e^{\frac{-v^2}{2t}} dt$$

$$\geq \int_1^{\infty} \frac{tv}{\sqrt{2\pi t^3}} e^{\frac{-v^2}{2t}} dt$$

Since $e^{\frac{-v^2}{2t}}$ is an increasing function of t,

$$\geq \int_1^{\infty} \frac{tv}{\sqrt{2\pi t^3}} e^{\frac{-v^2}{2}} dt$$

$$= \frac{v}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \int_1^{\infty} \frac{1}{\sqrt{t}} dt$$

$$= +\infty$$

So, $E(T_v) = +\infty$ for all v, thus $E(T_0) = +\infty$.

**2.**

Use optional stopping to prove that for some function $\theta$ of $u > 0$

$$(a) \quad E(e^{-uT_0}) = e^{\theta(u)v}$$

and find the function $\theta$. Use this formula to show that $E(T_0) = \infty$.

First, we want to find a martingale such that $E(M_{T_0}) = E(M_0)$ using Optional Stopping Theorem. Let $M_t = e^{\theta B_t - \theta^2 t/2}$ for any $\theta \in \mathbb{R}$, where $B_t$ is standard Brownian Motion. The stochastic process $M = \{M_t\}_{t \geq 0}$ is a martingale if $\forall t \geq 0$:

$$1)\ E(M_t | \{M_u\}_{0 \leq u \leq s}) = M_s, \text{for all } 0 \leq s < t < \infty$$

$$2)\ E(|M_t|) < \infty$$

Proof of 1)

$$E(M_t | \{M_u\}_{0 \leq u \leq s}) = E(M_t | M_s)$$

$$= E(e^{\theta B_t - \theta^2 t/2} | M_s) = e^{-\theta^2 t/2} E(e^{\theta B_t} | B_s) = e^{-\theta^2 t/2} E(e^{\theta(B_t - B_s) + \theta B_s} | B_s) = e^{\theta B_s - \theta^2 t/2} E(e^{\theta(B_t - B_s)})$$

$$\text{Since } B_t - B_s \sim N(0, t-s), \text{we have } E[e^{\theta(B_t - B_s)}] = e^{\theta^2(t-s)^2/2}$$

$$= e^{\theta B_s - \theta^2 t/2} e^{\theta^2(t-s)^2/2} = e^{\theta B_s - \theta^2 s/2} = M_s$$

Proof of 2)

$$E(|M_t|) = e^{-\theta^2 t/2} E(e^{\theta B_t}) = e^{-\theta^2 t/2} e^{\theta^2 t/2} = 1 < \inf$$

Now,

$$E(M_t) = E(e^{\theta B_t - \theta^2 t/2})$$

Since $\theta^2 t/2$ is constant,

$$= e^{-\theta^2 t/2} \cdot E(e^{\theta B_t})$$

$B_t \sim N(0, t)$ and $E(e^{\theta B_t})$ is simply the MGF, so

$$= e^{-\theta^2 t/2} \cdot e^{\theta^2 t/2} = 1$$

By Optional Stopping Theorem,

$$E(M_{T_0}) = E(M_0) = 1$$

$$1 = E(M_{T_0}) = E(e^{\theta B_{T_0} - \theta^2 T_0/2})$$

Since $B_{T_0} = v$ by the Reflection Principle,

$$= E(e^{\theta v - \theta^2 T_0/2})$$

$$= e^{\theta v} \cdot E(e^{-\theta^2 T_0/2})$$

$$\implies e^{-(\theta v)} = E(e^{-\theta^2 T_0/2})$$

We are given: $E(e^{-uT_0}) = e^{\theta(u)v}$. So, it must be that

$$u = \frac{\theta^2}{2}, \theta(u) = -\sqrt{2u}$$

$\theta = \pm\sqrt{2u} \to \theta = +\sqrt{2u}$ to match given formula.

Finally, we want to show $E(T_0) = \infty$.

$$E(T_0) = \frac{d}{du} E(e^{-uT_0})\Big|_{u=0}$$

$$= \lim_{h \to 0} \frac{e^{\sqrt{2(0+h)}v} - e^{\sqrt{2(0)}v}}{h}$$

$$= \lim_{h\to 0} \frac{e^{\sqrt{2h}v} - e^0}{h}$$

$$= \lim_{h\to 0} \frac{e^{\sqrt{2h}v} - 1}{h}$$

By L'Hopital's rule,

$$= \lim_{h\to 0} \frac{\frac{ve^{\sqrt{2h}v}}{\sqrt{2h}}}{1}$$

$$= \frac{v}{\sqrt{2}} \lim_{h\to 0} \frac{e^{\sqrt{2h}v}}{\sqrt{h}} = \infty = E(T_0)$$

Since $\sqrt{h} \to 0$ as $h \to 0$, $e^{\sqrt{2h}v} \to 1$ as $h \to 0$, and $v > 0$.

**3.**

Use the probability density of $T_0$ in part 1 to confirm $(a)$.

*Hint. You may use that* $\int_{-\infty}^{\infty} e^{-ax^2 - b/x^2} dx = \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}$ *for a,b > 0.*

**Extra credit.** Prove the integral identity in the hint.

$$E(e^{-uT_0}) = \int_0^\infty e^{-ut} \frac{v}{\sqrt{2\pi t^3}} e^{-\frac{v^2}{2t}}\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ut - \frac{v^2}{2t}} \frac{v}{\sqrt{t^3}}\ dt$$

$$\text{Let } t = \frac{v^2}{x^2},\ dt = \frac{-2v^2}{x^2}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_\infty^0 e^{-u(\frac{v^2}{x^2}) - \frac{v^2}{2(\frac{v^2}{x^2})}} \cdot \frac{v}{\sqrt{(\frac{v^2}{x^2})^3}} \cdot \frac{-2v^2}{x^3}\ dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-u(\frac{v^2}{x^2}) - \frac{x^2}{2}} \cdot \frac{-2v^3}{(\frac{v^3}{x^3})x^3}\ dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-(\frac{1}{2})x^2 - \frac{(uv^2)}{x^2}}\ dx$$

Using the hint with $a = \frac{1}{2}, b = uv^2$,

$$= \frac{2}{\sqrt{2\pi}} (\frac{1}{2} \cdot \sqrt{\frac{\pi}{\frac{1}{2}}} \cdot e^{-2\sqrt{(1/2)(uv^2)}})$$

$$= \frac{1}{\sqrt{2\pi}} \left(\sqrt{2\pi} \cdot e^{-2\sqrt{1/2u}v}\right)$$

$$= e^{-\sqrt{2u}v}$$

## Exercise 3

Suppose a population of wolves and rabbits that grow and decrease in numbers based on the following dynamics. Given $x_1$ rabbits a new one is born at rate $bx_1$. Given $x_2$ wolves, one dies of hunger with rate $dx_2$. Given $x_1$ rabbits and $x_2$ wolves, a rabbit is eaten by a wolf and a new wolf is born with rate $qx_1x_2$. These transition rates $\{R_{xy}\}_{x,y\in\mathbb{S}}$ define a continuous-time Markov chain $X = \{X_t^{(1)}, X_t^{(2)}\}_{t\geq 0}$ where $X_t^{(1)}$ and $X_t^{(2)}$ are the numbers of rabbits and wolves at time $t \geq 0$ respectively.

### 1.

Write down the state space for the continuous-time chain $X$ and compute the transition probabilities of the embedded chain for $X$.

The state space of $X$ is the set $\mathbb{S} = \{(x_1, x_2);\ x_1, x_2 \in \mathbb{N} \cup \{0\}\}$. The state of the Markov Chain at any time $t \geq 0$ is simply the number wolves and rabbits in the population.

There are three transitions that can occur:

A rabbit is born: $(x_1, x_2) \to (x_1+1, x_2)$ with rate $bx_1$. A wolf dies of hunger: $(x_1, x_2) \to (x_1, x_2-1)$ with rate $dx_2$. A rabbit is eaten by a wolf and a new wolf is born: $(x_1, x_2) \to (x_1 - 1, x_2 + 1)$ with rate $qx_1x_2$.

Let $p_{(x_1,x_2),(y_1,y_2)}$ denote the transition probability from state $(x_1, x_2)$ to state $(y_1, y_2)$ for the embedded chain for $X$. We use the transition rates found above:

$$p_{(x_1,x_2),(x_1+1,x_2)} = \frac{bx_1}{bx_1 + dx_2 + qx_1x_2}$$
$$p_{(x_1,x_2),(x_1,x_2-1)} = \frac{dx_2}{bx_1 + dx_2 + qx_1x_2}$$
$$p_{(x_1,x_2),(x_1-1,x_2+1)} = \frac{qx_1x_2}{bx_1 + dx_2 + qx_1x_2}$$

Now to deal with the edge cases. If $x_1 = 0$, then the first transition cannot occur, and no new rabbits will be born. If $x_2 = 0$, then the second and third transitions cannot occur, and there will be no wolves to eat the rabbits. If $x_1 > 0$ and $x_2 > 0$, then all three transitions are possible.

The embedded chain for $X$ is not irreducible, since there are states that cannot be reached from other states. For example, if $x_1 = 0$, then the state $(0, x_2)$ is an absorbing state, since no rabbits will be born and the wolves will starve and die out. Similarly, if $x_2 = 0$, then the state $(x_1, 0)$ is also absorbing, since there are no wolves to hunt the rabbits.

### 2.

State the rate parameter of the exponential distribution that determines how long we stay in each of the states of $X$.

As such, the rate parameter of the total exponential distribution is going to be the minimum of all 3 separate clocks. The first clock for the event where $x_1$ goes up by one has the rate parameter $bx_1$, the second clock has the exponential rate parameter $dx_2$, and the third clock has the exponential rate parameter $qx_1x_2$. Because the time step moves in regards to which over clock is the smallest, we want the distribution of the minimum of these three exponentials.

$$P(X_k, X_{k+1}) \rightarrow \begin{cases} y_1 & \text{exp rate } bx_1 \\ \\ y_2 & \text{exp rate } dx_2 \\ \\ y_3 & \text{exp rate } qx_1x_2 \\ \end{cases}$$

So the overall distribution of time between states will be:

$$min(y_1, y_2, y_3)$$

where:

$$y_1 \sim exp(bx_1)$$

$$y_2 \sim exp(dx_2)$$

$$y_3 \sim exp(qx_1x_2)$$

We know $min(y_1, y_2, y_3) = min(min(y_1, y_2), y_3)$

We also know the $min(y_1, y_2)$ where they are both exponential RV's.

B is also exponential with parameter $(\lambda_{y_1} + \lambda_{y_2})$, or alternatively:

distribution of $min(y_1, y_2)$

Let $P(Z > a)$, where $Z = min(y_1, y_2) = F_z(a)$
So, $min(y_1, y_2) > a \implies (y_1, y_2) > a$
because $y_1$ and $y_2$ are independent

$$\implies P(y_1 > a) * P(y_2 > a)$$

So, $e^{(-\lambda y_1 a)} \cdot e^{(-\lambda y_2 a)}$
$= e^{-a(\lambda y_1) + \lambda y_2)}$ which is exponentially distributed with parameter $(\lambda y_1 + \lambda y_2)$

$$min(y_1, y_2, y_3) = min(min(y_1, y_2), y_3)$$

$$= min(y_1, y_2) \sim exp(\lambda_{y_1} + \lambda_{y_2})$$

$$= min(exp(\lambda_{y_1} + \lambda_{y_2}), exp(\lambda_{y_3}))$$

$$= exp(\lambda_{y_1} + \lambda_{y_3} + \lambda_{y_3})$$

where,

$$\lambda_{y_1} = B \cdot x_1$$

$$\lambda_{y_2} = D \cdot x_2$$

$$\lambda_{y_3} = q \cdot x_1 \cdot x_2$$

so our overall time step is distributed exponentially with parameter $(Bx_1 + Dx_2 + qx_1x_2)$

### 3.

Write a program to simulate $X$ and plot three phase plots ($X^{(1)}$ on x-axis and $X^{(2)}$ on y-axis): (1) one in which the wolf population dies out, (2) one in which the rabbit population dies out and (3) one in which the populations move up and down together. You may select any time duration and rates $b, q, d > 0$ separately for each of the three plots. But start with 1000 wolves and 1000 rabbits in each of the three simulations.

In [1]:
```
import numpy as np
import math
import matplotlib.pyplot as plt
import statistics as st

# Allows to render plots directly within the notebook
%matplotlib inline
```

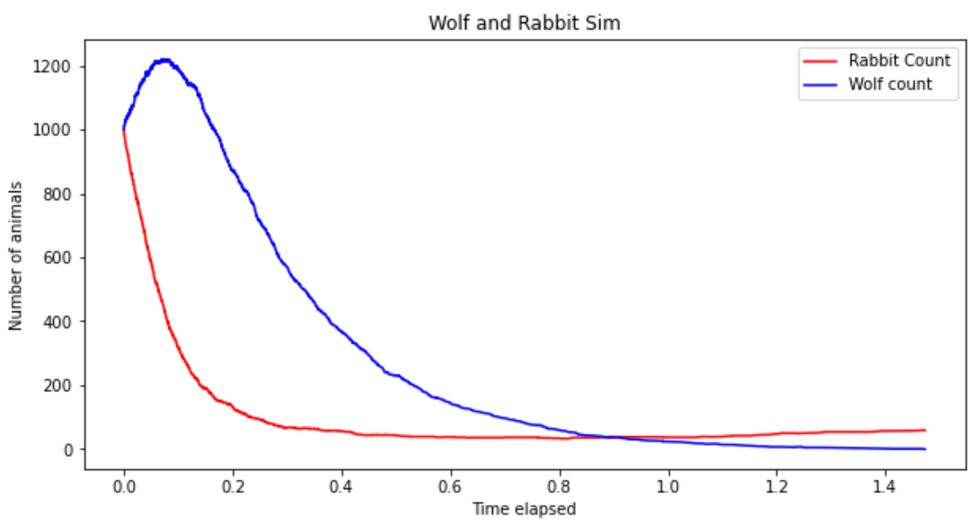
In [2]:
```
def exp_gen(lambda1):
    #I am using the inverse CDF method to generated Exponential samples
    samples = np.random.random_sample(1)

    return (math.log(1-samples)/(-lambda1))
```

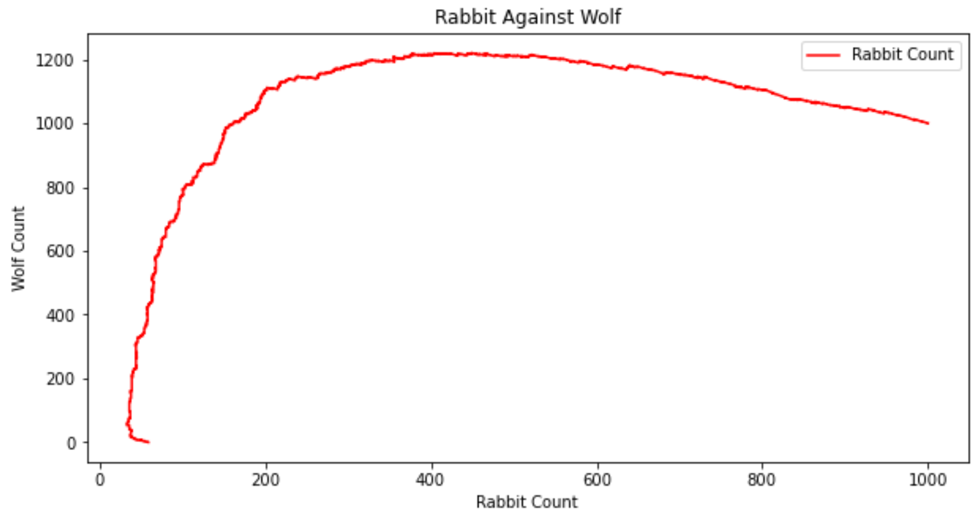
In [3]:
```
def simmer(x1, x2, b, d, q, total_time):
    #x1 = rabbits, x2 = wolves, bx1 = rabbit growth
    #dx2 = wolf death, qx1x2 = rabbit death and wolf growth
    rabbit = []
    wolf = []
    current_time = 0
    time = []
    while (total_time >= 0):
        rabbit.append(x1)
        wolf.append(x2)
        time.append(current_time)
        total_time = total_time -1

        if x1 <= 0 or x2 <= 0:
            break
        thing1 = exp_gen(b * x1)
        thing2 = exp_gen(d * x2)
        thing3 = exp_gen(q * x1 * x2)
        if thing1 <= thing2 and thing1 <= thing3:
            x1 = x1 + 1
            current_time = current_time + thing1
        elif thing2 <= thing1 and thing2 <= thing3:
            x2 = x2 - 1

            current_time = current_time + thing2
        else:
            x1 = x1 - 1
            x2 = x2 + 1
            current_time = current_time + thing3
    return rabbit , wolf, time
```

In [4]:
```
list1 , list2, list3 = simmer (1000,1000,1,5,0.01,1000000) #Wolf Dies
list111 , list222, list333 = simmer (1000,1000,3,3,0.01,1000000) #oscilating
list11 , list22, list33 = simmer (1000,1000,1,2,0.01,1000000) #rabbit die
```

In [5]:
```
#This is scenario 1, when the wolves die out, b = 1, d = 5, q = 0.01

plt.figure(figsize=(10,5))
plt.title("Wolf and Rabbit Sim")
plt.plot(list3,list1,'-',color="red",label="Rabbit Count")
plt.plot(list3,list2,'-',color="Blue",label="Wolf count")
plt.xlabel("Time elapsed")
plt.ylabel("Number of animals")
plt.legend(loc="upper right")

plt.figure(figsize=(10,5))
plt.title("Rabbit Against Wolf")
plt.plot(list1,list2,'-',color="red",label="Rabbit Count")
plt.xlabel("Rabbit Count")
plt.ylabel("Wolf Count")
plt.legend(loc="upper right")
```

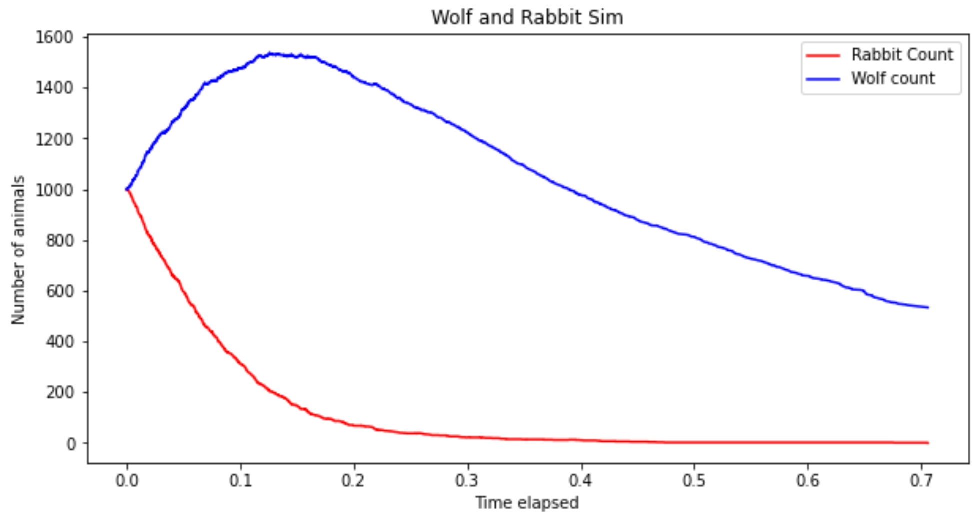
Out[5]:
```
<matplotlib.legend.Legend at 0x7fce80ec6220>
```

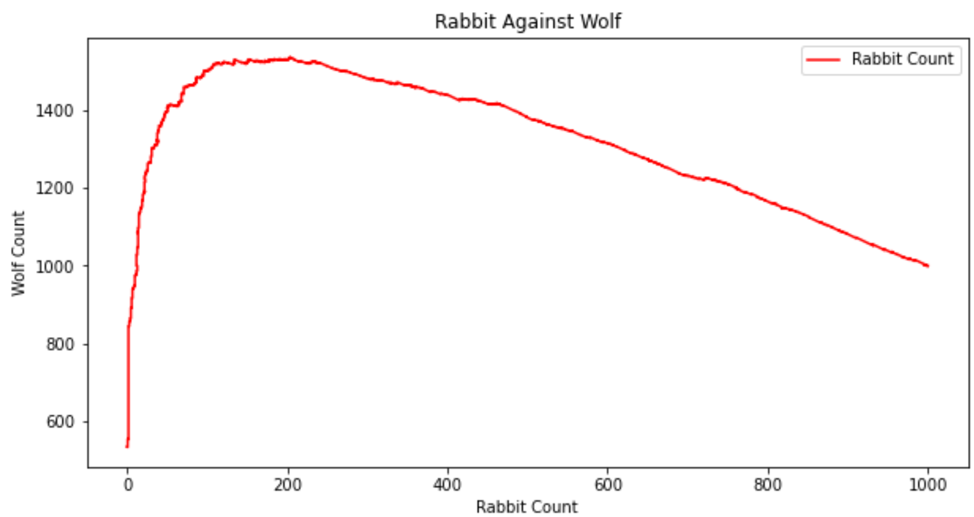





In [6]:
```
#This is scenario 2, when the rabbit die out, b = 1, d = 2, q = 0.01


plt.figure(figsize=(10,5))
plt.title("Wolf and Rabbit Sim")
plt.plot(list33,list11,'-',color="red",label="Rabbit Count")
plt.plot(list33,list22,'-',color="Blue",label="Wolf count")
plt.xlabel("Time elapsed")
plt.ylabel("Number of animals")
plt.legend(loc="upper right")

plt.figure(figsize=(10,5))
plt.title("Rabbit Against Wolf")
plt.plot(list11,list22,'-',color="red",label="Rabbit Count")
plt.xlabel("Rabbit Count")
plt.ylabel("Wolf Count")
plt.legend(loc="upper right")
```

Out[6]:
```
<matplotlib.legend.Legend at 0x7fce81187190>
```


Wolf and Rabbit Sim
Rabbit Count
Wolf count
Number of animals
Time elapsed
0
200
400
600
800
1000
1200
1400
1600
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

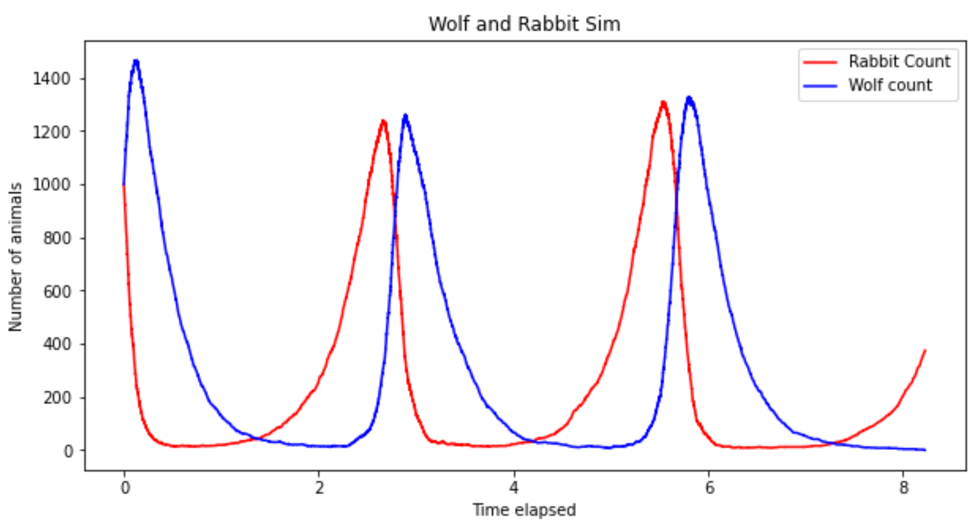

Rabbit Against Wolf
Rabbit Count
Wolf Count
Rabbit Count
600
800
1000
1200
1400
0
200
400
600
800
1000

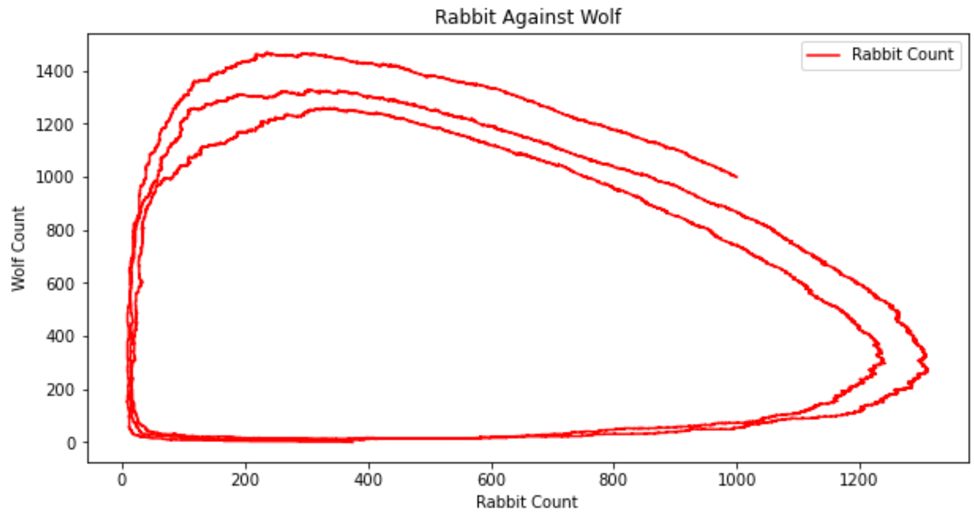
In [7]:

```
#This is scenario 3, when the rabbit and wolf population move together, b =


plt.figure(figsize=(10,5))
plt.title("Wolf and Rabbit Sim")
plt.plot(list333,list111,'-',color="red",label="Rabbit Count")
plt.plot(list333,list222,'-',color="Blue",label="Wolf count")
plt.xlabel("Time elapsed")
plt.ylabel("Number of animals")
plt.legend(loc="upper right")

plt.figure(figsize=(10,5))
plt.title("Rabbit Against Wolf")
plt.plot(list111,list222,'-',color="red",label="Rabbit Count")
plt.xlabel("Rabbit Count")
plt.ylabel("Wolf Count")
plt.legend(loc="upper right")
```

Out[7]:

```
<matplotlib.legend.Legend at 0x7fce5133b910>
```




Rabbit Against Wolf
Rabbit Count
Wolf Count
Rabbit Count
0
200
400
600
800
1000
1200
1400

```
In [ ]:
```

```
In [ ]:
```