LECTURE 2 SUMMARY **Math 20 Fall 2014, Dartmouth College**

Example 1 When we toss a coin, $\Omega = \{H, T\}$. Then either $X = H$ or $X = T$.

Example 2 When we roll two dice, red and blue, and write the result on the red die first:
$\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$
$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}.$

Example 3 When we roll two dice and record the sum of the two, $\Omega = \{2,3,4,5,6,7,8,9,10,11,12\}$.

Example 4 Toss a coin until the first time it comes up heads. When it does, stop tossing. The number of tosses you make is the outcome of the experiment.

$\Omega = \{1,2,3,4,5,6,\dots\}$
$P(1) = 1/2$
$P(2) = 1/4$
$P(3) = 1/8$
$\vdots$
$P(i) = 1/2^i$
And $P(1 \cup 2 \cup 3 \cup \dots) = 1/2 + 1/4 + 1/8 + \cdots = 1.$

1 Discrete probability distributions

Definition 1 Let X be a random variable with a finite sample space Ω. A *distribution function* m for X is a function from X to $[0,1]$ such that:

$$\sum_{\omega \in \Omega} m(\omega) = 1.$$

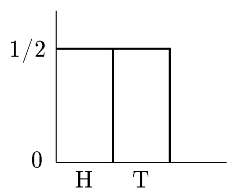
For any subset E of Ω (i.e. any event),

$$P(E) = \sum_{\omega \in E} m(\omega).$$

A special type of such probability distribution is *uniform distribution,* where m is constant. Examples 1 and 2 are such distributions. In that case,

$$m(\omega) = \frac{1}{|\Omega|} \text{ for all } \omega \in \Omega.$$

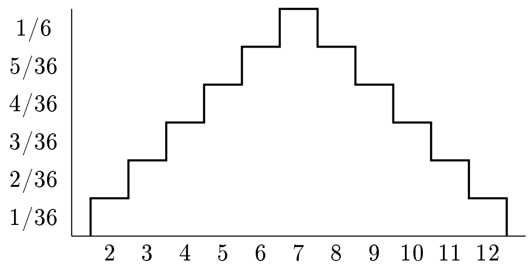
In order to picture a probability distribution function, we graph it with respect to the sample space Ω. Graphing a uniform distribution is easy, the following diagram refers to Example 1:



The area under the function is always equal to 1.
The following two diagrams refer to probability functions in Examples 3 and 4.

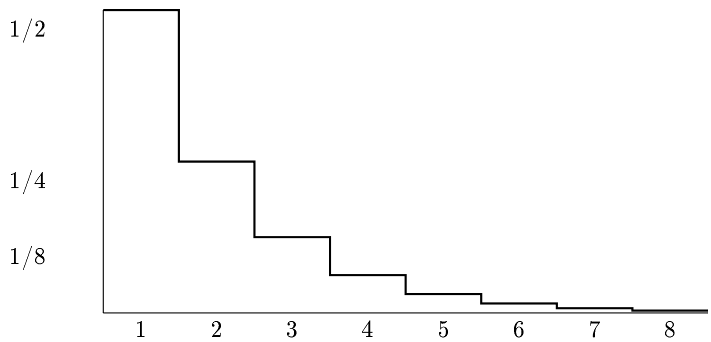
Example 3: sum on two dice.

The sample space are integers from 2 to 12, so that is what we put on the horizontal axis. Notice the shape of this distribution - it's centered around the most likely value and drops off symmetrically on both sides.



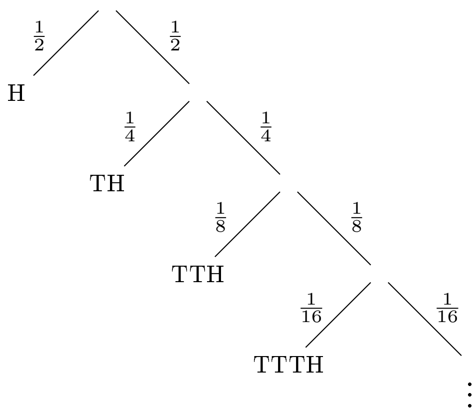
Example 4: number of coin tosses until it comes up "heads."

This distribution approximates exponential decay, which you have likely seen in calculus classes. In both of these pictures you can easily check that the area under the function is 1.



2 Tree Diagrams

In some circumstances when subsequent events depend on earlier ones, we can use a tree diagram to calculate the probability of a particular event. Consider Example 4 again, the number of coin tosses until the first time it comes up "heads." We toss the coin the first time, and chances are the whole experiment ends there. If not, we toss again. This can be represented as a tree:

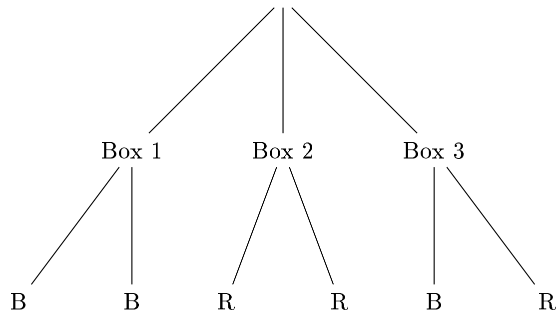


The labels on the edges in the tree represent the splitting of probability, in this case equally, and the sequences of Ts and Hs are the sequences of tosses the node corresponds to.

3 Balls in boxes

Question: I have three boxes and six balls. Three of the balls are blue and three are red. I put two blue balls in one box, two red balls in another box, and one red and one blue in the last. I pick a box at random, and then pick a ball from that box also at random. The ball is blue. What is the probability that the other ball in the box is also blue?

Most people assume that the answer is 1/2, but it's not so. To solve this, we will use a tree diagram again. First we pick a box, and then we pick a ball in that box:



There are three end events that result in picking a blue ball, all with the same probability. Two of them come from a box that had two blue balls. So the probability that the other ball in the box is blue is 2/3. this is not quite so counterintuitive when you think about it - we had the same probability of picking box 1 as box 3, but once we pick box 3 there is only 1/2 probability that we pick a blue ball, compared to certainly picking a blue ball if we picked box 1. Do you believe this?