Original Analysis Problems

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These problems, ordered by topic first and then by difficulty, cover a standard introductory series in undergraduate analysis. Some of my favorite problems are 7, 10, 18, 25, 34, 43, 44, 48, and 49.

Problem 1. Determine the common names of the following real numbers:

(a) $\left\{\frac{a}{b} \in \mathbb{Q} : b > 0, b > a\right\}$.

(b) $\left\{q \in \mathbb{Q} : (\exists n \in \mathbb{N}) \left[q < \sum_{k=0}^{n} \frac{1}{k!}\right]\right\}$.

(c) $\left\{q \in \mathbb{Q} : \text{the set } \{q, q^2, q^3, \dots\} \text{ does not have an infimum in } \mathbb{Q}\right\}$.

Problem 2. Fix a set X. We define a partial ordering on $\mathcal{P}(X)$ as follows: For any $A, B \in \mathcal{P}(X)$, we say $A \leq B$ if and only if $A \subseteq B$.

(a) Show that $\mathcal{P}(X)$ is Dedekind complete.

(b) Show that X is finite if and only if every totally ordered subset of $\mathcal{P}(X)$ contains its supremum.

Problem 3. Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$, and suppose that there exists some $k \in \mathbb{N}$ such that $|A \cap B| \leq k$ for any distinct elements $A, B \in \mathcal{F}$. Prove that $\mathcal{F}$ is countable.

Problem 4. Let X be a set, (E, d) be a metric space, and $f\colon X \to E$ be a function. Prove that the function $g\colon X \times X \to \mathbb{R}$ given by

$$g(x, y) = d(f(x), f(y))$$

is a metric on X if and only if f is injective.

Problem 5. Let (E, d) be a metric space and $\{a_n\}_{n=0}^{\infty} \subseteq E$ be a convergent sequence. Show that the set $\{a_n : n \in \mathbb{N}\}$ is totally bounded.

Problem 6. Show that a sequence $\{a_n\}_{n=0}^{\infty} \subseteq E$ is Cauchy if and only if for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $d(a_m, a_n) < \varepsilon$ whenever $m \geq n$.

Problem 7. Let (E, d) be a metric space. We call a set $C \subseteq E$ a *Cauchy set* if for all $\varepsilon > 0$, there exists $x \in E$ such that $C \setminus B_\varepsilon(x)$ is finite. Show that (E, d) is complete if and only if every infinite Cauchy set has a limit point in E.

Problem 8. (a) Show (with two examples) that an infinite union or intersection of clopen sets need not be clopen.

(b) Prove that such examples can only be found in metric spaces consisting of infinitely many connected components.

Problem 9. Let X be a set and $d\colon X \times X \to [0, \infty)$ be a function.

(a) Prove that the function $d_1\colon X \times X \to [0, \infty)$ given by

$$d_1(x, y) = \frac{d(x, y) + d(y, x)}{2}$$

is symmetric.

(b) Show that the function $d_2\colon X \times X \to [0, \infty)$ given by

$$d_2(x, y) = \inf \left\{ \sum_{i=1}^{n} d_1(x_{i-1}, x_i) : n \in \mathbb{N}, x_0, \ldots, x_n \in X, x_0 = x, x_n = y \right\}$$

satisfies the triangle inequality.

(c) Define the equivalence relation $\sim$ on X by $x \sim y$ whenever $d_2(x, y) = 0$, and let $\tilde{X}$ denote the set of equivalence classes $[x]$ for $x \in X$. Prove that the function $d_3\colon \tilde{X} \times \tilde{X} \to [0, \infty)$ given by $d_3([x], [y]) = d_2(x, y)$ is well-defined, and that $(\tilde{X}, d_3)$ is a metric space. We call $(\tilde{X}, d_3)$ the *metrification* of (X, d).

(d) Prove that, if d is a metric on X, then the metric spaces X and $\tilde{X}$ are isometric.

Problem 10. Suppose (X, d) is a metric space. For a set $T \subseteq X$, we define the *teleportation space* $(X, d)_T$ to be the metrification (as defined in the previous problem) of (X, f), where $f\colon X \times X \to [0, \infty)$ is defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x, y \in T \\ d(x, y) & \text{otherwise.} \end{cases}$$

Show that, if (X, d) is an unbounded metric space, then there exists an unbounded set $T \subseteq X$ such that the metric space $(X, d)_T$ is also unbounded.

Problem 11. (a) Find a complete metric space (E, d) and a set $S \subseteq E$ such that both of the following conditions hold:

1. For any $\varepsilon > 0$, there exists $x \in E$ such that $B_\varepsilon(x) \cap S$ is infinite.
2. S has no limit points in E.

(b) Show that this is impossible in $\mathbb{R}^n$ with the usual Euclidean metric.

Problem 12. Find a metric space (E, d) that can be covered by finitely many open balls of radius r if and only if $r > 1$.

Problem 13. Show that a set $K \subseteq \mathbb{R}$ is compact if and only if every cover of K by open intervals has a finite subcover.

Problem 14. Prove that the following are equivalent for a subset S of a metric space:

(i) S is finite.

(ii) S is totally bounded, and $\inf \{d(x, y) : x, y \in S, x \neq y\} > 0$.

Problem 15. Let (E, d) be a metric space and $D \subseteq E$ be discrete.

(a) Prove that there exists a family $\{U_d\}_{d \in D}$ of disjoint open sets such that $d \in U_d$ for each $d \in D$.

(b) Show that if E is compact, D must be countable.

Problem 16. We call a subset S of a metric space ε-*chainable* if, for any two points $x_0, y \in S$, there exists $N \in \mathbb{N}$ and points $x_1, \ldots, x_N \in S$ such that $x_N = y$ and $d(x_{n-1}, x_n) < \varepsilon$ for each $n = 1, \ldots, N$. We call S *chainable* if it is ε-chainable for any $\varepsilon > 0$.

(a) Prove that if S is compact and chainable, then S is connected.

(b) Prove that every closed and chainable subset of $\mathbb{R}^n$ is connected if and only $n = 1$.

(c) Prove that every compact and chainable subset of $\mathbb{R}^n$ is path-connected if and only if $n = 1$.

Problem 17. Let $\{a_n\}_{n=0}^{\infty} \subseteq [0, \infty)$ satisfy $\liminf_{n \to \infty} a_n = 0$. Prove that there exists a bijection $f \colon \mathbb{N} \to \mathbb{Q} \setminus \{0\}$ such that $|f(n)| > a_n$ for all $n \in \mathbb{N}$.

Problem 18. Let K be a connected subset of $\mathbb{R}$. Show that K is compact if and only if $\{f(a_n)\}_{n=0}^{\infty}$ converges for every monotonic sequence $\{a_n\}_{n=0}^{\infty} \subseteq K$ and increasing function $f \colon K \to \mathbb{R}$.

Problem 19. Let $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ be a Cauchy sequence. Show that there exists a subsequence $\{a_{n_k}\}_{k=0}^{\infty}$ of $\{a_n\}_{n=0}^{\infty}$ such that

$$\sum_{k=0}^{\infty} \left| a_{n_{k+1}} - a_{n_k} \right| < 1.$$

Problem 20. Let $\sum_{n=0}^{\infty} a_n$ be a conditionally convergent series. Show that there exists a rearrangement $\sum_{n=0}^{\infty} b_n$ of $\sum_{n=0}^{\infty} a_n$ whose partial sums form a dense subset of $\mathbb{R}$.

Problem 21. Show that the series $\sum_{n=0}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=0}^{\infty} a_n b_n$ converges for all sequences $\{b_n\}_{n=0}^{\infty}$ such that $\lim_{n \to \infty} b_n = 0$.

Problem 22. Show that a function $f \colon \mathbb{R} \to \mathbb{R}$ maps bounded sets to bounded sets if and only if each $x_0 \in \mathbb{R}$ has a neighborhood on which f is bounded.

Problem 23. For each part, either find a function $f \colon \mathbb{R} \to \mathbb{R}$ that satisfies the given criteria or prove that such a function does not exist:

(a) f is injective and continuous but not monotonic.

(b) f is bijective and is discontinuous at exactly one point.

(c) f is bijective and discontinuous everywhere.

(d) f is monotonic and discontinuous everywhere.

Problem 24. Prove that a function $f\colon \mathbb{R} \to \mathbb{R}$ mapping convergent sequences to convergent sequences (i.e. $\lim_{n\to\infty} f(x_n)$ exists whenever $\{x_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ converges) must be continuous.

Problem 25. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ maps convergent sequences consisting of distinct terms to convergent sequences.

(a) Prove that the set of discontinuities of f is at most countable.

(b) Show by example that the set of discontinuities of f can be dense in $\mathbb{R}$.

Problem 26. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable, let $a < b$, and assume that $f(a) = f(b) = c$. Show that, if $f'(a)$ and $f'(b)$ are either both positive or both negative, then there exists some $x \in (a, b)$ such that $f(x) = c$.

Problem 27. Let $f\colon [1, \infty) \to \mathbb{R}$ be twice-differentiable. Prove or disprove the following claims:

(a) If $\lim_{x\to\infty} f''(x) > 0$, then $\lim_{x\to\infty} f(x) = \infty$.

(b) If $\lim_{x\to\infty} f''(x) = 0$, then $\lim_{x\to\infty} f'(x)$ exists.

(c) If $\lim_{x\to\infty} f(x) = 0$, then $f'(x)$ is bounded.

(d) If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f''(x) = 0$, then $\lim_{x\to\infty} f'(x) = 0$.

Problem 28. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable with $f'(x_0) \neq 0$. Show that there exists $\delta > 0$ such that $f(x)$ is nonzero whenever $0 < |x - x_0| < \delta$.

(b) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be smooth functions, and suppose $f^{(n)}(x_0) \neq g^{(n)}(x_0)$ for at least one $n \in \mathbb{N}$. Show that there exists $\delta > 0$ such that $f(x) \neq g(x)$ whenever $0 < |x - x_0| < \delta$.

Problem 29. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x\to-\infty} |f(x)| = \lim_{x\to\infty} |f(x)| = \infty$, and choose $c \in \mathbb{R}$ such that $f'(x) \neq 0$ for all $x \in f^{-1}(c)$. Show that $f^{-1}(c)$ is finite. (Hint: show that $f^{-1}(c)$ is bounded and has no limit points.)

Problem 30. Define $g\colon [1, \infty) \to [1, \infty)$ by $g(t) = t + t^{-1}$.

(a) Show that $|g(x) - g(y)| < x - y$ for any $x > y > 1$.

(b) Show that g has no fixed points. Does this contradict the Banach fixed-point theorem?

(c) Prove that if $K \subseteq \mathbb{R}$ is a compact set and $f\colon K \to K$ satisfies the property that $|f(x) - f(y)| < |x - y|$ for any $x, y \in K$, then f has a unique fixed point. (Hint: consider the function $h(t) = |f(t) - t|$.)

(d) Is (c) still true if we only require K to be bounded?

Problem 31. Given $f\colon [0, \infty) \to \mathbb{R}$ such that $\int_0^\infty f(t)\,dt$ exists, show that

$$\lim_{x\to\infty} \int_x^\infty f(t)\,dt = 0.$$

Problem 32. Let $f\colon [0,\infty) \to \mathbb{R}$ be such that $\int_0^\infty f(t)\,dt$ exists.

(a) Show that if $\lim_{x\to\infty} f(x)$ exists, then it must be equal to zero.

(b) Show by example that $\lim_{x\to\infty} f(x)$ might not exist.

(c) Show that if f is differentiable and has a bounded derivative, then $\lim_{x\to\infty} f(x) = 0$.

Problem 33. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Riemann integrable function, define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_0^x f(t)\,dt,$$

and fix $x_0 \in \mathbb{R}$. We know by the fundamental theorem of calculus that, if f is continuous at x_0, then $g'(x_0)$ exists and is equal to $f(x_0)$.

(a) Show that the converse does not always hold.

(b) Show that the converse holds if we also require f to be monotonic.

Problem 34. Let $f\colon [0,\infty) \to \mathbb{R}$ be a bounded continuous function.

(a) Show that there exists $c \in [0,\infty)$ such that

$$f(c) = \int_0^c f(x)\,dx.$$

(b) Show by example that the conclusion might not hold when f is unbounded.

Problem 35. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuously differentiable and assume $f(0) = 0$. Show that

$$\int_0^1 f(x)^2\,dx \le \int_0^1 f'(x)^2\,dx.$$

(Hint: first use the Cauchy-Schwarz inequality to prove that $\left(\int_0^1 g(x)\,dx\right)^2 \le \int_0^1 g(x)^2\,dx$ for all continuous g.)

Problem 36. Suppose the function $f\colon [0,1) \times [0,1] \to \mathbb{R}$ is continuous, and define for $x \in [0,1)$ the function $g_x(y) = f(x,y)$. Prove that f is uniformly continuous if and only if the family of functions g_x converges uniformly as $x \to 1$.

Problem 37. Show that, for any $r > 0$ and continuous function $g\colon [0,r] \to \mathbb{R}$, the sequence of functions from $[0,r]$ to $\mathbb{R}$ defined by

$$f_0(x) = g(x), \qquad f_{n+1}(x) = \int_0^x f_n(t)\,dt$$

converges uniformly to the zero function.

Problem 38. Let $C_c(\mathbb{R})$ denote the set of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with compact support and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous. Show that $g^{-1}(\mathbb{R} \setminus (-\varepsilon,\varepsilon))$ is compact for each $\varepsilon > 0$ if and only if g can be expressed as a uniform limit of functions in $C_c(\mathbb{R})$.

Problem 39. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Show that there exists a sequence of polynomials converging pointwise to f.

Problem 40. Show that for any non-compact set $J \subseteq \mathbb{R}$, there exists a bounded continuous function $f\colon J \to \mathbb{R}$ which is not the uniform limit of a sequence of polynomials.

Problem 41. Prove or disprove each of the following claims:

(a) There exists a sequence $\{a_n\}_{n=0}^{\infty}$ such that for all $x \in \mathbb{R}$ we have

$$|x| = \sum_{n=0}^{\infty} a_n x^n.$$

(b) There exists a double sequence $\{a_{mn}\}_{m,n=0}^{\infty}$ such that for all $x \in \mathbb{R}$ we have

$$|x| = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^n.$$

Problem 42. Show that there exists a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ and point $a \in \mathbb{R}$ such that f has no roots and the sequence $\{x_n\}_{n=0}^{\infty}$ defined recursively by

$$x_0 = a, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is well defined and converges. (Hint: first prove that for any $\varepsilon > 0$ and real numbers x_1, x_2, y_1, y_2 with $x_1 \neq x_2$, there exists a function $f\colon \mathbb{R} \to (1-\varepsilon, 1+\varepsilon)$ such that $f(x_1) = f(x_2) = 1$, $f'(x_1) = y_1$, and $f'(x_2) = y_2$).

Problem 43. Show that every nonempty compact subset of the plane contains a vertical line segment with maximal length.

Problem 44. Let (E, d) be a path-connected metric space, and fix $x, y \in E$.

(a) Prove that if E is compact, then there exists a path in E connecting x and y with minimal (possibly infinite) length. (Hint: Use the ideas of the Arzelà-Ascoli theorem to find a uniformly convergent sequence of paths whose lengths approach the infimum.)

(b) Show by counterexample that (a) fails if we only require E to be complete.

Problem 45. Let X be an open bounded subset of $\mathbb{R}^n$ that has (Jordan) volume.

(a) Prove that the connected components of X all have strictly positive volume.

(b) Show that X is connected if and only if

$$\int_X f(x)\,dx = 0 \implies f^{-1}(0) \neq \emptyset$$

for all continuous bounded functions $f\colon X \to \mathbb{R}$.

Problem 46. For a set $X \subseteq \mathbb{R}^n$ with (Jordan) volume, we say that a function $f\colon X \to \mathbb{R}$ has *level-set volume* if $f^{-1}(\{y\})$ has volume for each $y \in \mathbb{R}$. Prove or disprove the following claims:

(a) Every bounded function with level-set volume is Riemann integrable.

(b) Every Riemann integrable function has level-set volume.

Problem 47. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that

$$\lim_{x\to\infty} \int_{[-x,x]^n} f(t)\,dt$$

exists and is finite but

$$\lim_{x\to\infty} \int_{[-x,x]^n} |f(t)|\,dt = \infty.$$

Prove that for any $a \in \mathbb{R}$, there exists an increasing sequence of subsets $\{A_k\}_{k=0}^{\infty}$ of $\mathbb{R}^n$ such that $\bigcup_{k\in\mathbb{N}} A_k = \mathbb{R}^n$,

$$\int_{A_k} f(t)\,dt$$

exists for each $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} \int_{A_k} f(t)\,dt = a.$$

Problem 48. Given a bounded function $f\colon \mathbb{R} \to \mathbb{R}$, we define the *filling* of f to be the function $F_f\colon \mathbb{R} \to \mathbb{R}$ given by

$$F_f(x) = \min\left(\sup_{t\le x} f(t),\ \sup_{t\ge x} f(t)\right).$$

(a) Prove that F_f is Riemann integrable whenever f is bounded and has compact support.

(b) Find a continuous function $f\colon (0,1] \to \mathbb{R}$ such exactly one of the limits

$$\lim_{x\to 0} \int_x^1 f(t)\,dt, \qquad \lim_{x\to 0} \int_x^1 F_f(t)\,dt$$

exists.

Problem 49. Given a bounded function $f\colon \mathbb{R}^n \to \mathbb{R}$, we define $F_f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$F_f(x) = \inf\left\{\sup_{t\in A} f(t) : x \in A,\ A \subseteq \mathbb{R}^n \text{ is connected and unbounded}\right\}.$$

(a) Prove that when $n = 1$, this definition is equivalent to that of the previous problem.

(b) Show by example that when $n \geq 2$, the filling of a bounded function with compact support might not be Riemann integrable.

(c) Prove that the filling of an Riemann integrable function is always Riemann integrable.