Deduction of PMSE in Predictive Mean Square Error and Stochastic Regressor Variables by SUBHASH C. NARULA

The response variable and the predictor variables follow a joint (k+ 1)-variate normal distribution with unknown mean vector $\mu^* = [\mu_0, \boldsymbol{\mu}']'$, and unknown covariance matrix $\boldsymbol{\Sigma^*} = \begin{bmatrix} \sigma_{00} & \boldsymbol{\sigma}' \\ \boldsymbol{\sigma} & \boldsymbol{\Sigma} \end{bmatrix}$. Let $\boldsymbol{z_1, z_2, ..., z_n}$ be n independent (k-component vector) observations on the predictor variables, $\boldsymbol{x_i = z_i - \bar{z}}$. Let $\boldsymbol{S^*} = \begin{bmatrix} s_{00} & \boldsymbol{s}' \\ \boldsymbol{s} & \boldsymbol{S} \end{bmatrix}$ be sample covariance matrix, where

$$s_{00} = \sum \frac{(y_i - \bar{y})^2}{n-1}, \boldsymbol{s} = \sum \frac{(y_i - \bar{y})\boldsymbol{x_i}}{n-1}, \boldsymbol{S} = \sum \frac{\boldsymbol{x_i x_i'}}{n-1}$$

We assume the correct the model(1.1)

$$\boldsymbol{y = \alpha + \beta_1 z_1 + \beta_2 z_2 + ... + \beta_k z_k + \epsilon}$$

The LSE prediction equation

$$\boldsymbol{\hat{y}} = \bar{y} + \boldsymbol{\hat{\beta}_1}(\boldsymbol{z_1} - \bar{z_1}) + \boldsymbol{\hat{\beta}_2}(\boldsymbol{z_2} - \bar{z_2}) + ... + \boldsymbol{\hat{\beta}_k}(\boldsymbol{z_k} - \bar{z_k}) = \bar{y} + \boldsymbol{X'\hat{\beta}}$$

$$\hat{y_i} = \bar{y} + \boldsymbol{x_i'\hat{\beta}}$$

For any given $\boldsymbol{z_i}$,

$$\begin{aligned} E(y_i|\boldsymbol{z_i}) &= \alpha + \boldsymbol{\beta z_i} \\ &= \mu_0 - \boldsymbol{\sigma}'\boldsymbol{\Sigma^{-1}\mu} + \boldsymbol{\sigma'\Sigma^{-1} z_i} \\ &= \mu_0 + \boldsymbol{\sigma}'\boldsymbol{\Sigma^{-1}(z_i - \mu)} \end{aligned}$$

where $\alpha = \mu_0 - \boldsymbol{\sigma}'\boldsymbol{\Sigma^{-1}\mu}, \boldsymbol{\beta = \sigma'\Sigma^{-1}}$. Thus $\hat{\alpha} = \bar{y} - \boldsymbol{s'S^{-1}\bar{x}}, \hat{\boldsymbol{\beta}} = \boldsymbol{S^{-1}s}$.

The conditional predictive mean square error by

$$\begin{aligned} E[(y_0 - \hat{y_0})^2|\boldsymbol{z_0}] &= E[(\alpha + \boldsymbol{(z_0 - \mu)'\beta} + \epsilon_0 - \bar{y} - \boldsymbol{x_0'}\hat{\boldsymbol{\beta}}|\boldsymbol{z_0})^2] \\ &= E[(\alpha + \boldsymbol{(z_0 - \mu)'\beta + \epsilon_0} - \alpha - (\boldsymbol{\bar{z} - \mu)'\beta - \bar{\epsilon} - x_0'}\beta|\boldsymbol{z_0})^2] \\ &= E[(\boldsymbol{x_0'}\beta - \boldsymbol{\bar{x}'\beta} + \epsilon_0 - \bar{\epsilon} - \boldsymbol{x_0'}\hat{\beta}|\boldsymbol{z_0})^2] \\ &= E[(\boldsymbol{x_0'\beta} + (\epsilon_0 - \bar{\epsilon}) - \boldsymbol{x_0'}\hat{\beta}|\boldsymbol{z_0})^2] \\ &= E[(\boldsymbol{x_0'\beta} + (\epsilon_0 - \bar{\epsilon}))^2 + (\boldsymbol{x_0'\hat{\beta}})^2 - 2(\boldsymbol{x_0'\beta} + (\epsilon_0 - \bar{\epsilon}))\boldsymbol{x_0'}\hat{\beta}|\boldsymbol{z_0}] \\ &= E[(\boldsymbol{x_0'\beta})^2 + (\epsilon_0 - \bar{\epsilon})^2 + (\boldsymbol{x_0'\hat{\beta}})^2 - 2\boldsymbol{x_0'\beta x_0'}\hat{\beta}|\boldsymbol{z_0}] \\ &= \boldsymbol{\beta'}E(\boldsymbol{x_0 x_0'}|\boldsymbol{z_0})\boldsymbol{\beta} + E[(\epsilon_0 - \bar{\epsilon})^2|\boldsymbol{z_0}] + E[(\boldsymbol{x_0'\hat{\beta}})^2|\boldsymbol{z_0}] - 2\boldsymbol{\beta'}E[\boldsymbol{x_0 x_0'}\hat{\beta}|\boldsymbol{z_0}] \end{aligned}$$

By Lemma A1, Lemma A3, Lemma A7

$$\begin{aligned} &E(\boldsymbol{\tilde{\beta}_1}|\boldsymbol{X_1}) = \boldsymbol{\beta_1} + \boldsymbol{\Sigma_{11}^{-1}\Sigma_{12}\beta_2} = \boldsymbol{\Phi_1} \\ &E(\boldsymbol{x_{01}x_{01}'}|\boldsymbol{z_0}) = (\boldsymbol{z_{01}} - \boldsymbol{\mu_1})(\boldsymbol{z_{01}} - \boldsymbol{\mu_1})' + \frac{\boldsymbol{\Sigma_{11}}}{n} \\ &E(\boldsymbol{x_0 x_0'}|\boldsymbol{z_0}) = (\boldsymbol{z_0} - \boldsymbol{\mu})(\boldsymbol{z_0} - \boldsymbol{\mu})' + \frac{\boldsymbol{\Sigma}}{n} \\ &E[(\boldsymbol{x_{01}'\tilde{\beta}_1})^2|\boldsymbol{z_0}] = \sigma_p^2 \left[\boldsymbol{(z_{01} - \mu_1)'\Sigma_{11}^{-1}(z_{01} - \mu_1)} + \frac{p}{n}\right] \frac{1}{n-p-2} \\ &\qquad + \boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}\frac{1}{n} + \boldsymbol{\Phi_1'(z_{01} - \mu_1)(z_{01} - \mu_1)'\Phi_1} \\ &\boldsymbol{\sigma'\Sigma^{-1}} \begin{bmatrix} \boldsymbol{\Sigma_{11}} \\ \boldsymbol{\Sigma_{21}} \end{bmatrix} = \boldsymbol{\sigma_1'} \end{aligned}$$

The conditional PMSE can be written as

$$
\begin{aligned}
E[(y_0-\hat{y_0})^2|\boldsymbol{z_0}] = &\boldsymbol{\beta' } E(\boldsymbol{x_0x_0'}|\boldsymbol{z_0})\boldsymbol{\beta} + E[(\epsilon_0-\bar{\epsilon})^2|\boldsymbol{z_0}] + E[(\boldsymbol{x_0'\hat{\beta}})^2|\boldsymbol{z_0}] - 2\boldsymbol{\beta'}E[\boldsymbol{x_0x_0'\hat{\beta}}|\boldsymbol{z_0}] \\
=&\boldsymbol{\beta'}\left[(\boldsymbol{z_0}-\boldsymbol{\mu})(\boldsymbol{z_0}-\boldsymbol{\mu})' + \frac{\boldsymbol{\Sigma}}{n}\right]\boldsymbol{\beta} + \sigma_k^2\left(1+\frac{1}{n}\right) \\
&+\sigma_k^2\left[(\boldsymbol{z_0}-\boldsymbol{\mu})'\boldsymbol{\Sigma^{-1}}(\boldsymbol{z_0}-\boldsymbol{\mu}) + \frac{k}{n}\right]\frac{1}{n-k-2} \\
&+\frac{1}{n}\boldsymbol{\Phi'\Sigma\Phi} + \boldsymbol{\Phi'}(\boldsymbol{z_0}-\boldsymbol{\mu})(\boldsymbol{z_0}-\boldsymbol{\mu})'\boldsymbol{\Phi} - 2\boldsymbol{\beta'}\left[(\boldsymbol{z_0}-\boldsymbol{\mu})(\boldsymbol{z_0}-\boldsymbol{\mu})' + \frac{\boldsymbol{\Sigma}}{n}\right]\boldsymbol{\beta} \\
=&\ \sigma_k^2\left(1+\frac{1}{n}\right) + \sigma_k^2\left[(\boldsymbol{z_0}-\boldsymbol{\mu})'\boldsymbol{\Sigma^{-1}}(\boldsymbol{z_0}-\boldsymbol{\mu}) + \frac{k}{n}\right]\frac{1}{n-k-2}
\end{aligned}
$$

Since Φ is the notation of expactation of $\tilde{\beta}_1$, when we are using all predictors, the LSE is unbiased, which means $\Phi=\beta$.

The unconditional PMSE

$$
\begin{aligned}
E[(y_0-\hat{y_0})^2] &= E\{E[(y_0-\hat{y_0})^2|\boldsymbol{z_0}]\} \\
&= E\left[\sigma_k^2\left(1+\frac{1}{n}\right) + \sigma_k^2\left[(\boldsymbol{z_0}-\boldsymbol{\mu})'\boldsymbol{\Sigma^{-1}}(\boldsymbol{z_0}-\boldsymbol{\mu}) + \frac{k}{n}\right]\frac{1}{n-k-2}\right] \\
&= \sigma_k^2\left(1+\frac{1}{n}\right) + \sigma_k^2 E\left[(\boldsymbol{z_0}-\boldsymbol{\mu})'\boldsymbol{\Sigma^{-1}}(\boldsymbol{z_0}-\boldsymbol{\mu}) + \frac{k}{n}\right]\frac{1}{n-k-2} \\
&= \sigma_k^2\left(1+\frac{1}{n}\right) + \sigma_k^2\left(k+\frac{k}{n}\right)\frac{1}{n-k-2} \\
&= \sigma_k^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n-k-2}\right) \\
&= \sigma_k^2\left(1+\frac{1}{n}\right)\left(\frac{n-2}{n-k-2}\right)
\end{aligned}
$$

For subset, we partition the k-component vector of predictor variables into two parts, $\boldsymbol{Z}=[\boldsymbol{Z_1},\boldsymbol{Z_2}], \boldsymbol{X}=[\boldsymbol{X_1},\boldsymbol{X_2}], \boldsymbol{x_1'}=[\boldsymbol{x_{i1}'},\boldsymbol{x_{i2}'}], \boldsymbol{\mu'}=[\boldsymbol{\mu_1'},\boldsymbol{\mu_2'}], \boldsymbol{\sigma'}=[\boldsymbol{\sigma_1'},\boldsymbol{\sigma_2'}], \boldsymbol{s'}=[\boldsymbol{s_1'},\boldsymbol{s_2'}]$, $\boldsymbol{\Sigma}=\begin{bmatrix}\boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}}\\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}, \boldsymbol{S}=\begin{bmatrix}\boldsymbol{S_{11}} & \boldsymbol{S_{12}}\\ \boldsymbol{S_{21}} & \boldsymbol{S_{22}}\end{bmatrix}$, so the subset prediction equation is given by

$$\tilde{y_i} = \bar{y} + \boldsymbol{x_{i1}'\tilde{\beta}_1}$$

where $\boldsymbol{\tilde{\beta_1}}=\boldsymbol{S_{11}^{-1}s_1}$. By LemmaA1, LemmaA3, LemmaA7, $\boldsymbol{\Phi_1}=\boldsymbol{\beta_1}+\boldsymbol{\Sigma_{11}^{-1}\Sigma_{12}\beta_2}$, the conditional PMSE at z_0 is given by

$$
\begin{aligned}
E[(y_0-\tilde{y_0})^2|\boldsymbol{z_0}] &= E[(\boldsymbol{x_0'\beta}+\epsilon_0-\bar{\epsilon}-\boldsymbol{x_{01}'\tilde{\beta}_1}|\boldsymbol{z_0})^2] \\
&= E[(\boldsymbol{x_0'\beta})^2 + (\boldsymbol{x_{01}'\tilde{\beta}_1})^2 - 2\boldsymbol{x_0'\beta x_{01}'\tilde{\beta}_1} + (\epsilon_0-\bar{\epsilon})^2 + 2(\boldsymbol{x_0'\beta}-\boldsymbol{x_{01}'\tilde{\beta}_1})(\epsilon_0-\bar{\epsilon})|\boldsymbol{z_0}] \\
&= E[(\boldsymbol{x_0'\beta})^2|\boldsymbol{z_0}] + E[\boldsymbol{x_{01}'\tilde{\beta}_1})^2|\boldsymbol{z_0}] - 2E[\boldsymbol{x_0'\beta x_{01}'\tilde{\beta}_1}|\boldsymbol{z_0}] + E[(\epsilon_0-\bar{\epsilon})^2|\boldsymbol{z_0}] \\
&\quad + 2E[(\boldsymbol{x_0'\beta}-\boldsymbol{x_{01}'\tilde{\beta}_1})(\epsilon_0-\bar{\epsilon})|\boldsymbol{z_0}]
\end{aligned}
$$

where

$$E[(\boldsymbol{x_0'\beta})^2|\boldsymbol{z_0}] = \boldsymbol{\beta}\left[(\boldsymbol{z_0}-\boldsymbol{\mu})(\boldsymbol{z_0}-\boldsymbol{\mu})' + \frac{\boldsymbol{\Sigma}}{n}\right]\boldsymbol{\beta}$$

$$E[(\boldsymbol{x'_{01}}\tilde{\boldsymbol{\beta}}_{\mathbf{1}})^2|\boldsymbol{z_0}] = \sigma_p^2\left[(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Sigma_{11}^{-1}}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})+\frac{p}{n}\right]\frac{1}{n-p-2} + \boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}\frac{1}{n}+\boldsymbol{\Phi_1'}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Phi_1},$$

$$E[(\epsilon_0-\bar{\epsilon})^2|\boldsymbol{z_0}] = \sigma_k^2+\frac{1}{n}\sigma_k^2,$$

$$E[(\boldsymbol{x_0'\beta}-\boldsymbol{x'_{01}}\tilde{\boldsymbol{\beta}}_{\mathbf{1}})(\epsilon_0-\bar{\epsilon})|\boldsymbol{z_0}] = E[(\boldsymbol{x_0'\beta}-\boldsymbol{x'_{01}}\tilde{\boldsymbol{\beta}}_{\mathbf{1}})\epsilon_0|\boldsymbol{z_0}] - E[(\boldsymbol{x_0'\beta}-\boldsymbol{x'_{01}}\tilde{\boldsymbol{\beta}}_{\mathbf{1}})\bar{\epsilon}|\boldsymbol{z_0}] = 0$$

(but $\bar{\epsilon}=0$ is not necessary)

Equation 3.6 could be written as

$$\begin{aligned}
&= \sigma_p^2\left[(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Sigma_{11}^{-1}}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})+\frac{p}{n}\right]\frac{1}{n-p-2}+\boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}\frac{1}{n}+\boldsymbol{\Phi_1'}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Phi_1}\\
&\quad+\boldsymbol{\beta}(\boldsymbol{z_0}-\boldsymbol{\mu})(\boldsymbol{z_0}-\boldsymbol{\mu})'\boldsymbol{\beta}+\boldsymbol{\beta'\Sigma\beta}\frac{1}{n}-2\boldsymbol{\beta'}E(\boldsymbol{x_0x'_{01}}|\boldsymbol{z_0})\boldsymbol{\Phi_1}+\sigma_k^2+\frac{1}{n}\sigma_k^2\\
&= \sigma_k^2+\frac{1}{n}(\sigma_p^2+\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}-\boldsymbol{\sigma'}\Sigma^{-1}\boldsymbol{\sigma})+[(\boldsymbol{z_0}-\boldsymbol{\mu})'\beta]^2+[(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Phi_1}]^2\\
&\quad+\sigma_p^2\left[(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Sigma_{11}^{-1}}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})+\frac{p}{n}\right]\frac{1}{n-p-2}\\
&\quad+\boldsymbol{\beta'\Sigma\beta}\frac{1}{n}+\boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}\frac{1}{n}-2\boldsymbol{\beta'}E(\boldsymbol{x_0x'_{01}}|\boldsymbol{z_0})\boldsymbol{\Phi_1}\\
&= \sigma_k^2+\frac{1}{n}\sigma_p^2+\sigma_p^2\left[(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Sigma_{11}^{-1}}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})+\frac{p}{n}\right]\frac{1}{n-p-2}+[(\boldsymbol{z_0}-\boldsymbol{\mu})'\beta]^2+[(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Phi_1}]^2\\
&\quad+\frac{1}{n}\boldsymbol{\beta'\Sigma\beta}+\frac{1}{n}\boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}+\frac{1}{n}\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}-\frac{1}{n}\boldsymbol{\sigma'\Sigma^{-1}\sigma}-2\boldsymbol{\beta'}E(\boldsymbol{x_0x'_{01}}|\boldsymbol{z_0})\boldsymbol{\Phi_1}
\end{aligned}$$

where

$$E(\boldsymbol{x_0x'_{01}}|\boldsymbol{z_0}) = E\left[\begin{bmatrix}\boldsymbol{x_{01}x'_{01}}\\ \boldsymbol{x_{02}x'_{01}}\end{bmatrix}|\boldsymbol{z_0}\right] = \begin{bmatrix}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'+\boldsymbol{\Sigma_{11}/n}\\ (\boldsymbol{z_{02}}-\boldsymbol{\mu_2})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'+\boldsymbol{\Sigma_{21}/n}\end{bmatrix}$$

$$\boldsymbol{\sigma'\Sigma^{-1}\sigma}=\boldsymbol{\beta'\Sigma\beta},\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}=\boldsymbol{\beta_1'\Sigma_{11}\beta_1}$$

so

$$\begin{aligned}
\boldsymbol{\beta'}E(\boldsymbol{x_0x'_{01}}|\boldsymbol{z_0})\boldsymbol{\Phi_1} &= [\boldsymbol{\beta_1'}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'+\boldsymbol{\beta_1'\Sigma_{11}}/n+\boldsymbol{\beta_2'}(\boldsymbol{z_{02}}-\boldsymbol{\mu_2})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'+\boldsymbol{\beta_2'\Sigma_{21}}/n]\boldsymbol{\Phi_1}\\
&= \boldsymbol{\beta_1'}(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Phi_1}+\boldsymbol{\beta_2'}(\boldsymbol{z_{02}}-\boldsymbol{\mu_2})(\boldsymbol{z_{01}}-\boldsymbol{\mu_1})'\boldsymbol{\Phi_1}+\boldsymbol{\sigma_1'\Phi_1}/n
\end{aligned}$$

In addition to a few terms appearing in 3.6a, other terms would be equale to

$$\begin{aligned}
&\frac{1}{n}\boldsymbol{\beta'\Sigma\beta}+\frac{1}{n}\boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}+\frac{1}{n}\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}-\frac{1}{n}\boldsymbol{\sigma'\Sigma^{-1}\sigma}-\frac{2}{n}\boldsymbol{\sigma_1'\Phi_1}\\
&= \frac{1}{n}\boldsymbol{\Phi_1'\Sigma_{11}\Phi_1}+\frac{1}{n}\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}-\frac{2}{n}\boldsymbol{\sigma_1'\Phi_1}\\
&= \frac{1}{n}\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}+\frac{1}{n}\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}-\frac{2}{n}\boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1}=0
\end{aligned}$$

in which $\Phi_1=\boldsymbol{\Sigma_{11}^{-1}\sigma_1}$

The conditional PMSE is equal to (3.6a)

$$
\begin{aligned}
E[(y_0 - \tilde{y_0})^2 | \boldsymbol{z_0}] &= \sigma_k^2 + \frac{\sigma_p^2}{n} + \sigma_p^2 \left[\boldsymbol{(z_{01} - \mu_1)' \Sigma_{11}^{-1} (z_{01} - \mu_1)'} + \frac{p}{n} \right] \frac{1}{n-p-2} \\
&\quad + [\boldsymbol{(z_0 - \mu)'\beta - (z_{01} - \mu_1)'\Phi_1}]^2 \\
&= \sigma_k^2 + \frac{\sigma_p^2}{n} + \sigma_p^2 \left[\boldsymbol{(z_{01} - \mu_1)' \Sigma_{11}^{-1} (z_{01} - \mu_1)'} + \frac{p}{n} \right] \frac{1}{n-p-2} \\
&\quad + [\boldsymbol{(z_{01} - \mu_1)'\beta_1 + (z_{02} - \mu_2)'} - \boldsymbol{(z_{01} - \mu_1)'}(\boldsymbol{\beta_1 + \Sigma_{11}^{-1}\Sigma_{12}\beta_2})]^2 \\
&= \sigma_k^2 + \frac{\sigma_p^2}{n} + \sigma_p^2 \left[\boldsymbol{(z_{01} - \mu_1)' \Sigma_{11}^{-1} (z_{01} - \mu_1)'} + \frac{p}{n} \right] \frac{1}{n-p-2} \\
&\quad + [\boldsymbol{(z_{02} - \mu_2)'} - \boldsymbol{(z_{01} - \mu_1)'\Sigma_{11}^{-1}\Sigma_{12}\beta_2}]^2
\end{aligned}
$$

Take expectation

$$
\begin{aligned}
E[(y_0 - \tilde{y_0})^2] &= E[E(y_0 - \tilde{y_0})^2 | \boldsymbol{z_0}] \\
&= E\{\sigma_k^2 + \frac{\sigma_p^2}{n} + \sigma_p^2 \left[\boldsymbol{(z_{01} - \mu_1)' \Sigma_{11}^{-1} (z_{01} - \mu_1)'} + \frac{p}{n} \right] \frac{1}{n-p-2} \\
&\quad + [\boldsymbol{(z_0 - \mu)'\beta - (z_{01} - \mu_1)'\Phi_1}]^2\} \\
&= \sigma_p^2 + \frac{\sigma_p^2}{n} + \boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1 - \sigma'\Sigma^{-1}\sigma} + \sigma_p^2 \left(p + \frac{p}{n} \right) \frac{1}{n-p-2} \\
&\quad + E[\boldsymbol{\beta'(z_0 - \mu)(z_0 - \mu)'\beta + \Phi_1'(z_{01} - \mu_1)(z_{01} - \mu_1)'\Phi_1} \\
&\quad - 2\boldsymbol{\beta'(z_0 - \mu)(z_{01} - \mu_1)'\Phi_1}]
\end{aligned}
$$

The expectation term is equal to

$$
\boldsymbol{\beta'\Sigma\beta + \Phi_1'\Sigma_{11}\Phi_1 - 2\beta'} \begin{bmatrix} \boldsymbol{\Sigma_{11}} \\ \boldsymbol{\Sigma_{21}} \end{bmatrix} \boldsymbol{\Phi_1}
$$

The unconditional PMSE

$$
\begin{aligned}
E[(y_0 - \tilde{y_0})^2] &= \sigma_p^2 (1 + \frac{1}{n})(n-2)/(n-p-2) \\
&\quad + \boldsymbol{\sigma_1'\Sigma_{11}^{-1}\sigma_1 - \sigma'\Sigma^{-1}\sigma} + \boldsymbol{\beta'\Sigma\beta} + \boldsymbol{\Phi_1'\Sigma_{11}\Phi_1 - 2\beta'} \begin{bmatrix} \boldsymbol{\Sigma_{11}} \\ \boldsymbol{\Sigma_{21}} \end{bmatrix} \boldsymbol{\Phi_1} \\
&= \sigma_p^2 (1 + \frac{1}{n})(n-2)/(n-p-2) \\
&\quad + \boldsymbol{\Phi_1'\Sigma_{11}\Phi_1} - \boldsymbol{\beta'\Sigma\beta} + \boldsymbol{\beta'\Sigma\beta} + \boldsymbol{\Phi_1'\Sigma_{11}\Phi_1 - 2\beta'} \begin{bmatrix} \boldsymbol{\Sigma_{11}} \\ \boldsymbol{\Sigma_{21}} \end{bmatrix} \boldsymbol{\Phi_1} \\
&= \sigma_p^2 (1 + \frac{1}{n})(n-2)/(n-p-2)
\end{aligned}
$$

Thus, the unconditional PMSE $= \sigma_p^2(1 + \frac{1}{n})(n-2)/(n-p-2)$