The Polynomial Method in Combinatorics

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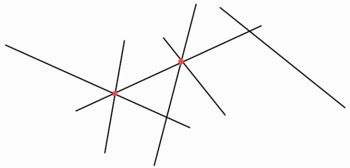
Roadmap

The joints problem

Statement of the problem

Given a set of lines $\mathcal{L}$ in $\mathbb{R}^3$, we call a point $p \in \mathbb{R}^3$ a *joint* if there are three distinct non-coplanar lines in $\mathcal{L}$ that intersect at p.



A set of 6 lines with 2 joints

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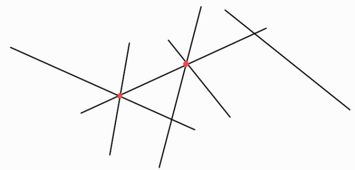

A set of 6 lines with 2 joints

Let $J(L)$ be the maximum number of joints that can be made with L lines.

The problem is to find a good upper and lower bound on $J(L)$.

History of the joints problem

Originally posed in 1990 by a large group of researchers.

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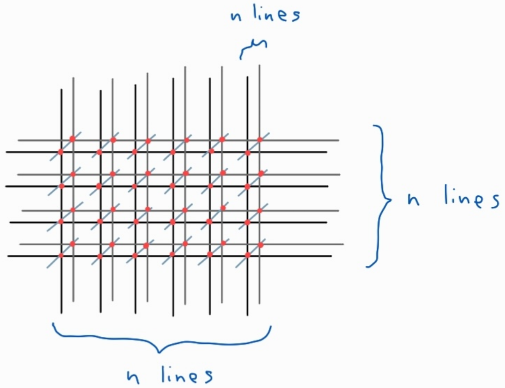
The upper bound slowly improved over time.

However, it took 20 years to close this gap and prove an upper bound of $O(n^{3/2})$.

A lower bound on the problem

Consider the following example with $L = 3n^2$ lines and $J = n^3$ joints:



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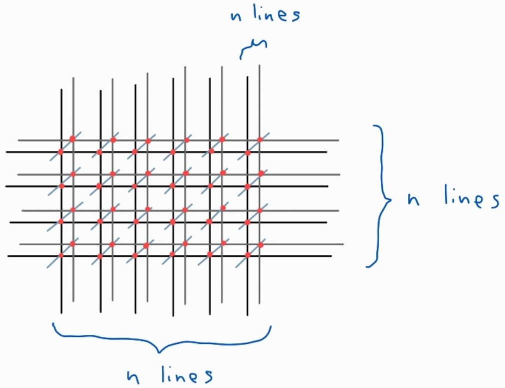

Since $L^{3/2} \leq 10n^3$, this means that $J(L) \geq 0.1L^{3/2}$.

The polynomial method

Polynomials in several variables

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We define the degree of a monomial to be the sum of the powers of each variable:

$$\deg(x^2y^3z) = 2 + 3 + 1 = 6$$

The degree of a polynomial is the maximum degree of its terms.

The vector space of polynomials

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Consider a set of points $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^3$. Then

$$p \mapsto (p(x_1), \ldots, p(x_n))$$

is a linear map from $\mathrm{Poly}_D(\mathbb{R}^3)$ to $\mathbb{R}^n$.

Finding a vanishing polynomial

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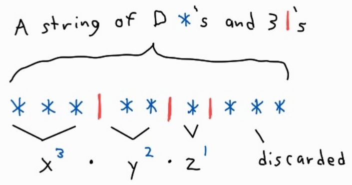
In other words, we have a nonzero polynomial with degree at most D that has $x_1, \ldots, x_n$ as roots.

How big must D be?

The dimension of $\mathrm{Poly}_D(\mathbb{R}^3)$ is $\binom{D+3}{3}$, as we can find by counting monomials with degree $\leq D$:



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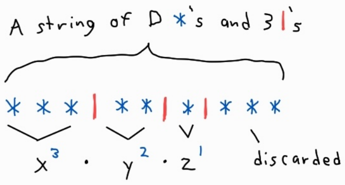

If $D \geq 3n^{1/3}$, then $D^3 \geq 27n$, so

$$\dim\left(\mathrm{Poly}_D(\mathbb{R}^3)\right) = \binom{D+3}{3} = \frac{(D+1)(D+2)(D+3)}{6} \geq \frac{1}{6}D^3 > n.$$

The existence lemma

All this work gives us the existence lemma:

Lemma (Existence Lemma)

If $\dim\left(\text{Poly}_D(\mathbb{R}^3)\right) > n$*, then there exists a nonzero polynomial with degree at most* D *that has* $x_1, \ldots, x_n$ *as roots.*

More specifically, there exists a polynomial with degree at most $3n^{1/3}$ *that vanishes on any set of* n *points.*

A nonzero polynomial from $\mathbb{R} \to \mathbb{R}$ with degree n has at most n roots.

This gives us the vanishing lemma:

Lemma (Vanishing Lemma)

If a degree D polynomial in one variable of has more than D roots, it must be the zero polynomial.

Outline of the method

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3. Use the vanishing lemma to show that the polynomial restricted to some set of lines must be uniformly zero.
4. Derive a contradiction.

Solving the joints problem

Our goal

Recall that a *joint* $p \in \mathbb{R}^3$ is an intersection point of three distinct non-coplanar lines in $\mathcal{L}$.

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$J(L)$ is the maximum number of joints that can be made with L lines.

We showed previously that $J(L) \geq 0.1L^{3/2}$. Our goal was to find an upper bound that differs only by a constant multiple.

To this end, we will prove that $J(L) \leq 10L^{3/2}$.

The main lemma

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Given a set of of lines $\mathcal{L}$ that determines J joints, there exists a line $L \in \mathcal{L}$ containing $3J^{1/3}$ or fewer joints.

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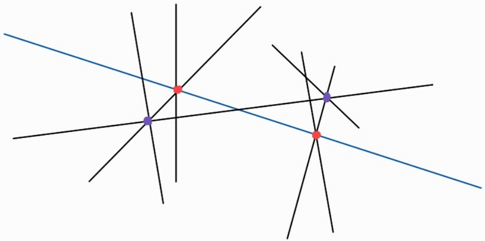
This implies that

$$J(L) \leq 3J(L)^{1/3} + J(L-1).$$



There are $\leq 3J(L)^{1/3}$ red joints and $\leq J(L-1)$ purple joints

The main lemma

Iterating the recursive inequality:

$$\begin{aligned} J(L) &\leq 3J(L)^{1/3} + J(L-1) \\ &\leq 2 \cdot 3J(L)^{1/3} + J(L-2) \\ &\leq 3 \cdot 3J(L)^{1/3} + J(L-3) \\ &\cdots \\ &\leq L \cdot 3J(L)^{1/3} \end{aligned}$$

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This means that

$$\begin{aligned} J(L)^{2/3} &\leq 3L \\ J(L) &\leq 10L^{3/2}. \end{aligned}$$

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Let $p : \mathbb{R}^3 \to \mathbb{R}$ be a nonzero polynomial **with minimal degree** that vanishes at every joint.

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Let $p : \mathbb{R}^3 \to \mathbb{R}$ be a nonzero polynomial **with minimal degree** that vanishes at every joint.

By the existence lemma, we have that $\deg(p) \leq 3J^{1/3}$.

Making p vanish

If we parametrize any line $L \in \mathcal{L}$ by $\gamma(t) = \mathsf{a} + t\mathsf{v}$, then $p(\gamma(t))$ is a polynomial in the t variable with degree at most $3J^{1/3}$.

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Since $p(\gamma(t))$ has $> 3J^{1/3}$ roots, it is the zero polynomial by the vanishing lemma.

Therefore, p vanishes on every line in $\mathcal{L}$.

Finding our contradiction

Fix a joint $\mathrm{j} \in \mathbb{R}^3$ that is the intersection of lines L_1, L_2, L_3. If v_1, v_2, v_3 are unit vectors pointing along these lines, then we have

$$\nabla_{v_1} p(\mathrm{j}) = \nabla_{v_2} p(\mathrm{j}) = \nabla_{v_3} p(\mathrm{j}) = 0.$$

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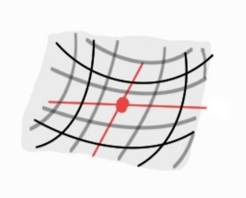
Since v_1, v_2, and v_3 are linearly independent, $\nabla_p(\mathbf{j}) = 0$. Thus $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, and $\frac{\partial p}{\partial z}$ vanish at every joint.



Because this polynomial is 0 on the red lines, its gradient is zero at the red point

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We know that p is not constant, so one of the $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, and $\frac{\partial p}{\partial z}$ must be nonzero.

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This contradicts the fact that p has minimal degree out all all nonzero polynomials that vanish at every joint!

The main lemma must therefore be true, so we indeed have that

$$J(L) \leq 10L^{3/2}.$$

Reflections

Why polynomials?

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1. $\dim\left(\mathrm{Poly}_D(\mathbb{R}^n)\right)$ grows quickly as D increases.
2. A multivariate polynomial with degree D, restricted to a line, is a single-variable polynomial with degree at most D.

In other words, there are lots of polynomials, but they behave rigidly on lines.

For this reason, the polynomial method is well-suited to counting sets defined in terms of lines.

Other problems

A set K is a *Kakeya set* if it contains a line in every direction.

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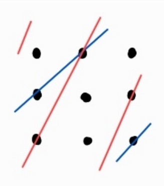

A set $K \subseteq \mathbb{F}_q^n$ is a *Kakeya set* if, for every $a \in \mathbb{F}_q^n \setminus \{0\}$, we have

$$\{at + b | t \in \mathbb{F}_q\} \subseteq K$$

for some $b \in \mathbb{F}_q^n$.

Other problems

A set N is a *Nikodym set* if, for every $x \in \mathbb{F}_q^n$, there is a line L containing x such that $L \setminus \{x\} \subseteq N$.

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The polynomial method can be used to easily prove a good lower bound for the size of these sets.

Specifically, for a fixed dimension n, these sets must contain a constant fraction of the total number of points, where the constant depends on n but not q.

Significance of the method

The finite-field Nikodym and Kakeya problems were unsolved until 2008, when Zeev Dvir published a proof using the polynomial method. The joints problem was proven in 2010 by Larry Guth and Nick Katz.

This method is quite new, and it is leading to some exciting results!

Questions?

Works cited

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Guth, Larry. *Polynomial Methods in Combinatorics.* American Mathematical Soc., 10 June 2016.

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