Joint modeling of wind speed and wind direction through a conditional approach
Supplementary Material

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December 14, 2022

Analysis using the mean square error (MSE) and the integrated mean square error (IMSE)

The BWHR and BPQR methods are evaluated using the MSE between the true and estimated conditional quantiles at a given quantile level and angle, over all 500 replicates. The MSE is computed by discretizing the wind direction domain using a set of grid points $\{\phi_1, \phi_2, \ldots, \phi_m\}$ with a sufficiently large m (we use $m = 629$ in our study) and for each angle ϕ_i, $i = 1, \ldots m$, using the following formula:

$$\mathrm{MSE}_{\phi_\mathrm{i}} = \frac{1}{500} \sum_{j=1}^{500} (q_{est,j}(\phi_i) - q_{true}(\phi_i))^2,$$

where $q_{true}(\phi_i)$ and $q_{est,j}(\phi_i)$, $j = 1, \ldots, 500$, denote the true and estimated conditional quantiles at a given quantile level and angle ϕ_i, $i = 1, \ldots, m$. Next, the MSE_{ϕ_i} values are averaged over the discretized domain using the following formula:

$$MSE = \frac{1}{m} \sum_{i=1}^{m} MSE_{\phi_i}$$

The MSE_{ϕ_i}, $i = 1, \ldots, m$, and MSE values considering the 95% quantile level are depicted in the top row of Fig. 16. A quick analysis of the graphs should reflect that for most angles the BWHR method has a smaller MSE than BPQR. However, there are angles for which the MSE of the BWHR method is higher than that of BPQR. Comparing these results to the estimated wind direction distribution, depicted in the middle row of Fig. 16, we see that these circumstances happen when the density of wind direction is very small. This

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suggests the inclusion of the density of wind direction when computing the error. Hence, the use of the integrated square error defined as follows:

$$\text{IMSE} = \int_0^{2\pi} MSE_\phi \cdot f_\Phi(\phi)\, d\phi,$$

where $f_\Phi(\phi)$ represents the density of wind direction (ϕ). To compute this value we proceed similarly to the MSE case, where we dicretize the wind direction domain to a set of grid points $\{\phi_1, \phi_2, \ldots, \phi_m\}$ and the IMSE_ϕ along with the IMSE is computed using the following formulas:

$$\begin{aligned} \text{IMSE}_{\phi_i} &= \frac{1}{500}\sum_{j=1}^{500}, f_{\Phi_i}(\phi_i)(q_{est,j}(\phi_i) - q_{true}(\phi_i))^2, \quad i = 1, \ldots m, \\ \text{IMSE} &= \frac{\sum_{i=1}^m \text{IMSE}_{\phi_i}}{\sum_{i=1}^m f_{\Phi_i}(\phi_i)}. \end{aligned}$$

The results of the IMSE_{ϕ_i}, $i = 1, \ldots n$, along with the IMSE values for the 95% quantile level are depicted in the bottom row of Fig. 16 (Fig. 17 for quantile differences) showing a decreased error in the `BWHR` model. Furthermore, in Fig. 18 we plot the distribution of the IMSE for the 95% quantiles of the `BWHR` and `BPQR` models. As the boxplots show, the distribution of IMSE of the `BWHR` method is less variable than that of `BPQR`. Finally, the scatter plots included in the bottom row of Fig. 18 tell us that `BWHR` model has a lower IMSE than `BPQR` 76.6% of times at the TX_GP location, 94% of times at ND_GP and 95.5% of times at NC_mountains.

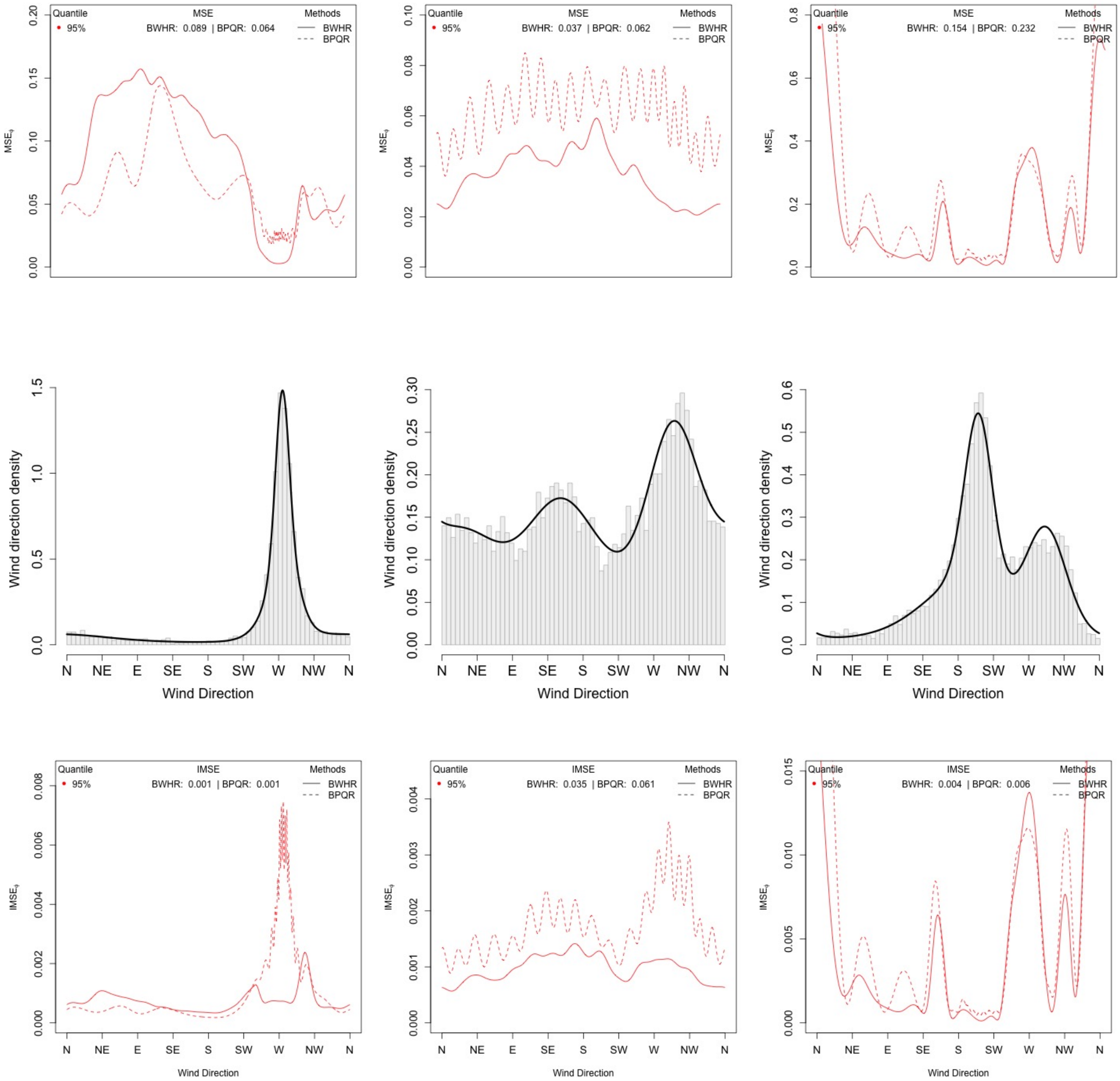

Figure 1: **Top row**: MSE_ϕ and MSE values for the 95% conditional quantile of the `BWHR` (solid line) and of the `BPQR` method (dashed line) at three different locations; **middle row:** estimates of the wind direction distribution at three different locations; **bottom row:** $\text{IMSE}_\phi$and IMSE values for the 95% conditional quantile of the `BWHR` method (solid line) and of the `BPQR` method (dashed line) at three different locations.

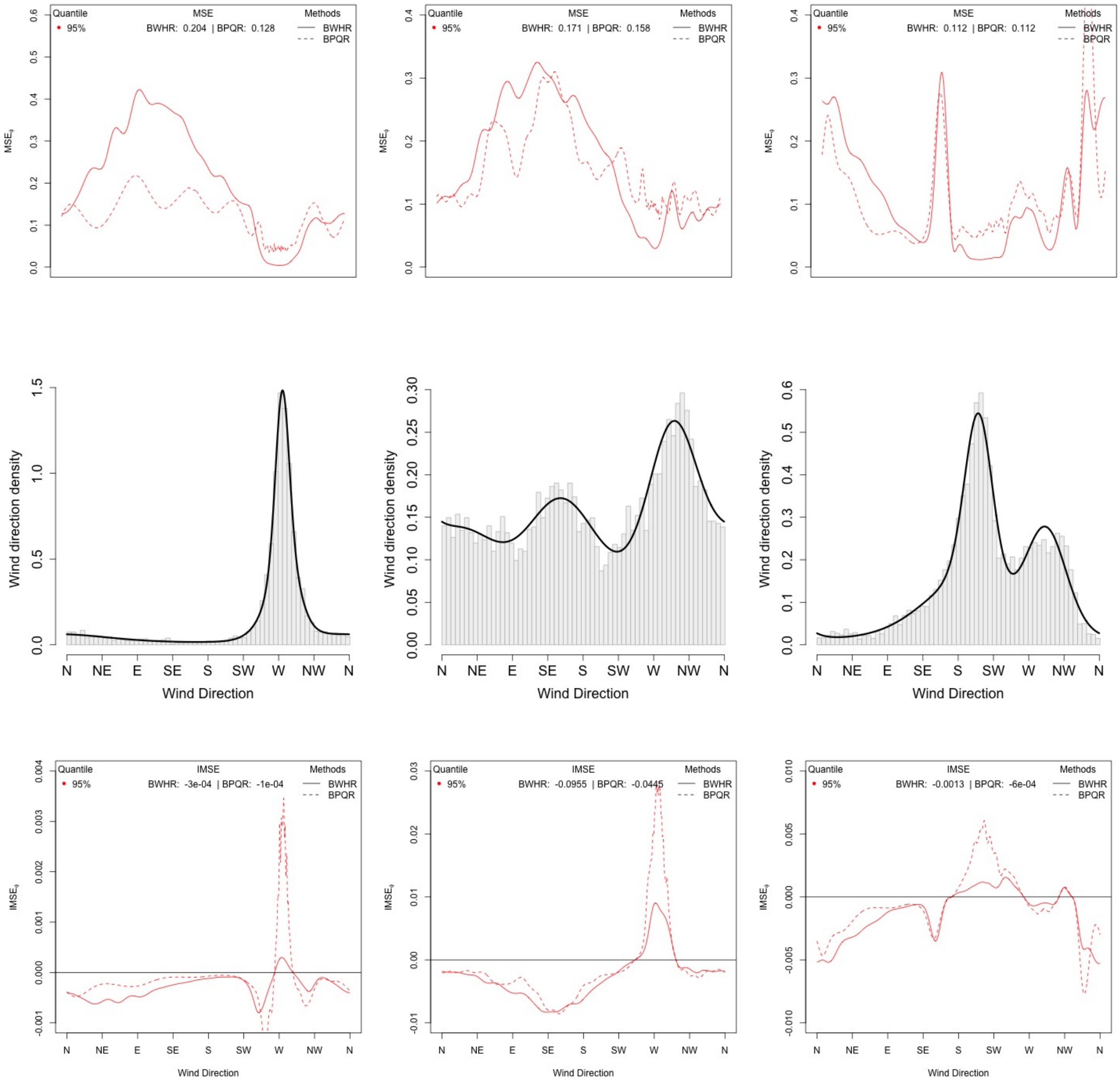

Figure 2: As in Fig 16 but for quantile differences.

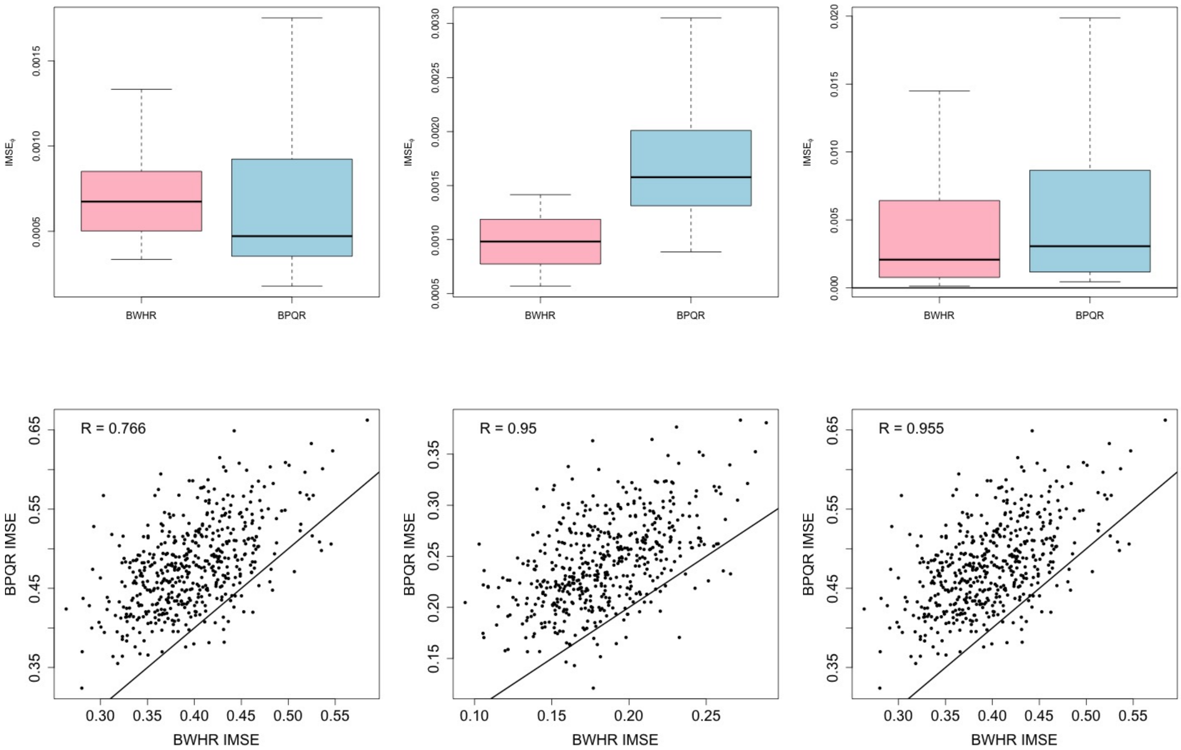

Figure 3: **Top row:** Boxplots of the distribution of the IMSE$_\phi$ of the 95% quantile curves; **bottom row:** Scatter plots of the `BWHR` vs the `BPQR` IMSE$_\phi$ where the solid line represents the line $y = x$ and R is the ratio of number of time the `BWHR` method has a lower IMSE compared to `BPQR`. Each column represents a different location.

The von Mises mixture distribution is evaluated using IMSE$_\phi$ and IMSE defined above. Specifically,

$$\mathrm{IMSE}_{\phi_i} = \frac{1}{500}\sum_{j=1}^{500} f_{true}(\phi_i)(f_{est,j}(\phi_i) - f_{true}(\phi_i))^2, \quad i = 1, \dots m,$$
$$\mathrm{IMSE} = \frac{\sum_{i=1}^{m} \mathrm{IMSE}_{\phi_i}}{\sum_{i=1}^{n} f_{true}(\phi_i)} ,$$

where $f_{true}(\phi_i)$ and $f_{est,j}(\phi_i)$, $j = 1, \dots, 500$, denote the true and estimated wind direction distribution at a given angle ϕ_i, $i = 1, \dots, n$. The results are plotted in Fig. 15 and show a small IMSE value suggesting that a von Mises mixture distribution is a good fit to wind direction.

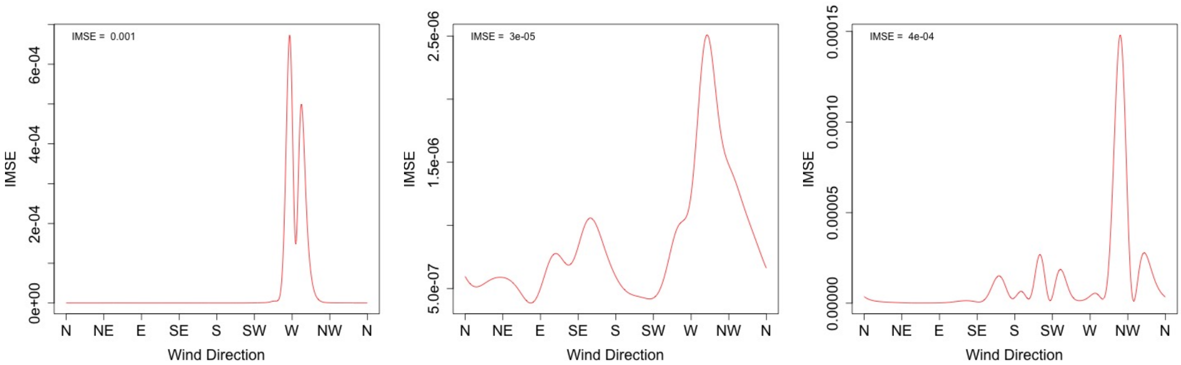

Figure 4: IMSE_ϕ values for a set of discretized wind directions along with the IMSE values at each of the three locations.

Analysis on the pairs of Fourier series using Mean Square Error (MSE)

In our work we regress the parameters of the Weibull distribution on fixed K_α and K_β pairs of trigonometric function, i.e. $K = K_\alpha = K_\beta = 8$. This choice was made after performing an analysis on the MSE of the BWHR model such that $K = 8$ and K is chosen using BIC. The results of this analysis are shown in Fig. 14 and it shows that the BWHR model with $K = 8$ has a smaller MSE than the BWHR model with K chosen using BIC.

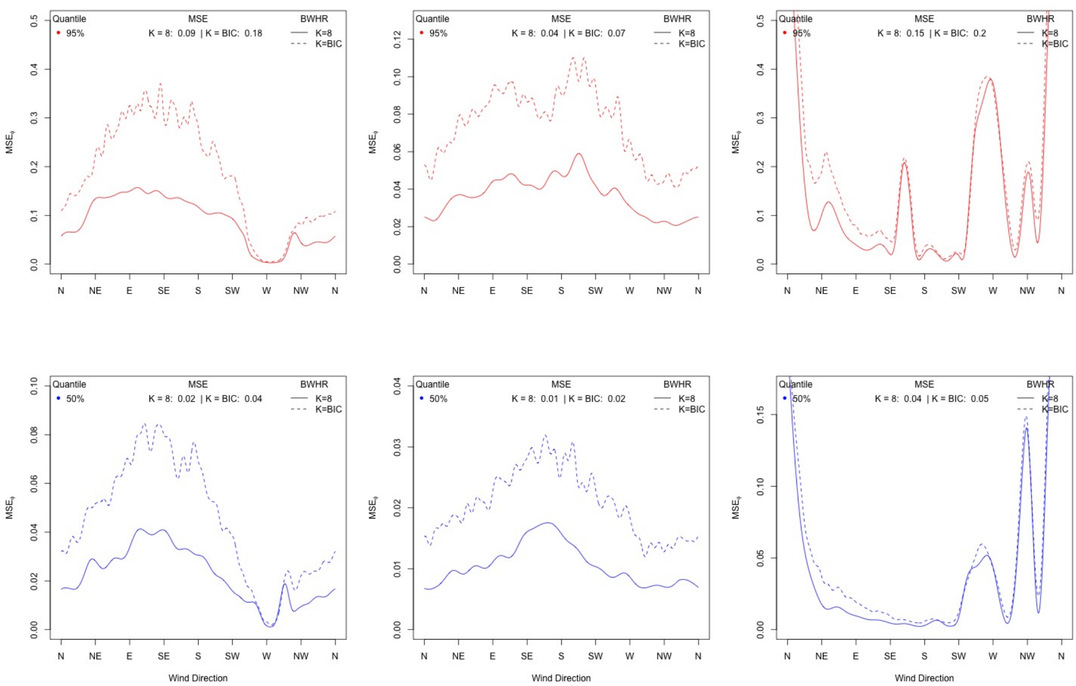

Figure 5: MSE and MSE_ϕ values of the 95% (**top row**) and 50% (**bottom row**) quantile curve of the BWHR model with $K = 8$ (solid line) and K chosen by BIC (dashed line); each column represents a location.

Analysis on the number of bins

We conduct a sensitivity analysis to understand how to bin the wind direction data. We look at different sample sizes ranging from 1000 to 70000 (r, ϕ) points, and use equal bins and unequal bins. In the case of equal bins, we divide the wind direction data so that each bin has a width of 20 degrees (i.e. $N = 18$) and 10 degrees (i.e. $N = 36$), respectively. The unequal bins are constructed such that each bin has the same amount of points and their total number matches the number of equal bins, i.e. $N = 18$ and $N = 36$, respectively. The analysis is plot in Fig. 6 helping us understand the followings:

1. When binning the wind direction with unequal bins some of the bins have a wider width causing the `BWHR` (also refered to as BinWeiHar) to break down (Fig. 11). This may happen due to the fact that fitting the wind speed data to a Weibull distribution in these large bins may be inappropriate.

2. When binning wind data that are sparse in some wind directions using equal bins, the number of bins matters. Specifically, having $N \geq 36$ can cause some bins to have no to small data points in which case the MLE optimizer might fail to converge. On the other hand, having $N < 18$ can lead to wider bins causing issues similar to that of unequal bins.

3. In the case of large sample sizes, i.e $n \geq 70000$, using $N = 36$ bins is already enough as shown in Fig. 7.

We conclude this analysis by suggesting to use equal bins to avoid having the `BWHR` method break down. Furthermore, the bins should be constructed such that they include on average at least 200 points (i.e. $N = 18$ or $N = 36$).

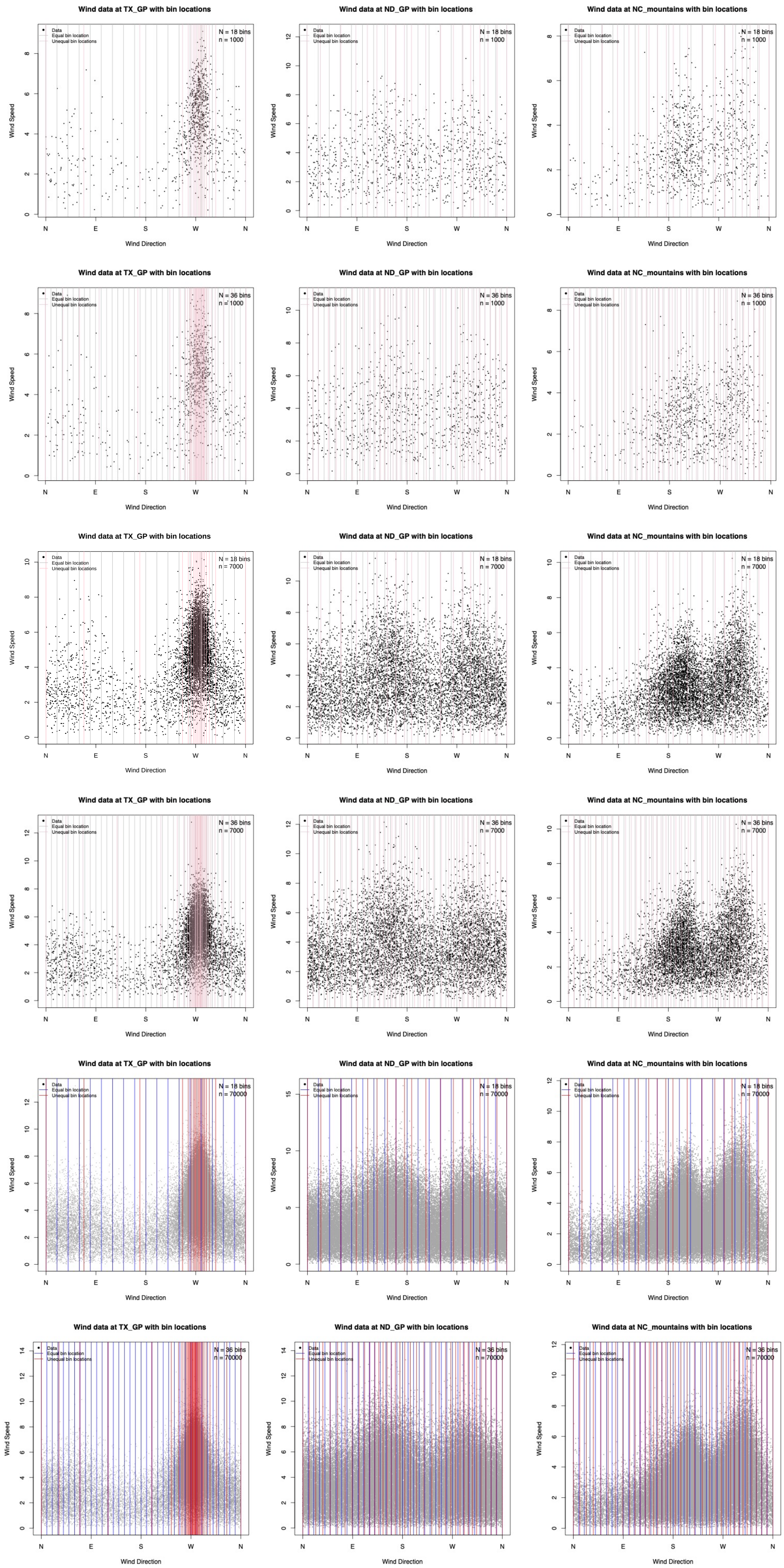

Figure 6: The wind direction domain is binned using equal bins (gray/blue lines) and unequal bins (pink/red lines) using N = 18 bins (rows 1, 3, 5) and N=36 bins (rows 2, 4, 6). Each column represents a different location.Every two rows represent a different sample size n.

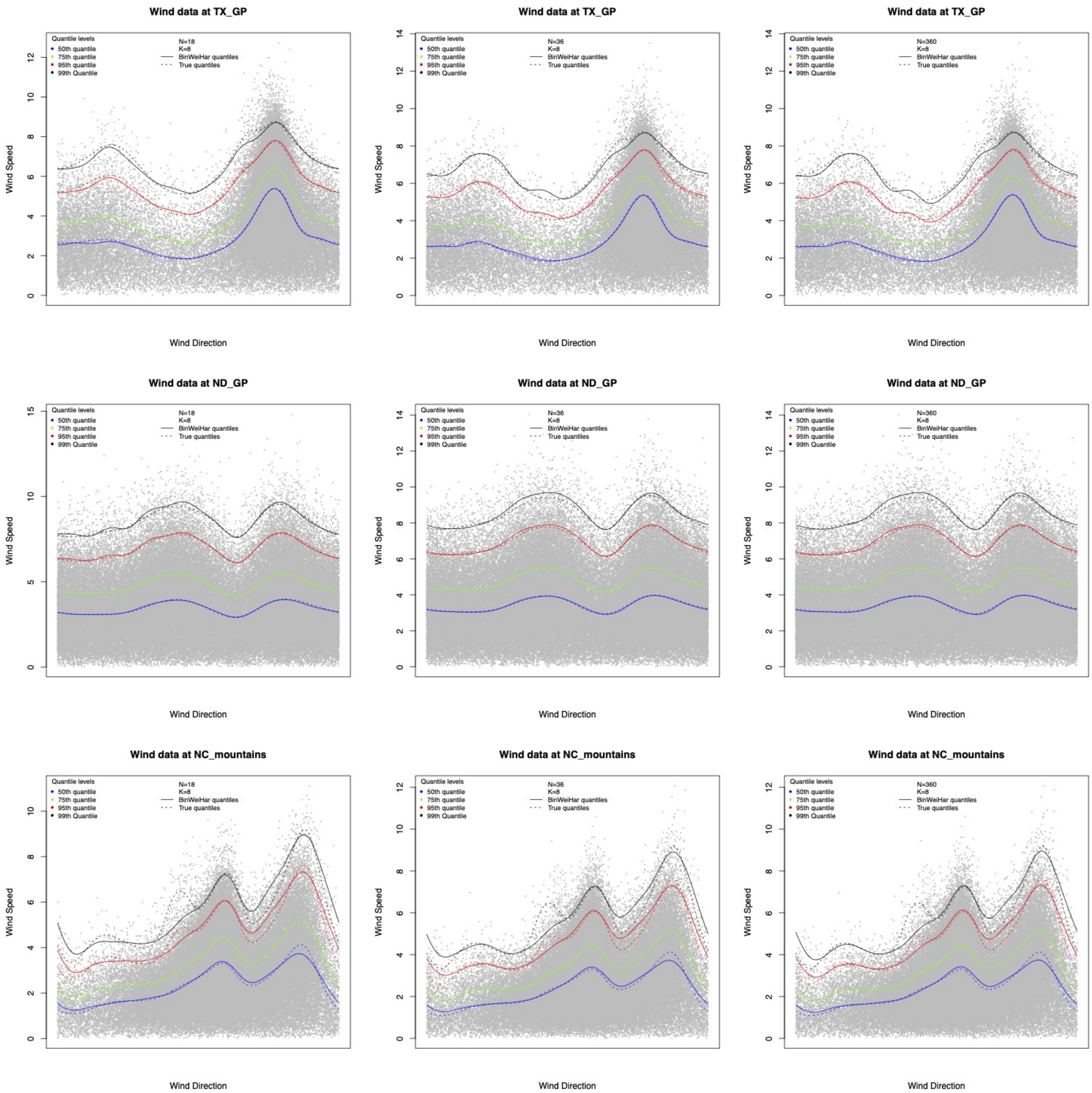

Figure 7: Qquantile curves of the directional wind speed distribution estimated using the BinWeiHar method when the sample size $n = 100000$ and $N = 18$ (left column), $N = 36$ (middle column), $N = 360$ (left columns). Each row represents a different location.

Analysis on choosing the number of Fourier series

In this section we perform an analysis on the choice of the number of pairs of Fourier series, K_α and K_β. In doing so we use $N = 18$ and $N = 36$ bins and in each case we analyze five different situations: (i) K_α and K_β are both chosen using BIC, (ii) $K_\alpha = 4$ and K_β is chosen using using BIC, (iii) $K_\alpha = 8$ and K_β is chosen using using BIC, (iv) $K_\alpha = K_\beta = 4$, (v) $K_\alpha = K_\beta = 8$. The fixed values of K_α and K_β are chosen such that the harmonic regression in Step 2 doesn't become underdetermined, i.e. we must have $N \geq 2K + 2$, where K represents K_α or K_β. We plot WIMRE for each of the five cases in Fig. 10.

We conclude this section by suggesting to use $K_\alpha = K_\beta = 8$ with $N = 36$ bins.

95% quantile BinWeiHar

	K_a, K_b from BIC	**K_a = 4, K_b from BIC**	**K_a = 8, K_b from BIC**	**K_a = K_b = 4**	**K_a = K_b = 8**
N = 18	**0.0111057405461189**	**0.0143681920625585**	**0.0111057405461189**	0.0151052293957332	**0.0112084742637167**
N = 36	0.0163102034248188	0.0168236441406529	0.0210556601856651	**0.0143046175867679**	0.0122377167251074

75% quantile BinWeiHar

	K_a, K_b from BIC	**K_a = 4, K_b from BIC**	**K_a = 8, K_b from BIC**	**K_a = K_b = 4**	**K_a = K_b = 8**
N = 18	**0.0133840201401709**	**0.0143833928998875**	**0.0133840201401709**	0.0191282493529481	**0.013316760257974**
N = 36	0.0184710706935268	0.019774937107861	0.0205630970715956	**0.0190401798894232**	0.0159884242667349

50% quantile BinWeiHar

	K_a, K_b from BIC	**K_a = 4, K_b from BIC**	**K_a = 8, K_b from BIC**	**K_a = K_b = 4**	**K_a = K_b = 8**
N = 18	0.0257743585387422	0.0254696248834977	0.0257743585387422	0.0276297977408933	0.0256166050777791
N = 36	**0.0213279442387626**	**0.0202043888120995**	**0.0205504205170147**	**0.0236621268955114**	**0.0193702477784595**

Figure 8: WIMRE values at Texas Great Plain (TX_GP) when K_α and K_β values are chosen using BIC (first column), K_α is fixed and K_β is chosen using BIC (second and third columns) and $K_\alpha = K_\beta$ are fixed (last two columns). Each table represents a different conditional quantile.

95% quantile BinWeiHar

	K_a, K_b from BIC	K_a = 4, K_b from BIC	K_a = 8, K_b from BIC	K_a = K_b = 4	K_a = K_b = 8
N = 18	**0.0205621832463836**	**0.019099306879011**	**0.0205621832463836**	0.0146796074610404	0.0205621832463836
N = 36	0.0253165508764861	0.0279329389324164	0.0269075494893134	**0.0136348390874929**	**0.0198510310896356**

75% quantile BinWeiHar

	K_a, K_b from BIC	K_a = 4, K_b from BIC	K_a = 8, K_b from BIC	K_a = K_b = 4	K_a = K_b = 8
N = 18	**0.0188532763005915**	**0.0187233121820366**	**0.0188532763005915**	**0.0125401809362793**	**0.0188532763005915**
N = 36	0.0254933718634748	0.0259272567762946	0.0258226033900824	0.0125624324969163	0.0191009964124994

50% quantile BinWeiHar

	K_a, K_b from BIC	K_a = 4, K_b from BIC	K_a = 8, K_b from BIC	K_a = K_b = 4	K_a = K_b = 8
N = 18	**0.0216836256122331**	**0.0215796825984641**	**0.0216836256122331**	0.016821834338898	**0.0216836256122331**
N = 36	0.0282671143398497	0.0280518219882675	0.0280697872821988	**0.0160582282594825**	0.0223069447337908

Figure 9: WIMRE values at North Dakota GP when K_α and K_β values are chosen using BIC (first column), K_α is fixed and K_β is chosen using BIC (second and third columns) and $K_\alpha = K_\beta$ are fixed (last two columns). Each table represents a different conditional quantile.

95% quantile BinWeiHar

	K_a, K_b from BIC	K_a = 4, K_b from BIC	K_a = 8, K_b from BIC	K_a = K_b = 4	K_a = K_b = 8
N = 18	0.0554739285694164	0.0515341615792185	0.0554739285694164	0.0624489651951732	0.0557380363671729
N = 36	**0.0512397117098671**	**0.0501693453338259**	**0.0509547271082063**	**0.0574833062375873**	**0.0484468021376172**

75% quantile BinWeiHar

	K_a, K_b from BIC	K_a = 4, K_b from BIC	K_a = 8, K_b from BIC	K_a = K_b = 4	K_a = K_b = 8
N = 18	**0.041541965992776**	**0.0410277999970727**	**0.041541965992776**	0.0498763790629277	0.0415973017671192
N = 36	0.0451700063999011	0.0465884217621606	0.0468538504243756	**0.0468238976653014**	**0.0404966690399305**

50% quantile BinWeiHar

	K_a, K_b from BIC	K_a = 4, K_b from BIC	K_a = 8, K_b from BIC	K_a = K_b = 4	K_a = K_b = 8
N = 18	**0.0482529347361654**	**0.0482281913704557**	**0.0482529347361654**	**0.0514493120791879**	**0.048295662382577**
N = 36	0.0568240685885806	0.0546545726739158	0.0549279679854316	0.0534650108411175	0.0505452257388911

Figure 10: WIMRE values at North Carolina mountains when K_α and K_β values are chosen using BIC (first column), K_α is fixed and K_β is chosen using BIC (second and third columns) and $K_\alpha = K_\beta$ are fixed (last two columns). Each table represents a different conditional quantile.

Estimating the directional wind speed distribution using BinWeiHar with equal bins, BinWeiHar with unequal bins and MLE

We perform a sensitivity analysis for estimating the directional wind speed distribution using the BinWeiHar method with equal and unequal bins and MLE. In the case of equal bins we use $N = 36$ and $K_\alpha = K_\beta = 8$. In the case of unequal bins, we bin the wind direction data so that each bin has the same amount of points. The MLE is done without any binning. The estimated 50%, 75% and 95% conditional quantiles for each method are plotted in Fig. 11 suggesting that the BinWeiHar with unequal bins and MLE methods are numerically unstable wen the wind direction data is sparse.

We conclude this section by suggesting to estimate the directional wind speed distribution using the BinWeiHar method with equal bins, i.e. $N = 36$.

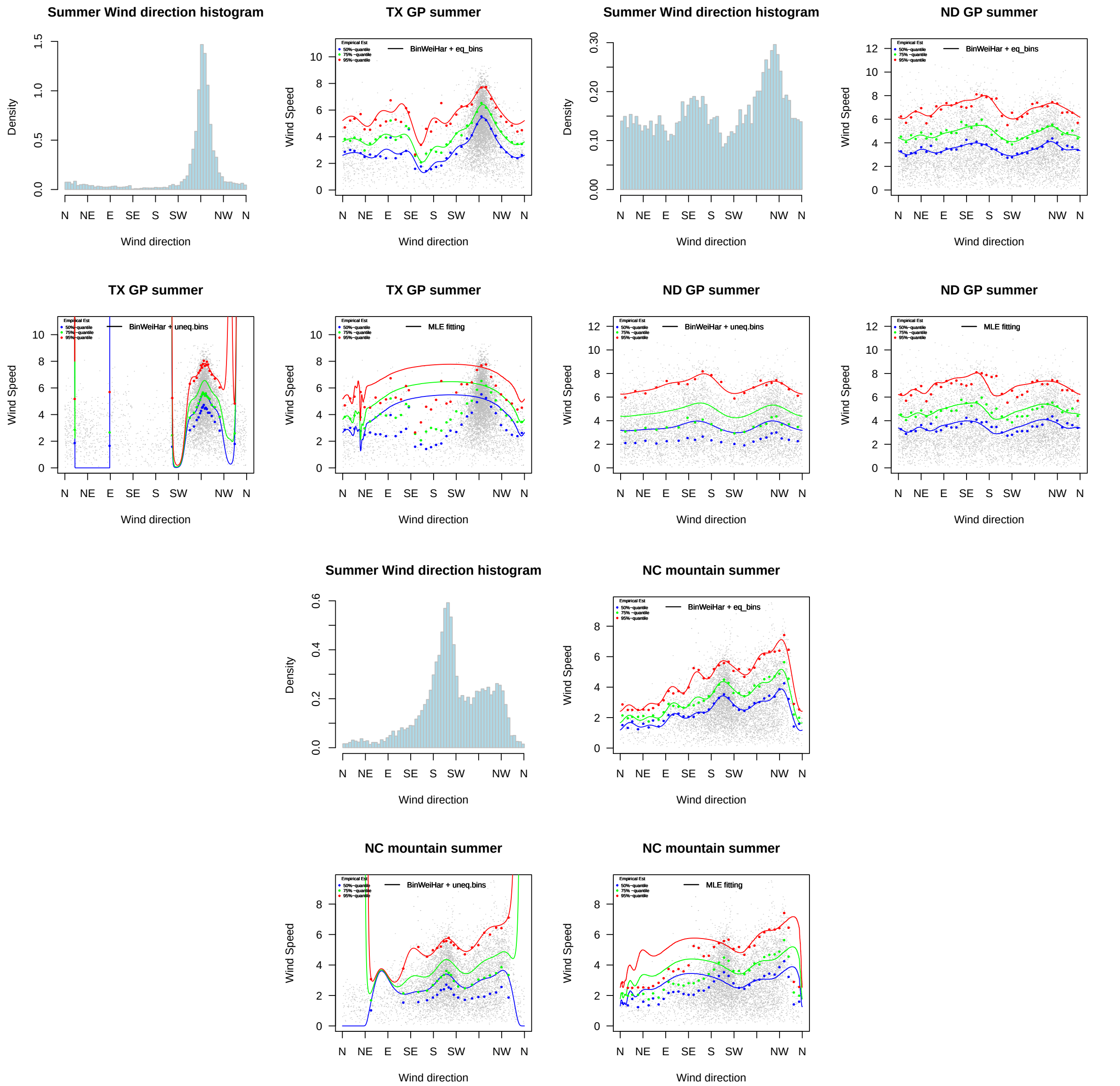

Figure 11: Sensitivity analysis for estimating the quantile curves using the BinWeiHar method with equal bins, unequal bins, and MLE at three different locations. The histograms represent the wind direction distribution at the given location.

Analysis of estimating the directional wind speed using BinWeiHar with K=4 pairs of Fourier series vs. BinWeiHar with `pbs()` and $df = 18$

We perform a study on estimating the directional wind speed distribution using BinWeiHar method with $K_\alpha = K_\beta = 8$ pairs of Fourier series and BinWeiHar where the parameter estimates are regressed on periodic B-splines with $Df = 18$. The WIMRE values are shown in Fig. 12 and they suggest that the BinWeiHar method using $K_\alpha = K_\beta = 8$ outperforms the BinWeiHar method using `pbs()` at all three locations.

	TX_GP	ND_GP	NC_mountain
BinWeiHar pbs	0.1126	0.2318	0.1101
BinWeiHar	0.0168	0.0210	0.0737

	TX_GP	ND_GP	NC_mountain
BinWeiHar pbs	0.0347	0.0807	0.0698
BinWeiHar	0.0143	0.0196	0.0695

	TX_GP	ND_GP	NC_mountain
BinWeiHar pbs	0.0364	0.0872	0.0956
BinWeiHar	0.0192	0.0231	0.0748

Figure 12: WIMRE values in estimating the conditional curves using the BinWeiHar method where the parameter estimates are regressed on periodic B-splines (**first row of each table**) and $K_\alpha = K_\beta = 8$ pairs of Fourier series (**second row of each table**). **Top table** represents the 95% conditional quantile curve, **middle table** represents the 75% conditional quantile curve and **third table** represents the 50% conditional quantile curve.

Sensitivity analysis on choosing the degrees of freedom (df) for BPQR

We estimate the directional wind speed distribution using `BPQR` with different df values and we show the plots of the estimated conditional quantile curves in Fig. 13. We observe that quantile curves estimated with lower df (5, 10) result in a smoothed curve while higher df value (30) leads to an overfitted curve. To balance the bias and variance we choose $df = 18$.

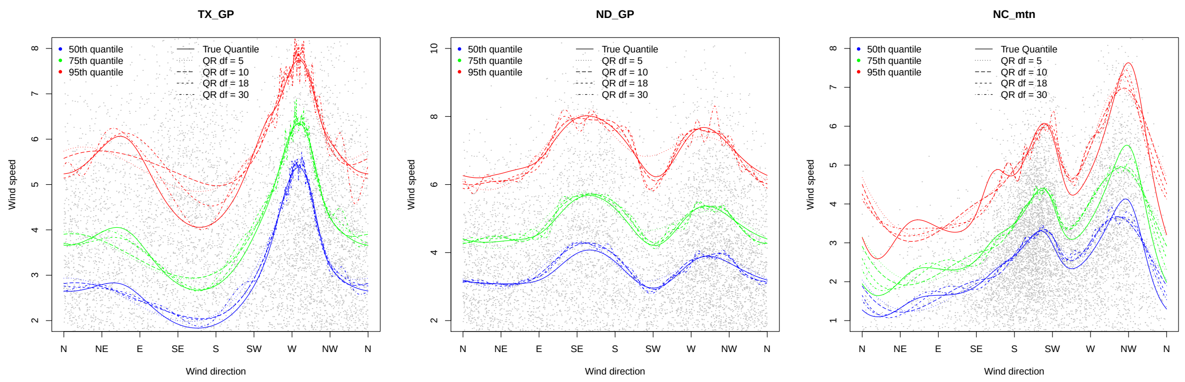

Figure 13: 95% conditional quantile (red), 75% conditional quantile (green) and 50% conditional quantile (blue) curves estimated with `BPQR` method using $df = 5$ (dotted line), $df = 10$ (long dashed line), $df = 18$ (dashed line) and $df = 30$ (dotted dashed line) along with the true quantile curves (solid line).

Analysis on the pairs of Fourier series using Mean Square Error (MSE) In our work we regress the parameters of the Weibull distribution on fixed K_α

and K_β pairs of trigonometric function, i.e. $K = K_\alpha = K_\beta = 8$. This choice was made after performing an analysis on the MSE of the BW-WLS model such that $K = 8$ and K is chosen using BIC. The results of this analysis are shown in Fig. 14 and it shows that the BW-WLS model with $K = 8$ has a smaller MSE than the BW-WLS model with K chosen using BIC.

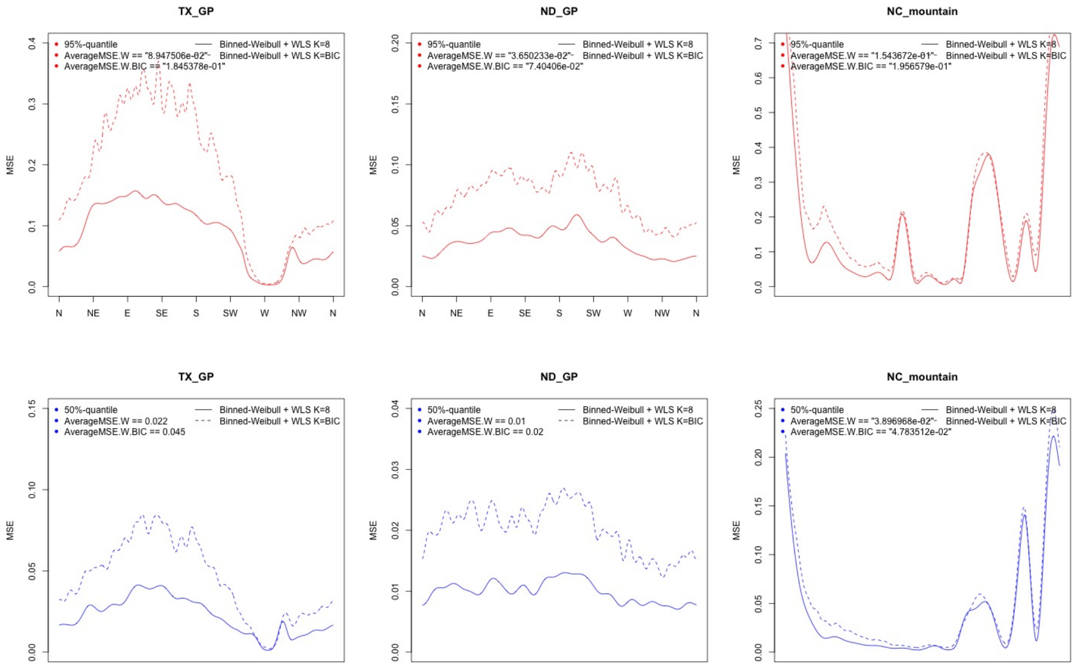

Figure 14: **Top row:** MSE of the 95% quantile curve of the `BWHR` model with $K = 8$ (solid line) and K chosen by BIC (dashed line) along with the corresponding average MSE values; **bottom row:** MSE of the 50% quantile curve of the `BWHR` model with $K = 8$ (solid line) and K chosen by BIC (dashed line) along with the corresponding average MSE values; each column represnts a location.

Analysis using MSE and the weighted integrated mean square error (WIMSE)

The von Mises mixture distribution is evaluated using WIMSE. This metric is computed by first discretizing the wind direction domain using a set of grid points $\{\phi_1, \phi_2, \ldots, \phi_n\}$ with a sufficiently large n (we use n = 629 in our study) and using the following formulas:

$$\text{WIMSE}_{\phi_i} = \frac{1}{500} \sum_{j=1}^{500}, f_{true}(\phi_i)(f_{mod,j}(\phi_i) - f_{true}(\phi_i))^2, \quad i = 1, \ldots n,$$

$$\text{WIMSE} = \frac{\sum_{i=1}^{n} \text{WIMSE}_{\phi_i}}{\sum_{i=1}^{n} f_{true}(\phi_i)}.$$

where $f_{true}(\phi_i)$ and $f_{mod,j}(\phi_i)$, $j = 1, \ldots, 500$, denote the true and estimated wind direction distribution at a given angle ϕ_i, $i = 1, \ldots, n$. The results are plotted in Fig. 15 and show a small WIMSE value suggesting that a von Mises mixture distribution is a good fit to wind direction.

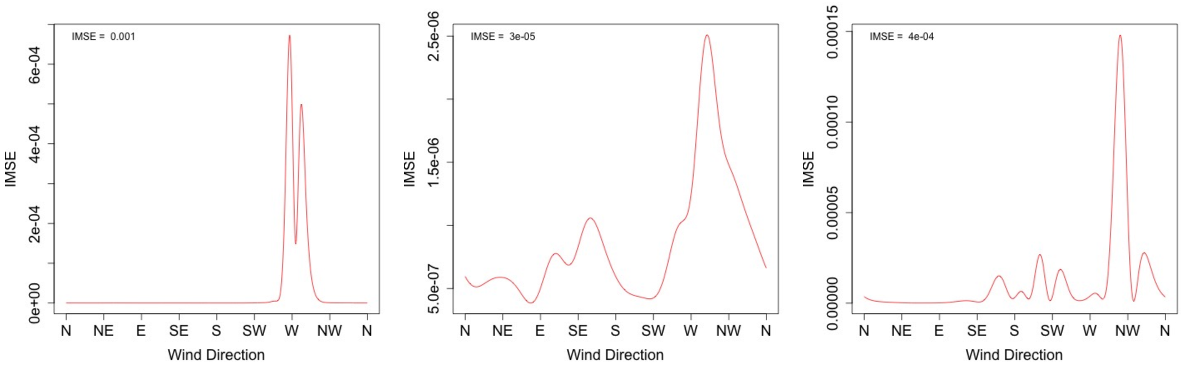

Figure 15: WIMSE_ϕ values for a set of discretized wind directions along with the WIMSE values at each of the three locations.

The `BWHR` and `BPQR` models are first evaluated using the MSE between the true and estimated conditional quantiles at a given quantile level and angle, over all 500 replicates. Similarly to the case of wind direction, the MSE is computed by discretizing the wind direction domain using a set of grid points $\{\phi_1, \phi_2, \ldots, \phi_n\}$ with a sufficiently large n (we use $n = 629$ in our study) and for each angle ϕ_i, $i = 1, \ldots n$, using the following formulas:

$$\text{MSE}_{\phi_i} = \frac{1}{500} \sum_{j=1}^{500} (q_{mod,j}(\phi_i) - q_{true}(\phi_i))^2,$$

where $q_{true}(\phi_i)$ and $q_{mod,j}(\phi_i)$, $j = 1, \ldots, 500$, denote the true and estimated conditional quantiles at a given quantile level and angle ϕ_i, $i = 1, \ldots, n$. Next the MSE_{ϕ_i} values are averaged over the discretized domain using the following formula:

$$MSE_{avg} = \frac{1}{n} \sum_{i=1}^{n} MSE_{\phi_i}$$

The MSE_{ϕ_i}, $i = 1, \ldots, n$, results considering the 95% quantile level are depicted in the top row of Fig. 16 along with the MSE_{avg} values. A quick analysis of the graphs should reflect that for the most angles the `BWHR` method has a smaller MSE than `BPQR`. However, there are angles for which the MSE of the `BWHR` method is higher than that of `BPQR`. Comparing these results to the estimated wind direction distribution, depicted in the middle row of Fig. 16, we see that these circumstances happen when the density of wind direction is very small. This suggests the inclusion of the density of wind direction when computing the error. Hence, the use of the weighted integrated square error, where the weights are the density of wind direction, i.e.,

$$\text{WIMSE} = \int_0^{2\pi} MSE_\phi \cdot f_\Phi(\phi)\, d\phi,$$

where $f_\Phi(\phi)$ represents the density of wind direction (ϕ). To compute this value we proceed similarly to the MSE case, where we dicretize the wind direction domain to a set of grid points $\{\phi_1, \phi_2, \ldots, \phi_n\}$ and the WIMSE_ϕ along with the WIMSE is computed using the following formulas:

$$
\begin{aligned}
\text{WIMSE}_{\phi_i} &= \frac{1}{500}\sum_{j=1}^{500}, f_{\Phi_i}(\phi_i)(q_{mod,j}(\phi_i) - q_{true}(\phi_i))^2, \quad i = 1, \dots n, \\
\text{WIMSE} &= \frac{\sum_{i=1}^{n} \text{WIMSE}_{\phi_i}}{\sum_{i=1}^{n} f_{\Phi_i}(\phi_i)}.
\end{aligned}
$$

The results of the WIMSE_{ϕ_i}, $i = 1, \dots n$, along with the WIMSE values for the 95% quantile level are depicted in the bottom row of Fig. 16 (Fig. 17 for quantile differences) and Table 1 (Table 2 for quantile differences) showing a decreased error in the `BWHR` model. Furthermore, in Fig. 18 we plot the distribution of WIMSE for the 95% quantiles of the `BWHR` and `BPQR` models. As the boxplots show, the distribution of the `BWHR` WIMSE has a smaller median and is less variable than that of `BPQR`. Also, the scatter plots included in the bottom row of Fig. 18 tell us that `BWHR` model has a lower WIMSE than `BPQR` 76.6% of times at the TX_GP location, 94% of times at ND_GP and 95.5% of times at NC_mountains.

Table 1: WIMSE values for the `BWHR` and `BPQR` methods at the three locations

Metric	**TX_GP**	**ND_GP**	**NC_mountains**
WIMSE `BWHR`	7.303083e-04	3.497012e-02	4.463615e-03
WIMSE `BPQR`	9.663166e-04	6.145615e-02	6.439603e-03

Table 2: WIMSE values for the differences in future and present 95% quantile curves of the Binned Weibul (BW) - WLS and QR models at the three locations

Metric	**TX_GP**	**ND_GP**	**NC_mountain**
WIMSE `BWHR`	-2.960907e-04	-9.550839e-02	-1.280624e-03
WIMSE `BPQR`	-1.390461e04	-4.445749e-02	-6.279233e-04

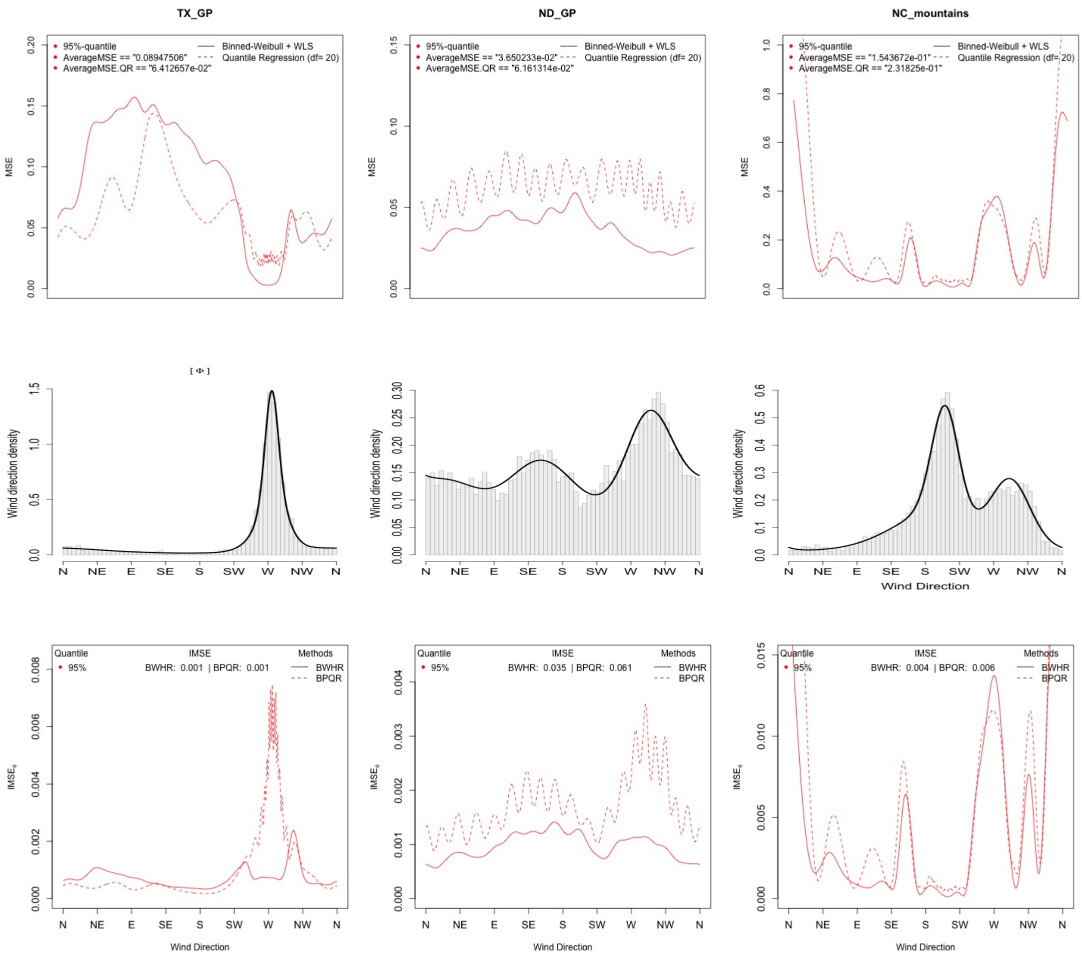

Figure 16: Top row: MSE values in the 95% conditional quantile of the BWHR (solid line) and of the BPQR method (dashed line) at three different locations and at different angles; middle row: estimates of the wind direction distribution at three different locations; bottom row: WISE values in the 95% conditional quantile of the BWHR method (solid line) and of the BPQR method (dashed line) at three different locations and at different angles.

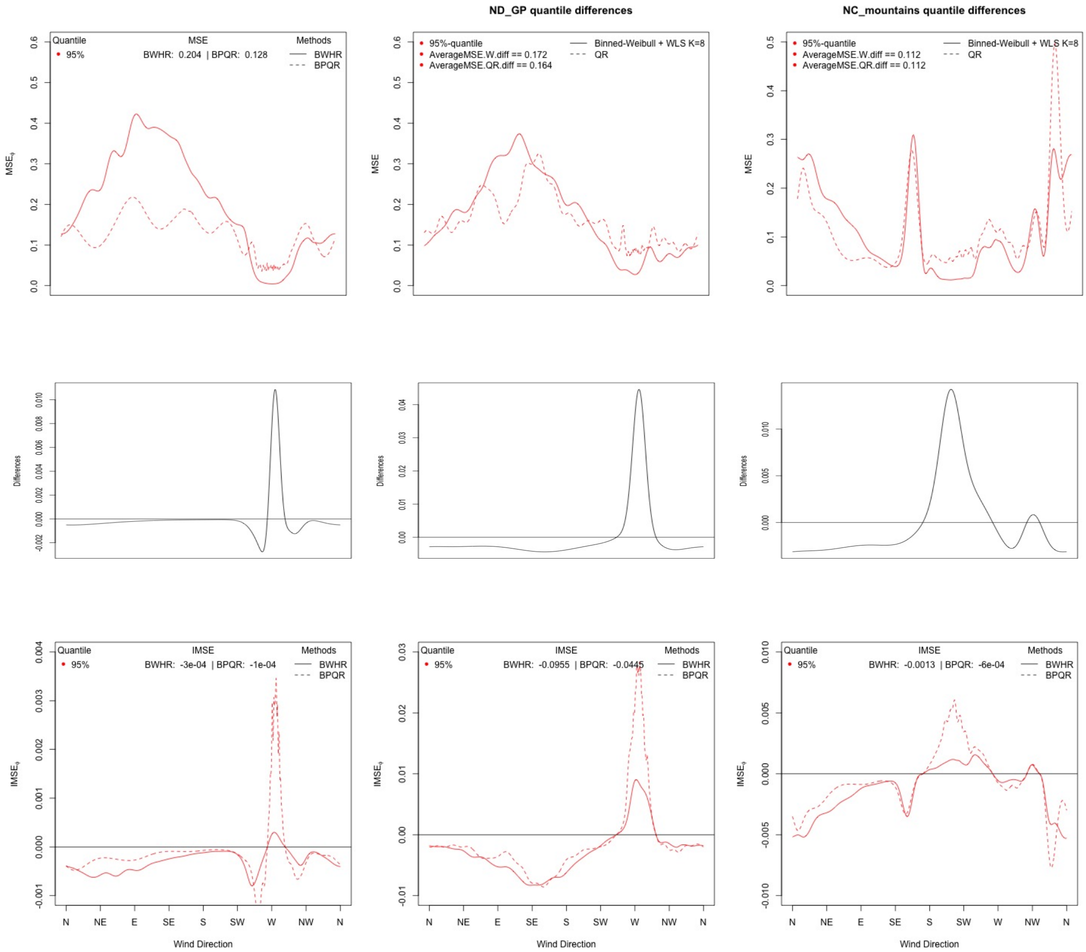

Figure 17: **Top row:** MSE values in the differences of the 95% conditional quantiles of the future and present `BWHR` (solid line) and `BPQR` (dashed line) at three different locations and at different angles; **middle row**: estimates of the differences in future and present wind direction distributions at three different locations; **bottom** row: WISE values in the differences of the 95% conditional quantile of the future and present `BWHR` (solid line) and `BPQR` (dashed line) at three different locations and at different angles

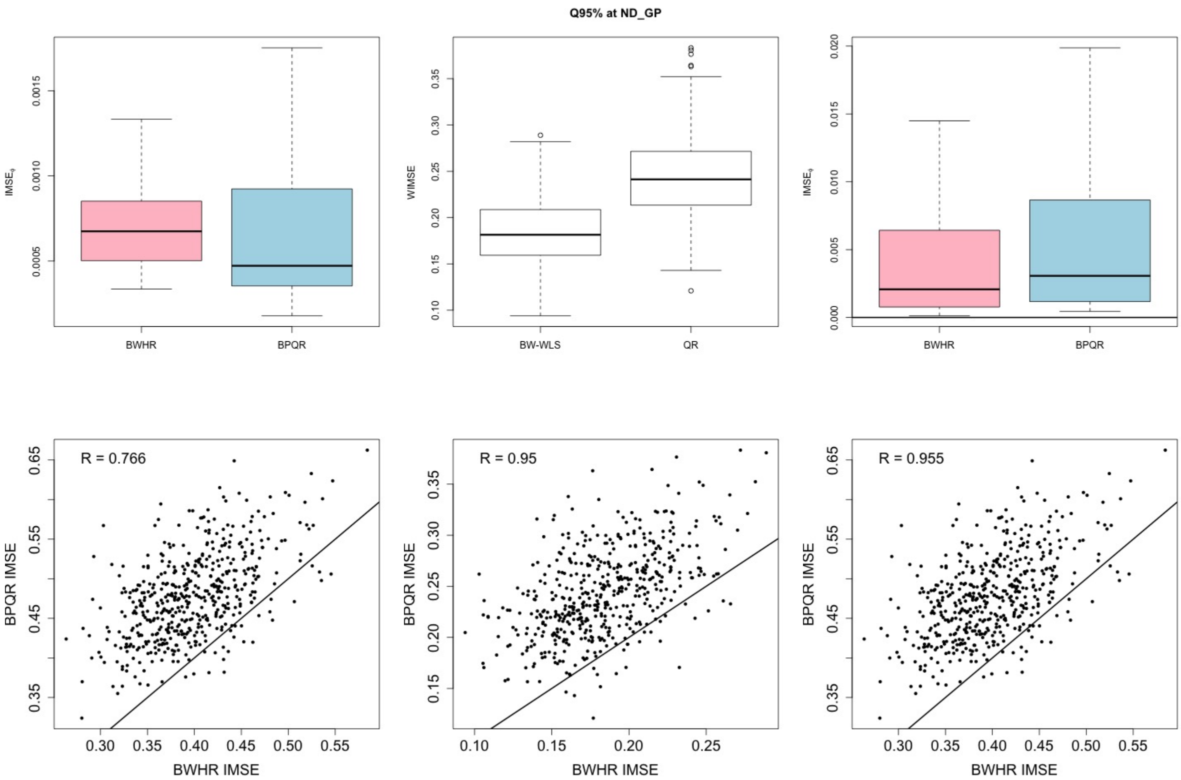

Figure 18: **Top row:** Boxplots of the WIMSE for the 95% conditional quantiles; **bottom row:** Scatter plots of the `BWHR` vs the `BPQR` WIMSE where the solid line represents the line $y = x$.