Tree Indexes

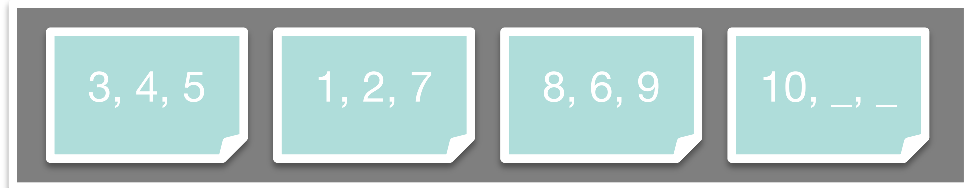
Alvin Cheung

Fall 2022

Reading: R & G Chapter 10

Simple Idea?

Input
Heap
File

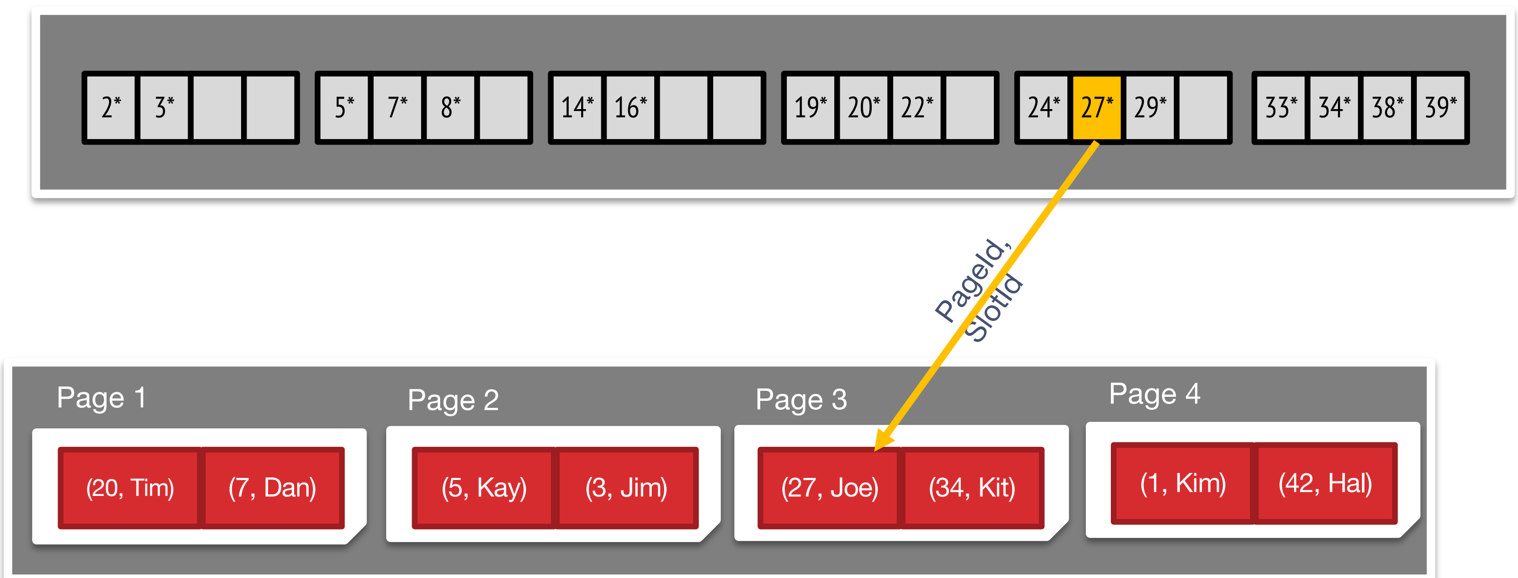


- **Step 1:** Sort heap file & leave some space
  - Pages physically stored in logical order (sequential access)
  - Maintenance as new records are added/deleted is a pain, can lead to B updates in the worst case (move everything down or up)

- **Step 2**: Use binary search on this sorted heap file: $\log_2(B)$ pages read
  - Fan-out of 2 → deep tree → lots of I/Os
  - Examine entire records just to read key during search: would prefer $\log_2(K)$ where K is number of pages to store keys << B

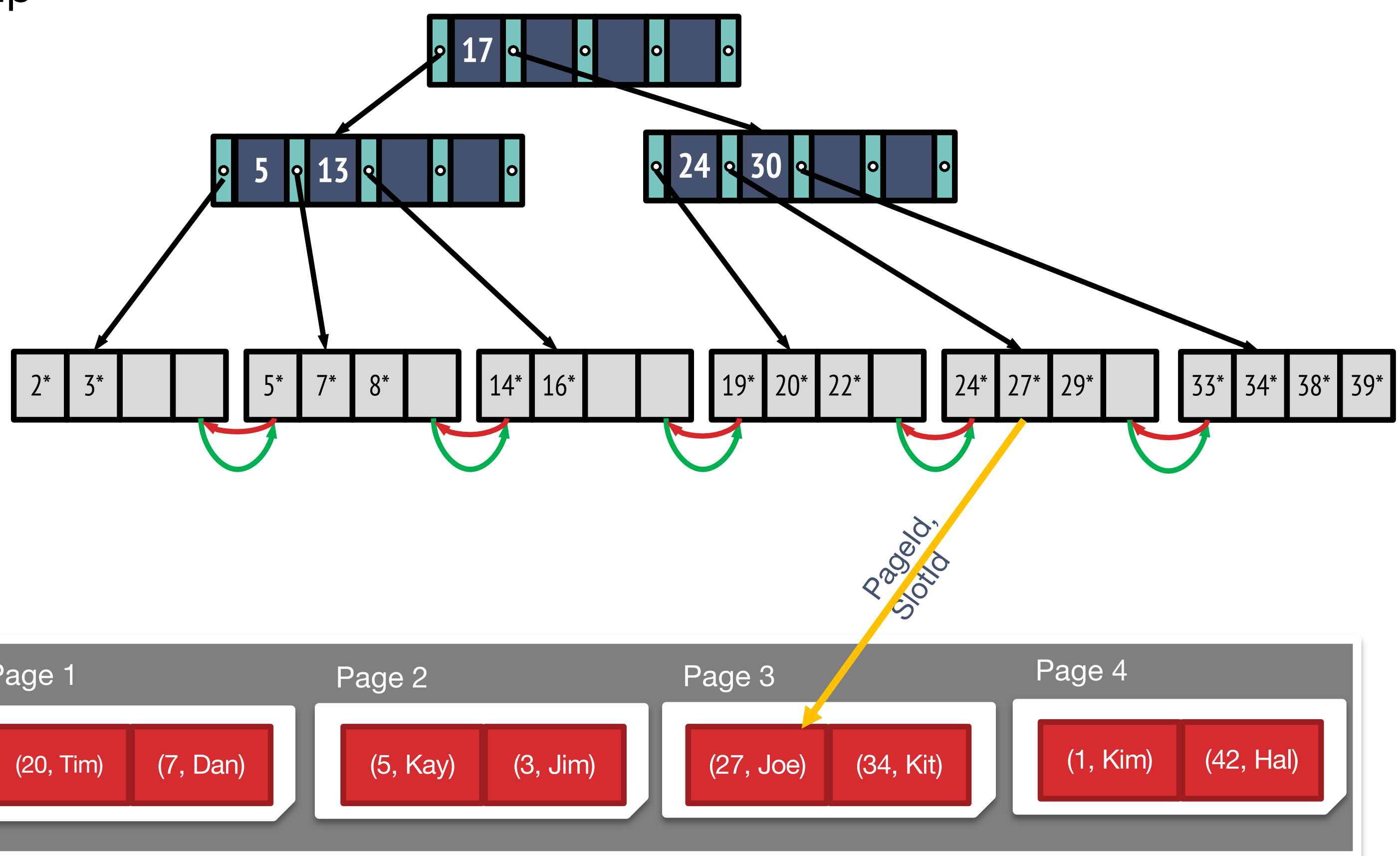
Let's fix these assumptions

- **Idea:** Keep separate (compact) key lookup pages, laid out sequentially
  - Maintain key ➔ recordID mapping [We'll revisit this later]
- No need to sort heap file anymore! Just sort key lookup pages
- Use binary search on **lookup pages** as opposed to on all of the data pages
  - Still have a deep tree due to fan-out of 2 → lots of I/Os
- Also, maintenance of the key lookup pages is a pain! Worst case K updates



Let’s fix these assumptions, take 2

- **Idea:** repeat the process!
- Lookup pages for the lookup pages
- And then lookup pages for the lookup pages for the lookup pages, ….
- Let’s set fanout to be >> 2
- That is essentially the idea behind B+ Trees …
- We’ll find out why the pointers are helpful later



Enter the B+ Tree, More Formally

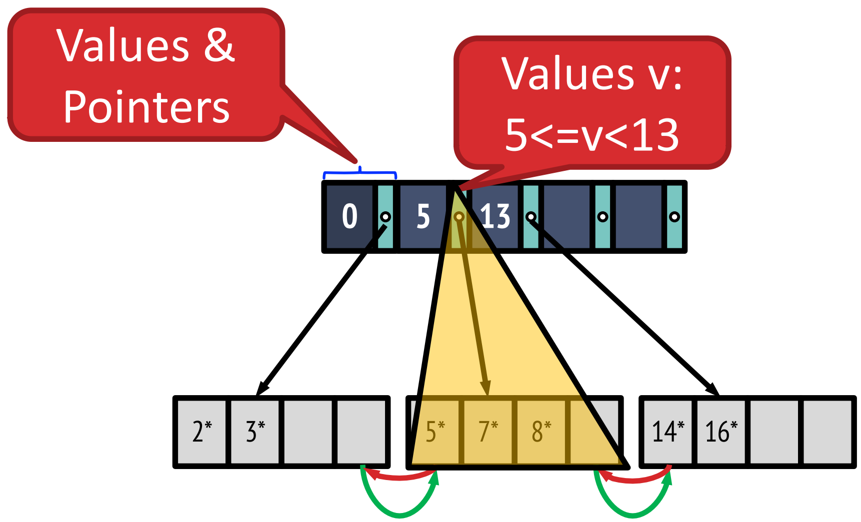
- **Dynamic Tree Index**
  - Always Balanced
  - High fanout
  - Support efficient insertion & deletion
    - Grows at root not leaves!
- “+”? B-tree that stores data entries in leaves only
  - Helps with range search

B+ Trees: How to Read an Interior Node

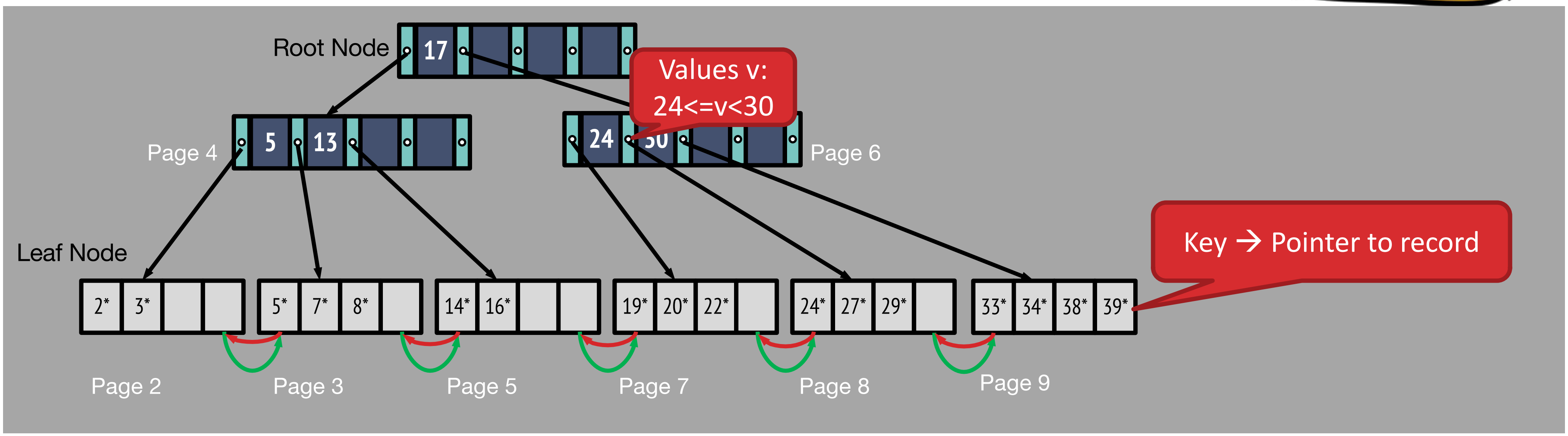
- Node[…, (K_L, P_L), (K_R, P_R)…]

  means that

  All tuples in range
  K_L <= K < K_R are in tree P_L

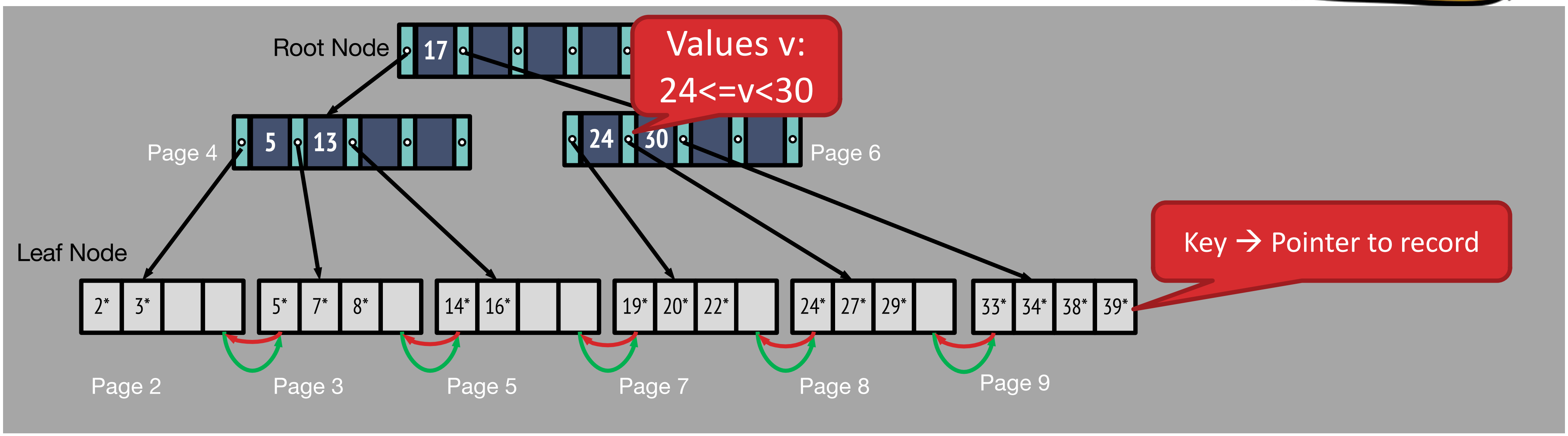


Example of a B+ Tree



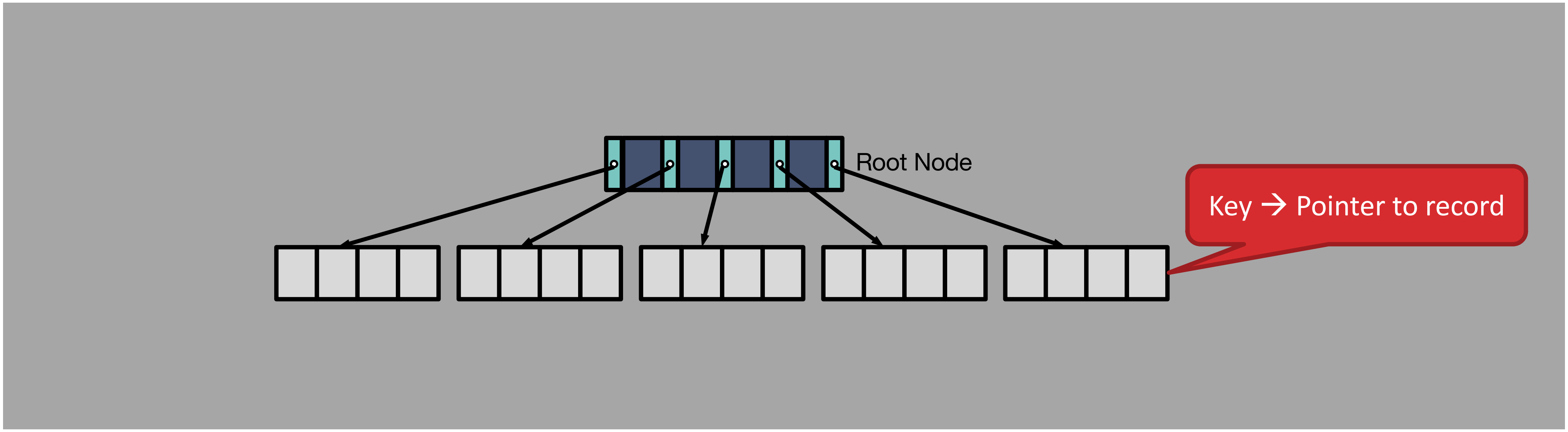
- Property 1: Nodes in a B+ tree must obey an **occupancy invariant**
  - Guarantees that lookup costs are bounded
  - Invariant: each interior node is full beyond a certain minimum: in this case [and typically], **at least half full**
    - This minimum, d, is called the **order of the tree**
    - Here, max # of entries = 4. Thus d = 2.
    - Guarantee: d <= # entries <= 2d. In this tree, 2 <= # entries <= 4
  - Root doesn't need to obey this invariant
  - Same invariant holds for leaf nodes: at least half full (d may differ, here it is the same)

Example of a B+ Tree



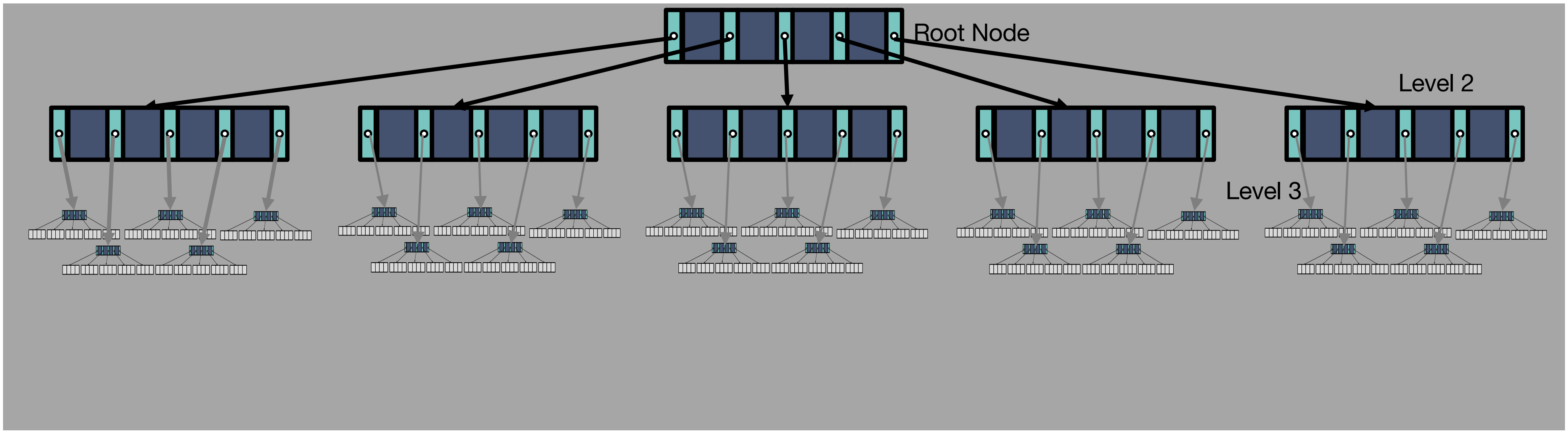
- Property 1: Nodes in a B+ tree must obey an **occupancy invariant**
  - Each interior/leaf node is full beyond a certain minimum **d**
- Property 2: Leaves need not be stored in sorted order (but often are)
  - Next and prev. pointers help examining them in sequence [useful as we will see soon]

B+ Trees and Scale



- How many records can this height 1 B+ tree index?
  - Max entries = 4; Fan-out (# of pointers) = 5
  - **Height 1:** 5 (pointers from root) x 4 (slots in leaves) = 20 Records

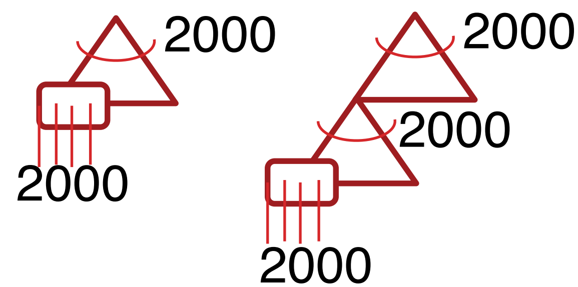
B+ Trees and Scale Part 2



- How many records can this height 3 B+ tree index?
  - Fan-out = 5; Max entries = 4
  - **Height 3:** 5 (root) x 5 (level 2) x 5 (level 3) x 4 (leaves) = 5^3 x 4 = 500 Records

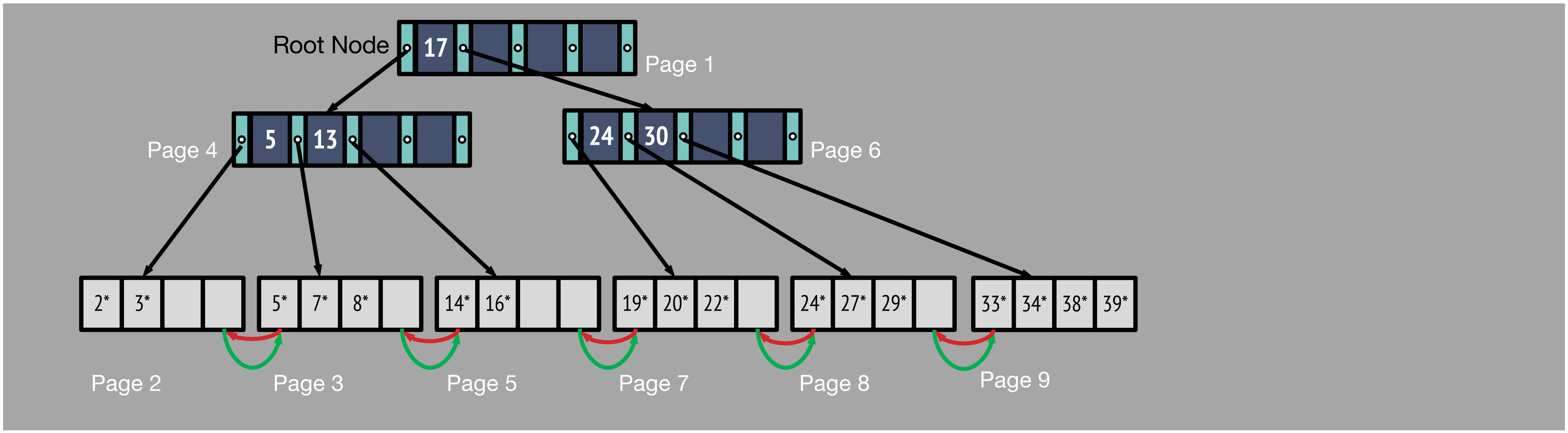
Extending this: B+ Trees in Practice

- (Warning: Sloppy back-of-the-envelope calculation!)
- Say 128KB pages, with around 40B per (val, ptr) pair
  - Max entries = roughly 128KB/40B = approx. 3000
  - Max fanout = 3000+1 = approx. 3000
  - Say 2/3 are filled on average
    - Average fan-out/entries = approx. 2000



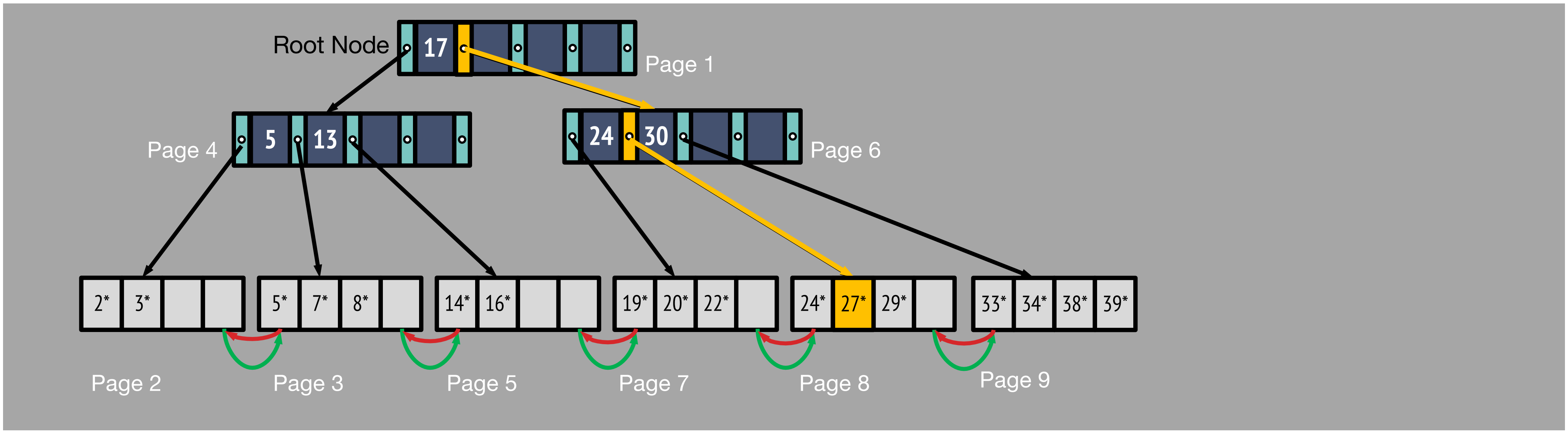
- At these capacities
  - Height 1: 2000 (pointers from root) x 2000 (entries per leaf) = 2000^2 = **4,000,000**
  - Height 2: 2000 (pointers from root) x 2000 (pointers from level 2) x 2000 (entries per leaf) = 2000^3 = **8,000,000,000 records!!**
- **Core takeaway: Even depths of 3 allow us to index a massive # of records!**

Searching the B+ Tree



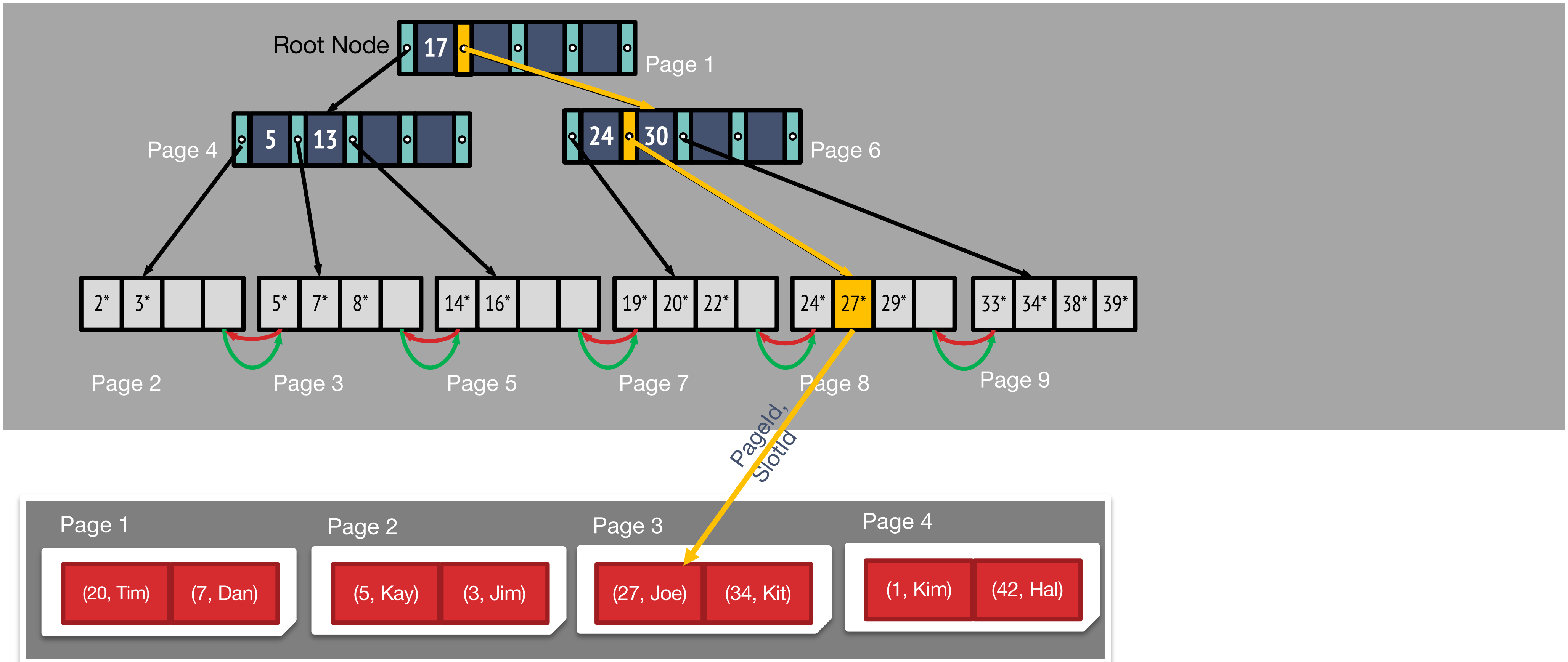
- Procedure:
  - Find split on each node (Binary Search)
  - Follow pointer to next node

Searching the B+ Tree: Find 27

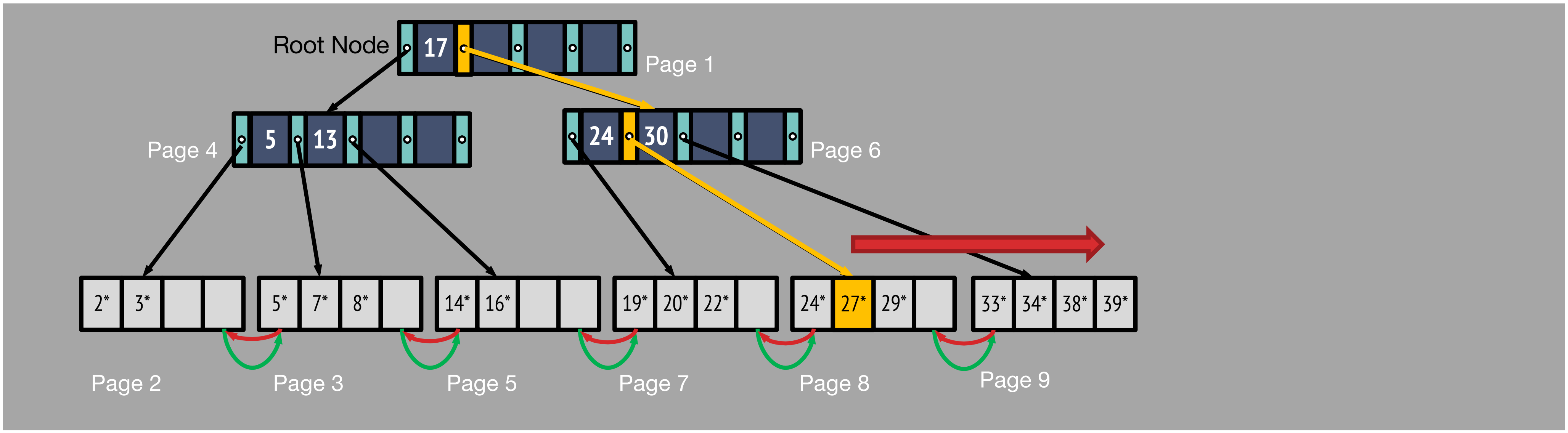


- Find key = 27
  - Find split on each node (Binary Search)
  - Follow pointer to next node

Searching the B+ Tree: Fetch Data

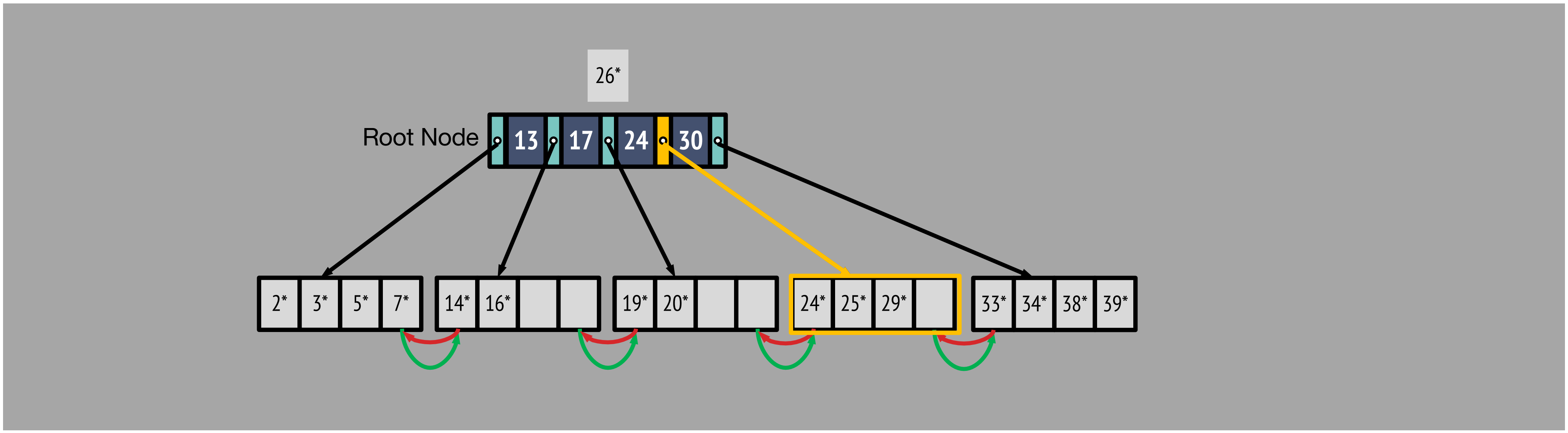


Searching the B+ Tree: Find 27 and up



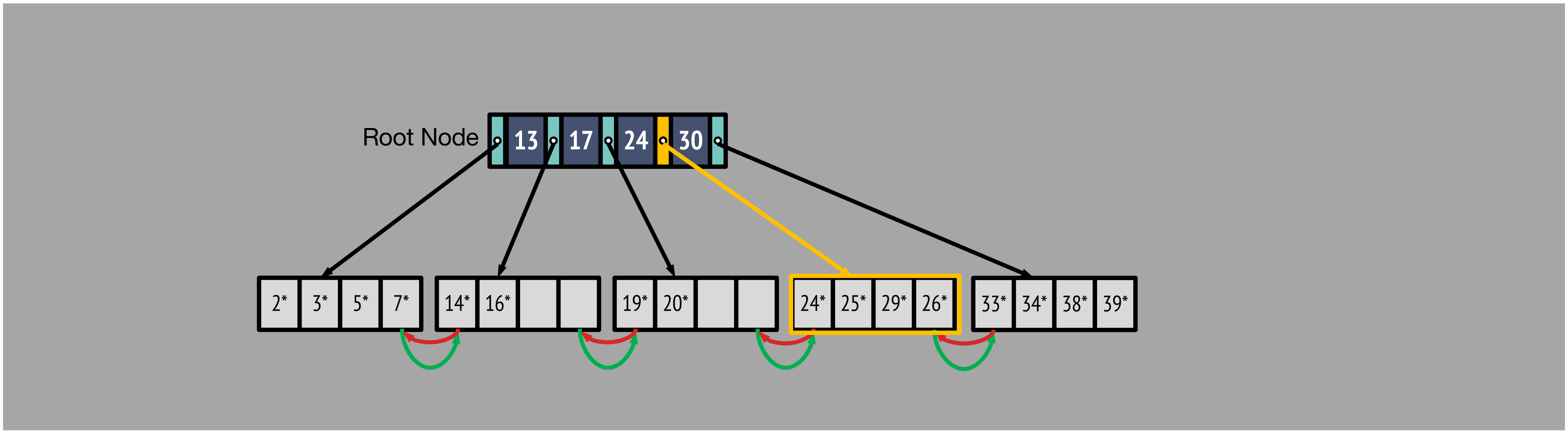
- Find keys >=27
  - Find 27 first, then traverse leaves following “next” pointers in leaf
  - This is an example of a *range scan*: find all values in [a, b]
  - Benefit: no need to go back up the tree! Saves I/Os

Inserting 26* into a B+ Tree Part 1



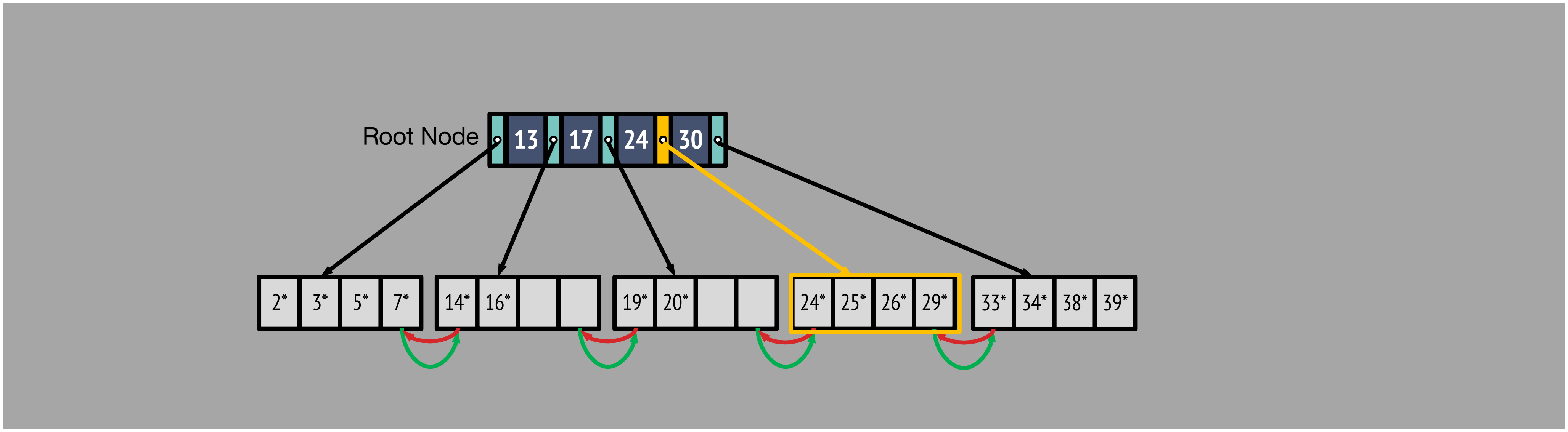
- Find the correct leaf

Inserting 26* into a B+ Tree Part 2



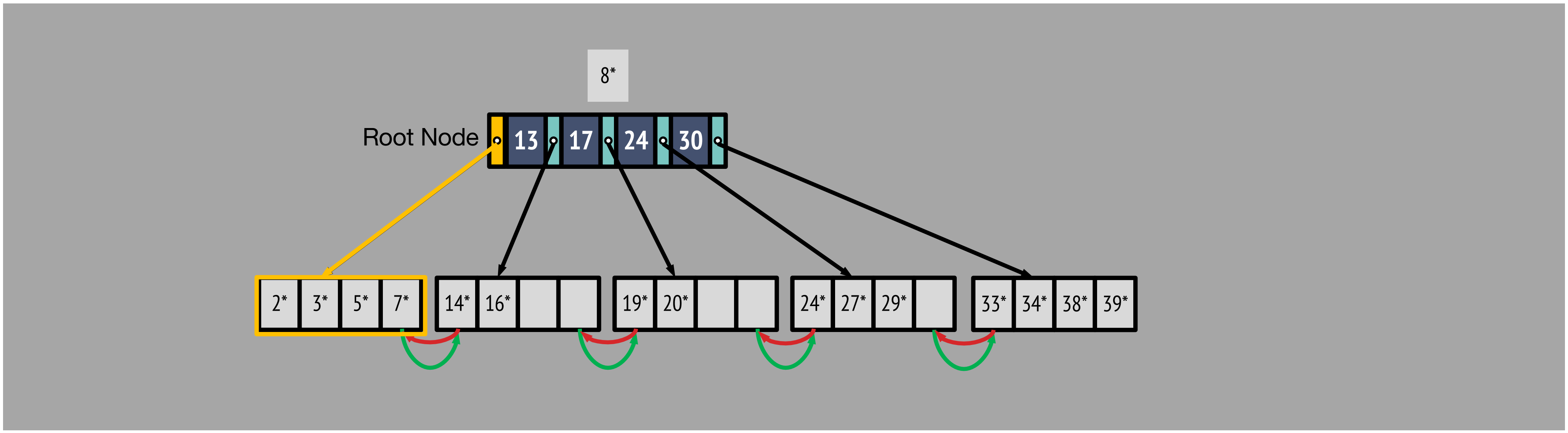
- Find the correct leaf
- If there is room in the leaf just add the entry

Inserting 26* into a B+ Tree Part 3



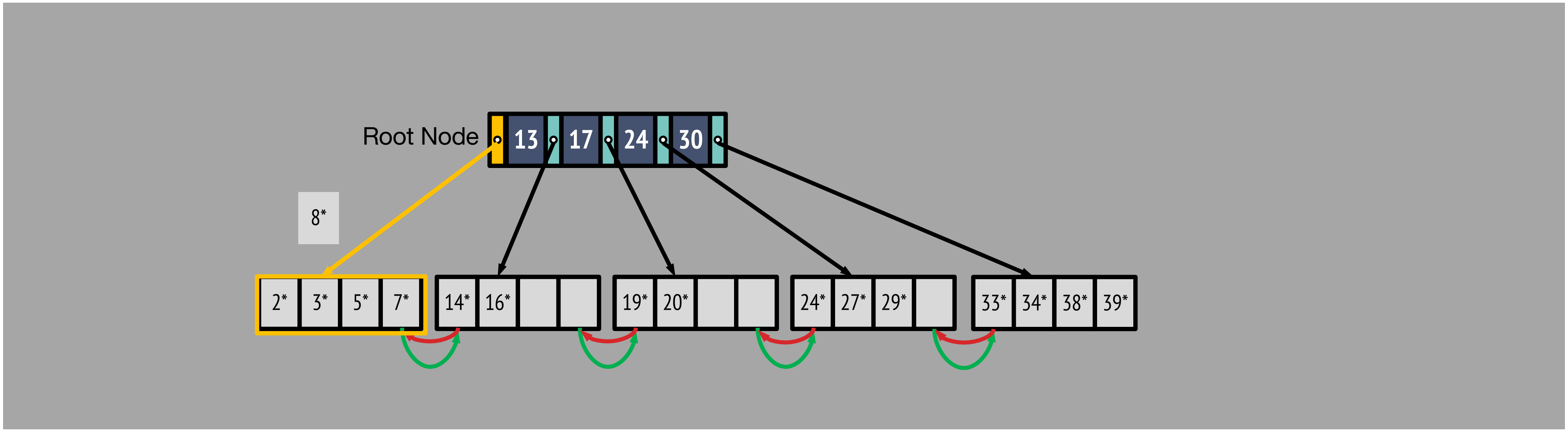
- Find the correct leaf
- If there is room in the leaf just add the entry
    - Sort the leaf page by key

Inserting 8* into a B+ Tree: Find Leaf



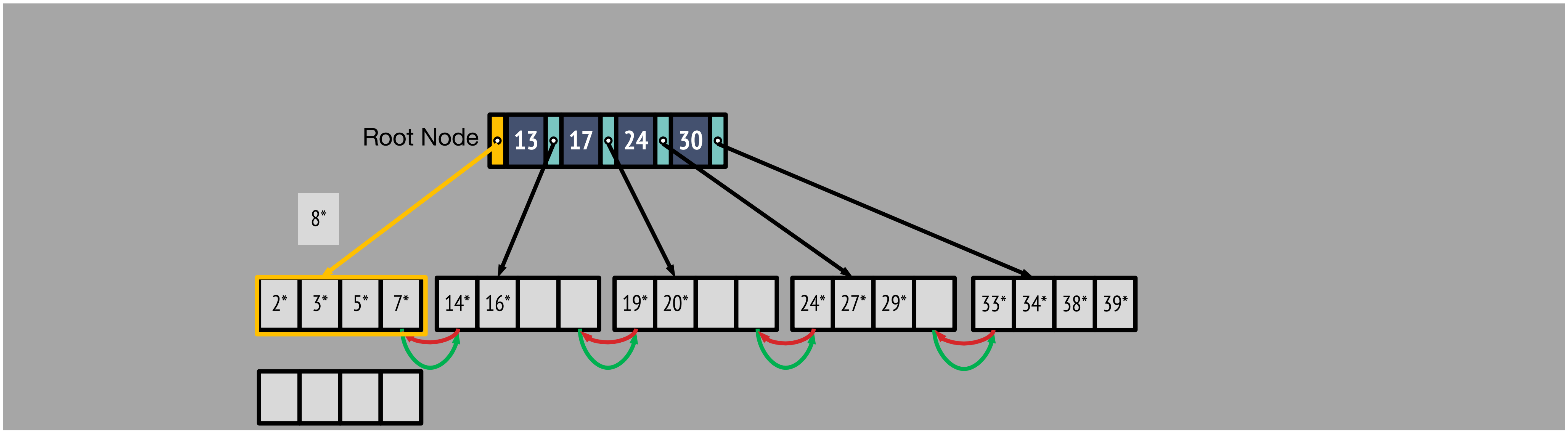
- Find the correct leaf

Inserting 8* into a B+ Tree: Insert



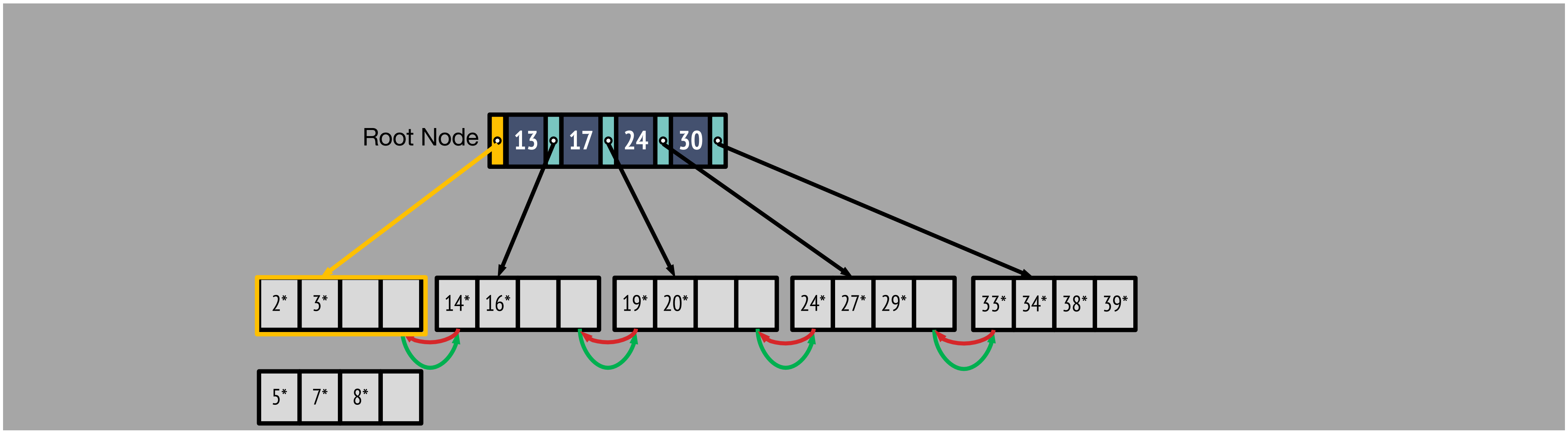
- Find the correct leaf
  - Split leaf if there is not enough room

Inserting 8* into a B+ Tree: Split Leaf



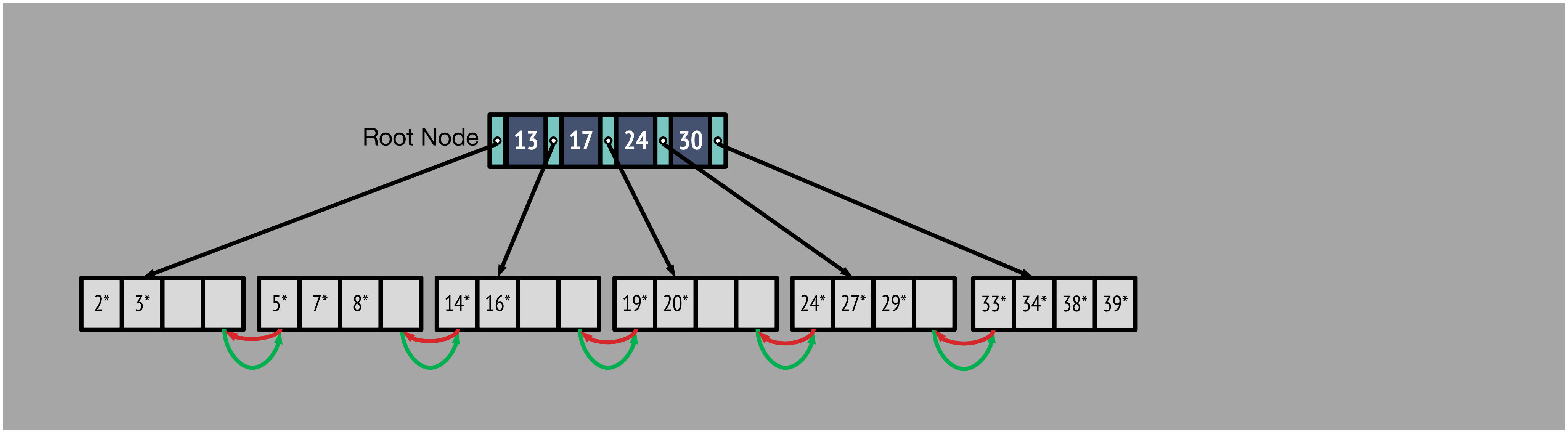
- Find the correct leaf
  - Split leaf if there is not enough room
  - Redistribute entries evenly

Inserting 8* into a B+ Tree: Split Leaf, cont



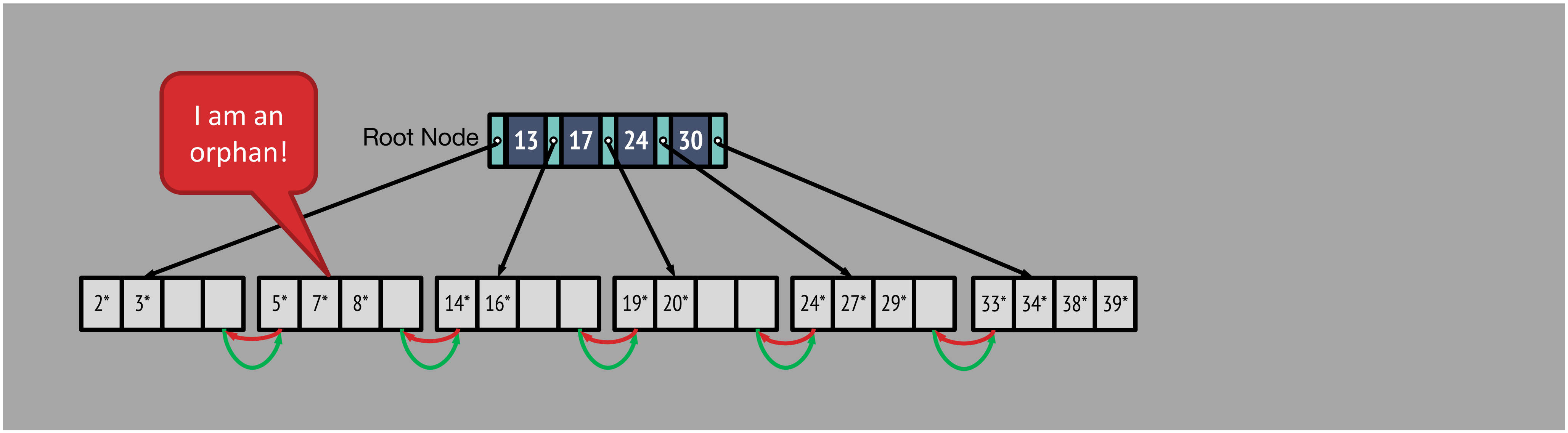
- Find the correct leaf
  - Split leaf if there is not enough room
  - Redistribute entries evenly
  - Fix next/prev pointers

Inserting 8* into a B+ Tree: Fix Pointers



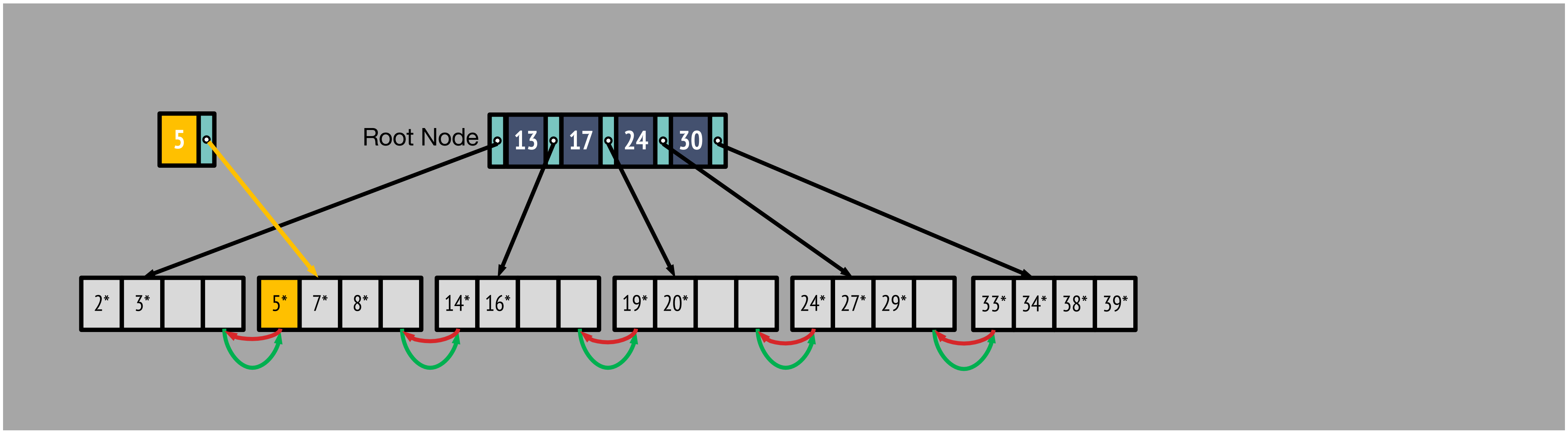
- Find the correct leaf
    - Split leaf if there is not enough room
    - Redistribute entries evenly
    - Fix next/prev pointers

Inserting 8* into a B+ Tree: Mid-Flight



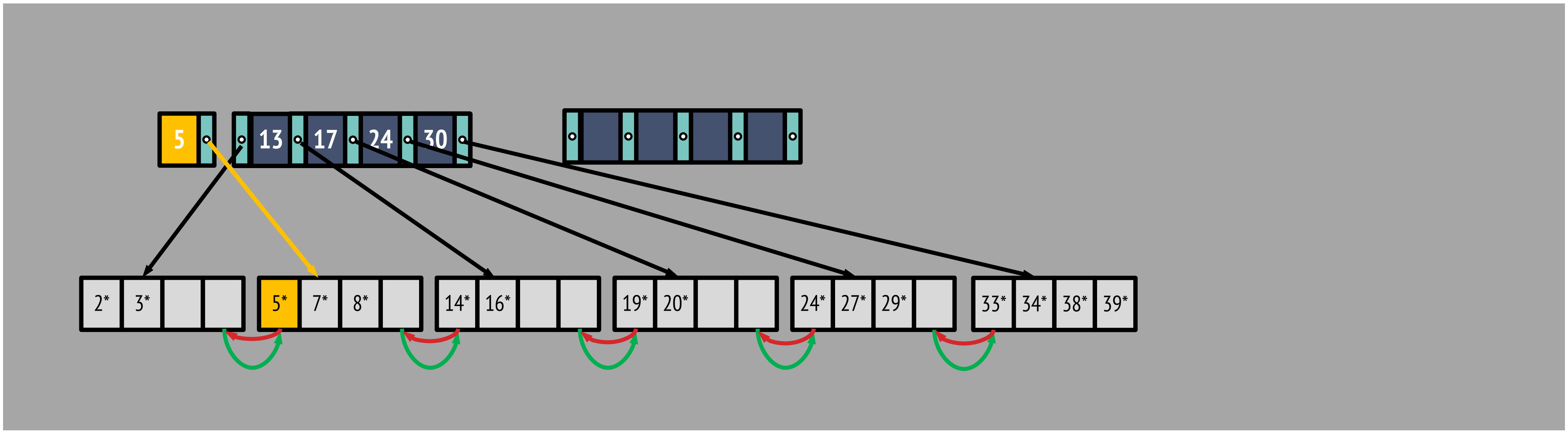
- Something is still wrong!

Inserting 8* into a B+ Tree: Copy Middle Key



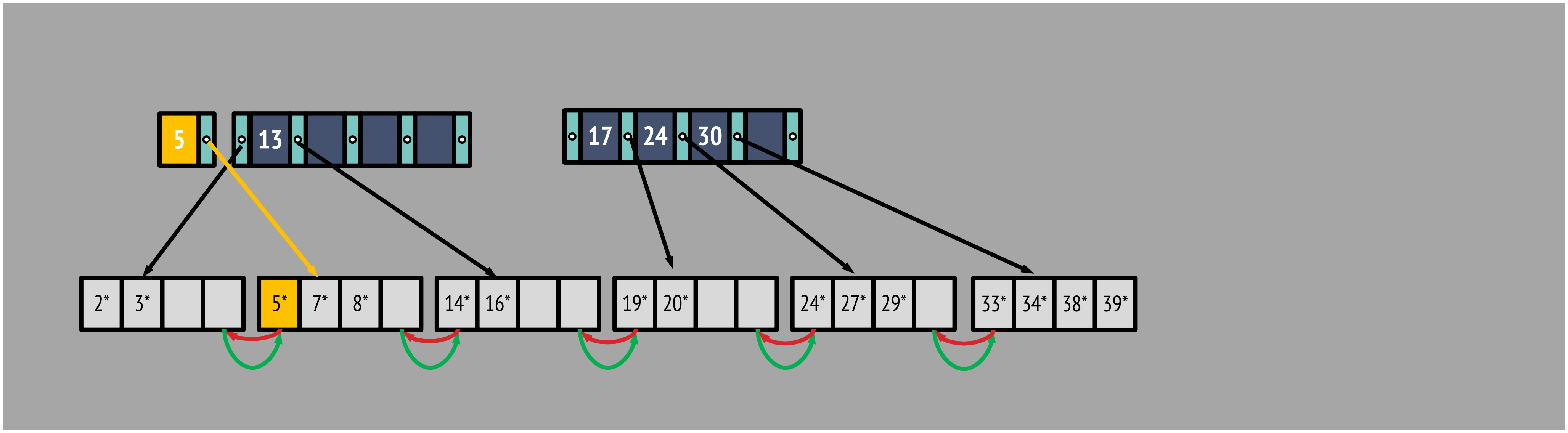
- **Copy up from leaf** the middle key and pointer to the orphan leaf
  - This is what we need to access it

Inserting 8* into a B+ Tree: Split Parent, Part 1



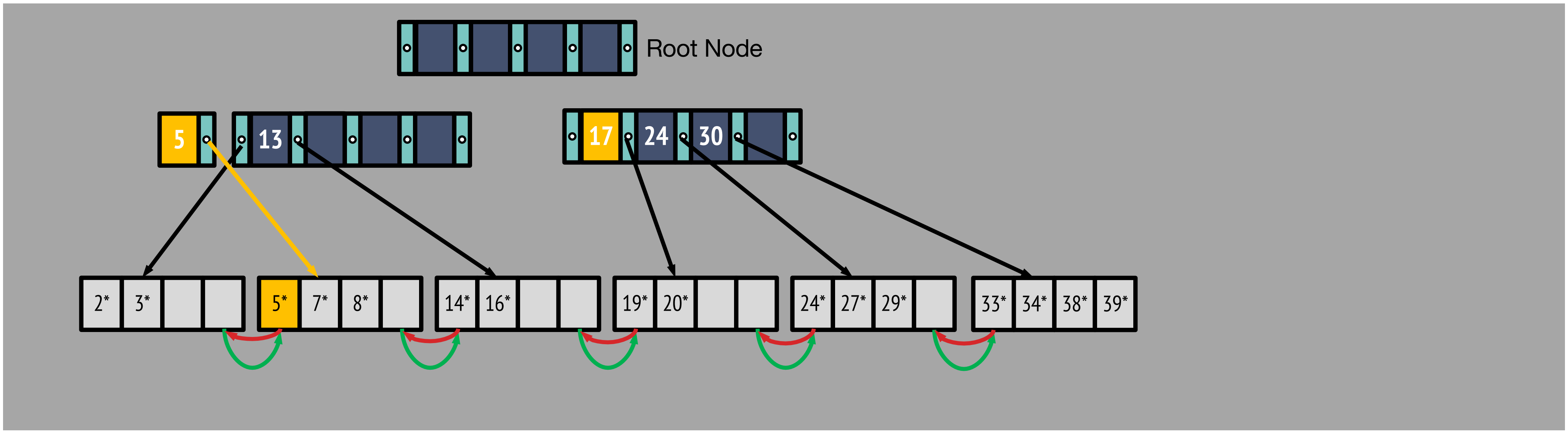
- **Copy up from leaf** the middle key and pointer to the orphan leaf
- No room in parent? (Parent now has 2d+1 instead of 2d)
  - Recursively split index nodes
  - Redistribute the rightmost d+1 keys

Inserting 8* into a B+ Tree: Split Parent, Part 2



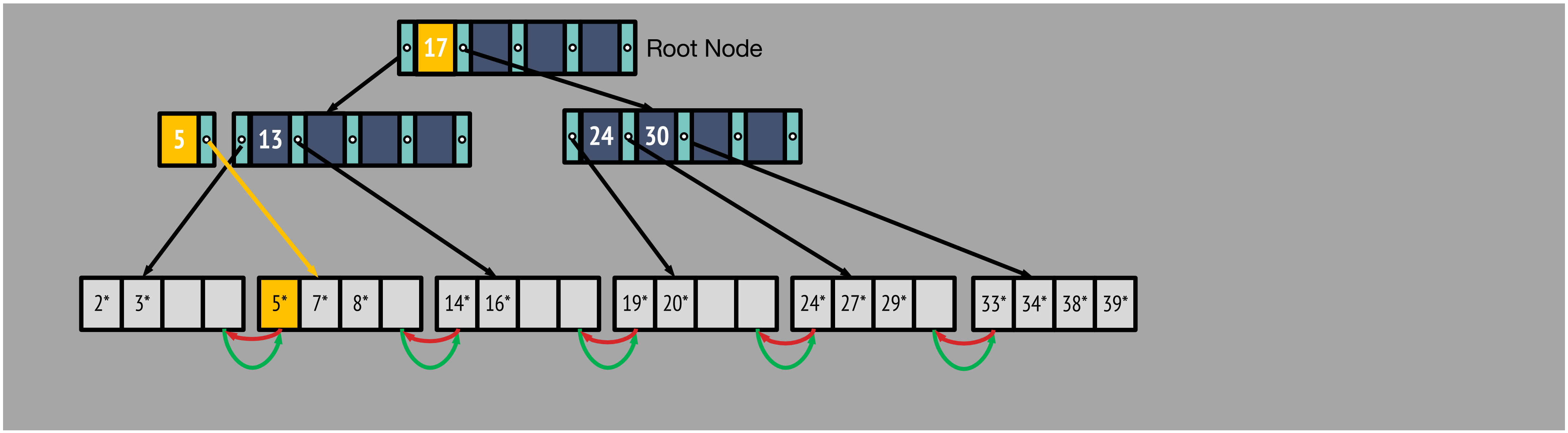
- **Copy up from leaf** the middle key and pointer to the orphan leaf
- No room in parent? Recursively split index nodes
  - Redistribute the rightmost d+1 keys
  - Not enough: we now have two roots!

Inserting 8* into a B+ Tree: Root Grows Up



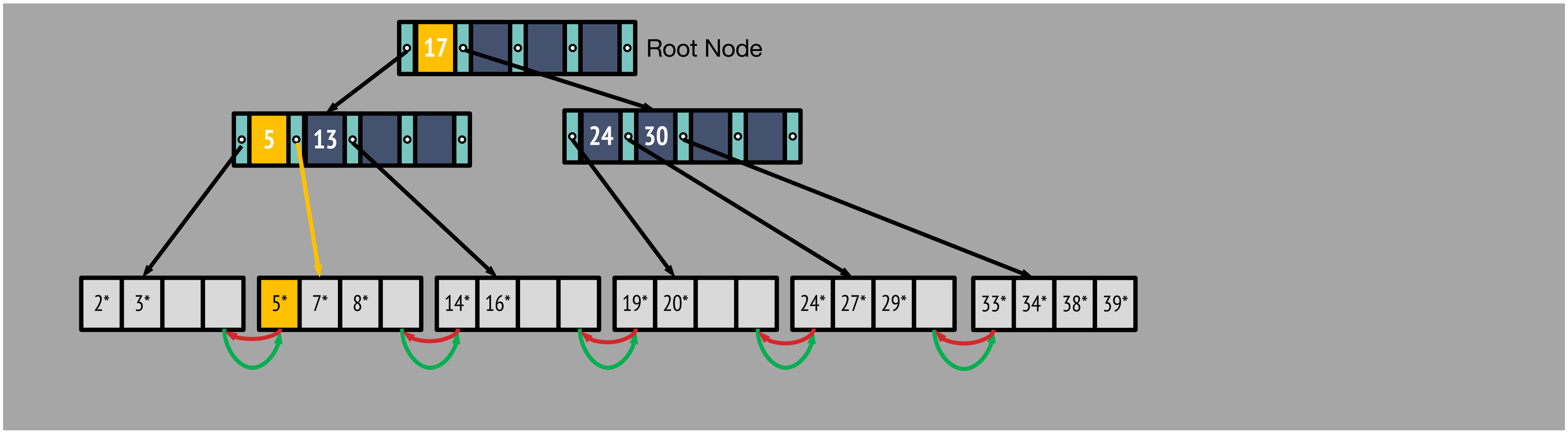
- No room in parent? Recursively split index nodes
  - Redistribute the rightmost d+1 keys
- **To fix, create a new root:**
  - **Push up from interior node** the middle key (and assoc. pointer)

Inserting 8* into a B+ Tree: Root Grows Up, Pt 2



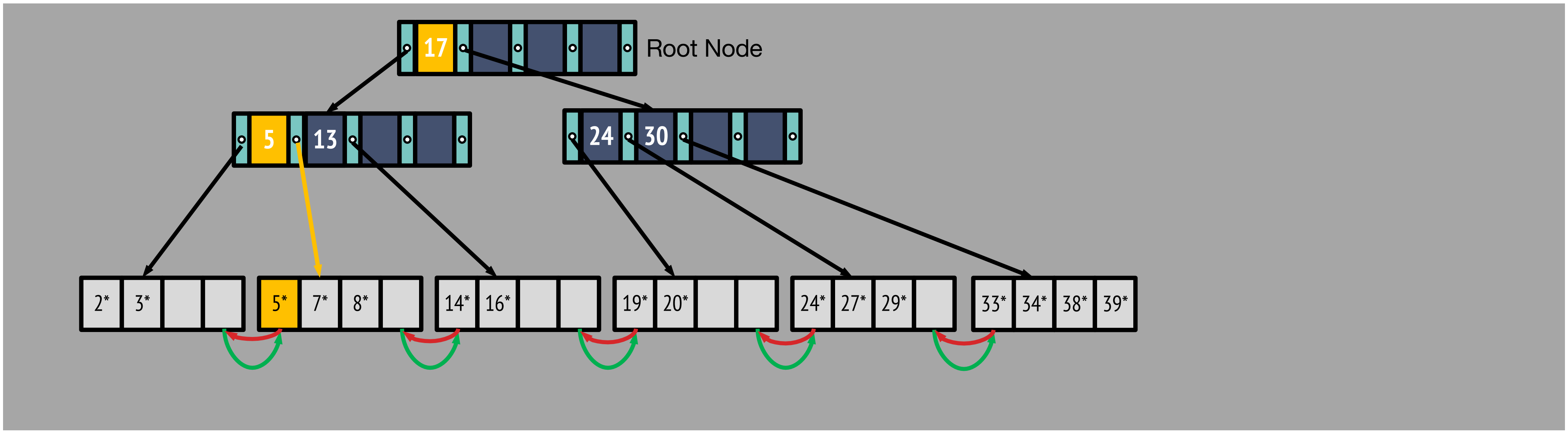
- Net effect
  - d keys on the left and right => invariant satisfied!
  - middle key pushed up
- Consolidate 5* into left node

Inserting 8* into a B+ Tree: Root Grows Up, Pt 3



- Net effect
  - d keys on the left and right
  - middle key pushed up
- **Here, we ended up creating a new root and increasing depth => rare**

Copy up vs Push up!



The **leaf** entry (5) was **copied** up

We can’t lose the original key: all keys must be in leaves

The **index** entry (17) was **pushed** up

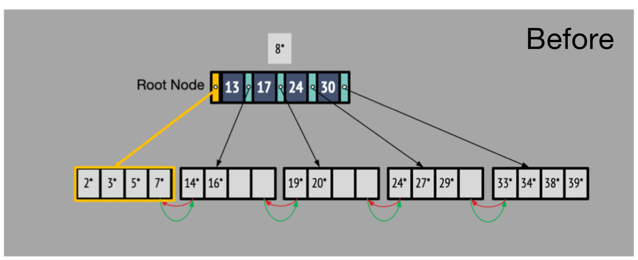
We don’t need it any more for routing => convince yourself!

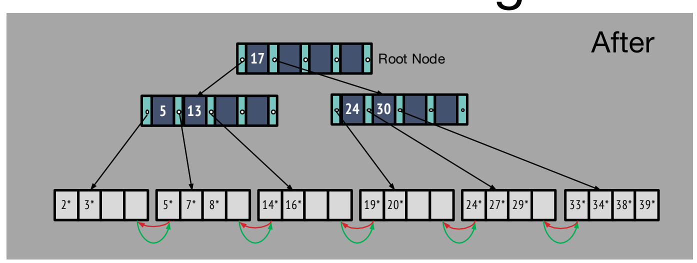
B+ Tree Insert: Algorithm Sketch

1. Find the correct leaf L.
2. Put data entry onto L.
   - If L has enough space, done!
   - Else, must split L (into L and a new node L2)
     - Redistribute entries evenly, copy up middle key (and ptr to L2)
     - Insert index entry pointing to L2 into parent of L.

B+ Tree Insert: Algorithm Sketch Part 2

- Step 2 can happen recursively
  - To split index node, redistribute entries evenly, but push up middle key (and ptr to new index node). (Contrast with leaf splits)
- Splits "grow" tree
  - Tree growth: gets wider if possible from bottom up
  - Worst case, adds another level with a new root
  - Ensures balance & therefore the logarithmic guarantee





We will skip deletion

- In practice, occupancy invariant often not enforced during deletion
- Just delete leaf entries and leave space
    - If new inserts come, great
        - This is common
- If page becomes completely empty, can delete
    - Parent may become underfull
    - That's OK too
- No need to delete inner pages even if empty
    - Only used for lookups
- Guarantees still attractive: $\log_F$(total number of inserts)

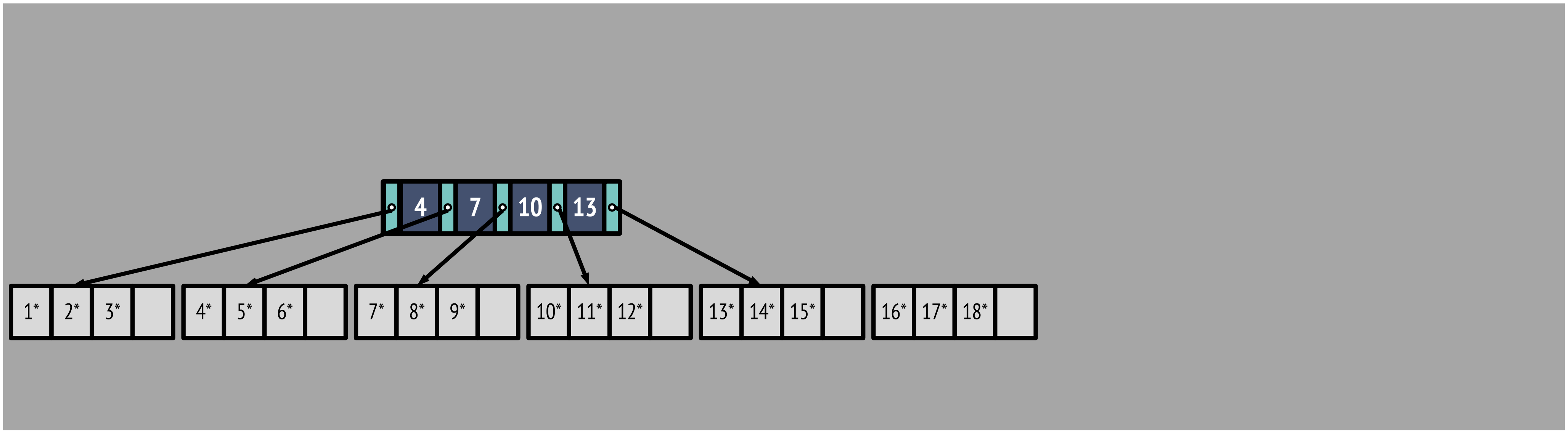
- Textbook describes algorithm for rebalancing and merging on deletes

BULK LOADING B+-TREES

Bulk Loading of B+ Tree

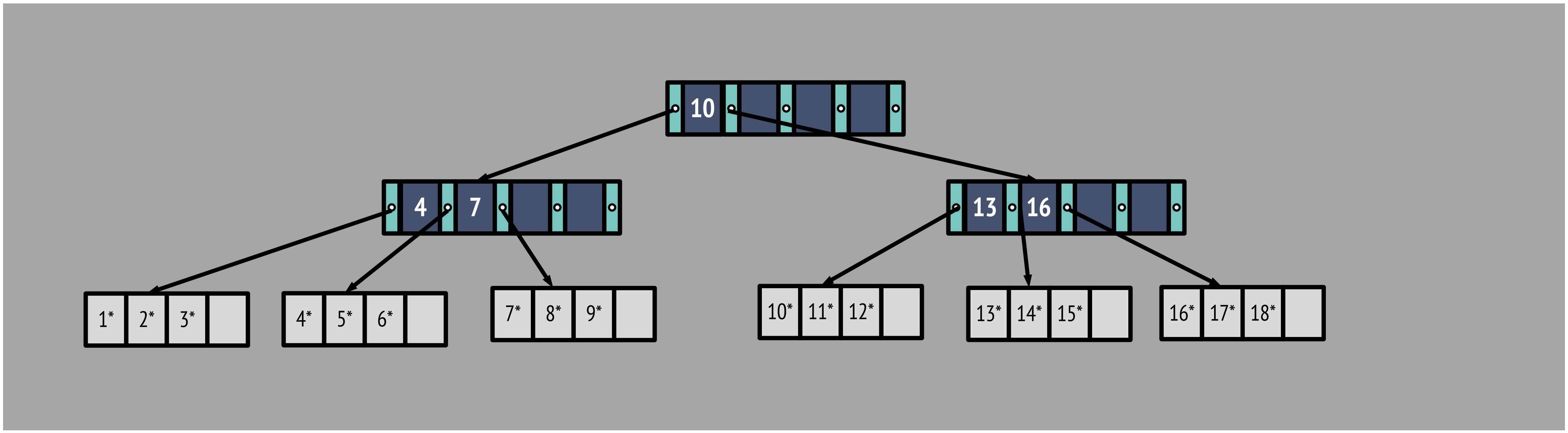
- Suppose we want to build an index on a large table from scratch
- Would it be efficient to just call insert repeatedly
    - Q: No … Why not?
    - Constantly need to search from root
    - Modifying random pages: poor cache efficiency
    - Leaves poorly utilized (typically half-empty)

Smarter Bulk Loading a B+ Tree



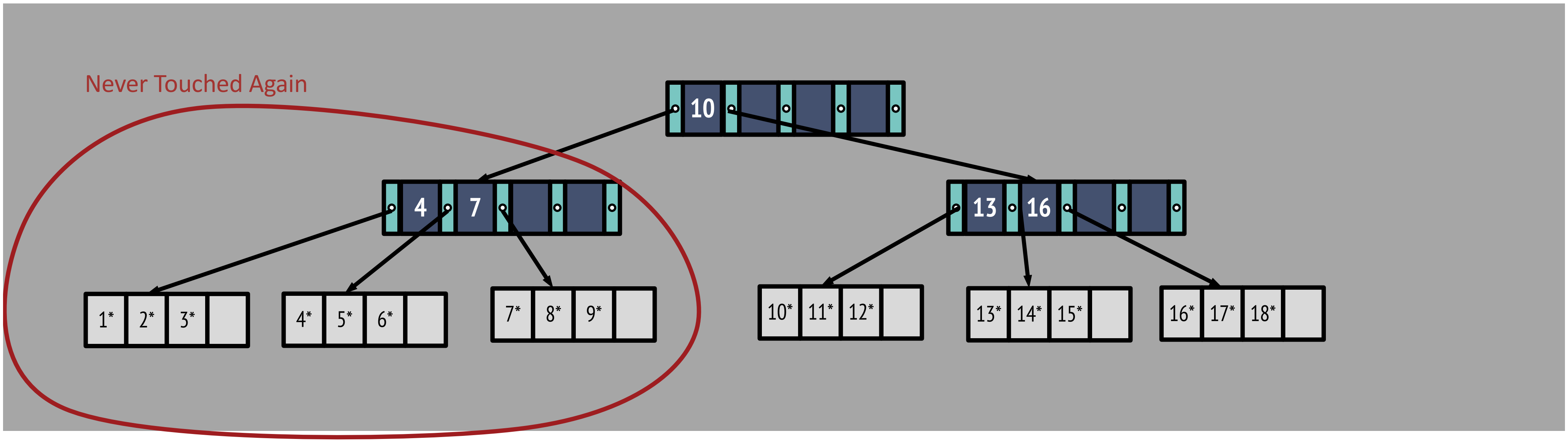
- Sort the input records by key:
  - 1*, 2*, 3*, 4*, …
  - We'll learn a good disk-based sort algorithm soon!
- Fill leaf pages to some fill factor > d (e.g. ¾)
  - Updating parent until full

Smarter Bulk Loading a B+ Tree Part 2



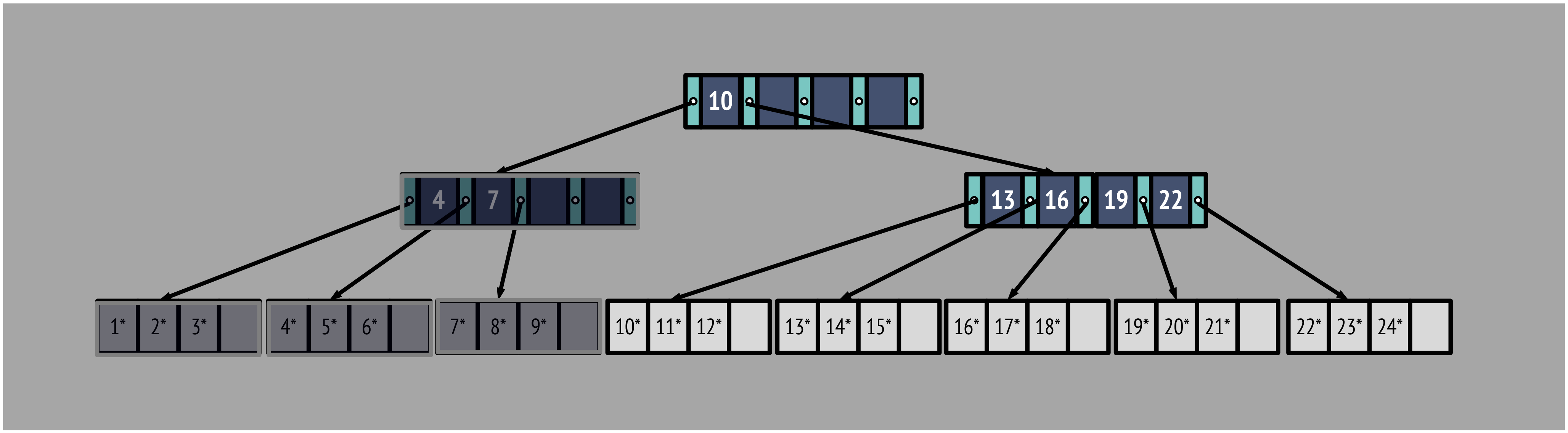
- Sort the input records by key:
  - 1*, 2*, 3*, 4*, …
- Fill leaf pages to some fill factor > d (e.g. ¾)
  - Update parent until full
  - Then create new sibling and copy over half: same as in index node splits for insertion

Smarter Bulk Loading a B+ Tree Part 3



- Lower left part of the tree is never touched again
- Occupancy invariant maintained:
  - leaves filled beyond d, rest of the nodes via insertion split procedure

Smarter Bulk Loading a B+ Tree Part 4



- Benefits: Better
  - Cache utilization than insertion into random locations
  - Utilization of leaf nodes (and therefore shallower tree)
  - Layout of leaf pages (more sequential)

Summary

- B+ Tree is a powerful dynamic indexing structure
  - Inserts/deletes leave tree height-balanced; $\log_F N$ cost
  - High fanout (F) means height rarely more than 3 or 4.
  - Higher levels stay in cache, avoiding expensive disk I/O
  - Almost always better than maintaining a sorted file.
  - Widely used in DBMSs!

- Bulk loading can be much faster than repeated inserts for creating a B+ tree on a large data set.