Quantized average consensus with a plateau escaping strategy in undirected graphs

Gabriele Oliva[1*], Camilla Fioravanti[1], Evagoras Makridis[2], and Themistoklis Charalambous[2,3]

***Abstract*— In this paper, the average consensus problem has been considered for undirected networks under finite bit-rate communication. While other algorithms reach approximate average consensus or require global information about the network for reaching the exact average consensus, we propose a fully distributed consensus algorithm that incorporates an adaptive quantization scheme and achieves convergence to the exact average while only requiring knowledge of an upper bound of the network diameter. Using Lyapunov stability analysis, we characterize the convergence properties of the resulting nonlinear quantized system. Moreover, we provide a fully distributed strategy to escape *plateaux*, i.e., situations where the Lyapunov function stops descending. Simulation results justify the performance of our proposed algorithm.**

***Index Terms*— Quantized average consensus, undirected graphs, adaptive quantization, plateau escaping strategy, multi-agent systems.**

I. INTRODUCTION

The coordination of autonomous agents, such as unmanned aerial vehicles [1] and self-driving cars [2], is essential for enabling a wide array of emerging technologies. A core primitive in this domain is the *average consensus* problem, wherein agents iteratively update their states to converge to the average of their initial conditions [3]. While many existing algorithms can achieve consensus under various network imperfections (e.g., packet losses or delays), most rely on the assumption that agents can exchange real numbers with infinite precision. In realistic settings, however, communication is limited by finite bit-rate digital channels, necessitating quantization of exchanged data. Quantization introduces two main challenges: (i) *saturation*, where signals exceed the quantizer's range, potentially destabilizing the system, and (ii) *precision loss*, which can stall convergence as states approach the desired value. These effects can prevent the system from achieving exact consensus.

A variety of methods have been proposed to address this problem [4]–[16]. Probabilistic gossip algorithms, such as those in [6], [7], preserve the average and converge up to a quantization bin, but exhibit slow convergence due to sequential link activations. Other approaches use randomized updates [8]–[10], yet cannot eliminate steady-state quantization errors. Event-driven strategies [11], [12] achieve finite-time approximate consensus, but still rely on quantized message exchange and converge only to nearby values.

Adaptive quantization schemes offer a promising route for achieving exact average consensus by adjusting quantization levels over time. However, existing approaches [13]–[16] require centralized coordination or knowledge of global parameters (such as the magnitude of the eigenvalues of the Perron or Laplacian matrix, and the norm of the initial conditions), which are nontrivial to estimate, especially in a quantized communication scenario, thus limiting their practical deployment. Despite sustained research effort, a fully distributed quantized average consensus algorithm that converges to the exact average–without requiring global information–has remained elusive.

This paper aims at closing this gap by proposing a scalable and fully distributed algorithm for exact average consensus using adaptive quantization, requiring only an upper bound on the network diameter, that can be easily computed in a distributed way within a finite number of steps (can be done as an initialization step) with quantized communications [17]. Within this algorithm, a novel *plateau escaping strategy* is proposed, ensuring that the proposed nonlinear quantized system always converges to the exact average. The validity of our proposed algorithm is demonstrated via simulations.

II. PRELIMINARIES

Notation. We denote vectors with boldface lowercase letters and matrices with uppercase letters. The transpose of matrix A and vector $\boldsymbol{x}$ are denoted as $A^\top$ and $\boldsymbol{x}^\top$, respectively. Given a matrix A we use $\texttt{row}_i(A)$ and $\texttt{col}_i(A)$ to denote the i-th row and column of A, respectively. We refer to the (i, j)-th entry of a matrix A by A_{ij}. We represent by $\mathbf{0}_n$ and $\mathbf{1}_n$ vectors with n entries, all equal to zero and to one, respectively. We use $\|\cdot\|_2$ to denote the Euclidean norm of a vector or the spectral norm of a matrix.

Graph Theoretic Concepts. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a graph with n nodes $\mathcal{V} = \{1, 2, \dots, n\}$ and e edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where $(j, i) \in \mathcal{E}$ captures the existence of a link from node i to node j. A graph is *undirected* if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. In this paper, we are considering undirected graphs only. An undirected graph is *connected* if each node can be reached by every other node via the edges. Let the neighborhood $\mathcal{N}_i$ of a node $i \in \mathcal{V}$ be the set of nodes $j \in \mathcal{V}$ such that $(i, j) \in \mathcal{E}$. The *degree* d_i of a node i is the number of its incoming (equivalently, outgoing) edges, *i.e.,* $d_i = |\mathcal{N}_i|$. The diameter δ of a graph $\mathcal{G}$ is the minimum distance between

[1]Department of Engineering, University Campus Bio-Medico of Rome, Via Alvaro del Portillo, 21 - 00128 Roma, Italy.

[2]Department of Electrical and Computer Engineering, School of Engineering, University of Cyprus, Nicosia, Cyprus.

[3]Department of Electrical Engineering and Automation, School of Electrical Engineering, Aalto University, Espoo, Finland.

[*] corresponding author. Email:g.oliva@unicampus.it

the two most distant nodes, i.e., the longest shortest path among any pair of nodes $i, j \in V$.

Quantized Communication Model. We consider a network depicted by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ in which agents are able to communicate only via quantized messages restricted by a given number of bits m. In particular, when a node i aims to transmit a value $x \in \mathbb{R}$ to a node j along a link, we assume that the agent is only able to transmit a message quantized according to the following quantization function

$$\mathrm{q}_{m,\Delta,\sigma}(x) = \begin{cases} S_{\max}\mathtt{sign}(x-\sigma)+\sigma, & \text{if } |x-\sigma| \geq S_{\max}, \\ \Delta \left\lfloor \dfrac{|x-\sigma|}{\Delta} + \dfrac{1}{2} \right\rfloor + \sigma, & \text{otherwise}, \end{cases}$$

where Δ is the quantization step, σ is an offset, and $S_{\max} = (2^{m-1}-1)\Delta$ is the largest representable magnitude.

III. Proposed Quantized Consensus Algorithm

Let us consider a network of n agents, interacting over an undirected and connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. Each agent i is provided with an initial condition[1] $x_{i0} \in \mathbb{R}$ and they aim to distributedly compute the average $x_{\mathtt{ave}} = \frac{1}{n}\sum_{i=1}^n x_{i0}$, by exchanging information over the graph in a synchronous and discrete-time fashion. In particular, in this paper, we consider a scenario where the agents are able to communicate only via messages encompassing a given number m of bits, and thus they resort to the quantization scheme discussed in § II in order to transmit their values. Interestingly, within the proposed scheme, the agents maintain time-varying quantization parameters $\Delta[k]$ and $\sigma[k]$ in order to adapt the quantization scheme and improve convergence. Hereafter, $\mathrm{q}_{m,\Delta[k],\sigma[k]}(x_i[k])$ is denoted by $\check{x}_i[k]$ for simplicity of exposition.

Within the proposed algorithm, the i-th agent adopts the following update rule

$$x_i[k+1] = x_i[k] + \sum_{j\neq i, j=1}^{n} P_{ij}[k](\check{x}_j[k] - \check{x}_i[k]), \;\; x_i[0] = x_{i0}.$$

Stacking the above equation for all agents, we obtain the following compact form

$$\boldsymbol{x}[k+1] = \boldsymbol{x}[k] + (P[k] - I_n)\check{\boldsymbol{x}}[k], \quad \boldsymbol{x}[0] = \boldsymbol{x}_0.$$

As noted above, agents exchange quantized values $\check{x}_i[k]$ with their neighbors, and these values are quantized with m bits. Further to that, the agents exchange the Boolean variables

$$\beta_i[k] = \begin{cases} 1 & \text{if } x_i[k] > \check{x}_i[k], \\ 0 & \text{otherwise}. \end{cases}$$

As discussed later in the paper, additional messages are used to implement $\min$- and $\max$-consensus iterations that drive the update of $\Delta[k]$ and $\sigma[k]$. Interestingly, since it receives both $\check{x}_j[k]$ and $\beta_j[k]$ from its neighbors j, at each step agent i is able to identify a lower bound on $|x_i[k] - x_j[k]|$. In order to characterize such a lower bound, we need to define the following logical conditions.

Definition 1: Each link $(i,j) \in \mathcal{E}$ such that at time k it holds $\check{x}_i[k] \neq \check{x}_j[k]$ is associated to a *condition* denoted by $\mathtt{c}_{ij}[k] \in \{\mathtt{c}^{\mathtt{I}}, \mathtt{c}^{\mathtt{II}}, \mathtt{c}^{\mathtt{III}}\}$, where $\mathtt{c}^{\mathtt{I}}, \mathtt{c}^{\mathtt{II}}, \mathtt{c}^{\mathtt{III}}$ represent three mutually exclusive situations for the link. In particular, $\mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{I}}$ if the segment joining $\check{x}_i[k]$ and $\check{x}_j[k]$ is contained in the one joining $x_i[k]$ and $x_j[k]$. Moreover, $\mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{II}}$ if the segment joining $x_i[k]$ and $x_j[k]$ is contained in the one joining $\check{x}_i[k]$ and $\check{x}_j[k]$. Finally, $\mathtt{c}[k] = \mathtt{c}^{\mathtt{III}}$ in all other cases, i.e., when the segment joining $x_i[k]$ and $x_j[k]$ has either $\check{x}_i[k]$ or $\check{x}_j[k]$ as an intermediate point (but not both).

Remark 1: Each agent i is able to determine the condition $\mathtt{c}_{ij}[k]$ of any link involving a neighboring agent j such that $\check{x}_i[k] \neq \check{x}_j[k]$. In particular, if $\check{x}_i[k] < \check{x}_j[k]$, $\beta_i[k] = 0$ and $\beta_j[k] = 1$ (or if $\check{x}_i[k] > \check{x}_j[k]$, $\beta_i[k] = 1$ and $\beta_j[k] = 0$), agent i can conclude that $\mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{I}}$. Similarly, if $\check{x}_i[k] < \check{x}_j[k]$, $\beta_i[k] = 1$ and $\beta_j[k] = 0$ (or if $\check{x}_i[k] > \check{x}_j[k]$, $\beta_i[k] = 0$ and $\beta_j[k] = 1$), agent i can conclude that $\mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{II}}$. Finally, in all other cases agent i can conclude that $\mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{III}}$.

Based on the above conditions, let us introduce the structure of the time-varying weights $P_{ij}[k]$ considered in this paper. In particular, for all $(i,j) \in \mathcal{E}$, if $\check{x}_i[k] \neq \check{x}_j[k]$, the agents set

$$P_{ij}[k] = \begin{cases} \dfrac{1}{\max\{d_i, d_j\}} & \text{if } \mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{I}}_{ij}, \\ \dfrac{\max\{0, \zeta_{ij}[k]\}}{\max\{d_i, d_j\}} & \text{if } \mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{II}}, \\ \dfrac{\Delta[k]}{2\max\{d_i, d_j\}|\check{x}_i[k] - \check{x}_j[k]|} & \text{if } \mathtt{c}_{ij}[k] = \mathtt{c}^{\mathtt{III}}, \end{cases} \tag{1}$$

where $\zeta_{ij}[k] = 1 - \dfrac{\Delta[k]}{|\check{x}_i[k] - \check{x}_j[k]|}$, while $P_{ij}[k] = 0$ if $(i,j) \notin \mathcal{E}$ or if $\check{x}_i[k] = \check{x}_j[k]$. Notice that, by construction, each agent i is able to verify locally whether $\mathtt{c}^{\mathtt{I}}, \mathtt{c}^{\mathtt{II}}$, or $\mathtt{c}^{\mathtt{III}}$ hold true at time k for any $j \in \mathcal{N}_i$. Due to the symmetry of the undirected graph, the same condition holds in both directions, i.e., $\mathtt{c}_{ij}[k] = \mathtt{c}_{ji}[k]$; hence, we have that $P_{ij}[k] = P_{ji}[k]$. Notice further that, to be able to compute $P_{ij}[k] = P_{ji}[k]$, agents i, j need to exchange their degrees d_i, d_j and quantized values $\check{x}_i[k], \check{x}_j[k]$.

A. Monotonicity of the Error

Let us now show that, with the above choice for the weights, the error with respect to (w.r.t.) the average of the initial conditions is nonincreasing. Moreover, let us show that, at any time k, either the error becomes smaller, or the distance among the projected states of any two agents connected by a link is smaller than or equal to $\Delta[k]$.

Let us now define the Lyapunov function $V(\boldsymbol{x}[k]) := \|\boldsymbol{x}[k] - x_{\mathtt{ave}}\mathbf{1}_n\|^2$, and let $\Delta V[k] := V(\boldsymbol{x}[k+1]) - V(\boldsymbol{x}[k])$ denote the Lyapunov drift of $V(\boldsymbol{x}[k])$ at time k. The next theorem shows that the Lyapunov drift $\Delta V[k]$ is always nonpositive and, in particular, that either $\Delta V[k]$ is negative

[1] In this paper initial conditions are assumed to be scalar for the sake of simplicity, but the approach can be extended to vector-valued states.

or the distance among the projected states of any two agents connected by a link is smaller than $\Delta[k]$.

Theorem 1: Let us choose the weights $P_{ij}[k]$ according to Eq. (1). Then, $\Delta V(\boldsymbol{x}[k]) \leq 0$ for all k. Moreover, at each time k, either $\Delta V(\boldsymbol{x}[k]) < 0$ or

$$|\breve{x}_i[k] - \breve{x}_j[k]| \leq \Delta[k], \quad \forall (i,j) \in \mathcal{E}.$$

Proof: See Appendix A. ■

The next corollary characterizes the case where $\Delta[k] = \Delta$ and $\sigma[k] = \sigma$ are fixed.

Corollary 1: Let the assumptions of Theorem 1 hold and suppose that, for all k, it holds $\Delta[k] = \Delta$ and $\sigma[k] = \sigma$. Then, $\lim_{k\to\infty} |\breve{x}_i[k] - \breve{x}_j[k]| \in \{0, \Delta\}$.

Proof: To prove the result we observe that, by Theorem 1, when $\Delta[k] = \Delta$ is fixed, in the limit of k approaching infinity all $|\breve{x}_i[k] - \breve{x}_j[k]|$ become smaller than or equal to Δ. Since the values $\breve{x}_i[k], \breve{x}_j[k]$ are quantized, the only option is that their absolute difference either approaches zero or Δ. This completes our proof. ■

B. Plateau Escaping Strategy

According to Corollary 1, when $\Delta[k] = \Delta$ and $\sigma[k] = \sigma$ are fixed, eventually we reach a situation where the distance among the projected states of any two agents connected by a link is smaller than Δ. When this situation occurs the algorithm stops, even thought the agents' state have not converged to the same value. In the following, we refer to this situation as a *plateau*, as the agents' state becomes constant. Also, we observe that if some agent is provided with an initial condition with magnitude larger than $S_{\max}$, then the algorithm will exhibit poor precision (i.e., as its state will be coarsely approximated by the quantizer).

This subsection is devoted to designing a mechanism to let the agents escape from these situations. To this end, let us make the following assumption.

Assumption 1: The agents know an upper bound $\bar{\delta}$ on the network diameter δ.

Notice that the estimation of the global parameter δ is amenable to being performed in a distributed fashion with quantization constraints, e.g., exploiting the protocol in [17]. Within the proposed strategy, at time instants $h\bar{\delta}$ that are integer multiples of $\bar{\delta}$ the agents initiate a Boolean min-consensus, using initial conditions

$$\eta_i[h\bar{\delta}] = \begin{cases} 1, & \text{if } \left|\dfrac{x_i[h\bar{\delta}] - x_i[h\bar{\delta}-1]}{x_i[h\bar{\delta}]}\right| \leq \psi\ , \\ 0, & \text{otherwise.} \end{cases}$$

After $\bar{\delta}$ steps the min-consensus procedure ends and, at time $(h+1)\bar{\delta}$, the agents know

$$\eta^*[h\bar{\delta}] = \begin{cases} 1, & \text{if } \min_i \left|\dfrac{x_i[h\bar{\delta}] - x_i[h\bar{\delta}-1]}{x_i[h\bar{\delta}]}\right| \leq \psi\ , \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\eta^*[h\bar{\delta}] = 1$ if for all $(i,j) \in \mathcal{E}$ the normalized minimum variation of a state at successive time instants is smaller than ψ. Hence, if $\psi \ll 1$, by knowing $\eta^*[h\bar{\delta}]$, the agents know at time $(h+1)\bar{\delta}$ if the main algorithm has halted at time $h\bar{\delta}$ or not. If $\eta^*[h\bar{\delta}] = 1$, the agents reduce $\Delta[k]$, scaling it by $k_{\mathtt{in}}$, and continue executing the main algorithm. Additionally, at time instants $h\bar{\delta}$ that are integer multiple of $\bar{\delta}$, the agents initiate a Boolean max-consensus over variables

$$\theta_i[h\bar{\delta}] = \begin{cases} 1, & \text{if } |x_i[h\bar{\delta}]| > S_{\max}[h\bar{\delta}] \\ 0, & \text{otherwise.} \end{cases}$$

After $\bar{\delta}$ steps the max-consensus procedure ends and, at time $(h+1)\bar{\delta}$, the agents know

$$\theta^*[h\bar{\delta}] = \begin{cases} 1, & \text{if } \exists i:\ |x_i[h\bar{\delta}]| > S_{\max}[h\bar{\delta}] \\ 0, & \text{otherwise.} \end{cases}$$

If $\theta^*[h\bar{\delta}] = 1$, the agents increase $\Delta[k]$ and continue executing the algorithm. In this way, eventually, all nodes will have a value that lies in the representable range. More formally, for $0 < k_{\mathtt{in}} < 1$ and $k_{\mathtt{out}} > 1$, the agents set

$$\Delta[k] = \begin{cases} k_{\mathtt{in}}\Delta[k-\bar{\delta}] & \text{if } \mathrm{mod}(k,\bar{\delta}) = 0 \bigwedge \eta^*[k-\bar{\delta}] = 1, \\ k_{\mathtt{out}}\Delta[k-\bar{\delta}] & \text{if } \mathrm{mod}(k,\bar{\delta}) = 0 \bigwedge \theta^*[k-\bar{\delta}] = 1, \\ \Delta[k-1], & \text{otherwise.} \end{cases} \tag{2}$$

A last measure to escape the plateau is to modify $\sigma[k]$ in order to center the quantizer around the range of values that are currently assumed by the agents' state. To this end, we let the agents execute a $\max$- and a $\min$-consensus procedure over the quantized states $\breve{x}_i[k]$ (with states $\sigma_i^+[k]$ and $\sigma_i^-[k]$ for the max- and min-consensus, respectively), which is re-initialized at each $\bar{\delta}$ steps. In this way, at time $(h+1)\delta$ the agents know $\sigma_i^+[(h+1)\bar{\delta}] = \max_i \breve{x}_i[h\bar{\delta}]$ and $\sigma_i^-[(h+1)\bar{\delta}] = \min_i \breve{x}_i[h\bar{\delta}]$ and are able to set

$$\sigma[(h+1)\bar{\delta}] = \frac{1}{2}\left(\sigma_i^+[(h+1)\bar{\delta}] + \sigma_i^-[(h+1)\bar{\delta}]\right).$$

In other words, the agents set

$$\sigma[k] = \begin{cases} \dfrac{\max_i \breve{x}_i[k-\bar{\delta}] + \min_i \breve{x}_i[k-\bar{\delta}]}{2}, & \text{if } \mathrm{mod}(k,\bar{\delta}) = 0, \\ \sigma[k-1], & \text{otherwise.} \end{cases} \tag{3}$$

By relying on the above strategy, the agents trigger a new reduction in $V(\boldsymbol{x}[k])$, thus achieving monotone asymptotic convergence to the average of the initial conditions.

Theorem 2: Let the assumptions of Theorem 1 and Assumption 1 hold true. Then, using the proposed plateau-escaping strategy, the agents' states converge monotonically to the average $x_{\mathtt{ave}}$ of the initial conditions.

Proof: See Appendix B. ■

A few remarks are now in order.

Remark 2: During each step of our algorithm, each agent sends three messages encompassing m bits each and three flags, with a total of $3m + 3$ bits per iteration.

Remark 3: The computational complexity at each round is $O(d_{\max})$, $d_{\max}$ being the maximum degree. In fact, each agent receives $O(1)$ messages from each neighbor.

IV. Simulations

We consider a random geometric graph with $n = 50$ nodes and $|E| = 492$ (the link density w.r.t. a complete graph is 20.08%, and the diameter is $\delta = 6$) and uniformly random initial conditions in $[0, 1]^n$. For the sake of simplicity, we choose $\bar{\delta} = \delta$. Moreover, let us consider the following parameters: $\Delta[0] = 10^{-4}$, $k_{\mathtt{in}} = 0.5$, $k_{\mathtt{out}} = 2$, $\psi = 10^{-7}$.

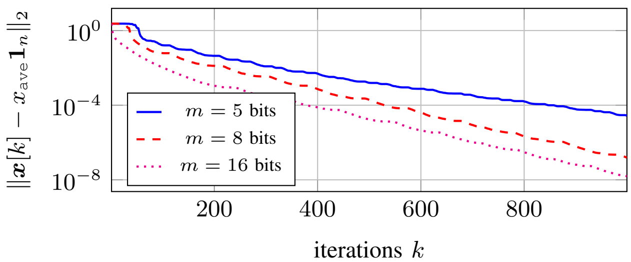

Fig. 1: Performance of the proposed algorithm for different choices of the number of quantization bits m.

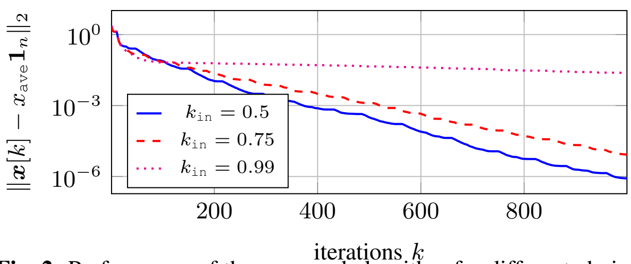

Fig. 2: Performance of the proposed algorithm for different choices of the gain $k_{\mathtt{in}}$.

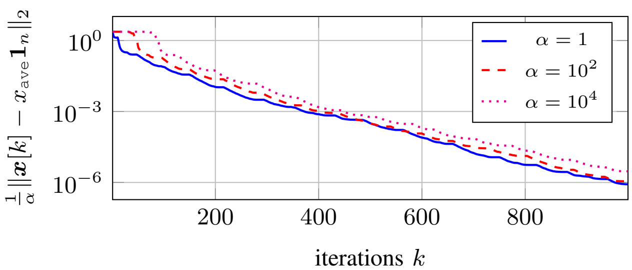

Fig. 3: Effect of the magnitude of the initial conditions on the convergence of the proposed algorithm.

Figure 1 evaluates the performance of our algorithm in terms of the error $\|\boldsymbol{x}[k] - x_{\mathtt{ave}}\mathbf{1}_n\|_2$ w.r.t. the average of the initial conditions, by considering quantization with $m \in \{5, 8, 16\}$ bits. According to the figure, we observe that convergence improves as m grows. In particular, when m is small, given the small value for $\Delta[0]$, the proposed algorithm experiences a phase of increase of $\Delta[k]$, thus delaying the start of the error reduction. In all cases, the trajectory of the error norm is monotone non-increasing. Figure 2 shows the dynamics of the norm of the error when different choices for $k_{\mathtt{in}}$ are chosen ($k_{\mathtt{out}}$ is set to $k_{\mathtt{out}} = 1/k_{\mathtt{in}}$, while $m = 6$ bits and $\Delta[0] = 10^{-2}$). According to the figure, the convergence rate appears to be inversely proportional[2] with

[2]The parameter k_{in} influences the aggressiveness of the quantization step size adjustment during the plateau-escaping phase, directly affecting the convergence rate of the algorithm.

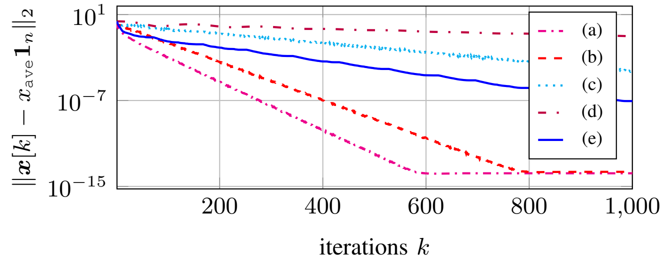

Fig. 4: Comparison against [13], [14]: [13] with carefully chosen parameters (a); [13] with slightly overestimated parameters (b); [13] with largely overestimated parameters (c); [14] with carefully chosen parameters (d); Proposed Algorithm (e).

$k_{\mathtt{in}}$. Choosing $k_{\mathtt{in}} = 0.5$, $m = 6$ bits, and $\Delta[0] = 10^{-2}$, we evaluate the effect of the magnitude of the initial conditions in Figure 3. Specifically, the initial conditions are scaled by $\alpha \in \{1, 10^2, 10^4\}$ and the plot reports the norm of the error, scaled down by $1/\alpha$. According to the figure, we observe that the initial plateau is directly proportional to α; this because the larger are the agents' states, the more time is required during the first iterations to obtain $\Delta[k]$ large enough to allow to represent correctly the values involved.

To conclude the section, in Figure 4 we present a comparison against [13], [14] which, we recall, require knowledge of global information. Notice that, for this instance, the second largest eigenvalue of the dynamical matrix, is $\rho = 0.9445$. The algorithm in [13] is guaranteed to converge for $m \geq 14$ bits and for carefully chosen parameters that depend on ρ, $\|\boldsymbol{x}[0]\|_2$, and n, i.e., on global information. Choosing $m = 14$ bits also for the our algorithm and for [14], and approximating $\rho \approx \rho' = 0.95$, we observe that the approach in [13], indeed, exhibits a fast convergence rate and outperforms our algorithm. When the parameters for [13] are slightly overestimated (i.e., $\rho' = 0.96$ and the initial conditions for the quantization gain are scaled by 10) we observe that in the first stages our algorithm exhibits better performance, while as iterations progress [13] exhibits faster convergence speed. When the parameters for the approach in [13] are largely overestimated (i.e., $\rho' = 0.99$ and the initial conditions for the quantization gain are scaled by 10), the performance is significantly degraded w.r.t. the proposed one. Finally, we observe that the proposed approach exhibits better performance than [14].

V. Conclusions and Future Work

In this paper, we proposed a fully distributed consensus algorithm that incorporates adaptive quantization and operates with a finite number of bits and still is able to achieve the exact average consensus without the need of centralized knowledge, except for an upper bound on the network diameter (which can be estimated in a distributed way. The analytical and simulation results demonstrate the algorithm's effectiveness.

There are still several open problems for this setup. In particular, the current version of the algorithm lacks robustness against packet losses and time delays, which can significantly hinder coordination among agents and degrade the algorithm's overall performance. Additionally, for a more realistic communication model, we could investigate the case for which the digital communication channel is noisy. In such more realistic settings, the considered plateau escaping strategy should be properly designed and incorporated within robustified frameworks that support packet retransmissions via acknowledgement messages or/and accumulated value broadcasting without acknowledgement messages.

APPENDIX

A. Proof of Theorem 1

First, let us introduce a few ancillary results. The following Proposition will be useful for proving Lemma 1.

Proposition 1: Suppose that $\mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{x}) \neq \mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{y})$ and that the segment joining $\boldsymbol{x}$ and $\boldsymbol{y}$ is contained in the one joining $\mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{x})$ and $\mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{y})$. Then, it must hold

$$|\boldsymbol{x}-\boldsymbol{y}| + \Delta \geq |\mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{x}) - \mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{x})|. \tag{4}$$

Proof: Let $\boldsymbol{q}_x, \boldsymbol{q}_y$ denote $\mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{x})$ and $\mathtt{q}_{m,\Delta,\sigma}(\boldsymbol{y})$, respectively. In order to prove the statement we observe that, under the above assumptions, by construction it holds $|\boldsymbol{x}-\boldsymbol{q}_x| \leq \Delta/2$ and $|\boldsymbol{y}-\boldsymbol{q}_y| \leq \Delta/2$ (i.e., as otherwise $\boldsymbol{x}$ and/or $\boldsymbol{y}$ would not be contained in the segment joining their quantizations $\boldsymbol{q}_x$ and $\boldsymbol{q}_x$. Hence,

$$\begin{aligned}|\boldsymbol{q}_x-\boldsymbol{q}_y| &= |\boldsymbol{q}_x-\boldsymbol{q}_y-\boldsymbol{x}+\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{y}| \\ &= |\boldsymbol{x}-\boldsymbol{y}| + |\boldsymbol{x}-\boldsymbol{q}_x| + |\boldsymbol{y}-\boldsymbol{q}_y| \leq |\boldsymbol{x}-\boldsymbol{y}| + \Delta.\end{aligned}$$

The proof is complete. ■

Specifically, let us now provide useful lower bounds on $|x_i[k]-x_j[k]|$ depending on $\mathtt{c}_{ij}[k]$.

Lemma 1: Suppose $\breve{x}_i[k] \neq \breve{x}_j[k]$. Then it holds

$$|x_i[k]-x_j[k]| > \begin{cases} |\breve{x}_i[k]-\breve{x}_j[k]|, & \text{if } \mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{I}}, \\ |\breve{x}_i[k]-\breve{x}_j[k]| - \Delta[k], & \text{if } \mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{II}}, \\ \dfrac{1}{2}\Delta[k], & \text{if } \mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{III}}. \end{cases}$$

Proof: In order to prove the statement we observe that, if $\mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{I}}$, then $x_i[k] \leq \breve{x}_i[k] < \breve{x}_j[k] < x_j[k]$ or $x_j[k] \leq \breve{x}_j[k] < \breve{x}_i[k] < x_i[k]$, from which we conclude that

$$|x_i[k]-x_j[k]| > |\breve{x}_i[k]-\breve{x}_j[k]|.$$

Let us now consider the case where $\mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{II}}$. In this case, since we assumed $x_i[k], x_j[k]$ are mapped into different quantized values, we have that $\breve{x}_i[k] < x_i[k] < x_j[k] \leq \breve{x}_j[k]$ or $\breve{x}_j[k] < x_j[k] < x_i[k] \leq \breve{x}_i[k]$, and thus it holds $|\breve{x}_i[k]-\breve{x}_j[k]| > |x_i[k]-x_j[k]|$. By Proposition 1, we have

$$|x_i[k]-x_j[k]| > |\breve{x}_i[k]-\breve{x}_j[k]| - \Delta[k].$$

Finally, when $\mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{III}}$, the segment joining $x_i[k]$ and $x_j[k]$ has either $\breve{x}_i[k]$ or $\breve{x}_j[k]$ as an intermediate point. In this case, since we assumed that $x_i[k]$ and $x_j[k]$ are mapped into different quantized values, we conclude that $|x_i[k]-x_j[k]| > \frac{1}{2}\Delta[k]$. The proof is complete. ■

Let us now provide a bound that will prove useful in establishing the monotone convergent behavior of our algorithm.

Lemma 2: Let us define

$$\gamma[k] = \sum_i \left(\sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k])\right)^2.$$

It holds $\gamma[k] \leq \gamma^\dagger[k] = \sum_i d_i \sum_{j\neq i} P_{ij}^2[k]\left(\breve{x}_j[k]-\breve{x}_i[k]\right)^2$, where d_i is the degree of node $i \in \mathcal{V}$.

Proof: To prove the lemma, let us define $\Omega[k]$ as the $n\times n$ matrix such that $\Omega_{ij}[k] = P_{ij}[k]\left(\breve{x}_j[k]-\breve{x}_i[k]\right)$. We have that $\gamma[k] = \|\Omega[k]\mathbf{1}_n\|_2^2 = \|\sum_i \boldsymbol{e}_i \mathtt{row}_i(\Omega[k])\mathtt{col}_i(A)\|_2^2$ where $\boldsymbol{e}_i$ is the i-th vector in the canonical basis in $\mathbb{R}^n$ and A is the adjacency matrix of the graph $\mathcal{G}$. At this point, we observe that

$$\gamma[k] \leq \sum_i \|\mathtt{row}_i(\Omega[k])\|_2^2 \underbrace{\|\mathtt{col}_i(A)\|_2^2}_{=d_i} = \gamma^\dagger[k].$$

The proof is complete. ■

Proof: [Proof of Theorem 1] In order to prove the result, let us consider the Lyapunov function $V(\boldsymbol{x}[k])$. We have that, by definition, it holds

$$V(\boldsymbol{x}[k]) = \sum_i x_i^2[k] + n x_{\mathtt{ave}}^2 - 2x_{\mathtt{ave}}\sum_i x_i[k].$$

Let us also consider

$$\begin{aligned} V(\boldsymbol{x}[k+1]) &= \sum_i \left(x_i[k+1]-x_{\mathtt{ave}}\right)^2 \\ &= \sum_i \Big(x_i[k] + \sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k]) - x_{\mathtt{ave}}\Big)^2 \\ &= \sum_i x_i^2[k] + \sum_i \Big(\sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k])\Big)^2 \\ &\quad + n x_{\mathtt{ave}}^2 + 2\sum_i x_i[k]\sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k]) \\ &\quad - 2x_{\mathtt{ave}}\sum_i x_i[k] - 2x_{\mathtt{ave}}\underbrace{\sum_i\sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k])}_{0}, \end{aligned}$$

where the last term is zero since $P[k]$ is by construction symmetric. At this point, let us consider the discrete derivative $\Delta V[k]$; we have that

$$\begin{aligned} \Delta V[k] &= \sum_i \Big(\sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k])\Big)^2 \\ &\quad + 2\sum_i x_i[k]\sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k]) \\ &= \underbrace{\sum_i \Big(\sum_{j\neq i} P_{ij}[k](\breve{x}_j[k]-\breve{x}_i[k])\Big)^2}_{\gamma[k]} \\ &\quad - 2\sum_i\sum_{j<i} P_{ij}[k](\breve{x}_i[k]-\breve{x}_j[k])(x_i[k]-x_j[k]) \end{aligned}$$

where the last term is the result of considering the (i,j)-th and the (j,i)-th terms together.

Using Lemma 2, we have that

$$\begin{aligned}\Delta V[k] &\leq \sum_i d_i \sum_{j\neq i} P_{ij}^2[k]\left(\check{x}_j[k]-\check{x}_i[k]\right)^2 \\ &\quad - 2\sum_i \sum_{j<i} P_{ij}[k](\check{x}_i[k]-\check{x}_j[k])(x_i[k]-x_j[k]) \\ &= 2\sum_i d_i \sum_{j<i} P_{ij}^2[k]\left(\check{x}_j[k]-\check{x}_i[k]\right)^2 \\ &\quad - 2\sum_i \sum_{j<i} P_{ij}[k](\check{x}_i[k]-\check{x}_j[k])(x_i[k]-x_j[k]) \\ &= 2\sum_i \sum_{j<i} P_{ij}[k]\phi_{ij}[k], \end{aligned} \tag{5}$$

with

$$\phi_{ij}[k]=(\check{x}_j[k]-\check{x}_i[k])[d_i P_{ij}[k](\check{x}_j[k]-\check{x}_i[k])-(x_i[k]-x_j[k])].$$

We observe that, if $\check{x}_i[k] = \check{x}_j[k]$, then $\phi_{ij}[k]=0$. If, conversely, $\check{x}_i[k] \neq \check{x}_j[k]$, we have that $\phi_{ij}[k]<0$ for

$$P_{ij}[k] < \frac{1}{d_i}\frac{|x_i[k]-x_j[k]|}{|\check{x}_i[k]-\check{x}_j[k]|}.$$

At this point, let us resort to Lemma 1. In particular we observe that, if $\mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{I}}$, then $\phi_{ij}[k]<0$ for $P_{ij}[k]\leq \frac{1}{d_i}$. Conversely, if $\mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{II}}$, we have that $\phi_{ij}[k]<0$ for

$$P_{ij}[k] \leq \underbrace{\frac{1}{d_i}\left(1-\frac{\Delta[k]}{|\check{x}_i[k]-\check{x}_j[k]|}\right)}_{\zeta_{ij}[k]},$$

provided that $\zeta_{ij}[k]>0$. Note that, by construction, $\zeta_{ij}[k]\geq 0$. Finally, if $\mathtt{c}_{ij}[k]=\mathtt{c}_{ij}^{\mathtt{III}}$, we have that $\phi_{ij}[k]<0$ for

$$P_{ij}[k] \leq \frac{\Delta[k]}{2d_i|\check{x}_i[k]-\check{x}_j[k]|}.$$

It can easily be noted that, if $P_{ij}[k]$ is chosen according to Eq. (1), then $\phi_{ij}[k]<0$, unless either $\check{x}_i[k]=\check{x}_j[k]$, or Condition $\mathtt{c}^{\mathtt{II}}[k]$ holds with $\zeta_{ij}[k]=0$; in both cases we have $\phi_{ij}[k]=0$. In particular, when $\zeta_{ij}[k]=0$ we have that $|\check{x}_i[k]-\check{x}_j[k]|=\Delta[k]$. In conclusion, we have that $\Delta V(\boldsymbol{x}[k])<0$ unless for all $(i,j)\in\mathcal{E}$ it holds $|\check{x}_i[k]-\check{x}_j[k]|\leq\Delta[k]$. The proof is complete. ■

B. Proof of Theorem 2

To prove asymptotic convergence we claim that, setting $\Delta[k]$ and $\sigma[k]$ according to Eqs. (2) and (3), respectively, if $\Delta V(\boldsymbol{x}[k^\dagger])=0$ for some $k^\dagger$ then there is a finite $\hat{k}>k^\dagger$ in which $\Delta V(\boldsymbol{x}[\hat{k}])<0$. To verify our claim, let us assume that $\Delta V(\boldsymbol{x}[k^\dagger])=0$ for some $k^\dagger$. Notably, according to our plateau-escaping strategy, the agents are able to detect a stall in at most 2δ steps (i.e., in the worst case the Boolean max- and min-consensus procedures are initialized just before the reach of a plateau, and have to be executed twice for the agents to detect such a situation). Interestingly, if some of the agents' states are outside the current quantization range, by construction there is a finite time $k^\ddagger$ after which $\Delta[k]$ stops growing. Thus, without loss of generality, let us assume that at time k' the agents' states are all within the quantization range. Since $\Delta V(\boldsymbol{x}[k'])=0$, we have that $x_i[k'+1]=\boldsymbol{x}_i[k']$ for all i. Hence, according to Eqs. (2) and (3), every δ steps, $\Delta[\cdot]$ is scaled down by $k_{\mathtt{in}}$ and $\sigma[\cdot]$ is centered at the mid-point between the minimum and maximum projected values. This, eventually, leads to a finite time $\hat{k}>k'$ in which $|\check{x}_i[\hat{k}]-\check{x}_j[\hat{k}]|>\Delta[\hat{k}]$ for at least one $(i,j)\in E$, thus triggering a new reduction in the Lyapunov-like function. Since we have established that every time a plateau is reached there is a finite time after which $\Delta V(\boldsymbol{x}[k])$ starts reducing again, we conclude that the sequence $\{V(\boldsymbol{x}[k])\}_{k=0}^{\infty}$ contains a monotonically converging subsequence, and since $V(\boldsymbol{x}[k])$ is bounded below by zero, this proves convergence of $V(\boldsymbol{x}[k])$ to zero.

References

[1] S. Javaid, N. Saeed, Z. Qadir, H. Fahim, B. He, H. Song, and M. Bilal, "Communication and control in collaborative UAVs: Recent advances and future trends," *IEEE Transactions on Intelligent Transportation Systems*, vol. 24, no. 6, pp. 5719–5739, 2023.

[2] V. Lesch, M. Breitbach, M. Segata, C. Becker, S. Kounev, and C. Krupitzer, "An overview on approaches for coordination of platoons," *IEEE Transactions on Intelligent Transportation Systems*, vol. 23, no. 8, pp. 10 049–10 065, 2021.

[3] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[4] R. Carli and F. Bullo, "Quantized coordination algorithms for rendezvous and deployment," *SIAM Journal on Control and Optimization*, vol. 48, no. 3, pp. 1251–1274, 2009.

[5] R. Carli, F. Bullo, and S. Zampieri, "Quantized average consensus via dynamic coding/decoding schemes," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 2, pp. 156–175, 2010.

[6] A. Kashyap, T. Başar, and R. Srikant, "Quantized consensus," *Automatica*, vol. 43, no. 7, pp. 1192–1203, 2007.

[7] K. Cai and H. Ishii, "Quantized consensus and averaging on gossip digraphs," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2087–2100, 2011.

[8] T. C. Aysal, M. Coates, and M. Rabbat, "Distributed average consensus using probabilistic quantization," in *IEEE/SP Workshop on Statistical Signal Processing*, pp. 640–644, 2007.

[9] M. El Chamie, J. Liu, and T. Başar, "Design and analysis of distributed averaging with quantized communication," in *IEEE Conference on Decision and Control*, 2014, pp. 3860–3865.

[10] L. Rong, P. Su, H. Gao, and S. Xu, "Distributed quantized consensus with recursive channel filters over directed networks," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 69, no. 3, pp. 1168–1172, 2022.

[11] A. I. Rikos and C. N. Hadjicostis, "Event-triggered quantized average consensus via ratios of accumulated values," *IEEE Transactions on Automatic Control*, vol. 66, no. 3, pp. 1293–1300, 2021.

[12] A. I. Rikos, C. N. Hadjicostis, and K. H. Johansson, "Finite time quantized average consensus with transmission stopping guarantees and no quantization error," *Automatica*, vol. 163, p. 111522, 2024.

[13] R. Carli, F. Fagnani, P. Frasca, S. Zampieri *et al.*, "Efficient quantized techniques for consensus algorithms," in *NeCST workshop*, pp. 1–8, 2007.

[14] T. Li, M. Fu, L. Xie, and J.-F. Zhang, "Distributed consensus with limited communication data rate," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 279–292, 2011.

[15] D. Thanou, E. Kokiopoulou, Y. Pu, and P. Frossard, "Distributed average consensus with quantization refinement," *IEEE Transactions on Signal Processing*, vol. 61, no. 1, pp. 194–205, 2012.

[16] F. F. Rego, Y. Pu, A. Alessandretti, A. P. Aguiar, and C. N. Jones, "A consensus algorithm for networks with process noise and quantization error," in *53rd Annual Allerton Conference on Communication, Control, and Computing*, pp. 488–495, 2015.

[17] A. I. Rikos, T. Charalambous, C. N. Hadjicostis, and K. H. Johansson, "Distributed computation of exact average degree and network size in finite time under quantized communication," *European Journal of Control*, vol. 74, p. 100848, 2023.