MTH314: Discrete Mathematics for Engineers

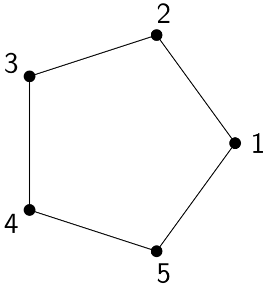
Lecture 9b: Introduction to Graph Theory

Dr Ewa Infeld

Ryerson University

Graph Theory Basics

A graph $G = (V, E)$ is a data structure/mathematical object that consists of a set of vertices/nodes V and a relation E (edges) on this set.

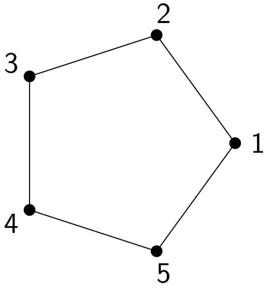


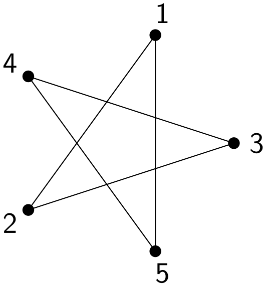
V is the vertex set
$V = \{1, 2, 3, 4, 5\}$

E is the set of edges

Graph Theory Basics

A graph $G = (V, E)$ is a data structure/mathematical object that consists of a set of vertices/nodes V and a relation E (edges) on this set.





These two are the same graph. Just two different ways to draw it.

Graph: Definition

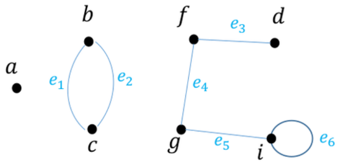
A graph $G = (V, E)$ is a data structure/mathematical object that consists of a set of vertices/nodes V and a relation E (edges) on this set.

Every edge $e \in E$ goes between two vertices, which we call *endpoints*. And edge from a vertex to itself is called a *loop*.

If there exists an $e \in E$ with endpoints $u, v \in V$ we say that u and v are adjacent, or that u is andjacent to v. We say that e is incident to both u and v.

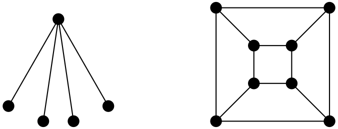
Example: $G=(V,E), V=\{a,b,c,d,f,g,i\},\ E=\{e_1,e_2,e_3,e_4,e_5,e_6\}$

Edge	e_1	e_2	e_3	e_4	e_5	e_6
Endpoints	$\{b,c\}$	$\{b,c\}$	$\{f,d\}$	$\{f,g\}$	$\{g,i\}$	$\{i\}$

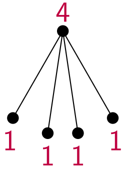


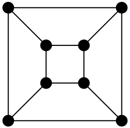
- e_6 is called a **loop**
- Vertex i is **adjacent** to itself
- Vertex a is **isolated**
- Edges e_1, e_2 are **parallel**
- Edges e_4, e_5 are **incident** to vertex g
- Vertices f, g are **connected** by edge e_4

The *degree* of a vertex is the number of edges coming out of that vertex. A loop will count twice, since both endpoints are at the same vertex.



The *degree* of a vertex is the number of edges coming out of that vertex. A loop will count twice, since both endpoints are at the same vertex.





The degree sequence of a graph is a list of vertex degrees, written in increasing order. The degree sequences of the graphs above are $1, 1, 1, 1, 4$ and $3, 3, 3, 3, 3, 3, 3, 3$ respectively.

Handshake Lemma

Lemma (The Handshake Lemma)

The sum of degrees of the vertices in an undirected graph is twice the number of edges in the graph.

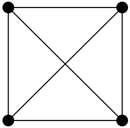
This is easy to see - every edge contributes 2 to the sum of vertex degrees.

Is it possible to connect 7 computers into a network by laying cables between (distinct) pairs of them so that every computer is connected to exactly three other computers?

No, $7 \cdot 3 = 21$ is an odd number.

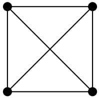
A graph $G = (V, E)$ is a *simple* if there are no edges from a vertex to itself ("loops") and between any two vertices there is at most one edge.

A *clique* is a simple graph where any two vertices are adjacent.

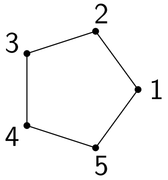


A graph $G = (V, E)$ is a *simple* if there are no edges from a vertex to itself ("loops") and between any two vertices there is at most one edge.

A *clique* is a simple graph where any two vertices are adjacent.

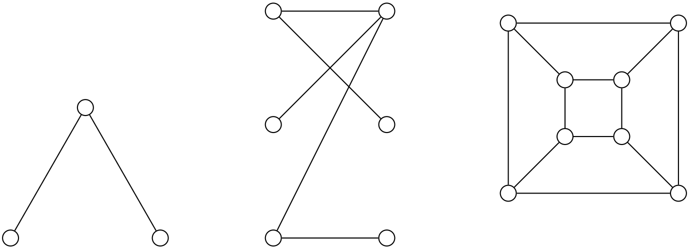


Not a clique:

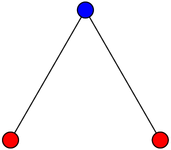


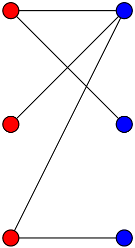
For example, vertices 1 and 3 are not adjacent.

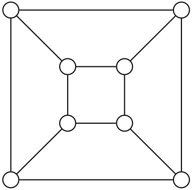
A graph $G = (V, E)$ is called *bipartite* if there exists a partition of the set of vertices into two sets A and B, such that no two vertices in A are adjacent and no two verices in B are adjacent.



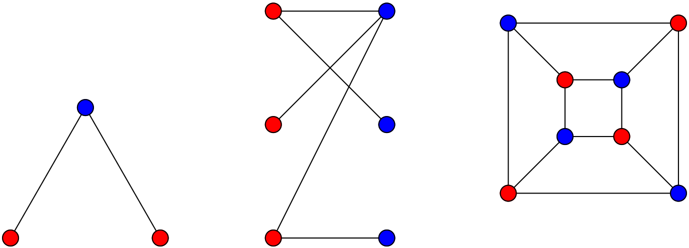
A graph $G = (V, E)$ is called *bipartite* if there exists a partition of the set of vertices into two sets A and B, such that no two vertices in A are adjacent and no two verices in B are adjacent.





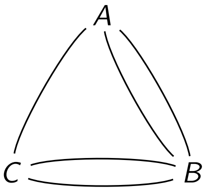


A graph $G = (V, E)$ is called *bipartite* if there exists a partition of the set of vertices into two sets A and B, such that no two vertices in A are adjacent and no two verices in B are adjacent.



Adjacency Matrix

An adjacency matrix is an integer matrix that encodes the graph. Rows correspond to vertices, and columns correspond to vertices. i, j-entry (ith row and jth column) is the integer representing how many edges connect vertices i and j.



$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 2 & 1 \\ B & 2 & 0 & 2 \\ C & 1 & 2 & 0 \end{array}$$

In a simple graph, all entries are either 0 or 1 and all diagonal entries are 0. (why?)