CO370, Winter 2022, Test #1 Solutions

Q1. Variables: $x_1 \equiv$ # units of Process 1
$x_2 \equiv$ # units of Process 2

Net revenue: Process 1: $4 * 130 + 3 * 110 - 90 = \$760$
Process 2: $3 * 130 + 4 * 110 - 80 = \$750$

LP model:
$$\begin{aligned} \max\ & 760x_1 + 750x_2 & \\ \text{s.t.}\ & 3x_1 + 5x_2 \le 16{,}000{,}000 & \text{(supply of Oil A)} \\ & 5x_1 + 3x_2 \le 10{,}000{,}000 & \text{(supply of Oil B)} \\ & x_1 \ge 0,\ x_2 \ge 0 & \end{aligned}$$

Common error: Attempting to model the problem in terms of the input oil to each process, rather than the # of units of the process. This makes it difficult to avoid infeasible solutions, such as using the same oil for both processes or to running a process with too much oil A and not enough oil B.

Q2. Let $t = 1/(2x_1 + x_2)$. Note that since $x_1 \geq 1$, we know $2x_1 + x_2 \geq 1$.

Using t, the objective becomes $3x_1 t + 6x_2 t + 4x_3 t$. We cannot multiply variables, so we make the substitution $w_1 = x_1 t$, $w_2 = x_2 t$, $w_3 = x_3 t$

$$\Rightarrow x_1 \geq 1 \iff \frac{w_1}{t} \geq 1 \iff w_1 \geq t \iff w_1 - t \geq 0$$

$$\Rightarrow 4x_1 + 2x_2 - x_3 = 8 \iff 4w_1 + 2w_2 - w_3 - 8t = 0$$

To enforce the substitution $t = 1/(2x_1 + x_2)$ we need $2x_1 t + x_2 t = 1 \iff 2w_1 + w_2 = 1$.

LP model:
$$\begin{aligned} \max \quad & 3w_1 + 6w_2 + 4w_3 \\ \text{s.t.} \quad & w_1 - t \geq 0 \\ & 4w_1 + 2w_2 - w_3 - 8t = 0 \\ & 2w_1 + w_2 = 1 \\ & w_1 \geq 0,\ w_2 \geq 0 \end{aligned}$$

Common error: Not including $2w_1 + w_2 = 1$ as a constraint. This is needed to enforce the relationship between t and x_1, x_2.

Q3. In an SVM, we want to find a separating line $a_1\bar{x}_1 + a_2\bar{x}_2 = b$, where the "Accept" points have $a_1\bar{x}_1 + a_2\bar{x}_2 \geq b$ and the "Reject" points have $a_1\bar{x}_1 + a_2\bar{x}_2 < b$.

LP Model: $\max \delta$

s.t. $\delta \leq \varepsilon_i \quad i = 1, \ldots, 6$

$$\left.\begin{aligned} \varepsilon_1 &= a_1 \cdot 1 + a_2 \cdot 2 - b \\ \varepsilon_4 &= a_1 \cdot 2 + a_2 \cdot 4 - b \\ \varepsilon_6 &= a_1 \cdot 4 + a_2 \cdot 4 - b \end{aligned}\right\} \text{"Accept" points}$$

$$\left.\begin{aligned} \varepsilon_2 &= -(a_1 \cdot 4 + a_2 \cdot 2 - b) \\ \varepsilon_3 &= -(a_1 \cdot 3 + a_2 \cdot 2 - b) \\ \varepsilon_5 &= -(a_1 \cdot 1 + a_2 \cdot 1 - b) \end{aligned}\right\} \text{"Reject" points}$$

$$\left.\begin{aligned} -1 &\leq a_1 \leq 1 \\ -1 &\leq a_2 \leq 1 \end{aligned}\right\} \text{Scale } a_1, a_2$$

b free

The variables are a_1, a_2 & b.

Common errors: Setting b equal to 0 or 1, rather than a variable. Optimizing the sum of ε_i rather than maximizing the min value. Having different a_1 & a_2 variables for each data point. Modeling L_1-regression rather than an SVM. Writing general form and not creating the model for the example.

Q4. A) $\bar{x}_B = \mathbb{B}^{-1} b = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 11 \\ 26 \end{bmatrix} \geq 0$ ✓

$$c_N^T - c_B^T \mathbb{B}^{-1} A_N = [-1 \ -2] - [-7 \ 2] \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 5 \end{bmatrix}$$

$$= [1 \ 1] \geq 0 \ ✓$$

B) $y_B^T = c_B^T \mathbb{B}^{-1} = [-7 \ 2] \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = [-2 \ -1]$

C) x_3 is a non-basic variable, so c_B does not change.
Need to compute the reduced cost $\bar{c}_3$ of x_3.
(This was already done in part A.)

$$\bar{c}_3 = c_3 - c_B^T \mathbb{B}^{-1} A_3 = 1.$$

$$\Rightarrow \Delta \geq -1$$

d) Need to keep primal feasibility.

$$\bar{x}_B = \mathbb{B}^{-1} b = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 7+\Delta \\ 11 \end{bmatrix} = \begin{bmatrix} 11 \\ 26-\Delta \end{bmatrix} \geq 0$$

$$\Rightarrow 26 - \Delta \geq 0 \iff \Delta \leq 26$$

<u>Common errors</u>: Writing the objective value in (B), rather than the dual solution. Changing c_B in part (C). Not checking primal feasibility in part (A).