HW #5: State Estimation in Oil Well Drilling

Due on Wednesday, May 1, 2024

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Problem 1: Dynamic System Modeling

(a).

The modeling objective is to estimate the drill bit speed by linking the measurements that are observable within the equations of motion for the drill bit assembly. This is to monitor for signs of improper rotation, from sticking or slipping, and allow for speed controllers to be designed to prevent any unwanted oscillations in the drill string.

Inputs:
The controllable input to this system is the applied table torque $T(t)$ which is driven by the motor.

The uncontrollable input is the frictional torque $T_f(t)$ at the bit/bottom portion.

Output:
The outputs are the rotational speed of the drill bit $\omega_B(t)$, and the measured table speed $\omega_T(t)$.

Parameters:

- Spring Constant: k
- Drag Coefficient: b
- Rotational Inertia (Top/Bottom): $J_{T/B}$

(b).

Using Newton's second law for rotation:

$$\sum_i T_i = J\alpha$$

Consider the upper part of the shaft, taking the summation of forces to equal the rotational inertia J_T times the angular acceleration of the top α_T:

$$T(t) - b\omega_T(t) - k[\theta_T(t) - \theta_B(t)] = J_T\alpha_T(t) \tag{1}$$

Next, consider the bottom of the system and set the summation equal to the bit/bottom acceleration $\alpha_B(t)$:

$$-T_f(t) - b\omega_B(t) + k[\theta_T(t) - \theta_B(t)] = J_B\alpha_B(t) \tag{2}$$

Given that angular acceleration is the time derivative of velocity, and velocity that of position $\alpha = \dot{\omega} = \ddot{\theta}$ we solve for the following states from Eq. 1 & 2 respectively:

$$\begin{aligned}
\dot{\omega}_T(t) &= -\frac{1}{J_T}b\omega_T(t) - \frac{1}{J_T}k[\theta_T(t) - \theta_B(t)] + \frac{1}{J_T}T(t) \\
\dot{\omega}_B(t) &= -\frac{1}{J_B}b\omega_B(t) + \frac{1}{J_B}k[\theta_T(t) - \theta_B(t)] - \frac{1}{J_B}T_{f(t)} \\
\dot{\theta}_T &= \omega_T \\
\dot{\theta}_B &= \omega_B
\end{aligned}$$

(c).

From the equations above we find $x = \begin{bmatrix}\omega_T(t) & \omega_B(t) & \theta_T(t) & \theta_B(t)\end{bmatrix}$ and the matrix A:

$$A = \begin{bmatrix} -\frac{b}{J_T} & 0 & -\frac{k}{J_T} & \frac{k}{J_T} \\ 0 & -\frac{b}{J_B} & \frac{k}{J_B} & -\frac{k}{J_B} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Considering the input is the applied table torque $T(t)$ and unmeasured frictional torque $T_f(t)$ matrix B is:

$$B = \begin{bmatrix} \frac{1}{J_T} & 0 \\ 0 & -\frac{1}{J_B} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For the matrix C we consider our measured output $y = \omega_T(t)$:

$$C = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{b}{J_T} & 0 & -\frac{k}{J_T} & \frac{k}{J_T} \\ 0 & -\frac{b}{J_B} & \frac{k}{J_B} & -\frac{k}{J_B} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -\frac{1}{J_T} & 0 \\ 0 & \frac{1}{J_B} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$$

Problem 2: Observability Analysis

(a).

Listing 1: Code Output

```
O4 =

    1.0000         0         0         0
   -0.0500         0   -0.0200    0.0200
   -0.0175    0.0200    0.0010   -0.0010
    0.0019   -0.0050    0.0020   -0.0020

Rank of Observability Matrix for four-state system

ans =

     3
```

The observability of the four-state system is 3, which does not equal the rank of the A matrix. This means we have too many unknowns compared to the number of state equations. This also means they could not be fully coupled equations.

(b).

For $\theta = \theta_T - \theta_B$:

$$\begin{aligned}
\dot{\omega}_T(t) =& -\frac{1}{J_T} b\omega_T(t) - \frac{1}{J_T} k\theta(t) + \frac{1}{J_T} T(t) \\
\dot{\omega}_B(t) =& -\frac{1}{J_B} b\omega_B(t) + \frac{1}{J_B} k\theta(t) - \frac{1}{J_B} T_{f(t)} \\
\dot{\theta} =& \omega_T - \omega_B
\end{aligned}$$

$$A = \begin{bmatrix} \frac{-b}{J_T} & 0 & \frac{k}{J_T} \\ 0 & \frac{-b}{J_B} & \frac{-k}{J_B} \\ 1 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{J_T} & 0 \\ 0 & \frac{-1}{J_B} \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

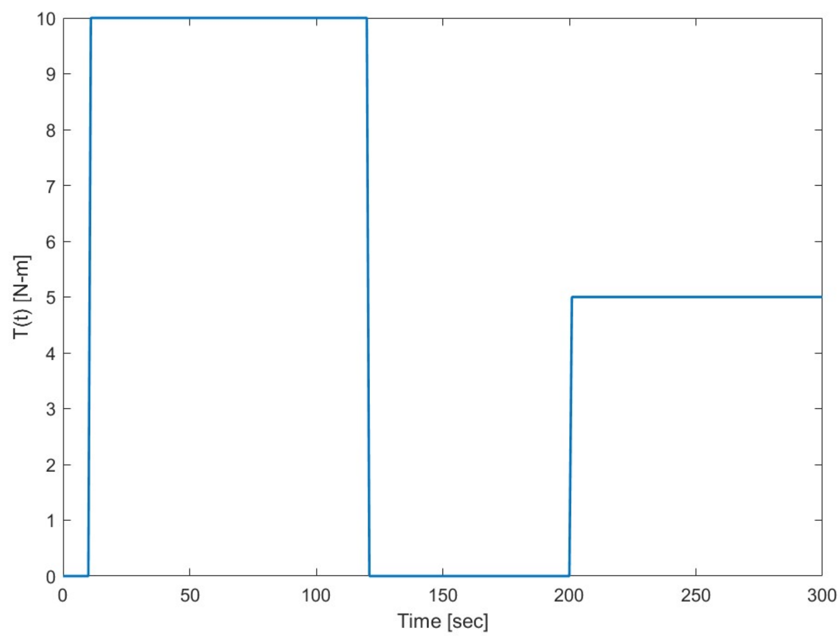
(c).

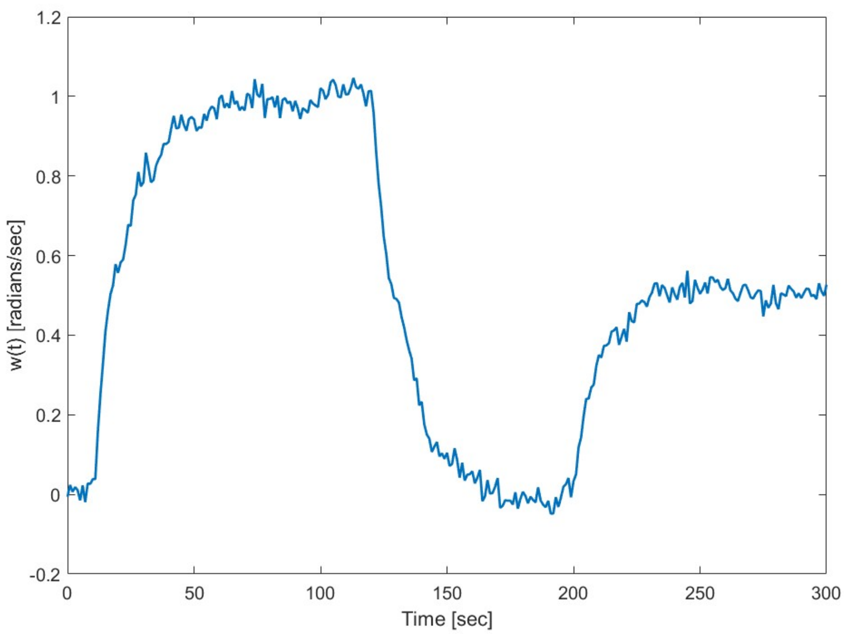
Listing 2: Code Output

```
O =
    1.0000         0         0
   -0.0500         0   -0.0200
   -0.0175    0.0200    0.0010
Rank of Observability Matrix for three-state system
ans =
     3
```

The observability of this three-state system is 3, which matches the rank of the A matrix. Thus this is now an observable system.

Problem 3: Measurement Data





Problem 4: Luenberger Observer

(a).

The following is the code output giving the system's eigenvalues:

Listing 3: Code Output

```
Eigenvalues of 3-state system:

Eg_A =

  -0.0832 + 0.0000i
  -0.0834 + 0.2986i
  -0.0834 - 0.2986i

-----------Stability-----------
Lambda = -0.083229, -0.083385, -0.083385
The system is Stable
```

The Luenberger Observer Equations are as follows:

$$\begin{aligned}\dot{\hat{x}}(t) =& A\hat{x}(t) + Bu(t) + \underbrace{L(y-\hat{y})}_{\text{Observer Gain}}\\ \hat{y}(t) =& C\hat{x}(t)\end{aligned}$$

As we disregard the D matrix, we also disregard the unmeasured frictional torque.

$$\begin{aligned}\dot{\omega}_T(t) =& -\frac{1}{J_T}b\omega_T(t) - \frac{1}{J_T}k\theta(t) + \frac{1}{J_T}T(t) + L(y-\hat{y})\\ \dot{\omega}_B(t) =& -\frac{1}{J_B}b\omega_B(t) + \frac{1}{J_B}k\theta(t)\\ \dot{\theta} =& \omega_T - \omega_B\end{aligned}$$

(b).

The design for the system is as follows:

Listing 4: Designing the output error injection gain

```
Re_Pole    = (Eg_A(1) + move )*Luen_s;
i_pole     = (Eg_A(2) + move )*Luen_s;
i_pole_n   = (Eg_A(3) + move )*Luen_s;

% Assign eigen values
lam = [Re_Pole, i_pole,  i_pole_n];

% Compute observer gain
% (See Remark 3.1 in Notes. Use "place" command)
L = place(A',C', lam)'

% State-space Matrices for Luenberger Observer
A_lobs = (A - L*C);
B_lobs = [B - L*D, L];
C_lobs = C;
D_lobs = [0,0];
```

(c).

First we disregard the frictional torque as such:

$$B = \begin{bmatrix} \frac{1}{J_T} & 0 \\ 0 & -\frac{1}{J_B} \\ 0 & 0 \end{bmatrix} \longrightarrow B = \begin{bmatrix} \frac{1}{J_T} \\ 0 \\ 0 \end{bmatrix}$$

Then we create the Luenberger observer such that:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B * u + L(y - y_hat)$$

From this we calcualate the A_{lobs} as:

$$A_{lobs} = (A - L \cdot C)$$

Leaving the terms:

$$Bu(t) + Ly(t)$$

Thus to account for this we consider:

$$B_{lobs} = \begin{bmatrix} B - LD, L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

The following are the equations encoded for the state-space matrices of the Luenberger observer:

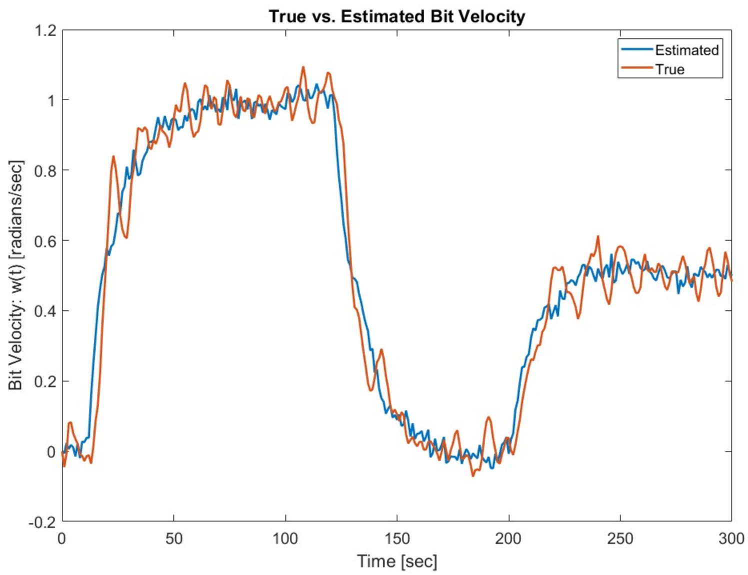
$$A_{lobs} = (A - L \cdot C) \qquad B_{lobs} = (B - L \cdot D) \qquad C_{lobs} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

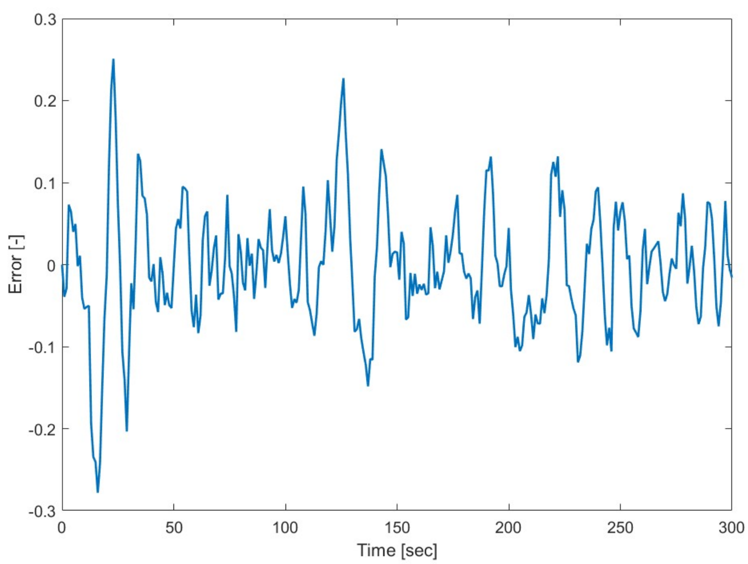
Luenberger Observer continued on next page...

(d).

I used the following eigen values calculated within the program, and a multiplication of 4 to the eigenvalues to create the graph below. This had nominal errors as compared to the un-tunend version which had a much more linear trend.

```
-----Chosen eigenvalues-----
-1.2929
-0.4935
-0.4935
----------------------------
```





Problem 5: Kalman Filter (KF) Design

(a).

The Kalman filter uses the following LTI energy system model:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \qquad x(0) = x_0, \qquad x, w \in \mathbb{R}^n, \qquad u \in \mathbb{R}^p$$
$$y_m(t) = Cx(t) + Du(t) + n(t), \qquad y_m, n \in \mathbb{R}^q$$

Next to complete the code `ode_kf.m` we use the state equation for the Kalman filter $\dot{\hat{x}}(t)$, and equaitons 47 & 48 in chapter 5:

$$L(t) = \Sigma(t)C^T N^{-1} \qquad \& \qquad \dot{\Sigma}(t) = \Sigma(t)A^T + A\Sigma(t) + W - \Sigma(t)C^T N^{-1} C\Sigma(t)$$

Listing 5: Kalman Filter Equations

```
% Compute Kalman Gain (Look at Chapter 3, Section 4)
L = Sig * C' * (1/N);

% Kalman Filter equations
x_hat_dot = A * x_hat + B * T + L * (y_m - (C * x_hat));

% Riccati Equation for Sigma
Sig_dot = Sig * A' + A * Sig + W - Sig * C' * (1/N) * C * Sig;
```

We assign the conditions of $N = 0.02$, $\Sigma_0 = I_{3\times3}$, and $W \in \mathbb{R}^{3\times3}$ with a value of 0.0042 shown below:

Listing 6: Noise Covariances

```
42 * eye(3); N = 0.02; Sig0 = eye(3);
```

The value was chosen as it was able to bring the maximum error well below 0.3, and thus was sufficient to produce the estimation.

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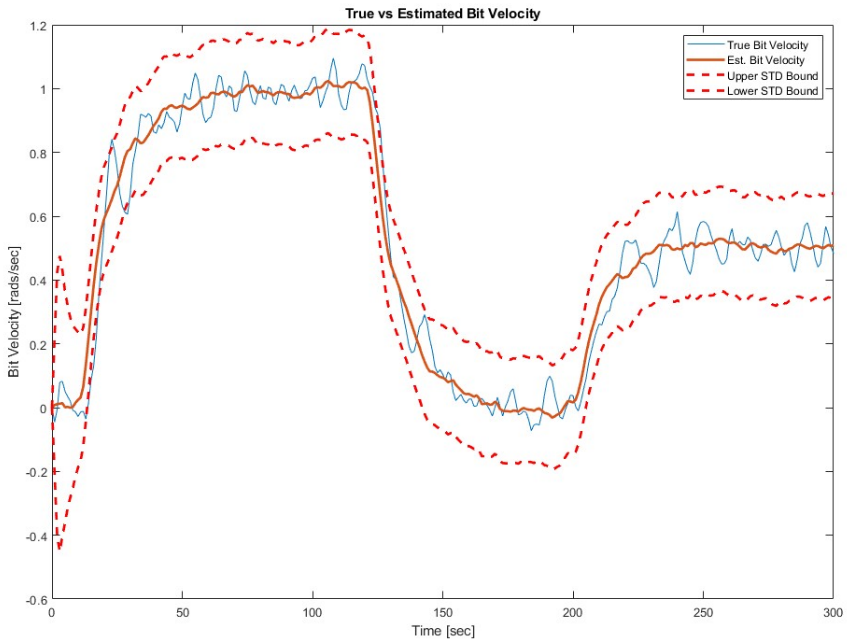
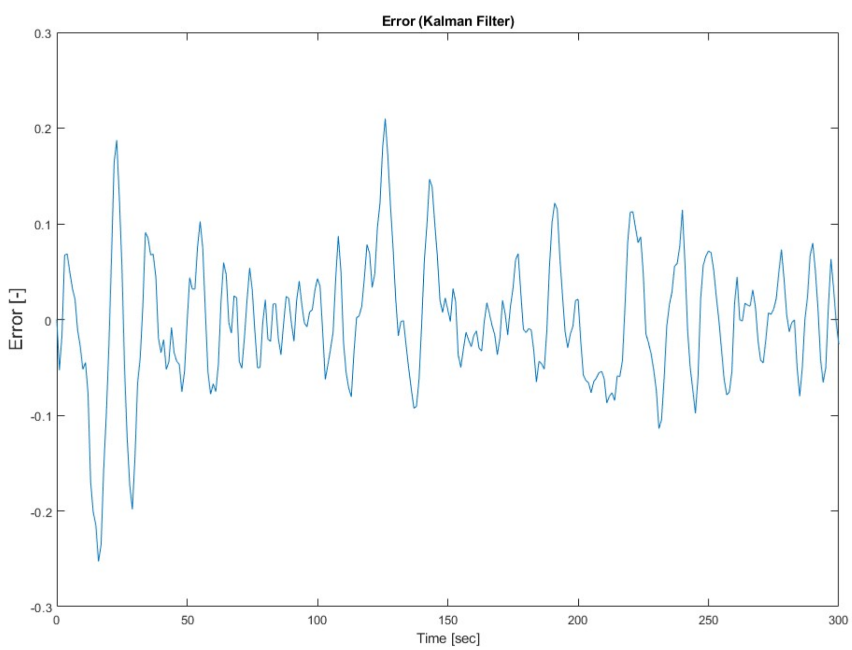
(b).


True vs Estimated Bit Velocity
True Bit Velocity
Est. Bit Velocity
Upper STD Bound
Lower STD Bound
Bit Velocity [rads/sec]
Time [sec]


Error (Kalman Filter)
Error [-]
Time [sec]

Problem 6: Extended Kalman Filter (EKF) Design

(a).

For the Jacobians:

$$F(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)) \qquad \& \qquad H(t) = \frac{\partial h}{\partial x}(\hat{x}(t), u(t))$$

were the position is $\theta(t)$ for the given states of the system, and considering the nonlinear spring torque $k_1\theta(t) + k_2\theta^3(t)$:

$$\begin{aligned}
\dot{\omega}_T(t) =& -\frac{1}{J_T}b\omega_T(t) - \frac{1}{J_T}[k_1\theta(t) + k_2\theta^3(t)] + \frac{1}{J_T}T(t) \\
\dot{\omega}_B(t) =& -\frac{1}{J_B}b\omega_B(t) + \frac{1}{J_B}[k_1\theta(t) + k_2\theta^3(t)] - \frac{1}{J_B}T_{f(t)} \\
\dot{\theta} =& \omega_T - \omega_B
\end{aligned}$$

Computing the Jacobians similarly to equation 69 within chapter 5, we find:

$$F(t) = \frac{d}{d\theta}\begin{bmatrix} 0 & 1 & -1 \\ \frac{-k_1}{J_T}\theta - \frac{k_2}{J_B}\theta^3 & \frac{-b}{J_T} & 0 \\ \frac{k_1}{J_T}\theta + \frac{k_2}{J_B}\theta^3 & 0 & \frac{-b}{J_B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ \frac{-k_1}{J_T} - \frac{3k_2}{J_B}\theta^2 & \frac{-b}{J_T} & 0 \\ \frac{k_1}{J_T} + \frac{3k_2}{J_B}\theta^2 & 0 & \frac{-b}{J_B} \end{bmatrix}$$

$$H(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

(b).

My best attempt at the extended Kalman filter: