Prediction Mean Square Error Calculation based on PMSE Improvement

Deduction of PMSE in Predictive Mean Square Error and Stochastic Regressor Variables by SUBHASH C. NARULA

The response variable and the predictor variables follow a joint (k+ 1)-variate normal distribution with unknown mean vector $\mu^* = [\mu_0, \boldsymbol{\mu}']'$, and unknown covariance matrix $\boldsymbol{\Sigma^*} = \begin{bmatrix} \sigma_{00} & \boldsymbol{\sigma}' \\ \boldsymbol{\sigma} & \boldsymbol{\Sigma} \end{bmatrix}$. Let $\boldsymbol{z_1, z_2, ..., z_n}$ be n independent (k-component vector) observations on the predictor variables, $\boldsymbol{x_i = z_i - \bar{z}}$. Let $\boldsymbol{S^*} = \begin{bmatrix} s_{00} & \boldsymbol{s}' \\ \boldsymbol{s} & \boldsymbol{S} \end{bmatrix}$ be sample covariance matrix, where

$$s_{00} = \sum \frac{(y_i - \bar{y})^2}{n-1}, \boldsymbol{s} = \sum \frac{(y_i - \bar{y})\boldsymbol{x_i}}{n-1}, \boldsymbol{S} = \sum \frac{\boldsymbol{x_i x_i'}}{n-1}$$

We assume the correct the model(1.1)

$$\boldsymbol{y = \alpha + \beta_1 z_1 + \beta_2 z_2 + ... + \beta_k z_k + \epsilon}$$

The LSE prediction equation

$$\boldsymbol{\hat{y}} = \bar{y} + \boldsymbol{\hat{\beta}_1}(\boldsymbol{z_1} - \bar{z}_1) + \boldsymbol{\hat{\beta}_2}(\boldsymbol{z_2} - \bar{z}_2) + ... + \boldsymbol{\hat{\beta}_k}(\boldsymbol{z_k} - \bar{z}_k) = \bar{y} + \boldsymbol{X'\hat{\beta}}$$

$$\hat{y}_i = \bar{y} + \boldsymbol{x_i'\hat{\beta}}$$

For any given $\boldsymbol{z_i}$,

$$\begin{aligned} E(y_i|\boldsymbol{z_i}) &= \alpha + \boldsymbol{\beta z_i} \\ &= \mu_0 - \boldsymbol{\sigma' \Sigma^{-1} \mu} + \boldsymbol{\sigma' \Sigma^{-1} z_i} \\ &= \mu_0 + \boldsymbol{\sigma' \Sigma^{-1} (z_i - \mu)} \end{aligned}$$

where $\alpha = \mu_0 - \boldsymbol{\sigma' \Sigma^{-1} \mu}, \boldsymbol{\beta = \sigma' \Sigma^{-1}}$. Thus $\hat{\alpha} = \bar{y} - \boldsymbol{s' S^{-1} \bar{x}}, \boldsymbol{\hat{\beta} = S^{-1} s}$.

The conditional predictive mean square error by

$$
\begin{aligned}
E[(y_0-\hat{y_0})^2|\boldsymbol{z_0}] &= E[(\alpha+\boldsymbol{(z_0-\mu)'\beta}+\epsilon_0-\bar{y}-\boldsymbol{x_0'\hat{\beta}}|\boldsymbol{z_0})^2] \\
&= E[(\alpha+\boldsymbol{(z_0-\mu)'\beta+\epsilon_0}-\alpha-\boldsymbol{(\bar{z}-\mu)'\beta-\bar{\epsilon}-x_0'}\beta|\boldsymbol{z_0})^2] \\
&= E[(\boldsymbol{x_0'\beta}-\boldsymbol{\bar{x}'\beta}+\epsilon_0-\bar{\epsilon}-\boldsymbol{x_0'\hat{\beta}}|\boldsymbol{z_0})^2] \\
&= E[(\boldsymbol{x_0'\beta}+(\epsilon_0-\bar{\epsilon})-\boldsymbol{x_0'\hat{\beta}}|\boldsymbol{z_0})^2] \\
&= E[(\boldsymbol{x_0'\beta}+(\epsilon_0-\bar{\epsilon}))^2+(\boldsymbol{x_0'\hat{\beta}})^2-2(\boldsymbol{x_0'\beta}+(\epsilon_0-\bar{\epsilon}))\boldsymbol{x_0'\hat{\beta}}|\boldsymbol{z_0}] \\
&= E[(\boldsymbol{x_0'\beta})^2+(\epsilon_0-\bar{\epsilon})^2+(\boldsymbol{x_0'\hat{\beta}})^2-2\boldsymbol{x_0'\beta x_0'\hat{\beta}}|\boldsymbol{z_0}] \\
&= \boldsymbol{\beta'}E(\boldsymbol{x_0x_0'}|\boldsymbol{z_0})\boldsymbol{\beta}+E[(\epsilon_0-\bar{\epsilon})^2|\boldsymbol{z_0}]+E[(\boldsymbol{x_0'\hat{\beta}})^2|\boldsymbol{z_0}]-2\boldsymbol{\beta'}E[\boldsymbol{x_0x_0'\hat{\beta}}|\boldsymbol{z_0}]
\end{aligned}
$$

By Lemma A1, Lemma A3, Lemma A7

$$
\begin{aligned}
&E(\boldsymbol{\tilde{\beta}_1}|\boldsymbol{X_1}) = \boldsymbol{\beta_1}+\boldsymbol{\Sigma_{11}^{-1}\Sigma_{12}\beta_2}=\boldsymbol{\Phi_1} \\
&E(\boldsymbol{x_{01}x_{01}'}|\boldsymbol{z_0}) = (\boldsymbol{z_{01}}-\boldsymbol{\mu_1})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'+\frac{\boldsymbol{\Sigma_{11}}}{n} \\
&E(\boldsymbol{x_0x_0'}|\boldsymbol{z_0}) = (\boldsymbol{z_0-\mu})(\boldsymbol{z_0}-\boldsymbol{\mu})'+\frac{\boldsymbol{\Sigma}}{n} \\
&E[(\boldsymbol{x_{01}'\tilde{\beta}_1})^2|\boldsymbol{z_0}] = \sigma_p^2\left[\boldsymbol{(z_{01}-\mu_1)'\Sigma_{11}^{-1}(z_{01}-\mu_1)}+\frac{p}{n}\right]\frac{1}{n-p-2} \\
&\qquad\qquad + \boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}\frac{1}{n}+\boldsymbol{\Phi_1'(z_{01}-\mu_1)(z_{01}-\mu_1)'\Phi_1} \\
&\boldsymbol{\sigma'\Sigma^{-1}}\begin{bmatrix}\boldsymbol{\Sigma_{11}} \\ \boldsymbol{\Sigma_{21}}\end{bmatrix} = \sigma_1'
\end{aligned}
$$

The conditional PMSE can be written as

$$
\begin{aligned}
E[(y_0-\hat{y_0})^2|\boldsymbol{z_0}] &= \boldsymbol{\beta'}E(\boldsymbol{x_0x_0'}|\boldsymbol{z_0})\boldsymbol{\beta}+E[(\epsilon_0-\bar{\epsilon})^2|\boldsymbol{z_0}]+E[(\boldsymbol{x_0'\hat{\beta}})^2|\boldsymbol{z_0}]-2\boldsymbol{\beta'}E[\boldsymbol{x_0x_0'\hat{\beta}}|\boldsymbol{z_0}] \\
&= \boldsymbol{\beta'}\left[\boldsymbol{(z_0-\mu)(z_0-\mu)'}+\frac{\boldsymbol{\Sigma}}{n}\right]\boldsymbol{\beta}+\sigma_k^2\left(1+\frac{1}{n}\right) \\
&\quad+\sigma_k^2\left[\boldsymbol{(z_0-\mu)'\Sigma^{-1}(z_0-\mu)}+\frac{k}{n}\right]\frac{1}{n-k-2} \\
&\quad+\frac{1}{n}\boldsymbol{\Phi'\Sigma\Phi}+\boldsymbol{\Phi'(z_0-\mu)(z_0-\mu)'\Phi}-2\boldsymbol{\beta'}\left[\boldsymbol{(z_0-\mu)(z_0-\mu)'}+\frac{\boldsymbol{\Sigma}}{n}\right]\boldsymbol{\beta} \\
&= \sigma_k^2\left(1+\frac{1}{n}\right)+\sigma_k^2\left[\boldsymbol{(z_0-\mu)'\Sigma^{-1}(z_0-\mu)}+\frac{k}{n}\right]\frac{1}{n-k-2}
\end{aligned}
$$

Since $\boldsymbol{\Phi}$ is the notation of expectation of $\boldsymbol{\tilde{\beta}_1}$, when we are using all predictors, the LSE is unbiased, which means $\boldsymbol{\Phi}=\boldsymbol{\beta}$.

The unconditional PMSE

$$\begin{aligned}
E[(y_0-\hat{y_0})^2] &= E\{E[(y_0-\hat{y_0})^2|\boldsymbol{z_0}]\} \\
&= E\left[\sigma_k^2\left(1+\frac{1}{n}\right)+\sigma_k^2\left[\boldsymbol{(z_0-\mu)'\Sigma^{-1}(z_0-\mu)}+\frac{k}{n}\right]\frac{1}{n-k-2}\right] \\
&= \sigma_k^2\left(1+\frac{1}{n}\right)+\sigma_k^2 E\left[\boldsymbol{(z_0-\mu)'\Sigma^{-1}(z_0-\mu)}+\frac{k}{n}\right]\frac{1}{n-k-2} \\
&= \sigma_k^2\left(1+\frac{1}{n}\right)+\sigma_k^2\left(k+\frac{k}{n}\right)\frac{1}{n-k-2} \\
&= \sigma_k^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n-k-2}\right) \\
&= \sigma_k^2\left(1+\frac{1}{n}\right)\left(\frac{n-2}{n-k-2}\right)
\end{aligned}$$

For subset, we partition the k-component vector of predictor variables into two parts, $\boldsymbol{Z} = [\boldsymbol{Z_1}, \boldsymbol{Z_2}], \boldsymbol{X} = [\boldsymbol{X_1}, \boldsymbol{X_2}], \boldsymbol{x_1'} = [\boldsymbol{x_{i1}'}, \boldsymbol{x_{i2}'}], \boldsymbol{\mu'} = [\boldsymbol{\mu_1'}, \boldsymbol{\mu_2'}], \boldsymbol{\sigma'} = [\boldsymbol{\sigma_1'}, \boldsymbol{\sigma_2'}], \boldsymbol{s'} = [\boldsymbol{s_1'}, \boldsymbol{s_2'}]$, $\boldsymbol{\Sigma} = \begin{bmatrix}\boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}, \boldsymbol{S} = \begin{bmatrix}\boldsymbol{S_{11}} & \boldsymbol{S_{12}} \\ \boldsymbol{S_{21}} & \boldsymbol{S_{22}}\end{bmatrix}$
, so the subset prediction equation is given by

$$\tilde{y_i} = \bar{y} + \boldsymbol{x_{i1}'}\boldsymbol{\tilde{\beta_1}}$$

where $\boldsymbol{\tilde{\beta_1}} = \boldsymbol{S_{11}^{-1}s_1}$. By LemmaA1, LemmaA3, LemmaA7, $\boldsymbol{\Phi_1} = \boldsymbol{\beta_1} + \boldsymbol{\Sigma_{11}^{-1}\Sigma_{12}\beta_2}$, the conditional PMSE at z_0 is given by

$$\begin{aligned}
E[(y_0-\tilde{y_0})^2|\boldsymbol{z_0}] &= E[(\boldsymbol{x_0'\beta}+\epsilon_0-\bar{\epsilon}-\boldsymbol{x_{01}'\tilde{\beta_1}}|\boldsymbol{z_0})^2] \\
&= E[(\boldsymbol{x_0'\beta})^2+(\boldsymbol{x_{01}'\tilde{\beta_1}})^2-2\boldsymbol{x_0'\beta x_{01}'\tilde{\beta_1}}+(\epsilon_0-\bar{\epsilon})^2+2(\boldsymbol{x_0'\beta}-\boldsymbol{x_{01}'\tilde{\beta_1}})(\epsilon_0-\bar{\epsilon})|\boldsymbol{z_0}] \\
&= E[(\boldsymbol{x_0'\beta})^2|\boldsymbol{z_0}]+E[\boldsymbol{x_{01}'\tilde{\beta_1}})^2|\boldsymbol{z_0}]-2E[\boldsymbol{x_0'\beta x_{01}'\tilde{\beta_1}}|\boldsymbol{z_0}]+E[(\epsilon_0-\bar{\epsilon})^2|\boldsymbol{z_0}] \\
&\quad +2E[(\boldsymbol{x_0'\beta}-\boldsymbol{x_{01}'\tilde{\beta_1}})(\epsilon_0-\bar{\epsilon})|\boldsymbol{z_0}]
\end{aligned}$$

where

$$E[(\boldsymbol{x_0'\beta})^2|\boldsymbol{z_0}] = \boldsymbol{\beta}\left[\boldsymbol{(z_0-\mu)(z_0-\mu)'}+\frac{\boldsymbol{\Sigma}}{n}\right]\boldsymbol{\beta}$$

$$\begin{aligned}
E[(\boldsymbol{x_{01}'\tilde{\beta_1}})^2|\boldsymbol{z_0}] =& \sigma_p^2\left[\boldsymbol{(z_{01}-\mu_1)'\Sigma_{11}^{-1}(z_{01}-\mu_1)}+\frac{p}{n}\right]\frac{1}{n-p-2} \\
&+\boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}\frac{1}{n}+\boldsymbol{\Phi_1'(z_{01}-\mu_1)(z_{01}-\mu_1)'\Phi_1},
\end{aligned}$$

$$E[(\epsilon_0-\bar{\epsilon})^2|\boldsymbol{z_0}] = \sigma_k^2+\frac{1}{n}\sigma_k^2,$$

$$E[(\boldsymbol{x_0'\beta}-\boldsymbol{x_{01}'\tilde{\beta_1}})(\epsilon_0-\bar{\epsilon})|\boldsymbol{z_0}] = E[(\boldsymbol{x_0'\beta}-\boldsymbol{x_{01}'\tilde{\beta_1}})\epsilon_0|\boldsymbol{z_0}]-E[(\boldsymbol{x_0'\beta}-\boldsymbol{x_{01}'\tilde{\beta_1}})\bar{\epsilon}|\boldsymbol{z_0}] = 0$$

(but $\bar{\epsilon} = 0$ is not necessary)

Equation 3.6 could be written as

$$
\begin{aligned}
&= \sigma_p^2 \left[(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Sigma_{11}^{-1}} (\boldsymbol{z_{01}} - \boldsymbol{\mu_1}) + \frac{p}{n} \right] \frac{1}{n-p-2} + \boldsymbol{\Phi_1' \Sigma_{11} \Phi_1} \frac{1}{n} + \boldsymbol{\Phi_1'} (\boldsymbol{z_{01}} - \boldsymbol{\mu_1})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Phi_1} \\
&\quad + \boldsymbol{\beta} (\boldsymbol{z_0} - \boldsymbol{\mu})(\boldsymbol{z_0} - \boldsymbol{\mu})' \boldsymbol{\beta} + \boldsymbol{\beta' \Sigma \beta} \frac{1}{n} - 2\boldsymbol{\beta'} E(\boldsymbol{x_0 x_{01}'} | \boldsymbol{z_0}) \boldsymbol{\Phi_1} + \sigma_k^2 + \frac{1}{n} \sigma_k^2 \\
&= \sigma_k^2 + \frac{1}{n} (\sigma_p^2 + \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} - \boldsymbol{\sigma' \Sigma^{-1} \sigma}) + [(\boldsymbol{z_0} - \boldsymbol{\mu})' \beta]^2 + [(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Phi_1}]^2 \\
&\quad + \sigma_p^2 \left[(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Sigma_{11}^{-1}} (\boldsymbol{z_{01}} - \boldsymbol{\mu_1}) + \frac{p}{n} \right] \frac{1}{n-p-2} \\
&\quad + \boldsymbol{\beta' \Sigma \beta} \frac{1}{n} + \boldsymbol{\Phi_1' \Sigma_{11} \Phi_1} \frac{1}{n} - 2\boldsymbol{\beta'} E(\boldsymbol{x_0 x_{01}'} | \boldsymbol{z_0}) \boldsymbol{\Phi_1} \\
&= \sigma_k^2 + \frac{1}{n} \sigma_p^2 + \sigma_p^2 \left[(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Sigma_{11}^{-1}} (\boldsymbol{z_{01}} - \boldsymbol{\mu_1}) + \frac{p}{n} \right] \frac{1}{n-p-2} + [(\boldsymbol{z_0} - \boldsymbol{\mu})' \beta]^2 + [(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Phi_1}]^2 \\
&\quad + \frac{1}{n} \boldsymbol{\beta' \Sigma \beta} + \frac{1}{n} \boldsymbol{\Phi_1' \Sigma_{11} \Phi_1} + \frac{1}{n} \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} - \frac{1}{n} \boldsymbol{\sigma' \Sigma^{-1} \sigma} - 2\boldsymbol{\beta'} E(\boldsymbol{x_0 x_{01}'} | \boldsymbol{z_0}) \boldsymbol{\Phi_1}
\end{aligned}
$$

where

$$
E(\boldsymbol{x_0 x_{01}'} | \boldsymbol{z_0}) = E \left[\begin{bmatrix} \boldsymbol{x_{01} x_{01}'} \\ \boldsymbol{x_{02} x_{01}'} \end{bmatrix} | \boldsymbol{z_0} \right] = \begin{bmatrix} (\boldsymbol{z_{01}} - \boldsymbol{\mu_1})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' + \boldsymbol{\Sigma_{11}/n} \\ (\boldsymbol{z_{02}} - \boldsymbol{\mu_2})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' + \boldsymbol{\Sigma_{21}/n} \end{bmatrix}
$$

$$
\boldsymbol{\sigma' \Sigma^{-1} \sigma} = \boldsymbol{\beta' \Sigma \beta}, \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} = \boldsymbol{\beta_1' \Sigma_{11} \beta_1}
$$

so

$$
\begin{aligned}
\boldsymbol{\beta'} E(\boldsymbol{x_0 x_{01}'} | \boldsymbol{z_0}) \boldsymbol{\Phi_1} &= [\boldsymbol{\beta_1'} (\boldsymbol{z_{01}} - \boldsymbol{\mu_1})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' + \boldsymbol{\beta_1' \Sigma_{11}}/n + \boldsymbol{\beta_2'} (\boldsymbol{z_{02}} - \boldsymbol{\mu_2})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' + \boldsymbol{\beta_2' \Sigma_{21}}/n] \boldsymbol{\Phi_1} \\
&= \boldsymbol{\beta_1'} (\boldsymbol{z_{01}} - \boldsymbol{\mu_1})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Phi_1} + \boldsymbol{\beta_2'} (\boldsymbol{z_{02}} - \boldsymbol{\mu_2})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' \boldsymbol{\Phi_1} + \boldsymbol{\sigma_1' \Phi_1}/n
\end{aligned}
$$

In addition to a few terms appearing in 3.6a, other terms would be equal to

$$
\begin{aligned}
&\frac{1}{n} \boldsymbol{\beta' \Sigma \beta} + \frac{1}{n} \boldsymbol{\Phi_1' \Sigma_{11} \Phi_1} + \frac{1}{n} \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} - \frac{1}{n} \boldsymbol{\sigma' \Sigma^{-1} \sigma} - \frac{2}{n} \boldsymbol{\sigma_1' \Phi_1} \\
&= \frac{1}{n} \boldsymbol{\Phi_1' \Sigma_{11} \Phi_1} + \frac{1}{n} \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} - \frac{2}{n} \boldsymbol{\sigma_1' \Phi_1} \\
&= \frac{1}{n} \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} + \frac{1}{n} \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} - \frac{2}{n} \boldsymbol{\sigma_1' \Sigma_{11}^{-1} \sigma_1} = 0
\end{aligned}
$$

in which $\Phi_1 = \boldsymbol{\Sigma_{11}^{-1} \sigma_1}$

The conditional PMSE is equal to (3.6a)

$$
\begin{aligned}
E[(y_0-\tilde{y_0})^2|\boldsymbol{z_0}] &= \sigma_k^2+\frac{\sigma_p^2}{n}+\sigma_p^2\left[\boldsymbol{(z_{01}-\mu_1)'\Sigma_{11}^{-1}(z_{01}-\mu_1)'}+\frac{p}{n}\right]\frac{1}{n-p-2}\\
&\quad+[\boldsymbol{(z_0-\mu)'\beta-(z_{01}-\mu_1)'\Phi_1}]^2\\
&=\sigma_k^2+\frac{\sigma_p^2}{n}+\sigma_p^2\left[\boldsymbol{(z_{01}-\mu_1)'\Sigma_{11}^{-1}(z_{01}-\mu_1)'}+\frac{p}{n}\right]\frac{1}{n-p-2}\\
&\quad+[\boldsymbol{(z_{01}-\mu_1)'\beta_1+(z_{02}-\mu_2)'-(z_{01}-\mu_1)'(\beta_1+\Sigma_{11}^{-1}\Sigma_{12}\beta_2)}]^2\\
&=\sigma_k^2+\frac{\sigma_p^2}{n}+\sigma_p^2\left[\boldsymbol{(z_{01}-\mu_1)'\Sigma_{11}^{-1}(z_{01}-\mu_1)'}+\frac{p}{n}\right]\frac{1}{n-p-2}\\
&\quad+[\boldsymbol{(z_{02}-\mu_2)'-(z_{01}-\mu_1)'\Sigma_{11}^{-1}\Sigma_{12}\beta_2}]^2
\end{aligned}
$$

Take expectation

$$
\begin{aligned}
E[(y_0-\tilde{y_0})^2] &= E[E(y_0-\tilde{y_0})^2|\boldsymbol{z_0}]\\
&=E\{\sigma_k^2+\frac{\sigma_p^2}{n}+\sigma_p^2\left[\boldsymbol{(z_{01}-\mu_1)'\Sigma_{11}^{-1}(z_{01}-\mu_1)'}+\frac{p}{n}\right]\frac{1}{n-p-2}\\
&\quad+[\boldsymbol{(z_0-\mu)'\beta-(z_{01}-\mu_1)'\Phi_1}]^2\}\\
&=\sigma_p^2+\frac{\sigma_p^2}{n}+\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1-\sigma'\Sigma^{-1}\sigma}+\sigma_p^2\left(p+\frac{p}{n}\right)\frac{1}{n-p-2}\\
&\quad+E[\boldsymbol{\beta'(z_0-\mu)(z_0-\mu)'\beta+\Phi_1'(z_{01}-\mu_1)(z_{01}-\mu_1)'\Phi_1}\\
&\quad-2\boldsymbol{\beta'(z_0-\mu)(z_{01}-\mu_1)'\Phi_1}]
\end{aligned}
$$

The expectation term is equal to

$$
\boldsymbol{\beta'\Sigma\beta+\Phi_1'\Sigma_{11}\Phi_1-2\beta'}\begin{bmatrix}\boldsymbol{\Sigma_{11}}\\ \boldsymbol{\Sigma_{21}}\end{bmatrix}\boldsymbol{\Phi_1}
$$

The unconditional PMSE

$$
\begin{aligned}
E[(y_0-\tilde{y_0})^2] &= \sigma_p^2(1+\frac{1}{n})(n-2)/(n-p-2)\\
&\quad+\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1-\sigma'\Sigma^{-1}\sigma}+\boldsymbol{\beta'\Sigma\beta+\Phi_1'\Sigma_{11}\Phi_1-2\beta'}\begin{bmatrix}\boldsymbol{\Sigma_{11}}\\ \boldsymbol{\Sigma_{21}}\end{bmatrix}\boldsymbol{\Phi_1}\\
&=\sigma_p^2(1+\frac{1}{n})(n-2)/(n-p-2)\\
&\quad+\boldsymbol{\Phi_1'\Sigma_{11}\Phi_1-\beta'\Sigma\beta}+\boldsymbol{\beta'\Sigma\beta+\Phi_1'\Sigma_{11}\Phi_1-2\beta'}\begin{bmatrix}\boldsymbol{\Sigma_{11}}\\ \boldsymbol{\Sigma_{21}}\end{bmatrix}\boldsymbol{\Phi_1}\\
&=\sigma_p^2(1+\frac{1}{n})(n-2)/(n-p-2)
\end{aligned}
$$

Thus, the unconditional PMSE $= \sigma_p^2(1+\frac{1}{n})(n-2)/(n-p-2)$