APPM 2360 Project 1

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1 Introduction

Suppose you and your enemy are on a space station orbiting a black hole. Naturally, you launch your enemy out of the space station towards the black hole. From their perspective, they will get pulled towards the black hole and eventually pass the event horizon. However, from your safe point of view on the space station, your enemy will move slower and slower as they approach the event horizon. In fact, they will never appear to cross the event horizon from your point of view! This is due to the distortion of time caused by the immense gravitational field caused by the black hole.

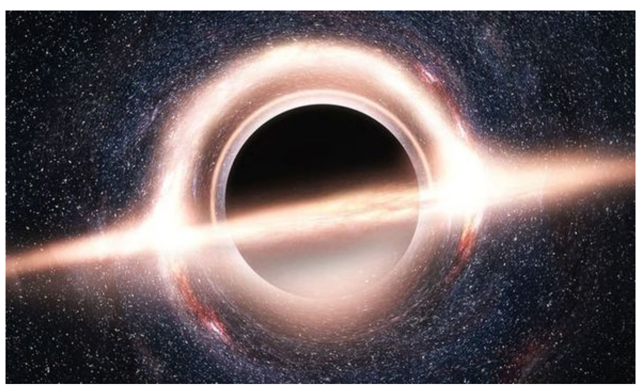


Figure 1: Black Hole

2 Modeling objects falling towards a black hole

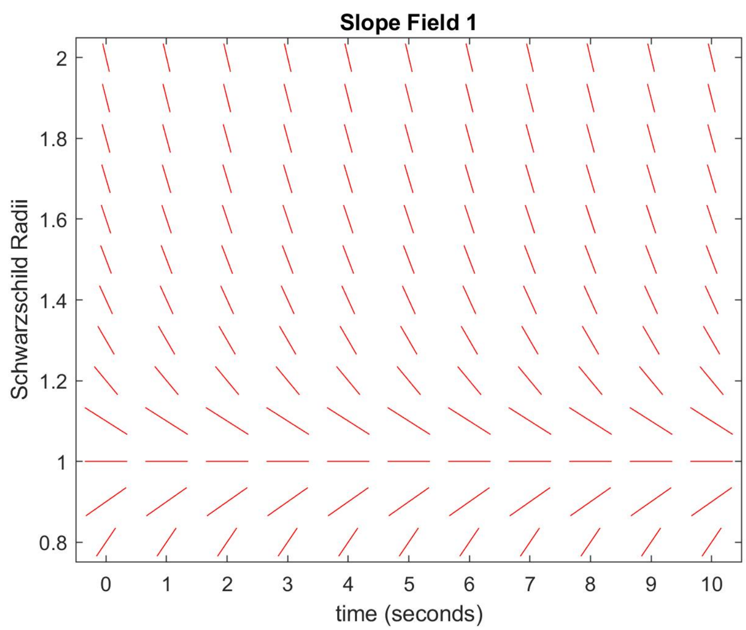
Consider equation 1 to model your perspective of your enemy as they fly towards the black hole. Let x(t) represents the radial distance from the center of the black hole to your enemy in Schwarzschild radii. The space ship is at 2 Schwarzschild radii and the event horizon is at x(t) = 1. Time is measured in seconds a time from $0 < t < 10$ will be considered

$$(1) \quad dx/dt = e^{-x+1} - 1 \qquad x(0) = 2$$

The Differential Equation 1 is autonomous, nonlinear, order 1, and is separable
Differential Equation Equilibrium Solutions:

$$0 = e^{-x+1} - 1$$
$$1 = e^{-x+1}$$
$$ln(1) = -x + 1$$
$$x = 1$$

Differential Equation Direction Field



The Differential Equation (equation 1) depicts the physics of the situation in a range of $1 < x < 2$.

The initial condition x(0) = 2 comes from when you push your enemy out of the spaceship, the space station is at x = 2 and starting the time there (t=0)

Using Picard's theorem, the initial condition on the domain exists and within the box R: $1.9 < x < 2.1, -0.1 < t < 0.1$is continuous therefore a solution exists. The derivative with respect to x also exists at the initial condition and is continuous within the region R therefore it is a unique solution

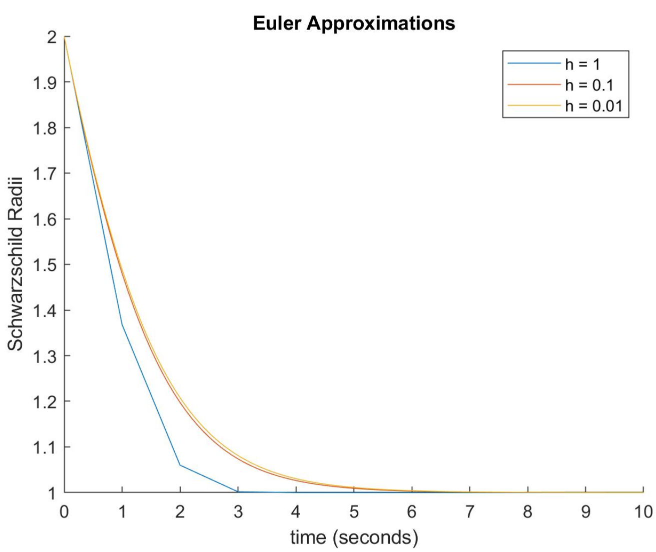
If x(t) is position then the derivative dx/dt is the velocity. Therefore the differential equation is equal to the velocity and you can determine the velocity based on different x values. At x=0 the velocity = e - 1

Analytic solution of the Differential Equation (1)
See Reference 1

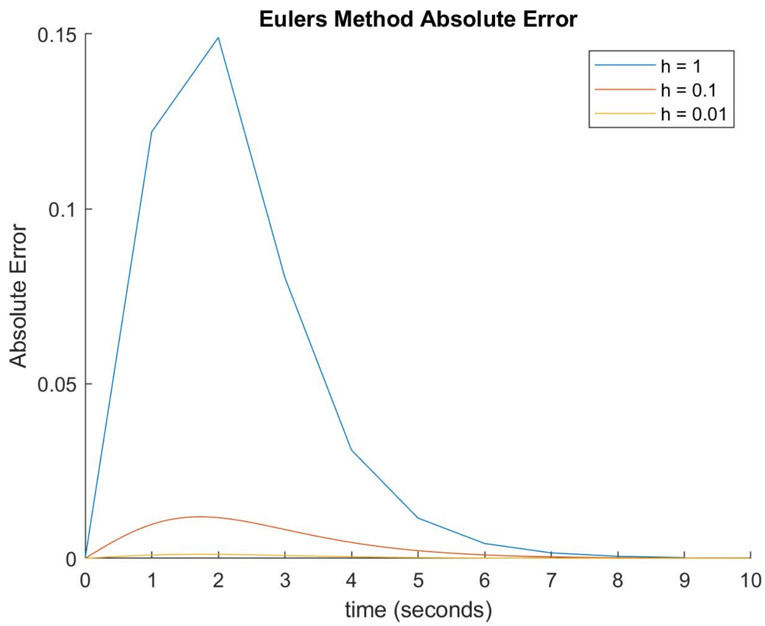
$$\int \frac{1}{e^{-x+1}-1}dx = \int dt$$

$$x(t) = ln(\frac{e^2-e}{e^t}+e)$$

The analytical solution of equation 1 shows the path that your enemy takes towards the black hole.
We used Euler approximations to model the path of your enemy. We included multiple step sizes to show the accuracy increase with a smaller step size. We use h = 1, 0.1, and 0.01

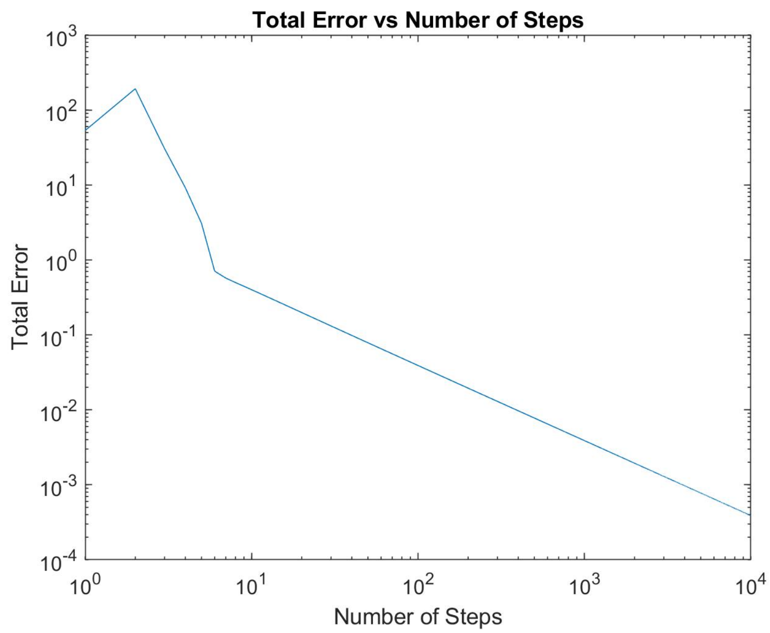


The difference between the Euler approximation and the true analytical solution gives an absolute error for each point plotted. Below is a graph of the absolute error with the three previous step sizes.



By adding up all the differences between the Euler Approximations and the analytical solution for each step size, we calculate the accumulated total error for each step size.
Total Error when h = 1 = 0.400495
Total Error when h = 0.1 = 0.039045
Total Error when h = 0.01 = 0.003886

To minimize the number of computations for the accuracy, we plotted a total error function based on the number of steps on a logarithmic scaled graph. This shows the the change of the error based on an exponentially growing number of step sizes.

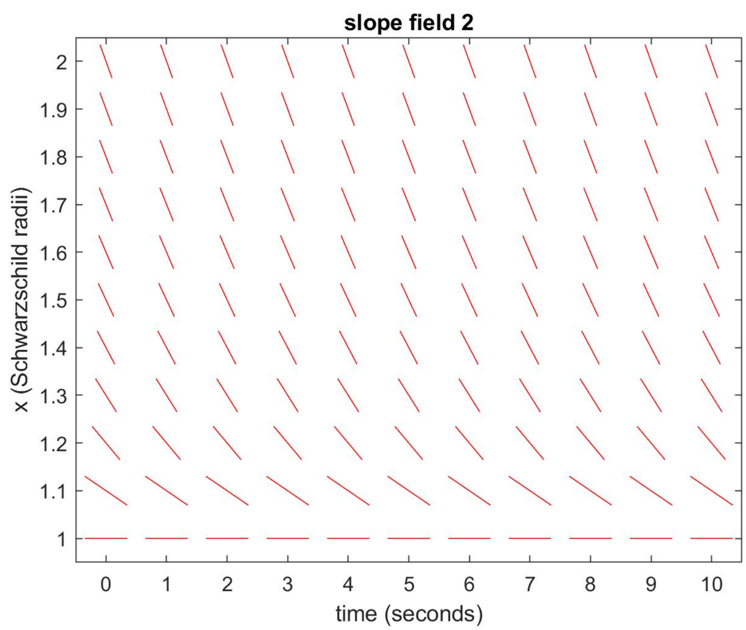


We determined the best number of steps for this situation to minimize the number of computations in a Euler approximation is 6 steps. After 6, the rate of change of the total error decreases as the number of steps increase exponentially. Meaning by increasing the number of steps after 6, the total error can only decrease by very little which would be 0.7.

3 Remorse

$$dx/dt = (\frac{1}{x} - 1) * \frac{1}{\sqrt{x}} \qquad x(0) = 2$$

This differential equation is autonomous because dx/dt is only dependent on x and not by t. The differential equation is nonlinear because x has a different power then 1. It is order 1 because it is an equation with the highest derivative being one. It is also separable because there are no t's in the equation. It has one equilibrium solution at $x = 1$

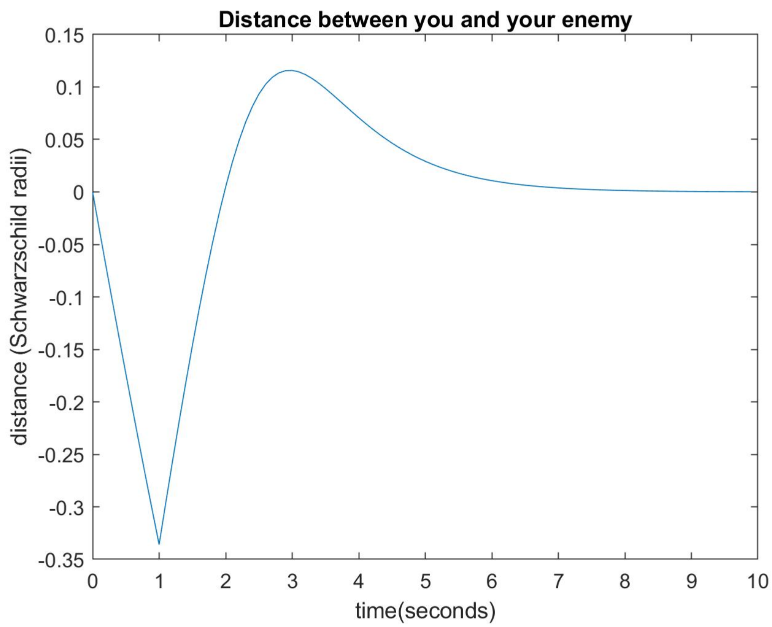


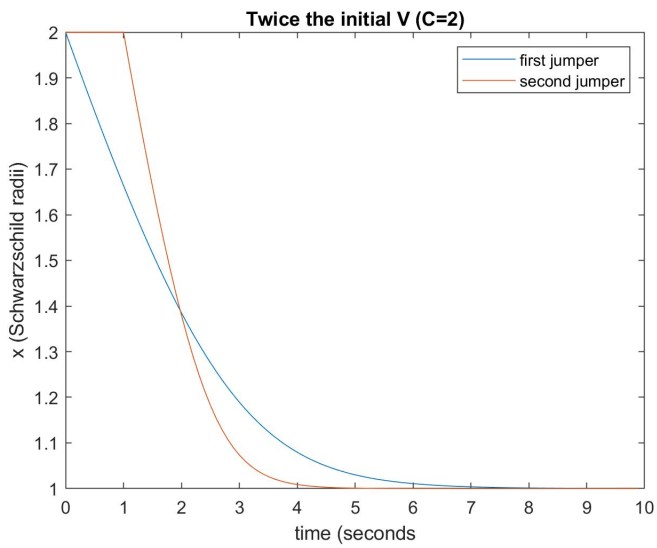
The places where Picard's theorem does not guarantee uniqueness or a solution is at $x = (-\infty, 0]$, because $\frac{\delta f}{\delta x} = \frac{-3}{2x^{\frac{5}{3}}} - \frac{-1}{2x^{\frac{3}{2}}}$ and $dx/dt = (\frac{1}{x} - 1) * \frac{1}{\sqrt{x}}$ only exist when x is positive and nonzero.

4 Redemption

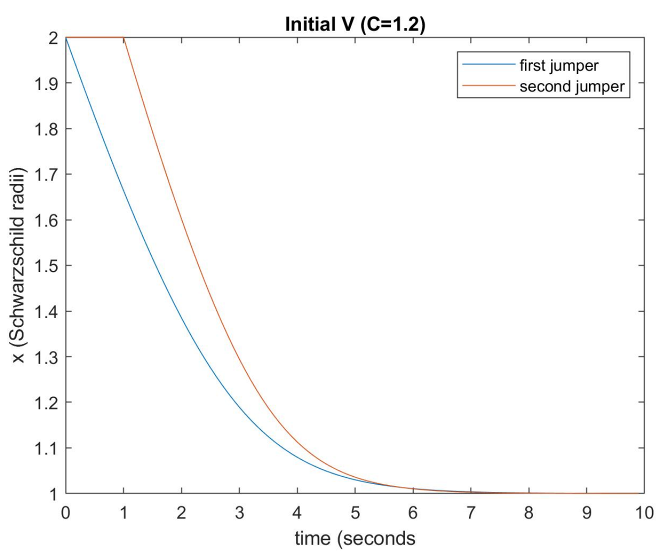
$dx/dt = (\frac{1}{x_e} - 1) * \frac{1}{\sqrt{x_e}} \qquad x(0) = 2 \qquad dx/dt = C(\frac{1}{x_y} - 1) * \frac{1}{\sqrt{x_y}} \qquad x(1) = 2$

If you wait one second and then jump out after them at twice the initial speed you will catch up to them at time t=1.8 seconds, but unfortunately you do not make it to them before x=1.5. This can be seen by the two graphs below one being the distance between you and your enemy and the second one being the separate plots on the same graph.





Surprisingly, when you jump out slower it takes even more time to catch your enemy it would be around t=5.8 seconds and you do not make it before x=1.5, which is visible .



References

Reference 1: 2.1.8 Analytic Solution

$$\int \frac{1}{e^{-x+1} - 1} dx = \int dt$$

$$\int \frac{e^x}{e - e^x} dx = t + c \quad Let \quad u = e - e^x \quad du = e^x dx$$

$$\int \frac{-du}{u} = t + c$$

$$-ln|e - e^x| = t + c$$

$$\frac{1}{e - e^x} = c * e^t$$

$$\frac{1}{c * e^t} = e - e^x$$

$$e^x = \frac{-1}{c * e^t} + e$$

$$ln(\frac{-1}{c * e^t} + e) = x$$

apply Initial Condition x(0) = 2

$$2 = ln(\frac{-1}{c} + e)$$

$$e^2 = \frac{-1}{c} + e$$

$$c = \frac{-1}{e^2 - e}$$

$$x(t) = ln(\frac{e^2 - e}{e^t} + e)$$

Reference 2: Question 2 Code