Problem Set 5

Evan Arch (https://github.com/evan-nyu/phys-ga2000)

October 8, 2024

Problem 1

The derivative of

$$1 + \frac{1}{2}\tanh(2x) \tag{1}$$

can be numerically generated using the central difference method. This gives a highly accurate result as can be seen from figure 1 where the numerical result is plotted along with the analytic derivative and the jax auto derivative.

Problem 2

The gamma function is given by

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx. \tag{2}$$

Plots of the integrand for $a = 2$, 3, and 4 for $x = [0, 5]$ are shown in figure 2. The maximum value of the integrand of equation 2 can be found by taking the derivative and setting it equal to zero:

$$\frac{d}{dx}\left(x^{a-1}e^{-x}\right) = 0 \tag{3a}$$

$$(a-1)x^{a-2}e^{-x} - x^{a-1}e^{-x} = 0 \tag{3b}$$

$$x^{a-1}\left((a-1) - x\right) = 0 \tag{3c}$$

$$a - 1 = x. \tag{3d}$$

To change the limits of integration of equation 2 from $[0, \infty)$ to $[0, 1]$, we will use the change of variables

$$z = \frac{x}{x+c}. \tag{4}$$

We want to place the peak of the integrand at $z = 1/2$ for the best integration result. Thus, we choose, $c = a - 1$.

To combat numerical instability, we rewrite the integrand as

$$e^{(a-1)\log(z)-z} \tag{5}$$

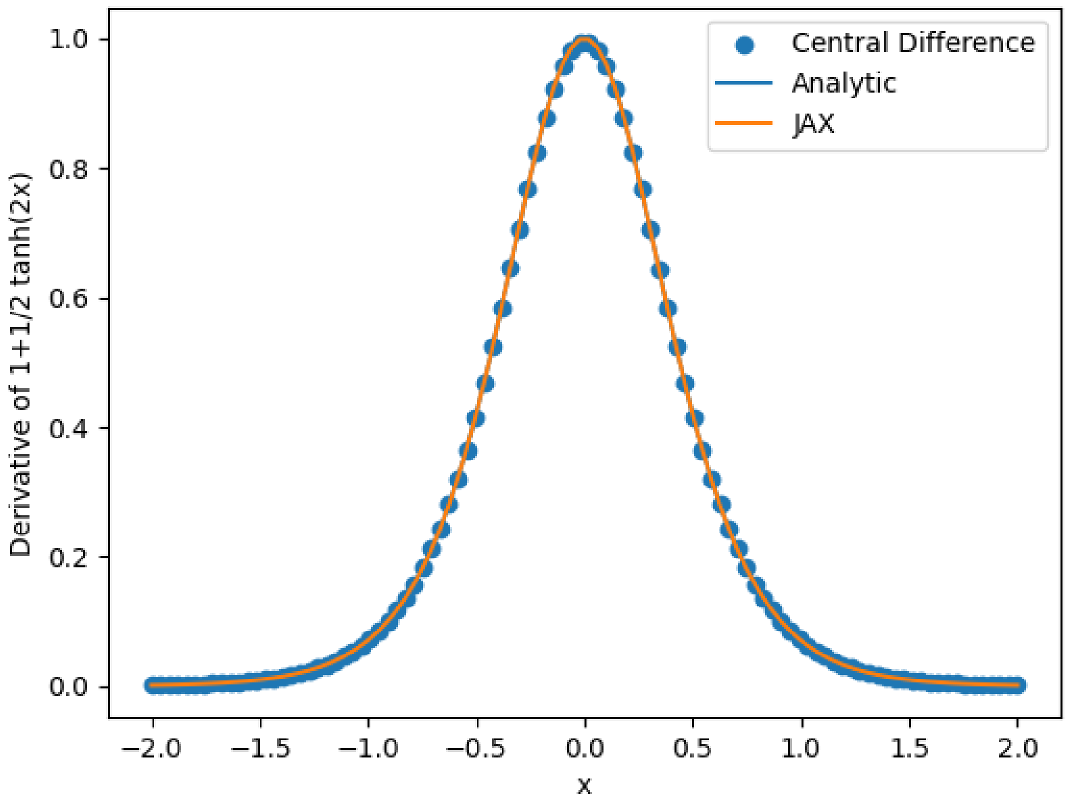

Figure 1: The derivative of equation 1 obtained via three different methods: central difference, jax autodiff, and analytic. Note that the jax autodiff and the analytic lines overlap to the point where they cannot be distinguished.

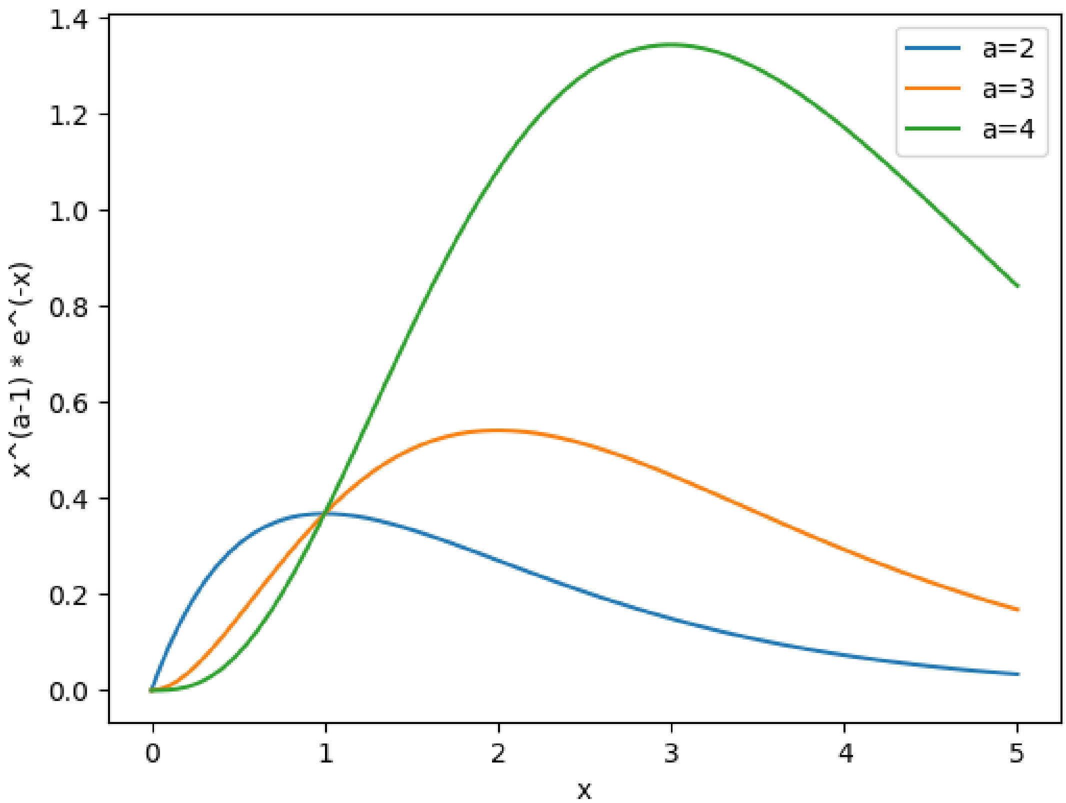

Figure 2: Plot of the integrand of equation 2 for multiple values of a.

This keeps the removes the multiplication of the exponential which can cause problems. The equation for the gamma function becomes:

$$\Gamma(a) = \int_0^1 e^{(a-1)*\log(x)-x} dx \tag{6}$$

where $dx = (a-1)/(z-1)^2 dz$.

Integrating our new gamma function with the change of variables using scipy.integrate.quad, we find that $\Gamma(\frac{3}{2}) = 0.886$ which is the expected result. Further, for integers, we find that $\Gamma(a) = (a-1)!$

$$\Gamma(4) = 6 = 3! \tag{7a}$$
$$\Gamma(6) = 120 = 5! \tag{7b}$$
$$\Gamma(10) = 362880 = 9! \tag{7c}$$

Problem 3

The data in the signal.dat file is plotted in figure 3. We can find a polynomial fit for this data using SVD. The resultant third order fit and residuals are plotted in figure 4. The fit is quite bad as the residuals are on the same order as the data and no useful information about the period is obtained. These residuals are far larger than the measurement uncertainties. A better fit can be obtained by going to higher order, 23rd order produces a nice plot, however, the condition number becomes far too high for a numerically stable fit (see figure 5). For 23rd order the condition number is 4503046701.257012.

A better result can be found by switching from a polynomial fit to a fit of a series of sines and cosines. We fit sines and cosines of the form $\sin(2\pi n \frac{t}{P})$ where n is an integer and P is half of the time length of the data. This gives better fits while having few terms in the fit. A $n \leq 4$ fit is shown in figure 6. The periodicity of the data is clearly shown by the fit, however residuals indicate that the amplitude information in not captured to much accuracy. The periodicity can be determined to be about 0.15 from the fit.

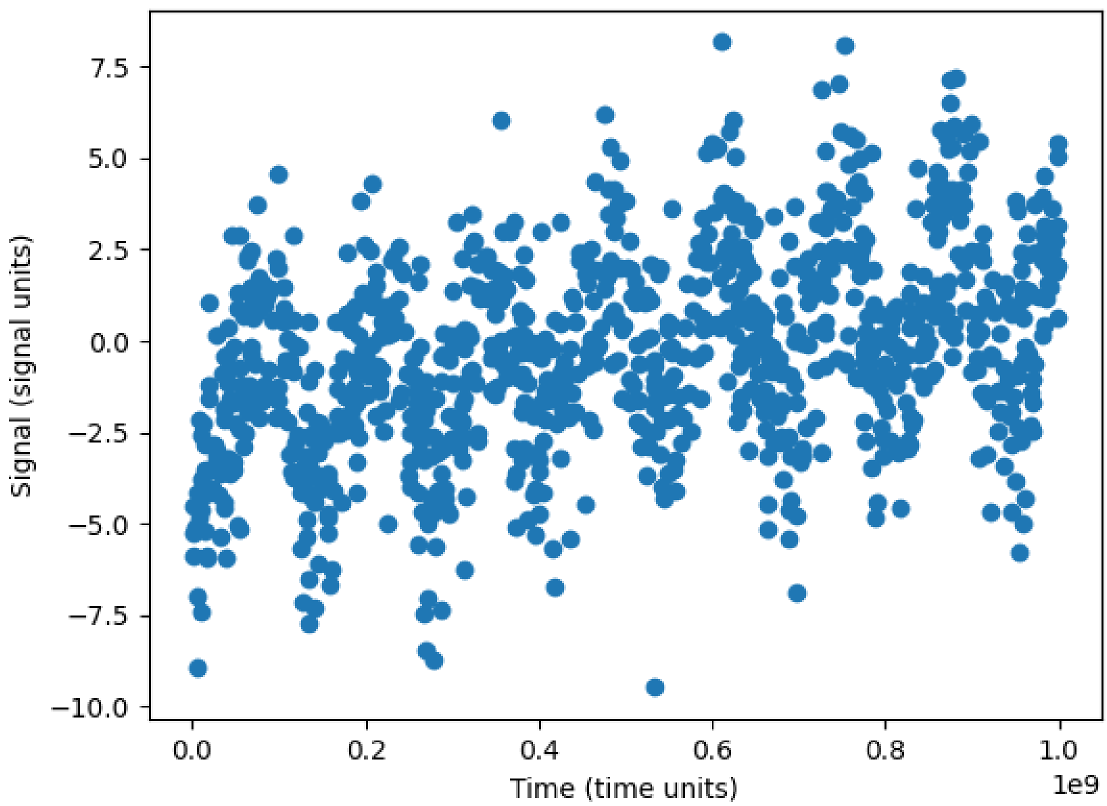

Figure 3: Scatter plot of signal versus time data.

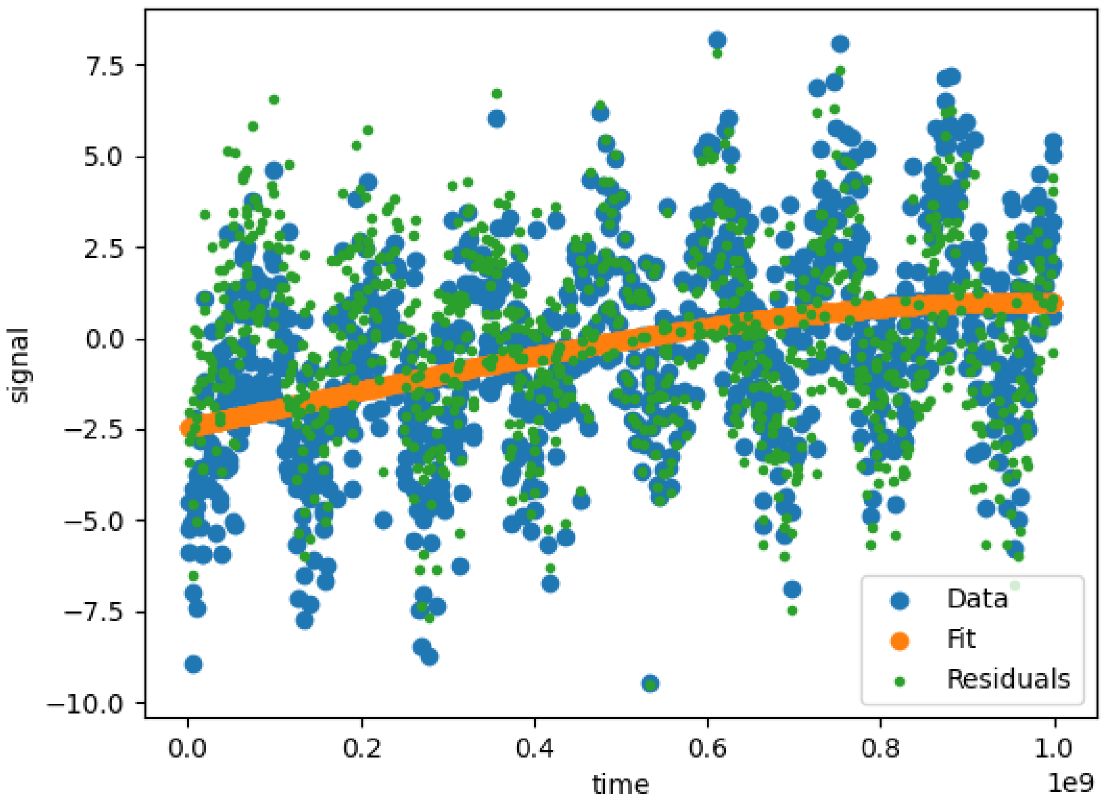

Figure 4: Third order polynomial fit using SVD

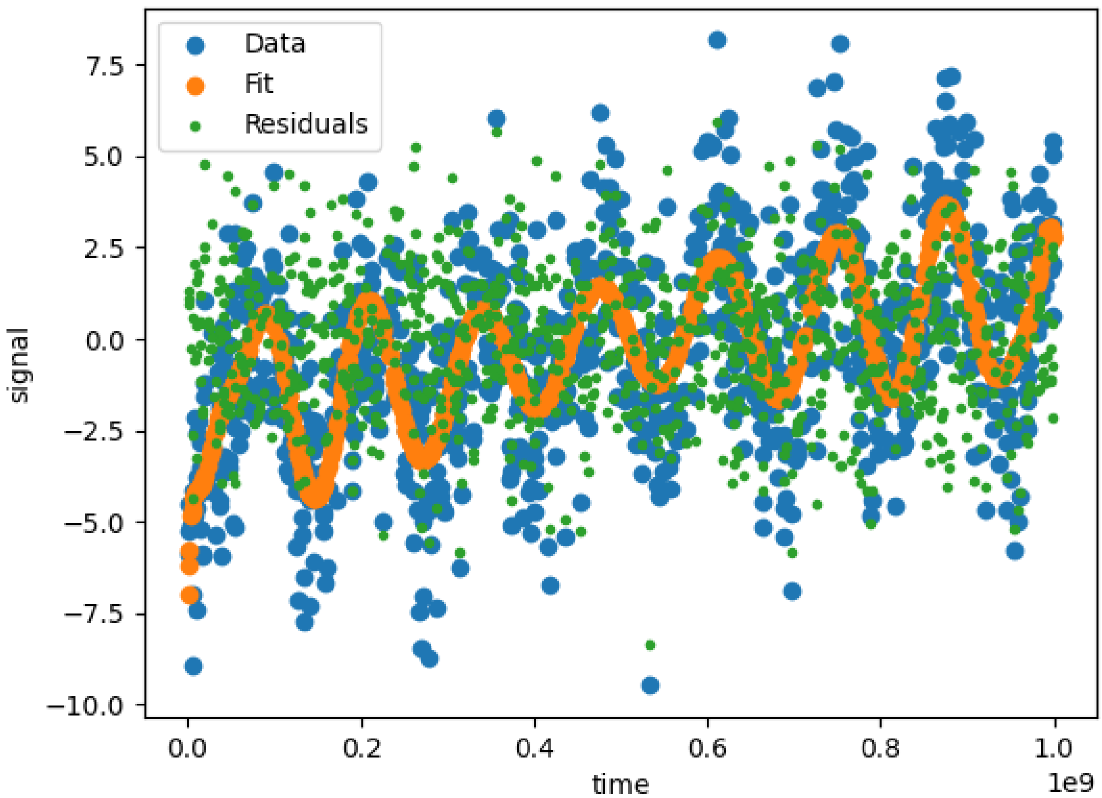

Figure 5: 23rd order polynomial fit.

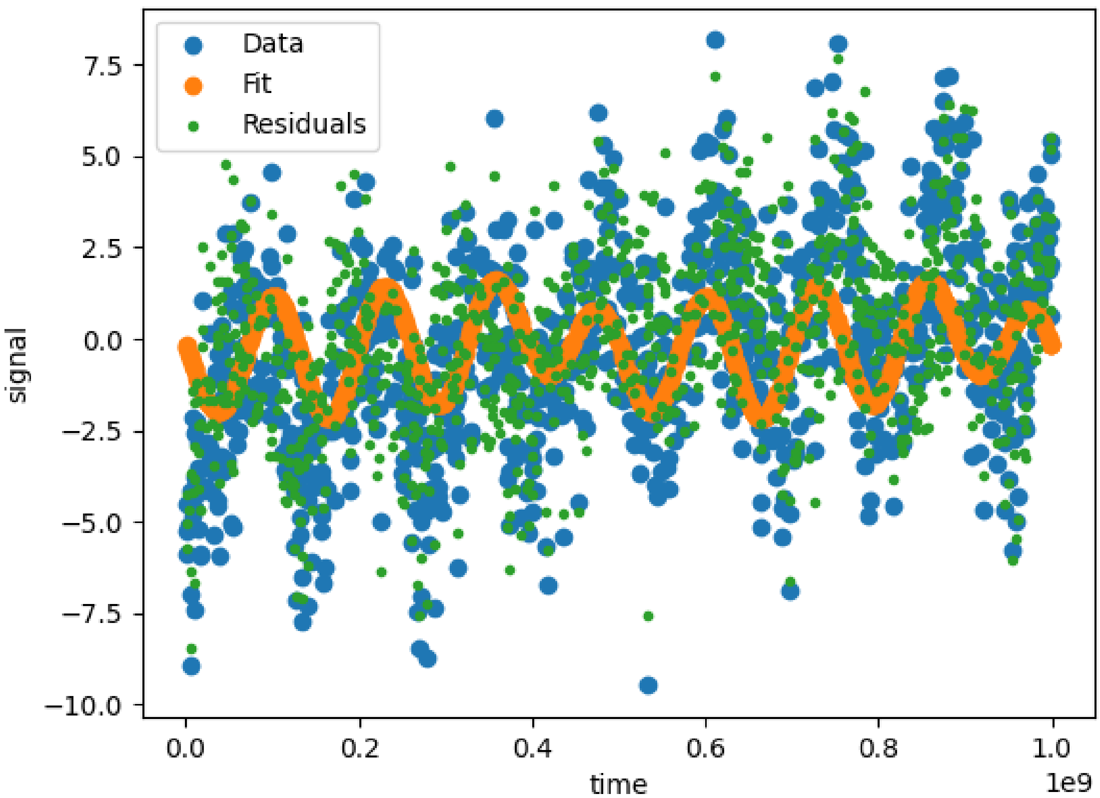

Figure 6: Fit of the data using a series of sines and cosines with $n \leq 4$.