PSTAT 160B Homework 1

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Exercise 1

Design the sample space Ω, the state space $\mathbb{S}$ and explain why each X_k is a random variable. Why is $X = \{X_k\}_{k\in\mathbb{N}}$ a stochastic process?

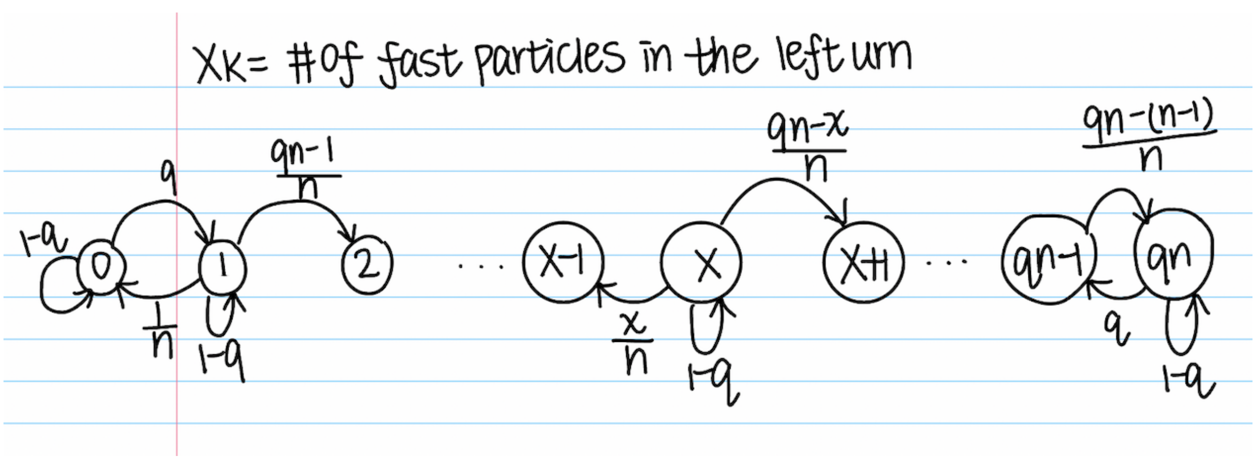
$$\text{The sample space: } \Omega = \{0,1\}^{qn}$$
$$\text{The state space: } \mathbb{S} = \{0,1,2,...,qn\}$$
$$\text{with } \omega \in \Omega, \quad X_k(\omega) = \sum_{i=1}^{qn} \omega_i$$

X_k denotes the number of fast particles in the left urn at time step k, thus it is a random variable as it is a measurable function $X_k : \Omega \to \mathbb{S}$ from the sample space Ω to the state space $\mathbb{S}$. Since $X = \{X_k\}_{k\in\mathbb{N}}$ is a sequence of random variables indexed by k, it is a stochastic process.

Exercise 2

State the (possible) transitions of X for any state $x \in \mathbb{S}$ and draw a transition diagram. For each transition describe what even occurs.

$$x \to \begin{cases} x-1 & \text{w.p } \frac{x}{n} \\ x & \text{w.p } 1-q \\ x+1 & \text{w.p } \frac{qn-x}{n} = q - \frac{x}{n} \end{cases}$$



The probability of moving from state x to $x-1$ is $\frac{x}{n}$, which is the probability of selecting a fast particle from the left urn and moving it to the right urn. The probability of moving from state x to x is $1-q$, which is the probability of selecting a slow particle from either urn. The probability of moving from state x to $x+1$ is $\frac{x}{n}$, which is the probability of selecting a fast particle from the right urn and moving it to the left urn.

Exercise 3

Calculate the transition probabilities for $X = \{X_k\}_{k\in\mathbb{N}}$ and explain why X is a Markov chain on the state space $\mathbb{S}$.

$$\forall y \in \mathbb{S},\ P(X_{k+1} = y|X_k = x) = p(x, y) = \begin{cases} \frac{x}{n} & y = x-1 \\ 1-q & y = x \\ \frac{qn-x}{n} & y = x+1 \end{cases}$$

Thus satisfying the Markov property which states:

$$P(X_{k+1} = y|X_0, X_1, ..., X_k) = P(X_{k+1} = y|X_k)$$

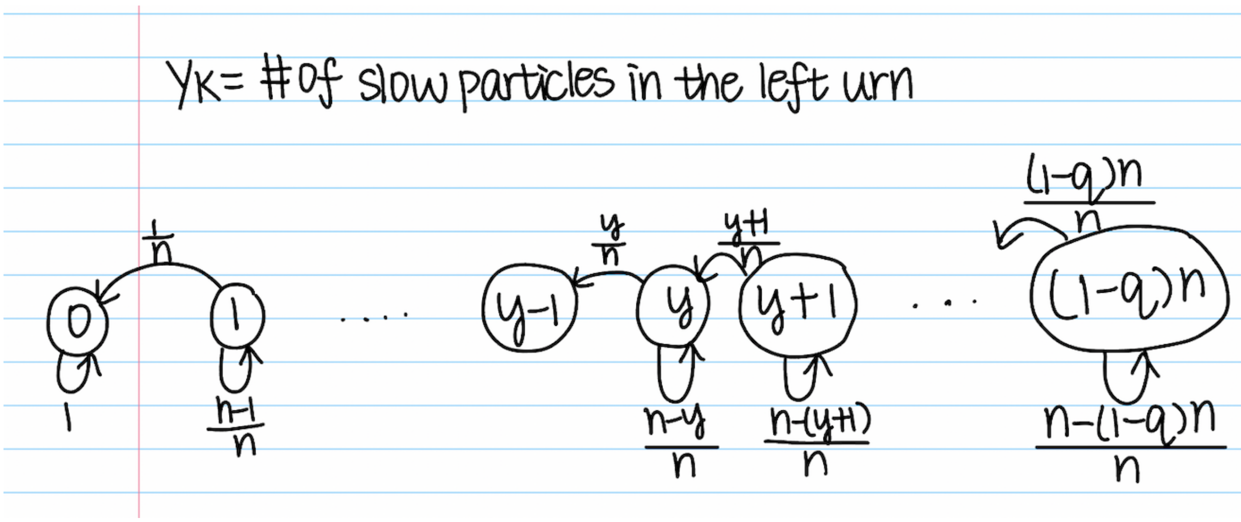
for any state $y \in \mathbb{S}$ and any time $k \geq 0$. These probabilities depend only on x and constants, in this case nq, the number of fast particles. Therefore X is a Markov chain on the state space $\mathbb{S}$.

Exercise 4

Let Y_k denote the number of slow particles present in left urn at time k. Write down the state space $\mathbb{S}$ for the process $Y = \{Y_k\}_{k\in\mathbb{N}}$, state the (possible) transitions of Y for any state $y \in \mathbb{S}$, draw the transition diagram, calculate the transition probabilities and show that Y is a Markov chain.

The sample space: $\Omega = \{0, 1\}^{(1-q)n}$
The state space: $\mathbb{S} = \{0, 1, 2, ..., (1-q)n\}$
with $\omega \in \Omega$, $\quad Y_k(\omega) = \sum_{i=1}^{(1-q)n} \omega_i$

Y_k denotes the number of slow particles in the left urn at time step k, thus it is a random variable as it is a measurable function $Y_k : \Omega \to \mathbb{S}$ from the sample space Ω to the state space $\mathbb{S}$. Since $Y = \{Y_k\}_{k\in\mathbb{N}}$ is a sequence of random variables indexed by k, it is a stochastic process.



$$y \to \begin{cases} y-1 & \text{w.p } \frac{y}{n} \\ y & \text{w.p } \frac{n-y}{n} = 1 - \frac{y}{n} \end{cases}$$

or

$$\forall x \in \mathbb{S},\ P(Y_{k+1} = x | Y_k = y) = p(y, x) = \begin{cases} \frac{y}{n} & x = y-1 \\ \frac{n-y}{n} & x = y \end{cases}$$

Thus satisfying the Markov property which states:

$$P(Y_{k+1} = x | Y_0, Y_1, ..., Y_k) = P(Y_{k+1} = x | Y_k)$$

for any state $x \in \mathbb{S}$ and any time $k \geq 0$. These probabilities depend only on x and constants, in this case nq, the number of fast particles. Therefore Y is a Markov chain on the state space $\mathbb{S}$.

Exercise 5

Let L_k be the total number of particles at time k in the left urn. Is the process $L = \{L_k\}_{k \in \mathbb{N}}$ a Markov chain? Justify your answer in detail.

A Markov chain is defined as a sequence of random variables constructed such that the probability distribution of each variable depends only on the value of the previous variable and constants. In this case, the transition probabilities for L depend on both more than just the previous iteration of the variable, it also depends on the distribution of fast particles within L. In other words, for $x \in \mathbb{S}$, each transition $x \to y$ cannot be assigned a probability in terms of x and any constants and is therefore not a Markov chain.

Exercise 6

Compute $E(X_k)$ for $X_0 = z$ for a fixed $0 \leq z \leq qn$. Compute the limit $\lim_{k\to\infty} E(X_k)$ and discuss the intuition behind the resulting expression for the case $q \in (0,1)$ and the two corner cases $q \in \{0,1\}$.

$$E(X_{k+1}) = E[E(X_{k+1}|X_k)]$$

$$E(X_{k+1}|X_k) = (X_k - 1)(\frac{X_k}{n}) + X_k(1-q) + (X_k+1)(q - \frac{X_k}{n})$$

$$= q + X_k - X_k\frac{2}{n}$$

$$= q + X_k(1 - \frac{2}{n})$$

$$E[E(X_{k+1}|X_k)] = E(X_{k+1}) = q + (1 - \frac{2}{n})E(X_k)$$

$$\implies \mu_{k+1} = q + (1 - \frac{2}{n})\mu_k$$

Let $a = q, b = (1 - \frac{2}{n})$, so:

$$\mu_x = a + b\mu_{x-1}$$

$$= a + b(a + b\mu_{x-2})$$

$$= a + ab + b^2\mu_{x-2}$$

$$= a + ab + b^2(a + b\mu_{x-3})$$

$$= a + ab + ab^2 + b^3\mu_{x-3}$$

$$= a(1 + b + b^2 + ... + b^{k-1}) + b^k z =, \quad \mu_0 = z = E(X_0)$$

This can be represented as a geometric series with $a = q, b = (1 - \frac{2}{n})$ s.t

$$a(1 + b + b^2 + ... + b^{k-1}) + b^k z = \frac{a(1-b^k)}{1-b} + b^k z$$

$$= \frac{q(1-(1-\frac{2}{n})^k)}{1-(1-\frac{2}{n})} + (1-\frac{2}{n})^k z$$

$$= q\frac{(1-(1-\frac{2}{n})^k)}{\frac{2}{n}} + (1-\frac{2}{n})^k z$$

$$= nq\frac{(1-(1-\frac{2}{n})^k)}{2} + (1-\frac{2}{n})^k z$$

$$= \frac{nq * [1-(1-\frac{2}{n})^k]}{2} + \frac{2(1-\frac{2}{n})^k z}{2}$$

$$= \frac{nq - nq(1-\frac{2}{n})^k + 2z(1-\frac{2}{n})^k}{2}$$

$$= \frac{(1-\frac{2}{n})^k(2z-nq)}{2} + \frac{nq}{2}$$

$$\implies E(X_k) = \frac{(1-\frac{2}{n})^k(2z-nq)}{2} + \frac{nq}{2} \quad \text{for } x_0 = z$$

So, as $k \to \infty$, $(1-\frac{2}{n})^k \to 0$, thus $\frac{(1-\frac{2}{n})^k(2z-nq)}{2} \to 0$ as well.

Hence, $\lim_{k\to\infty} E(X_k) = \frac{nq}{2}$ for $q \in (0,1)$

When $q = 0$, there are no fast particles as $qn = 0$. So, there are n slow particles. Since there are only slow particles, as $k \to \infty$, the expected number of fast particles in the left urn is 0. Or,

$$\lim_{k\to\infty} E(X_k) = 0 \quad \text{for } q = 0$$

When $q = 1$, there will be n fast particles, so this becomes the traditional Ehrenfest urn problem and the particles will be evenly distributed in the 2 urns as $k \to \infty$. Or,

$$\lim_{k\to\infty} E(X_k) = \frac{n}{2} \quad \text{for } q = 1$$

Exercise 7

Let L_k be the total number of particles at time k in the left urn. Construct the sample/state spaces Ω and $\mathbb{S}$ for $(X, L) = \{(X_k, L_k)\}_{k\in\mathbb{N}}$ Show that (X, L) is a Markov chain by finding its transition probabilities.

The sample space: $\Omega = \{(\Omega_x, \Omega_L) : \Omega_x \in (0,1)^{qn}, \Omega_L \in (0,1)^n\}$
The state space: $\mathbb{S} = \{(x,l) : x \in S_k, l \in S_L, x \le l\}$
with $\omega \in \Omega$, $\quad X_k(\omega) = \sum_{i=1}^{qn} \omega_i$

$$(x,l) \to \begin{cases} (x-1, l-1) & \text{w.p } \frac{x}{n} \\ (x, l-1) & \text{w.p } \frac{l-x}{n} \\ (x,l) & \text{w.p } \frac{n-l}{n} \\ (x+1, l+1) & \text{w.p } \frac{qn-x}{n} \end{cases}$$

Thus satisfying the Markov property which states:

$$P(L_{k+1} = (x,l)|L_0, L_1, ..., L_k) = P(L_{k+1} = (x,l)|L_k)$$

for any state $(x,l) \in \mathbb{S}$ and any time $k \geq 0$. These probabilities depend only on (x,l) and constants, in this case q and n. Therefore (X, L) is a Markov chain on the state space $\mathbb{S}$.

Exercise 8

Compute the stationary distribution of $X = \{X_k\}_{k\in\mathbb{N}}$.

$$\text{Our guess: } \pi(x) = \frac{\binom{qn}{x}}{2^{qn}}$$

We verify with the local balance equation:

$$\mu(x)p(x,y) = \mu(y)p(y,x)$$

with $p(x, x+1) = q - \frac{x}{n}$ and $p(x+1, x) = \frac{x+1}{n}$. Thus,

$$\frac{\binom{qn}{x+1}}{2^{qn}}\frac{x+1}{n} = \frac{\binom{qn}{x}}{2^{qn}}(q - \frac{x}{n})$$

$$\implies \frac{qn!}{(x+1)!(qn-x-1)!}\frac{x+1}{n} = (q - \frac{x}{n})\frac{qn!}{x!(qn-x)!}$$

$$\implies \frac{1}{n} = (q - \frac{x}{n})\frac{1}{qn-x}$$

$$\implies \frac{1}{n} = (\frac{qn-x}{n})\frac{1}{qn-x}$$

$$\implies \frac{1}{n} = \frac{1}{n}$$

$$\implies 1 = 1 \checkmark$$

Thus our guess for the stationary distribution satisfies the local balance equation, therefore it is a stationary measure of X. We must now normalize to the stationary distribution.

$$\text{Lemma: } \sum_0^n \binom{n}{i} = 2^n$$

$$\text{Lemma: } \implies \frac{\sum_0^n \binom{nq}{x}}{2^{nq}} = \frac{2^{nq}}{2^{nq}} = 1$$

Thus

$$\pi(x) = \frac{\binom{qn}{x}}{2^{qn}}$$

is already normalized, and is the true stationary distribution.

Exercise 9

Compute $E(L_k)$ with L_k as in Exercise 7. Find the stationary distribution of the Markov chain (X, L) (i.e. $\pi(x, l)$ for each state (x, l)).

$$E(L_k) = E[E(L_k|L_{k-1}), X_{k-1}]$$

By Exercise 7,

$$E(L_{k+1}|L_k, X_k) = \frac{qn - X_k}{n}(L_k+1) + \frac{(n - L_k) - (qn - X_k)}{n}L_k + \frac{L_k - X_k}{n}(L_k-1) = q - \frac{X_k}{n} + (1 - \frac{1}{n})L_k$$

so,

$$E(L_k) = E[E(L_k|L_{k-1}), X_{k-1}] = q - \frac{E(X_{k-1})}{n} + (1 - \frac{1}{n})E(L_{k-1})$$

$$= (q - \frac{E(X_{k-1})}{n}) + (1 - \frac{1}{n})(q - \frac{E(X_{k-2})}{n}) + (1 - \frac{1}{n}^2)E(L_{k-2})$$

$$= (1 - \frac{1}{n})^k E(L_0) + \sum_{m=0}^{k-1}(1 - \frac{1}{n})^m (q - \frac{E(X_{k-m-1})}{n})$$

$$= q\sum_{m=0}^{k-1}(1 - \frac{1}{n})^m - \sum_{m=0}^{k-1}(1 - \frac{1}{n})^m * \frac{E(X_{x-m-1})}{n}$$

By exercise 6,

$$E(X_{k-m-1}) = \frac{nq}{2} + (z - \frac{nq}{2})(1 - \frac{2}{n})^{k-m-1}$$

So,

$$\frac{E(X_{k-m-1})}{n} = \frac{q}{2} + \frac{(z - \frac{nq}{2})(1 - \frac{2}{n})^{k-m-1}}{n}$$

Therefore,

$$\sum_{m=0}^{k-1}(1 - \frac{1}{n})^m (q - \frac{E(X_{k-m-1})}{n})$$

$$= \frac{q}{2}\sum_{m=0}^{k-1}(1 - \frac{1}{n})^m - \frac{z - \frac{nq}{2}}{n}\sum_{m=0}^{k-1}(1 - \frac{1}{n})^m(1 - \frac{2}{n})^{k-m-1}$$

Finally we have,

$$E(L_k) = E[E(L_k|L_{k-1}), X_{k-1}] = \frac{qn}{2} - (\frac{qn}{2}(1 - \frac{1}{n})^k)E(L_0) - \frac{2z - nq}{2}(n(1 - \frac{1}{n})^k - (1 - \frac{2}{n})^k)$$

Stationary distribution of the Markov chain (X, L)

$$\text{Our guess: } \pi(x) = \frac{\binom{qn}{l_k}}{2^{qn}} \text{when } L_k = X_k$$

$$P(L, L+1) = \frac{qn - x}{n}$$

$$P(L+1,L) = \frac{x}{n} + \frac{l+1-x}{n} = \frac{l+1}{n}$$
$$P(L+1,L)\pi(x) = P(L,L+1)\pi(x+1)$$
$$\frac{l+1}{n}\tbinom{qn}{l_k+1} = \frac{qn-x}{n}\tbinom{qn}{l_k}$$
$$\implies \frac{l+1}{n}\frac{qn!}{(l_k+1)!(qn-l_k-1)!} = \frac{qn-x}{n}\frac{qn!}{l_k!(qn-l_k)!}$$
$$\implies \frac{l+1}{n}\frac{1}{l_k+1} = \frac{qn-x}{n}\frac{1}{qn-l_k}$$
$$\implies 1 = 1 \text{ when } L_k = X_k$$

Thus our guess for the stationary distribution satisfies the local balance equation, therefore

$$\pi(x) = \frac{\binom{qn}{l_k}}{2^{qn}}$$

is the true stationary distribution.When $X_k \neq L_k$ there is no stationary distribution.But eventually X_k will equal to L_k and it will have the stationary distribution above.

Exercise 10

Implement our modified Ehrenfest urn in either R or Python. Fix $n = 100$ particles total with any q which makes qn an integer. Your code should simulate the process (X, Y) where X_k and Y_k record the number of fast and slow particles in the left urn at time k. Compute the entropy

$$S_k = \log\binom{qn}{X_k} + \log\binom{(1-q)n)}{Y_k}$$

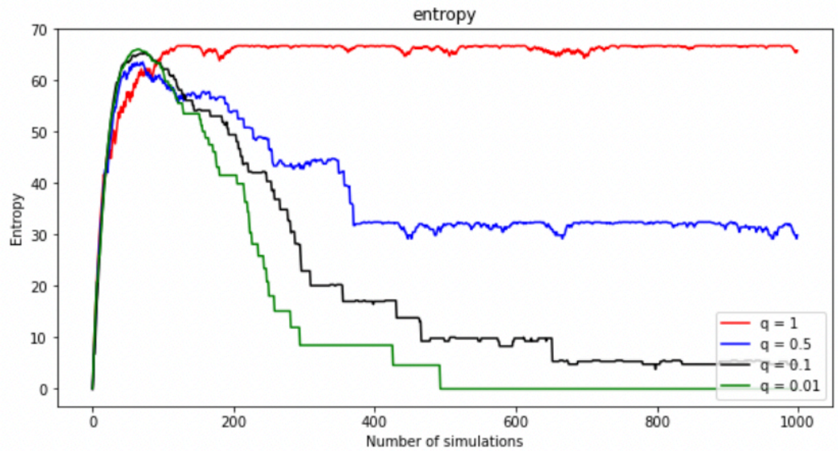
Simulate a path of S over 10^3 time steps and plot it (S on the y-axis and times $k = 0, 1, ...10^3$ on the x-axis. Produce plots for three more values of $q : 0.5, 0.1, 0.01$. Explain the behaviour of the entropy that you see for these four values of q in term of the definition of entropy (i.e. an increasing entropy corresponds to an increasing disorder in the system). You can start the system in the state in which all particles in the left urn.

In [1]:
```
import numpy as np
import math
import matplotlib.pyplot as plt
def urn(probQ,n):
    leftFast = int(probQ*n)
    leftSlow = int((1-probQ) * n)
    rightFast = 0
    rightSlow = 0
    samples = np.random.random_sample(1000)
    entropy = []
    for x in samples:
        entropy.append(math.log(math.comb(int(n*probQ), leftFast) ) +
        if x <= leftSlow/n:
            leftSlow = leftSlow - 1
            rightSlow = rightSlow + 1
        elif x <= (leftSlow/n + leftFast/n):
            leftFast = leftFast - 1
            rightFast = rightFast +1
        elif x <= (leftSlow/n + leftFast/n + rightFast/n):
            leftFast = leftFast + 1
            rightFast = rightFast - 1
    return entropy
```

In [2]:
```
plt.figure(figsize=(10,5))
plt.title("entropy")
ent1 = urn(1,100)
ent2 = urn(0.5, 100)
ent3 = urn(0.1, 100)
ent4 = urn(0.01, 100)
plt.plot(np.arange(1000),ent1,'-',color="red",label="q = 1")
plt.plot(np.arange(1000),ent2,'-',color="blue",label="q = 0.5")
plt.plot(np.arange(1000),ent3,'-',color="black",label="q = 0.1")
plt.plot(np.arange(1000),ent4,'-',color="green",label="q = 0.01")

plt.xlabel("Number of simulations")
plt.ylabel("Entropy")
plt.legend(loc="lower right")
```

Out[2]: <matplotlib.legend.Legend at 0x7fbc62c59910>



In [3]:
```
probQ = 0.25
n=100
entropy = []
leftFast = 12
leftSlow = 40
temp = math.log(math.comb(int(n*probQ), leftFast) ) + math.log(math.cc
print(temp)
```

```
64.89781376547391
```

In [ ]:

Explanation of results: These graphs make sense to us as entropy is a measure of how "random" or spread out the particles in the urn are. Because this implementation of the Ehrenfest urn has a subset of the particles with an absorbing state on the right side, it makes sense that as you lower q, you raise the amount of particles in this subset, which ultimately lowers the overall entropy of the system after a while as they stay absorbed in one side and are no longer random. However, in the beginning part of the graph, it makes sense that they all peak around the same entropy, as the number of slow particles on the right will always deteriorate, peaking at the point where the combination of $(1-q)n$ choose left slow is maximized.