Solutions for *Trigonometry* by Gelfand & Saul

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Introduction

Trigonometry by Gelfand and Saul is often recommended as a precalculus text for self-study. However, those who are learning without the help of a teacher can struggle with the lack of solutions to exercises in the text. Partial sets of solutions for *Trigonometry* have been published by John Beach[1] and Fardeen Ashraf[2]. It is hoped that this document will eventually contain a complete set of solutions. Contributions are welcome. These can take the form of pull requests or issues submitted to the project's GitHub repository[3].

Chapter 0: Trigonometry

Page 8

1. Statement I applies:

$$\begin{aligned} c^2 &= a^2 + b^2 = 10^2 + 24^2 = 100 + 576 = 676 \\ c &= \sqrt{676} = 26 \end{aligned}$$

2. Statement I applies:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 9^2 &= 41^2 \\ a^2 + 81 &= 1681 \\ a^2 &= 1600 \\ a &= \sqrt{1600} = 40 \end{aligned}$$

[1] https://jbeach50.weebly.com/gelfand–saul-trig-solutions.html
[2] https://archive.org/details/gelfand-trigonometry-solutions-manual
[3] https://github.com/philip-healy/gelfand-trigonometry-solutions

3. $5^2 + 12^2 = 25 + 144 = 169 = 13^2$. By Statement II, a right triangle exists with legs of length 5 and 12, and hypotenuse of length 13.

4. Statement I applies:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 1^2 &= 3^2 \\ a^2 + 1 &= 9 \\ a^2 &= 8 \\ a &= \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \end{aligned}$$

5. Statement I applies, where $a = b$:

$$\begin{aligned} a^2 + a^2 &= c^2 \\ a^2 + a^2 &= 1^2 \\ 2a^2 &= 1 \\ a^2 &= \frac{1}{2} \\ a &= \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \end{aligned}$$

6. From the diagram at the bottom of Page 11, we can see the shorter leg is half the length of the hypotenuse. So in this instance the shorter leg has length 1/2. We can use Statement 1 to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + \left(\frac{1}{2}\right)^2 &= 1^2 \\ a^2 + \frac{1}{4} &= 1 \\ a^2 &= \frac{3}{4} \\ a &= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} \end{aligned}$$

7. For any point Y, we can draw a triangle with sides AY, BY and AB. Let a be the length of side AY, b be the length of side BY and c be the length of side AB. According to Statement II, the subset of these triangles where $a^2 + b^2 = c^2$ are right triangles with legs of length a and b and hypotenuse c. Let X be the subset of Y that are vertices of these right triangles. This set of points describes a circle with its centre at the midpoint of AB, and radius $AB/2$.

8.

Page 9

1. $6^2 + 8^2 = 36 + 64 = 100 = 10^2$. By Statement II on Page 7 (converse of the Pythagorean Theorem), this is a right triangle.

2. 10-24-26 (Exercise 1), 9-40-41 (Exercise 2), 5-12-13 (Exercise 3)

3. Using the Pythagorean Theorem:

$$\begin{aligned} c^2 &= a^2 + b^2 = 8^2 + 15^2 = 64 + 225 = 289 \\ c &= \sqrt{289} = 17 \end{aligned}$$

4. The first column in the table increases by 3, the second increases by 4 and the third increases by 5. Continuing to add rows yields triangles 12-16-20, 15-20-25 and 18-24-30.

5. Shortest side with length 10: 10-24-26. Shortest side with length 15: 15-36-39.

6. Multiplying all sides by the common denominator (5), we get a similar triangle with sides $15/5 = 3$, $20/5 = 4$ and 5. We know that this is a right triangle from the table in Question 4.

7. To find a similar triangle with shorter leg 1, divide all sides by 3, resulting in sides 1-4/3-5/3. To find a similar triangle with longer leg 1, divide all sides by 4, resulting in sides 3/4-1-5/4.

8. To find a similar triangle with hypotenuse 1, divide all sides by 13, resulting in sides 5/13-12/13-1. To find a similar triangle with shorter leg 1, divide all sides by 5, resulting in sides 1-12/5-13/5. To find a similar triangle with longer leg 1, divide all sides by 12, resulting in sides 5/12-1-13/12.

9. To formula for the area of a triangle is $\frac{1}{2}bh$ where b is the length of the base and h is the height. For right triangles, finding the area is easy: one leg is the base and the other leg is the height. For other triangles, finding the height is more difficult: we need to find the length of the altitude drawn from the base. The triangles with sides 5-12-13 and 9-12-15 are both right triangles: see Exercise 3 on Page 8 and Exercise 4 on Page 9. The triangle with sides 13-14-15 is not a right triangle. We can confirm this using Statement I: $a^2 + b^2 = 13^2 + 14^2 = 365$, $c^2 = 15^2 = 225, a^2 + b^2 \neq c^2$. However, if we join the 5-12-13 and 9-12-15 triangles using their equal legs, the resulting triangle has the dimensions we are looking for: 13-14-15. The base of this combined triangle has length $5 + 9 = 14$. We also know the length of the altitude from the base of the combined triangle: 12. So, the area of the 13-14-15 triangle is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ units squared.

10. (a)

    (b)

Page 11

1. $\frac{1}{\sqrt{2}}$ (see the solution for Question 5 on page 8).
   Challenge: $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ (multiplying above and below by $\sqrt{2}$). $\sqrt{2}$ is given to 9 decimal places in the diagram on the top of page 11: 1.4141213562373. Dividing this decimal representation by 2 (using long division if necessary) yields a figure of 0.707060678.

2. $c^2 = a^2 + b^2 = 3^2 + 3^2 = 9 + 9 = 18$. $c = \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$.

3. The hypotenuse of a 30° right triangle is double the length of the shorter leg. In this instance the hypotenuse is 10 units long. We can use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
a^2 + 5^2 &= 10^2 \\
a^2 + 25 &= 100 \\
a^2 &= 75 \\
a &= \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3}
\end{aligned}$$

4. We can solve these by finding similar triangles to the 30° right triangle with sides 1-$\sqrt{3}$-2, or the 45° right triangle with sides 1-1-$\sqrt{2}$.
   (a) $x = \sqrt{3}$, $y = 2$
   (b) $x = \frac{1}{\sqrt{3}}$, $y = \frac{2}{\sqrt{3}}$
   (c) $x = 1/2$, $y = \sqrt{3}/2$
   (d) $x = 4\sqrt{3}$, $y = 8$
   (e) $x = y = 2\sqrt{2}$
   (f) $x = 5$, $y = 5\sqrt{2}$

Page 14 (Examples)

1. Why didn't we need to compare 3^2 with $2^2 + 4^2$, or 2^2 with $3^2 + 4^2$?
   The obtuse angle will always be opposite the longest side.

2. This conclusion is *incorrect.* Why?
   From the footnote at the beginning of Chapter 0: *"Given three arbitrary lengths... they form a triangle if and only if the sum of any two of them is greater than the third."* In this case $1 + 2 = 3$ which is equal to (not greater than) the third side.

Page 14 (Exercise)

1. (a) $c^2 = 8^2 = 64$. $a^2 + b^2 = 6^2 + 7^2 = 36 + 49 = 85$. $c^2 < a^2 + b^2$, so the triangle is acute.
   (b) $c^2 = 10^2 = 100$. $a^2 + b^2 = 6^2 + 8^2 = 36 + 64 = 100$. $c^2 = a^2 + b^2$, so the triangle is a right triangle.
   (c) a and b are the same as in question b), but c is smaller, so the triangle is acute.
   (d) a and b are the same as in question b), but c is larger, so the triangle is obtuse.
   (e) $c^2 = 12^2 = 144$. $a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$. $c^2 < a^2 + b^2$, so the triangle is acute.
   (f) $c^2 = 14^2 = 196$. $a^2 + b^2 = 169$, as above. $c^2 > a^2 + b^2$, so the triangle is obtuse.
   (g) The sum of two sides must be larger than the third, but $12 + 5 = 17$ in this case.

Chapter 1: Trigonometric Ratios in a Triangle

Page 23

1. (a) $\sin \alpha = 5/13$
   (b) $\sin \alpha = 4/5$
   (c) $\sin \alpha = 5/13$
   (d) $c = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$. $\sin \alpha = 8/10$.
   (e) $\sin \alpha = 3/5$
   (f) $\sin \alpha = 12/13$
   (g) $\sin \alpha = 3/5$
   (h) $c = \sqrt{7^2 + 3^2} = \sqrt{58}$. $\sin \alpha = 7/\sqrt{58}$.

2. (a) $\sin \beta = 12/13$
   (b) $\sin \beta = 3/5$
   (c) $\sin \beta = 12/13$
   (d) $\sin \beta = 6/10$
   (e) $\sin \beta = 4/5$
   (f) $\sin \beta = 5/13$
   (g) $\sin \beta = 4/5$
   (h) $\sin \beta = 3/\sqrt{58}$

3. The example 30-60-90 triangle given on page 11 has sides 1, $\sqrt{3}$, 2. Let β represent the 60°angle. The opposite leg b has length $\sqrt{3}$. The hypotenuse c has length 2. So, $\sin\beta = b/c = \sqrt{3}/2 \approx 1.732/2 = 0.866$.
Crossing off the numbers listed:
~~0.1~~ ~~0.2~~ ~~0.3~~ ~~0.4~~ ~~0.5~~ ~~0.6~~ ~~0.7~~ ~~0.8~~ 0.9

Page 25

1. The Altitude-on-Hypotenuse Theorem tells us that when an altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other. Therefore, the triangles with sides a-b-c, a-p-d and d-b-q are similar, and the ratio for $\sin\alpha$ appears in all of them:
   (a) b/c
   (b) d/a
   (c) q/b
2. (a) $\sin\alpha = h/b$
   (b) Multiplying both sides of formula above by b: $h = b\sin\alpha$
   (c) Substituting $b\sin\alpha$ for h, the formula for the area of ABC can be rewritten as: $bc\sin\alpha/2$.
   (d) $\sin\beta = h/a$. Rewriting this in terms of h: $h = a\sin\beta$. Substituting this for h in the area formula: $ac\sin\beta/2$.
   (e) Let h_2 represent the altitude from A to BC. $sin\beta = h_2/c$. Rewriting in terms of h_2, we get $h_2 = c\sin\beta$.
3. (a) Expressing h in terms of $\sin\alpha$ and b:

$$\sin\alpha = \frac{h}{b}$$
$$h = b\sin\alpha$$

   Expressing h in terms of $\sin\beta$ and a:

$$\sin\beta = \frac{h}{a}$$
$$h = a\sin\beta$$

   (b) Both expresssions are equal to h:

$$a\sin\beta = h = b\sin\alpha$$

   (c) Expressing h_2 in terms of $\sin\beta$ and c:

$$\sin\beta = \frac{h_2}{c}$$
$$h_2 = c\sin\beta$$

Expressing h_2 in terms of $\sin\gamma$ and b:

$$\sin\gamma = \frac{h_2}{b}$$
$$h_2 = b\sin\gamma$$

Both expressions are equal to h_2:

$$b\sin\alpha = h_2 = c\sin\gamma$$

(d) i. We can rewrite the result from part (b) so that the expressions on each side are fractions with sine denominators:

$$a\sin\beta = b\sin\alpha$$
$$\frac{a\sin\beta}{\sin\alpha\sin\beta} = \frac{b\sin\alpha}{\sin\alpha\sin\beta}$$
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

ii. We can rewrite the result from part (c) similarly:

$$c\sin\beta = b\sin\gamma$$
$$\frac{c\sin\beta}{\sin\beta\sin\gamma} = \frac{b\sin\gamma}{\sin\beta\sin\gamma}$$
$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

We can derive the Law of Sines by combining results i. and ii. using the common expression $b/\sin\beta$:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

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1. (a) $\cos\alpha = 12/13$. $\cos\beta = 5/13$.
   (b) $\cos\alpha = 3/5$. $\cos\beta = 4/5$.
   (c) $\cos\alpha = 12/13$. $\cos\beta = 5/13$.
   (d) $\cos\alpha = 6/10$. $\cos\beta = 8/10$.
   (e) $\cos\alpha = 4/5$. $\cos\beta = 3/5$.
   (f) $\cos\alpha = 5/13$. $\cos\beta = 12/13$.
   (g) $\cos\alpha = 4/5$. $\cos\beta = 3/5$.
   (h) $\cos\alpha = 3/\sqrt{58}$. $\cos\beta = 7/\sqrt{58}$.
2. (a) $c = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$. $\cos\alpha = 8/10$. $\cos\beta = 6/10$.

(b) $c = \sqrt{5^2 + 12^2} = \sqrt{25 + 155} = \sqrt{169} = 13$. $\cos\alpha = 12/13$. $\cos\beta = 5/13$.

(c) Scaling up the 1-$\sqrt{3}$-2 30° triangle gives us a value of 20 units for the length of c. Next, we will use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 10^2 + b^2 &= 20^2 \\ b^2 &= 400 - 100 = 300 \\ b &= \sqrt{300} = \sqrt{100}\sqrt{3} = 10\sqrt{3} \end{aligned}$$

We can now find $\cos\alpha$ and $\cos\beta$:

$$\begin{aligned} \cos\alpha &= \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2} \\ \cos\beta &= \frac{10}{20} = \frac{1}{2} \end{aligned}$$

(d) The triangle is congruent to the one above, so the solution is the same.

(e) Consider the 45° right triangle with legs of length 1 and hypotenuse $\sqrt{2}$. $\cos\alpha = \cos\beta = 1/\sqrt{2}$.

(f) $c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. $\cos\alpha = 3/5$. $\cos\beta = 4/5$.

(g) $b = x\sqrt{3}$. $\cos\alpha = x\sqrt{3}/2x = \sqrt{3}/2$. $\cos\beta = x/2x = 1/2$.

3. The Altitude-on-Hypotenuse Theorem tells us that when an altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other. Therefore, the triangles with sides a-b-c, a-p-d and d-b-q are similar, and the ratio for $\cos\alpha$ appears in all of them:

   (a) a/c

   (b) p/a

   (c) d/b

Page 28

1. In this instance, $\alpha = 29°$, $\beta = 61°$, and $\alpha + \beta = 90°$. According to the theorem above, if $\alpha + \beta = 90°$, then $\sin\alpha = \cos\beta$.

2. $x = 90 - 35 = 55°$

3. If $\alpha + \beta = 90°$, then $\beta = 90° - \alpha$. According to the theorem above, $\sin\alpha = \cos\beta$. Substituting $(90 - \alpha)$ for β: $\sin\alpha = \cos(90 - \alpha)$.

Page 29

First, we need to find the length of the hypotenuse: $c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

1. $\sin^2 \alpha = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$
2. $\sin^2 \beta = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$
3. $\cos^2 \alpha = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$ (same as $\sin^2 \beta$)
4. $\cos^2 \beta = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$ (same as $\sin^2 \alpha$)
5. $\sin^2 \alpha + \cos^2 \alpha = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$
6. $\sin^2 \alpha + \cos^2 \beta = \frac{16}{25} + \frac{16}{25} = \frac{32}{25}$
7. $\cos^2 \alpha + \sin^2 \beta = \frac{9}{25} + \frac{9}{25} = \frac{18}{25}$

Page 30

1. $\sin^2 \alpha + \cos^2 \alpha = \left(\dfrac{4}{5}\right)^2 + \left(\dfrac{3}{5}\right)^2 = \dfrac{16}{25} + \dfrac{9}{25} = \dfrac{25}{25} = 1$
2. It's not an error. According to the corollary of the Pythagoream Theorem, this a right triangle: $a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 = c^2$.
3. $\sin^2 \beta + \cos^2 \beta = \left(\dfrac{3}{5}\right)^2 + \left(\dfrac{4}{5}\right)^2 = \dfrac{9}{25} + \dfrac{16}{25} = \dfrac{25}{25} = 1$
4. $\cos^2 \alpha + \sin^2 \alpha = 1$
$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169} = \frac{144}{169}$$
$$\cos \alpha = \sqrt{\frac{144}{169}} = \frac{12}{13}$$
5. $\cos^2 \alpha + \sin^2 \alpha = 1$
$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{5}{7}\right)^2 = 1 - \frac{25}{49} = \frac{24}{49}$$
$$\cos \alpha = \sqrt{\frac{24}{49}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{49}} = \frac{2\sqrt{6}}{7}$$

6. We will follow the proof at the bottom of Page 29:

$$\begin{aligned} sin^2\alpha + \sin^2\beta &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\ &= \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{a^2+b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

7. Again, we will follow the proof at the bottom of Page 29:

$$\begin{aligned} cos^2\alpha + \cos^2\beta &= \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 \\ &= \frac{b^2}{c^2} + \frac{a^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{a^2+b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

Page 31

1.

angle x	$\sin x$	$\cos x$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
α	$\frac{4}{5}$	$\frac{3}{5}$
β	$\frac{3}{5}$	$\frac{4}{5}$

2. $\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°$

3. $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

4. We can observe from the table that $\sin x$ increases with the size of an acute angle ($\sin 30^\circ < \sin 45^\circ < \sin 60^\circ$), while $\cos x$ decreases with the size of an acute angle. You can compare the fractions or convert to decimal make sure. We know that $\sin\alpha = \frac{4}{5}$. We also know that α is an acute angle.
Is it larger or smaller than 30°*?* Larger, $\frac{4}{5} > \frac{1}{2}$ so $\sin\alpha > \sin 30^\circ$.
Than 45°*?* Larger, $\frac{4}{5} > \frac{1}{\sqrt{2}}$ so $\sin\alpha > \sin 45^\circ$.
Than 60°*?* Smaller, $\frac{4}{5} < \frac{\sqrt{3}}{2}$ so $\sin\alpha < \sin 60^\circ$.

Page 33 (First)

1. As the angle α get smaller, the ratio of the opposite side to the hypotenuse approaches 0.

2. Recall from the theorem on page 28 that if $\alpha + \beta = 90^\circ$, then $\sin\alpha = \cos\beta$ and $\cos\alpha = \sin\beta$. So, if $\sin 90^\circ = 1$, then $\cos 0^\circ = 1$.

3. $\sin^2 0^\circ + \cos^2 0^\circ = 0^2 + 1^2 = 0 + 1 = 1$

4. $\sin^2 90^\circ + \cos^2 90^\circ = 1^2 + 0^2 = 1 + 0 = 1$

5. Our friend is mistaken; the sine of an angle can never be greater than 1.

Page 33 (Second)

1.

$\sin 0^\circ + \cos 0^\circ$	$0+1$	1
$\sin 30^\circ + \cos 30^\circ$	$\frac{1}{2} + \frac{\sqrt{3}}{2}$	1.366 (approx.)
$\sin 45^\circ + \cos 45^\circ$	$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$	1.414 (approx.)
$\sin 60^\circ + \cos 60^\circ$	$\frac{\sqrt{3}}{2} + \frac{1}{2}$	1.366 (approx.)
$\sin 90^\circ + \cos 90^\circ$	$1+0$	1
$\sin\alpha + \cos\alpha$, where α is the smaller...	$\frac{3}{5} + \frac{4}{5}$	1.4
$\sin\alpha + \cos\alpha$, where α is the larger...	$\frac{4}{5} + \frac{3}{5}$	1.4

2. If $\sin\alpha = 1$, then $\cos\alpha = 0$ and $\sin\alpha + \cos\alpha = 1$. If $\cos\alpha = 1$, then $\sin\alpha = 0$ and $\sin\alpha + \cos\alpha = 1$. Otherwise, $\sin\alpha < 1$ and $\cos\alpha < 1$, so $\sin\alpha + \cos\alpha < 2$.

3. First we will expand and simplify $(\sin\alpha + \cos\alpha)^2$:

$$\begin{aligned}(\sin\alpha + \cos\alpha)^2 &= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha \\ &= (\sin^2\alpha + \cos^2\alpha) + 2\sin\alpha\cos\alpha \\ &= 1 + 2\sin\alpha\cos\alpha\end{aligned}$$

   We know that $0 \le \sin\alpha \le 1$ and $0 \le \cos\alpha \le 1$ because α is acute. So $2\sin\alpha\cos\alpha$ is the product of three nonnegative numbers, and is itself a nonnegative number. A nonnegative number added to 1 results in a number ≥ 1. Therefore, $1+2\sin\alpha\cos\alpha \ge 1$. The square root of a number ≥ 1 is itself ≥ 1. Therefore, $\sqrt{1+2\sin\alpha\cos\alpha} \ge 1$. Rewriting the expression on the left: $\sqrt{1+2\sin\alpha\cos\alpha} = \sqrt{(\sin\alpha+\cos\alpha)^2} = \sin\alpha + \cos\alpha$. So, $\sin\alpha + \cos\alpha \ge 1$.

4. $\sin 45° + \cos 45° = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

5. You should notice that the values for $\sin\alpha+\cos\alpha$ increases with larger *alpha* when $0° \le \alpha < 45°$, reaches a maximum value when $\alpha = 45°$, then decreases with larger α when $45° < \alpha \le 90°$.

Page 35

1.

$(\sin 0°)(\cos 0°)$	$0 \cdot 1$	0
$(\sin 30°)(\cos 30°)$	$\frac{1}{2} \cdot \frac{\sqrt{3}}{2}$	0.433 (approx.)
$(\sin 45°)(\cos 45°)$	$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$	0.5
$(\sin 60°)(\cos 60°)$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$	0.433 (approx.)
$(\sin\alpha)(\cos\alpha)$, where α is the smaller...	$\frac{3}{5} \cdot \frac{4}{5}$	0.48
$(\sin\alpha)(\cos\alpha)$, where α is the larger...	$\frac{4}{5} \cdot \frac{3}{5}$	0.48

   How large can the product $(\sin\alpha)(\cos\alpha)$ *get?* We can see from the table that the maximum value of the product appears to be when $\alpha = 45°$.

Page 37

1. $\cos\alpha = 3/5$, $\cos\beta = 4/5$, $\sin\alpha = 4/5$, $\sin\beta = 3/5$, $\tan\alpha = 4/3$, $\tan\beta = 3/4$, $\cot\alpha = 3/4$, $\cot\beta = 4/3$.

2. We can show that this assumption is correct using the corollary of the Pythagorean Theorem: $a^2 + b^2 = 3^2 + 4^2 = 25 = c^2$.

3. $\cos\alpha = a/c$, $\cos\beta = b/c$, $\sin\alpha = b/c$, $\sin\beta = a/c$, $\tan\alpha = b/a$, $\tan\beta = a/b$, $\cot\alpha = a/b$, $\cot\beta = b/a$.

4. $c = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$. $\cos\alpha = 12/13$. $\cos\beta = 5/13$. $\cot\alpha = 12/5$. $\cot\beta = 5/12$.

5. First, we will use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 7^2 &= 25^2 \\ a^2 + 49 &= 625 \\ a^2 &= 576 \\ a &= 24 \end{aligned}$$

   We can now find the numerical values that were asked for: $\cos\alpha = 24/25$, $\cos\beta = 7/25$, $\cot\alpha = 24/7$, $\cot\beta = 7/24$.

6. $\dfrac{a}{c} = \sin\alpha = \cos\beta$
   $\dfrac{b}{c} = \cos\alpha = \sin\beta$
   $\dfrac{a}{b} = \tan\alpha = \cot\beta$
   $\dfrac{b}{a} = \cot\alpha = \tan\beta$

7. First, we will use the Pythagorean Theorem to find the length of the other leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 3^2 &= 5^2 \\ a^2 + 9 &= 25 \\ a^2 &= 16 \\ a &= 4 \end{aligned}$$

   We can now find the numerical values that were asked for: $\cos\alpha = 4/5$, $\cot\alpha = 4/3$.

8. If $\tan\alpha = 1$, then $a/b = 1$, implying that $a = b$ and $\alpha = 45°$. $\cos\alpha = \cos 45° = 1/\sqrt{2}$. $\cot\alpha = 1/1 = 1$.

9. $\tan 45° = 1/1 = 1$.

10. $\tan 30° = 1/\sqrt{3} \approx 0.57735$.

11. $\tan 45° + \sin 30° = 1 + \frac{1}{2} = \frac{3}{2}$. We don't need a calculator because both numbers are rational.

Chapter 2: Relations among Trigonometric Ratios

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1. $\cos\alpha = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \sqrt{1 - \frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$

   $\tan\alpha = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}$

   $\cot\alpha = \frac{15}{8}$

2. Let the length of the adjacent leg a be $\frac{3}{7}$ and the length of the hypotenuse be 1 (see the first triangle diagram on page 44).

   $$\sin\alpha = \sqrt{1-a^2} = \sqrt{1 - \left(\frac{3}{7}\right)^2} = \sqrt{1 - \frac{9}{49}} = \sqrt{\frac{40}{49}} = \frac{\sqrt{4}\sqrt{10}}{\sqrt{49}} = \frac{2\sqrt{10}}{7}$$

   $$\tan\alpha = \frac{\sqrt{1-a^2}}{a} = \frac{\frac{2\sqrt{10}}{7}}{\frac{3}{7}} = \frac{2\sqrt{10}}{3}$$

   $$\cot\alpha = \frac{a}{\sqrt{1-a^2}} = \frac{3}{2\sqrt{10}}$$

3. $\sin\alpha = \sqrt{1-b^2}$, $\tan\alpha = \frac{\sqrt{1-b^2}}{b}$, $\cot\alpha = \frac{b}{\sqrt{1-b^2}}$

4. $\sin\alpha = \frac{d}{\sqrt{1+d^2}}$, $\cos\alpha = \frac{1}{\sqrt{1+d^2}}$, $\cot\alpha = \frac{1}{d}$

5.

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	a	$\sqrt{1-a^2}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{\sqrt{1-a^2}}{a}$
$\cos\alpha$	$\sqrt{1-a^2}$	a	$\frac{\sqrt{1-a^2}}{a}$	$\frac{a}{\sqrt{1-a^2}}$
$\tan\alpha$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	a	$\frac{1}{a}$
$\cot\alpha$	$\frac{1}{\sqrt{1+a^2}}$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{a}$	a

Page 45 (First)

1. Given in text

2. $\sin^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

3.

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	$\sin\alpha$	$\sqrt{1-\sin^2\alpha}$	$\frac{a}{\sqrt{1-\sin^2\alpha}}$	$\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$
$\cos\alpha$	$\sqrt{1-\cos^2\alpha}$	$\cos\alpha$	$\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$	$\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}$
$\tan\alpha$	$\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$	$\frac{1}{\sqrt{1+\tan^2\alpha}}$	$\tan\alpha$	$\frac{1}{\tan\alpha}$
$\cot\alpha$	$\frac{1}{\sqrt{1+\cot^2\alpha}}$	$\frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$	$\frac{1}{\cot\alpha}$	$\cot\alpha$

Page 45 (Second)

1. $\tan\alpha = \frac{a}{b} = \cot\beta$

2. $\cot\alpha = \frac{b}{a} = \tan\beta$

3. $\sec\alpha = \frac{c}{a} = \csc\beta$

4. $\csc\alpha = \frac{c}{b} = \sec\beta$

Page 47

1. (a) $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

   (b) $\sin^2 45° + \cos^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

   (c) $\sin^2 60° + \cos^2 60° = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

2. $\sin^2\alpha + \cos^2\alpha = 1$

$$\left(\frac{\sqrt{5}}{4}\right)^2 + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \left(\frac{\sqrt{5}}{4}\right)^2 = 1 - \frac{5}{16} = \frac{11}{16}$$

$$\cos\alpha = \sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4}$$

3. $\sin^2\alpha + \cos^2\alpha = 1$

$$\sin^2\alpha + \left(\frac{2}{3}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\sin\alpha = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

4. $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{1}{\sqrt{3}}$

$$\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{3}$$

$$\frac{\sin^2\alpha}{1-\sin^2\alpha} = \frac{1}{3}$$

$$3\sin^2\alpha = 1 - \sin^2\alpha$$

$$4\sin^2\alpha = 1$$

$$\sin^2\alpha = \frac{1}{4}$$

$$\sin\alpha = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\begin{aligned}
\frac{\sin\alpha}{\cos\alpha} &= \tan\alpha = \frac{1}{\sqrt{3}}\\
\frac{\sin^2\alpha}{\cos^2\alpha} &= \frac{1}{3}\\
\frac{1-\cos^2\alpha}{\cos^2\alpha} &= \frac{1}{3}\\
3 * \frac{1-\cos^2\alpha}{\cos^2\alpha} &= 3 * \frac{1}{3}\\
\frac{3*(1-\cos^2\alpha)}{\cos^2\alpha} &= 1\\
3*(1-\cos^2\alpha) &= \cos^2\alpha\\
3-3\cos^2\alpha &= \cos^2\alpha\\
3 &= 4\cos^2\alpha\\
\frac{3}{4} &= \cos^2\alpha\\
\frac{\sqrt{3}}{2} &= \cos\alpha
\end{aligned}$$

And then to check our solution we can calculate the fraction we are given $\frac{1}{\sqrt{3}}$ from our $\cos\alpha$ and $\sin\alpha$ fractions.

$$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$\frac{2}{2*\sqrt{3}}$$

$$\frac{1}{\sqrt{3}}$$

5. (a) $\cot x \sin x = \left(\frac{1}{\tan x}\right)\sin x = \frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}} = \frac{\sin x \cos x}{\sin x} = \cos x$

   (b) $\frac{\tan x}{\sin x} = \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{\frac{\sin x}{\cos x}\cdot\frac{1}{\sin x}}{\sin x \cdot \frac{1}{\sin x}} = \frac{\frac{\sin x}{\sin x \cos x}}{1} = \frac{\sin x}{\sin x \cos x} = \frac{1}{\cos x}$

   (c) $\cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha) = \cos^2\alpha - 1 + \cos^2\alpha = 2\cos^2\alpha - 1$

   (d) This one is tricky. You might need to try a few different approaches (squaring above and below, multiplying above and below by $\cos\alpha\sin\alpha$). Eventually it becomes clear that you need to multiply above and below by $(1-\cos\alpha)$ and find a way to cancel out the $\sin\alpha$ factor in the numerator:

$$\begin{aligned}
\frac{\sin\alpha}{1+\cos\alpha} &= \frac{\sin\alpha(1-\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)} = \frac{\sin\alpha(1-\cos\alpha)}{1-\cos\alpha+\cos\alpha-\cos^2\alpha}\\
&= \frac{\sin\alpha(1-\cos\alpha)}{1-\cos^2\alpha} = \frac{\sin\alpha(1-\cos\alpha)}{1-(1-\sin^2\alpha)} = \frac{\sin\alpha(1-\cos\alpha)}{1-1+\sin^2\alpha}\\
&= \frac{\sin\alpha(1-\cos\alpha)}{\sin^2\alpha} = \frac{1-\cos\alpha}{\sin\alpha}
\end{aligned}$$

(e) $$\frac{\sin^2\alpha+2\cos^2\alpha-1}{\cot^2\alpha}=\frac{1-\cos^2\alpha+2\cos^2\alpha-1}{\cot^2\alpha}=\frac{\cos^2\alpha}{\left(\frac{\cos\alpha}{\sin\alpha}\right)^2}$$
$$=\frac{\cos^2\alpha}{\frac{\cos^2\alpha}{\sin^2\alpha}}=\frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha}$$
$$=\sin^2\alpha$$

(f) $$\cos^2\alpha=\frac{\cos^2\alpha}{1}=\frac{\cos^2\alpha}{\cos^2\alpha+\sin^2\alpha}$$
$$=\frac{\frac{\cos^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha+\sin^\alpha}{\cos^2\alpha}}=\frac{1}{\frac{\cos^2\alpha}{\cos^2\alpha}+\frac{\sin^2\alpha}{\cos^2\alpha}}$$
$$=\frac{1}{1+\tan^2\alpha}$$

(g) $$\sin^2\alpha=\frac{\sin^2\alpha}{1}=\frac{\cos^2\alpha}{\cos^2\alpha+\sin^2\alpha}$$
$$=\frac{\frac{\sin^2\alpha}{\sin^2\alpha}}{\frac{\cos^2\alpha+\sin^\alpha}{\sin^2\alpha}}=\frac{1}{\frac{\cos^2\alpha}{\sin^2\alpha}+\frac{\sin^2\alpha}{\sin^2\alpha}}$$
$$=\frac{1}{\cot^2\alpha+1}$$

(h) $$\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{(1-\cos\alpha)(1+\cos\alpha)}{(1+\cos\alpha)(1+\cos\alpha)}=\frac{1+\cos\alpha-\cos\alpha-\cos^2\alpha}{(1+\cos\alpha)^2}$$
$$=\frac{1-\cos^2\alpha}{(1+\cos\alpha)^2}=\frac{\sin^2\alpha}{(1+\cos\alpha)^2}$$
$$=\left(\frac{\sin\alpha}{1+\cos\alpha}\right)^2$$

(i) The key to solving this one is the formula for factoring a difference of cubes: $a^3-b^3=(a-b)(a^2+ab+b^2)$.

$$\frac{\sin^3\alpha-\cos^3\alpha}{\sin\alpha-\cos\alpha}=\frac{(\sin\alpha-\cos\alpha)(\sin^2\alpha+\sin\alpha\cos\alpha+\cos^2\alpha)}{\sin\alpha-\cos\alpha}$$
$$=\sin^2\alpha+\sin\alpha\cos\alpha+\cos^2\alpha$$
$$=1+\sin\alpha\cos\alpha$$

6. (a) We can rewrite the LHS to show that $\sin^4\alpha-\cos^4\alpha=\cos^2\alpha-\sin^2\alpha$:

$$\sin^4\alpha-\cos^4\alpha=(\sin^2\alpha+\cos^2\alpha)(\sin^2\alpha-\cos^2\alpha)=1(\sin^2\alpha-\cos^2\alpha)$$
$$=\sin^2\alpha-\cos^2\alpha$$

Answer: There are no angles α for which $\sin^4\alpha-\cos^4\alpha>\cos^2\alpha-\sin^2\alpha$ because the expressions on either side of the inequality are equivalent.

(b) $\sin^4\alpha-\cos^4\alpha>=\cos^2\alpha-\sin^2\alpha$ for all angles α because the expressions on either side of the inequality are equivalent.

7. If we rewrite $2\sin\alpha\cos\alpha$ as a fraction, we can divide above and below by $\cos\alpha$ to convert the numerator and denominator into expressions in terms of $\tan\alpha$:

$$\begin{aligned}
2\sin\alpha\cos\alpha &= \frac{2\sin\alpha\cos\alpha}{1} = \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha+\cos^2\alpha} \\
&= \frac{\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha}}{\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{\frac{\sin^2\alpha}{\cos^2\alpha}+\frac{\cos^2\alpha}{\cos^2\alpha}} \\
&= \frac{2\tan\alpha}{\tan^2\alpha+1}
\end{aligned}$$

Now we can plug in the given value for $\tan\alpha$ to find the value of $2\sin\alpha\cos\alpha$ in this instance:

$$2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{\tan^2\alpha+1} = \frac{2(\frac{2}{5})}{(\frac{2}{5})^2+1} = \frac{\frac{4}{5}}{\frac{4}{25}+1} = \frac{\frac{4}{5}}{\frac{4}{25}+\frac{25}{25}} = \frac{\frac{20}{25}}{\frac{29}{25}} = \frac{20}{29}$$

8. First, we will rewrite the expresssion $\cos^2\alpha-\sin^2\alpha$ in terms of $\tan\alpha$:

$$\begin{aligned}
\cos^2\alpha-\sin^2\alpha &= \frac{\cos^2\alpha-\sin^2\alpha}{1} = \frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha+\sin^2\alpha} = \frac{\frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}} \\
&= \frac{1-\tan^2\alpha}{1+\tan^2\alpha}
\end{aligned}$$

   (a) To find the numerical value of $\cos^2\alpha-\sin^2\alpha$ when $\tan\alpha=\frac{2}{5}$ we can substitute $\frac{2}{5}$ for $\tan\alpha$ in the formula above:

$$\cos^2\alpha-\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-\left(\frac{2}{5}\right)^2}{1+\left(\frac{2}{5}\right)^2} = \frac{1-\frac{4}{25}}{1+\frac{4}{25}} = \frac{\frac{21}{25}}{\frac{29}{25}} = \frac{21}{29}$$

   (b) Substituting r for $\tan\alpha$ in the formula above:

$$\cos^2\alpha-\sin^2\alpha = \frac{1-r^2}{1+r^2}$$

9. First, we will rewrite the expresssion in terms of $\tan\alpha$:

$$\frac{\sin\alpha-2\cos\alpha}{\cos\alpha-3\sin\alpha} = \frac{\frac{\sin\alpha-2\cos\alpha}{\cos\alpha}}{\frac{\cos\alpha-3\sin\alpha}{\cos\alpha}} = \frac{\frac{\sin\alpha}{\cos\alpha}-\frac{2\cos\alpha}{\cos\alpha}}{\frac{\cos\alpha}{\cos\alpha}-\frac{3\sin\alpha}{\cos\alpha}} = \frac{\tan\alpha-2}{1-3\tan\alpha}$$

Next, we substitute $\frac{2}{5}$ for $\tan\alpha$:

$$\frac{\tan\alpha-2}{1-3\tan\alpha} = \frac{\frac{2}{5}-2}{1-3\left(\frac{2}{5}\right)} = \frac{\frac{2}{5}-\frac{10}{5}}{\frac{5}{5}-\frac{6}{5}} = \frac{-\frac{8}{5}}{-\frac{1}{5}} = 8$$

10. First, we will rewrite the expresssion in terms of $\tan\alpha$:

$$\frac{a\sin\alpha + b\cos\alpha}{c\cos\alpha + d\sin\alpha} = \frac{\frac{a\sin\alpha}{\cos\alpha} + \frac{b\cos\alpha}{\cos\alpha}}{\frac{c\cos\alpha}{\cos\alpha} + \frac{d\cos\alpha}{\cos\alpha}} = \frac{a\tan\alpha + b}{c + d\tan\alpha}$$

Next, we substitute $\frac{2}{5}$ for $\tan\alpha$ and simplify:

$$\frac{a\tan\alpha + b}{c + d\tan\alpha} = \frac{a\left(\frac{2}{5}\right) + b\left(\frac{5}{5}\right)}{c\left(\frac{5}{5}\right) + d\left(\frac{2}{5}\right)} = \frac{\frac{2a+5b}{5}}{\frac{5c+2d}{5}} = \frac{2a + 5b}{5c + 2d}$$

Now we can see why the problem included the restriction that $5c + 2d \neq 0$; the value of the expression is undefined if the denominator is zero. The sum of two rational numbers is a rational number. Therefore the numerator and denominator in the expression are both rational numbers. The quotient of two rational numbers is a rational number. Therefore, the entire expression evaluates to a rational number for arbitratrary rational values of a, b, c and d.

11. We can expand and simplify the expression:

$$\begin{aligned}
&(\sin\alpha + \cos\alpha)^2 + (\sin\alpha - \cos\alpha)^2 \\
&= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha \\
&= 2\sin^2\alpha + 2\cos^2\alpha \\
&= 2(\sin^2\alpha + \cos^2\alpha) \\
&= 2(1) \\
&= 2
\end{aligned}$$

As the expression evaluates to a constant, it is as large as possible for all values of α.

Page 49

1. Rewriting any instances of $\sec\alpha$ or $\csc\alpha$ on either side of the identities:

(a) $\tan\alpha\csc\alpha = \sec\alpha$

$$\begin{aligned}
\tan\alpha\frac{1}{\sin\alpha} &= \frac{1}{\cos\alpha} \\
\frac{\tan\alpha}{\sin\alpha} &= \frac{1}{\cos\alpha}
\end{aligned}$$

(b) $\cot\alpha\csc\alpha = \sec\alpha$

$$\begin{aligned}
\cot\alpha\frac{1}{\cos\alpha} &= \frac{1}{\sin\alpha} \\
\frac{\cot\alpha}{\cos\alpha} &= \frac{1}{\sin\alpha}
\end{aligned}$$

(c) $$\frac{1}{\sec\alpha}\csc\alpha = \cot\alpha$$
$$\frac{1}{\frac{1}{\cos\alpha}} \cdot \frac{1}{\sin\alpha} = \cot\alpha$$
$$\cos\alpha\frac{1}{\sin\alpha} = \cot\alpha$$
$$\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$$

(d) $$\tan^2\alpha = (\sec\alpha + 1)(\sec\alpha - 1)$$
$$\tan^2\alpha = \sec^2\alpha - 1$$
$$\tan^2\alpha = \frac{1}{\cos^2\alpha} - 1$$

(e) $$\csc^2\alpha = 1 + \cot^2\alpha$$
$$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$$

2. Rewriting any instances of $\sin\alpha$ or $\cos\alpha$ on either side of the identities, and eliminating fractions:

(a) $$\frac{\tan\alpha}{\sin\alpha} = \frac{1}{\cos\alpha}$$
$$\tan\alpha\frac{1}{\sin\alpha} = \sec\alpha$$
$$\tan\alpha\csc\alpha = \sec\alpha$$

(b) $$\frac{1}{\sin\alpha}\cos\alpha = \cot\alpha$$
$$\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$$
$$\cot\alpha = \cot\alpha$$

(c) $$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}$$
$$\tan^2\alpha + 1 = \sec^2\alpha$$

(d) $$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$$
$$\csc^2\alpha = 1 + \cot^2\alpha$$

Page 50

1. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= (\cos^2\alpha - \sin^2\alpha)^2 + (2\sin\alpha\cos\alpha)^2 \\ &= \cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha + 4\sin^2\alpha\cos^2\alpha \\ &= \cos^4\alpha + 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha \\ &= (\cos^2\alpha + \sin^2\alpha)^2 \\ &= (1)^2 \\ &= 1 \end{aligned}$$

   According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

2. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= \left(\sqrt{\frac{1+\cos\alpha}{2}}\right)^2 + \left(\sqrt{\frac{1-\cos\alpha}{2}}\right)^2 \\ &= \frac{1+\cos\alpha}{2} + \frac{1-\cos\alpha}{2} \\ &= \frac{1+\cos\alpha+1-\cos\alpha}{2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

3. First, we will rewrite a and b to eliminate the cube exponents:

$$\begin{aligned} a &= 4\cos^3\alpha - 3\cos\alpha \\ &= 4\cos\alpha\cos^2\alpha - 3\cos\alpha \\ &= 4\cos\alpha(1-\sin^2\alpha) - 3\cos\alpha \\ &= 4\cos\alpha - 4\sin^2\alpha\cos\alpha - 3\cos\alpha \\ &= \cos\alpha - 4\sin^2\alpha\cos\alpha \end{aligned}$$

$$\begin{aligned} b &= 3\sin\alpha - 4\sin^3\alpha \\ &= 3\sin\alpha - 4\sin\alpha\sin^2\alpha \\ &= 3\sin\alpha - 4\sin\alpha(1-\cos^2\alpha) \\ &= -\sin\alpha + 4\sin\alpha\cos^2\alpha \end{aligned}$$

   Next, we will expand a^2 and b^2:

$$\begin{aligned} a^2 &= (\cos\alpha - 4\sin^2\alpha\cos\alpha)^2 \\ &= \cos^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha \end{aligned}$$

$$\begin{aligned}b^2 &= (-\sin\alpha + 4\sin\alpha\cos^2\alpha)^2\\ &= \sin^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^2\alpha\cos^4\alpha\end{aligned}$$

Next, we add the expressions for a^2 and b^2 and simplify to 1:

$$\begin{aligned}a^2 + b^2 &= \cos^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha + \sin^2\alpha - 8\sin^2\alpha\cos^2\alpha +\\ &\qquad 16\sin^2\alpha\cos^4\alpha\\ &= \cos^2\alpha + \sin^2\alpha - 16\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha + 16\sin^2\alpha\cos^4\alpha\\ &= \cos^2\alpha + \sin^2\alpha + 16\sin^2\alpha\cos^2\alpha(-1 + \sin^2\alpha + \cos^2\alpha)\\ &= 1 + 16\sin^2\alpha\cos^2\alpha(0)\\ &= 1\end{aligned}$$

According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

4. First, we find the value of $a^2 + b^2$:

$$\begin{aligned}a^2 + b^2 &= \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2\\ &= \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2}\\ &= \frac{(1-t^2)^2 + (2t)^2}{(1+t^2)^2}\\ &= \frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)(1+t^2)}\\ &= \frac{(1+t^2)(1+t^2)}{(1+t^2)(1+t^2)}\\ &= 1\end{aligned}$$

According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

5. We expand $(p^2 - q^2)^2 + (2pq)^2$ and use the fact that $p^2 + q^2 = 1$ to simplify to 1:

$$\begin{aligned}(p^2 - q^2)^2 + (2pq)^2 &= p^4 - 2p^2q^2 + q^4 + 4p^2q^2\\ &= p^4 + 2p^2q^2 + q^4\\ &= (p^2 + q^2)^2\\ &= (1)^2\\ &= 1\end{aligned}$$

This is similar to Exercise 1 above.

Page 51

1. $\sin\alpha < 1$ when α is acute, therefore $1-\sin\alpha > 0$ when α is acute. $1-\sin\alpha = 0$ when $\sin\alpha = 1$, i.e., $\alpha = 90°$.

2. $\cos\alpha < 1$ when α is acute, therefore $1-\cos\alpha > 0$ when α is acute. $1-\cos\alpha = 0$ when $\cos\alpha = 1$, i.e., $\alpha = 0°$.

3. Statement a) is always true. Statements b) and c) both include the case that $\sin^2\alpha + \cos^2\alpha = 1$, which is always true.

4. Let x be the maximum cost of the items in a supermarket. In Supermarket A, $x \leq \$1$. In Supermarket B, $x < \$1$. In Supermarket C, $x \leq \$1$. In Supermarket D, $x > \$1$. We can see that Supermarkets A and C are offering the same terms.

5. Inequality a) is correct. For b) to be correct, an angle α would have to exist such that $\sin\alpha + \cos\alpha = 2$. We know that this is not the case. When $\alpha = 90°$, $\sin\alpha = 1$ and $\cos\alpha = 0$. When $\alpha = 0°$, $\sin\alpha = 0$ and $\cos\alpha = 1$. When $0° < \alpha < 90°$, $\sin\alpha < 1$ and $\cos\alpha < 1$. In all cases, $\sin\alpha + \cos\alpha < 2$.

6. The largest possible value of $\sin\alpha$ is 1, and occurs when $\alpha = 90°$. The largest possible value of $\cos\alpha$ is 1, and occurs when $\alpha = 0°$. See Page 32.

Page 52

1. $\sin 30° = 0.5$, $\sin 45° = 0.707$, $\sin 60° = 0.866$.

2. By using the `tan` button to calculate $\tan 60°$, and the `sqrt` button to calculate $\sqrt{3}$, Betty can compare the results: both are 1.732.

3. Press `tan`, then enter the angle degree measure, then press `1/`x

4.

in radical or rational form				
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$

in decimal form, from calculator				
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$30°$	0.5	0.866	0.577	1.732
$45°$	0.707	0.707	1	1
$60°$	0.866	0.5	1.732	0.577

Page 53

1. The sine of the larger angle is $4/5 = .8$. We can use the inverse sine function to find the angle: $\arcsin .8 = 53.1301°$. The sum of the three angles in the triangles is: $\arcsin .6 + \arcsin .8 + 90° = 36.8699° + 53.1301° + 90° = 180°$.
2. (a) $\arcsin 1 = 90°$

   (b) $\arccos 0.7071067811865 = 45°$
3. $\arccos 0.8 = 36.8699°$
4. $\arcsin 0.6 = 36.8699°$
5. Half of $\sin 30°$ (0.25) seems like a reasonable estimate. The actual value is 0.2588.
6. Entering $\arcsin 0.3$ in calculator, we get $17.458°$

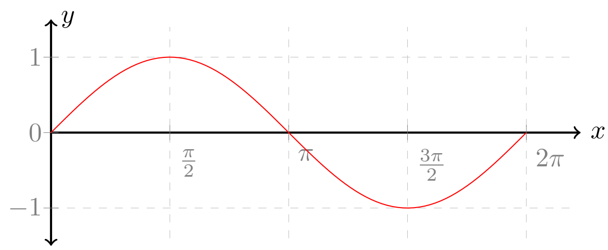
   If we enter $\sin 17.458°$
7. With $\arcsin x = 53°$, to find value of x, we do $x = \sin 53°$, which gives us $x = 0.7986$

   If we take $\arcsin 0.7986$, we will get 52.9966
8. $x = \sqrt{3}/2$
9. $\arcsin(\sin 17\circ) = 17\circ$
10. $\sin(\arcsin 0.4) = 0.4$
11. $\arcsin(\sin 30°)) = 30°$

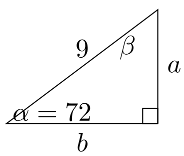
    $\arcsin 1/2 = 30^{\{}circ\}$. Since $\sin 30° = 0.5$ as in question 5
12. $\cos^2 20° + \sin^2 20° = 1$

    $\cos^2 80° + \sin^2 80° = 1$
13. $\tan 20° = 0.36397$ and $\frac{\sin 20°}{\cos 20°} = \frac{0.34202}{0.93969} \approx 0.36397$

    Note $\tan 80° = 5.67128$ and $\frac{\sin 80°}{\cos 80°} = \frac{0.98481}{0.17365} \approx 5.67128$

14. 

Page 55

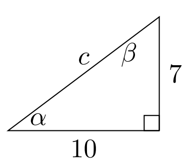
1. 

   To find a

$$
\begin{aligned}
\sin\alpha &= \frac{a}{c} \\
\sin 72^\circ &= \frac{a}{9} \\
a &= 9 \times \sin 72^\circ \\
a &\approx 8.5595
\end{aligned}
$$

   To find b

$$
\begin{aligned}
\cos\alpha &= \frac{b}{c} \\
\cos 72^\circ &= \frac{b}{9} \\
b &= 9 \times \cos 72^\circ \\
b &\approx 2.7812
\end{aligned}
$$

2. 

   To find the hypotenuse c, use pythagoras theorem

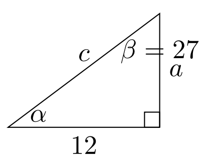
$$
\begin{aligned}
c &= \sqrt{a^2 + b^2} \\
c &= \sqrt{149}
\end{aligned}
$$

To find α

$$\begin{aligned}
\tan\alpha &= \frac{b}{a} \\
\tan\alpha &= \frac{7}{10} \\
\alpha &= \arctan\frac{7}{10} \\
\alpha &\approx 34.992^\circ
\end{aligned}$$

To find β

$$\begin{aligned}
\tan\beta &= \frac{a}{b} \\
\tan\beta &= \frac{10}{7} \\
\beta &= \arctan\frac{10}{7} \\
\beta &\approx 55.008^\circ
\end{aligned}$$

3. 

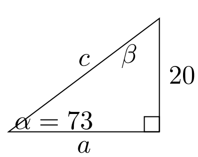
To other acute angle will be $90 - 27 = 63$ degrees

To find length of hypotenuse c

$$\begin{aligned}
\sin\beta &= \frac{12}{c} \\
c &= \frac{12}{\sin 27} \\
&= 26.432
\end{aligned}$$

To find length of the other leg a

$$\begin{aligned}
\tan\beta &= \frac{12}{a} \\
a &= \frac{12}{\tan 27} \\
&= 23.551
\end{aligned}$$

4. 

To other acute angle will be $90 - 73 = 17$ degrees

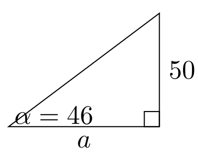
To find length of hypotenuse c

$$\begin{aligned} \sin\beta &= \frac{20}{c} \\ c &= \frac{20}{\sin 73} \\ &= 20.914 \end{aligned}$$

To find length of the other leg a

$$\begin{aligned} \tan\beta &= \frac{20}{a} \\ a &= \frac{20}{\tan 73} \\ &= 6.115 \end{aligned}$$

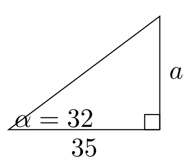
Page 56

1. 

To find length of shadow a

$$\begin{aligned} \tan\alpha &= \frac{50}{a} \\ a &= \frac{50}{\tan 46} \\ a &= 48.284 \end{aligned}$$

Therefore the length of shadow is 48.284 feet

2. 

To find length of shadow a

$$\begin{aligned} \tan\alpha &= \frac{a}{35} \\ a &= 35\tan 32 \\ a &= 21.87 \end{aligned}$$

Therefore the height of flagpole is 21.87 feet

3. Shadow is the longest in the morning as the sun is at lower altitude

   Shadow is shortest during mid day as the sun is exactly above the pole

4. There will be no shadow when the sun is exactly above the flag pole, which happens at a specific time during 'zero shadow day' (search online for exact time and day for your location)

Page 59

1. Draw point C on the left side of circle, by circle geometry, the angle in that triangle vertex will be $\theta/2$. Then apply the relation $\sin\alpha = \frac{AB}{2r}$, where AB is the opposite side of angle α. Therefore $\sin\theta/2 = AB$ since radius is 1/2

2. Draw point C on the left side of circle, by circle geometry, the angle in that triangle vertex will be $\theta/2$. Then apply the relation $\sin\alpha = \frac{AB}{2r}$, where AB is the opposite side of angle α. Therefore $\sin\theta/2 = \frac{AB}{2}$ since radius is 1, rearranging the equation gives $2\sin\theta/2 = AB$

3. From question 1, for circle of diameter 1, we have the relation $\sin\theta/2 = AB$. Therefore if angle $\phi > \theta$ then the opposite side of angle ϕ will be longer too, therefore $\sin\phi > \sin\theta$

4. Assuming the three points are on a circle, and realising that the sides 6, 8 and 10 form a right angle triangle, therefore the side of length 10 would have been the diameter of the circle. Therefore the radius for the circle is 5

5. Since $\sin\alpha = \frac{PB}{2r}$, re-arranging the equation gives $\frac{PB}{\sin\alpha} = 2r$. That is, the ratio of $a : \sin\alpha$ is equal to the diameter of circle (double of radius)

6. let a be opposite side of angle α let b be opposite side of angle β let c be opposite side of angle γ

   Exercise shows that $\frac{a}{\sin\alpha} = 2r$, $\frac{b}{\sin\beta} = 2r$, $\frac{c}{\sin\gamma} = 2r$. Since all are equal to $2r$ then they are ratios with same value, that is

$$\frac{a}{\sin\alpha} = \frac{b}{\beta} = \frac{c}{\gamma}$$

7. Since $\sin\alpha = \frac{AB}{2\times 1/2}$, then chord $AB = \sin\alpha$. Since arc AC is double of arc AB, then chord $AC = 2\sin\alpha$

   But the diagram shows that in triangle ABC and using triangle inequality, $AB + BC \geq AC$. Therefore $\sin 2\alpha < 2\sin\alpha$

8. Using $\sin\alpha = \frac{AB}{2r}$, if $\alpha = 60°$ and radius is 10/2=5

   The chord AB would be $2 \times 5 \times \sin 60° = 5\sqrt{3}$ units long

9. Using $\sin\alpha = \frac{AB}{2r}$, if $\alpha = 30°$ (as central angle is double of angle on circle) and radius is 10/2=5

   The chord AB would be $2 \times 5 \times \sin 30° = 5$ units long

10. The diagonal of a square inside a circle will be the diameter of circle and the angle of each triangle corner is 45°

    Using $\sin\alpha = \frac{AB}{2r}$, where 2r is 10 (diameter) and angle $\alpha = 45°$, we get $AB = 10\sin 45° = 5\sqrt{2}$ units long for each side of square inside a circle

11. Using $\sin\alpha = \frac{AB}{2r}$, then letting one side of pentagon $AB = 2r \times \sin\alpha$. Nothing that a pentagon can be cut into three distinct triangles, therefore the internal angles add up to $180 \times 3 = 540$ making each internal angle $540 \div 5 = 108$ degrees large

    As each vertex is divided into three equal angles, it will give $\alpha = 108 \div 3 = 36$ degrees. As diameter is 10, each side of pentagon will be $10 \times \sin 36$

Page 62

1. The degree measure of a semicircle is 180°. The degree measure of a quarter circle is 90°.

2. The measure of arc cut off by one side of regular pentagon inscribed in a circle is $360°/5 = 72°$. For a regular hexagon: $360°/6 = 60°$. For a regular octagon: $360°/8 = 45°$.

Page 64

1. Let AB be an arc on the circle and points Q and R are two other points on the circle

   The angle in the centre will be double of angle AQB, which will be double of angle ARB as well, therefore the both aangles AQB and ARB must be equal

2. A regular pentagon can be divided into three equal circles, therefore a pentagon's total internal angle is $180 \times 3 = 540$ degrees.

   Since the intenal angles are equal size for a regular pentagon, then each angle is $540 \div 5 = 108$ degrees large

3. Let points ABCD be points on a circle, then draw line AC. Note that the angle AOC will be double of ABC

   Meawhile on the other side of AOC will be double of ADC, these two sides makes a full revolution (360 degrees). Therefore we have $2\angle ABC + 2\angle ADC = 360$, that is $\angle ABC + \angle ADC = 180$ degrees

Page 65

1. As it is larger than 60 degrees, then the points will be inside the circle

2. As it is 90 degrees, then the points will be on the circle with AB being the diameter of the circle

3. Triangle AQB would have a larger inscribed circle of radius R, therefore the relation $\sin\alpha = \frac{AB}{2R}$ that is $2R = \frac{AB}{\sin\alpha}$

   Triangle APB is inside a smaller circle of radius r, therefore the relation $\sin Q = \frac{AB}{2r}$. That is $2r = \frac{AB}{\sin Q}$

   The ratio r:R will be $\frac{AB}{\sin Q} \div \frac{AB}{\sin\alpha} = \frac{\sin\alpha}{\sin Q}$. That is $\frac{r}{R} = \frac{\sin\alpha}{\sin Q}$

   if $r < R$ then $\sin\alpha < \sin Q$. That is $\alpha < Q$ for the left diagram if $r > R$ then $\sin\alpha > \sin Q$. That is $\alpha > Q$ for the right diagram

4. Following above proof, let angle Q be 0.5α. Since $\sin\alpha > \sin 0.5\alpha$, then $r > R$, so it will be inscribed in the circle of original arc

5. On the same side, the angle subtended will be α, where angle subtended in the centre would be 2α. The other side of the centre angle would be $360-2\alpha$, which gives $180-\alpha$ as size of subtended angle on the other arc side

   Note, this is proof for opposite angles of a cyclic quadrilateral are supplementary

Chapter 3: Relationships in a Triangle

Page 68

1.

	Data	Determine a triangle?	Restrictions?
1	ABa	Yes	$0° < m\angle A, m\angle B, m\angle A + m\angle B < 180°$
2	ABb	Yes (Duplicate of 1)	$0° < m\angle A, m\angle B, m\angle A + m\angle B < 180°$
3	ABc	Yes	$0° < m\angle A, m\angle B, m\angle A + m\angle B < 180°$
4	AbC	Yes (Duplicate of 3)	$0° < m\angle A, m\angle B, m\angle A + m\angle B < 180°$
5	ABC	No	One side length must also be given.
6	Abc	Yes	$0° < m\angle A < 180°$
7	Bbc	No	$b \geq c\sin B$. Also, $m\angle A$ must be given in some cases.
8	Cbc	No (Duplicate of 7)	$c \geq b\sin C$. Also, $m\angle A$ must be given in some cases.

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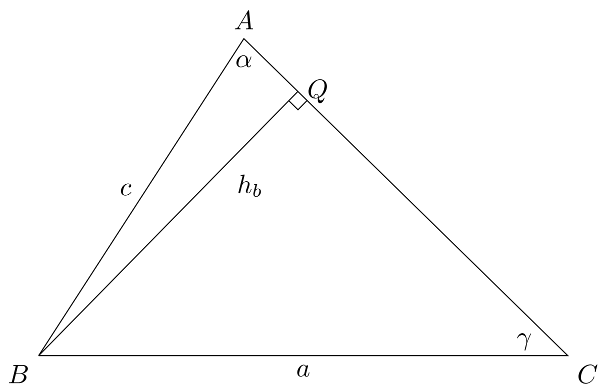
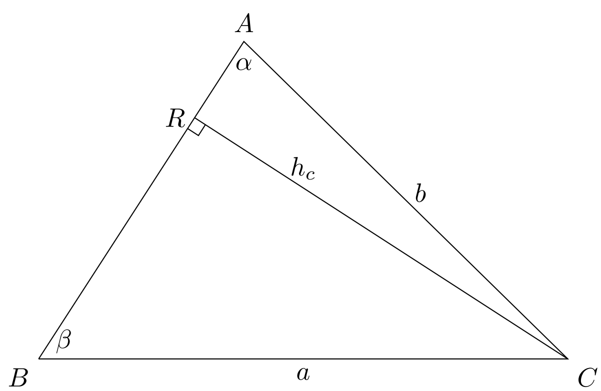
1.

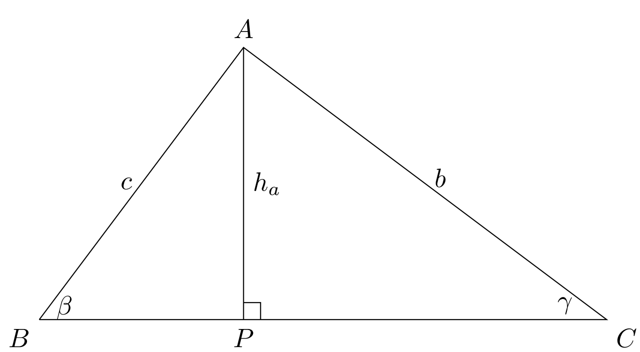




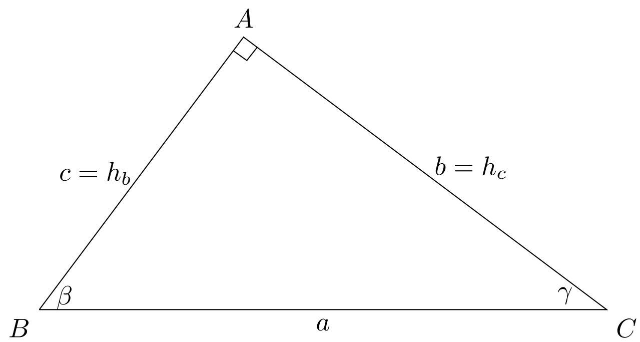
2.

$$\begin{aligned} h_c &= b \sin \alpha \\ &= 12 \sin 70^\circ \\ &\approx 11.28 \end{aligned}$$

3. Let us consider a right triangle ABC with a right angle at A.

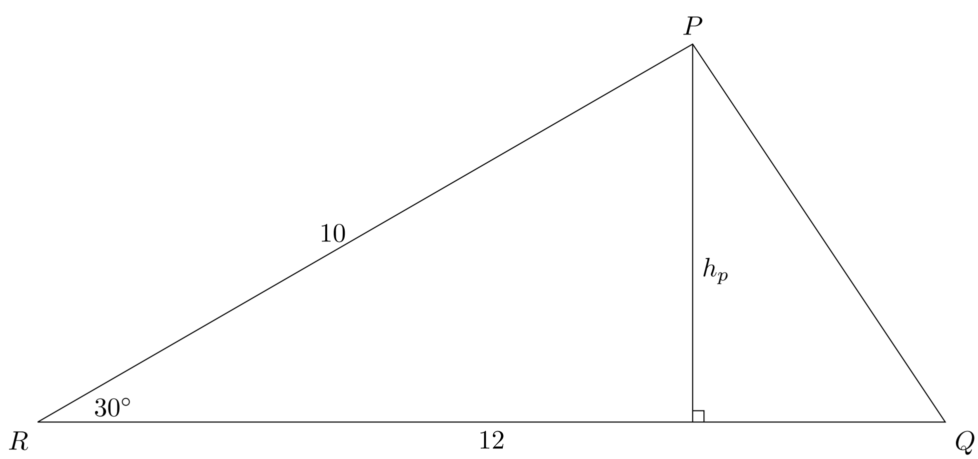


Analyzing the right triangle APB in the above diagram gives us that $h_a = c \sin \beta$. Similarly, analyzing the right triangle APC in the above diagram gives us that $h_a = b \sin \gamma$. This is consistent with our formulas for h_a.



Since A is a right angle, b and c are themselves altitudes, as shown by the diagram above. Since the sine of a right angle is equal to 1, this is consistent with the formulas $h_b = c \sin \alpha$ and $h_c = b \sin \alpha$. Additionally, by analyzing the sines at angles B and C, we find that $h_b = a \sin \gamma$ and $h_c = a \sin \beta$, which again agrees with the formulas for h_b and h_c.

4.



$$
\begin{aligned}
A &= \frac{1}{2}ph_p \\
&= \frac{1}{2}pq\sin R \\
&= \frac{1}{2}(12)(10)\sin 30^\circ \\
&= \frac{1}{2}(12)(10)\left(\frac{1}{2}\right) \\
&= 30
\end{aligned}
$$

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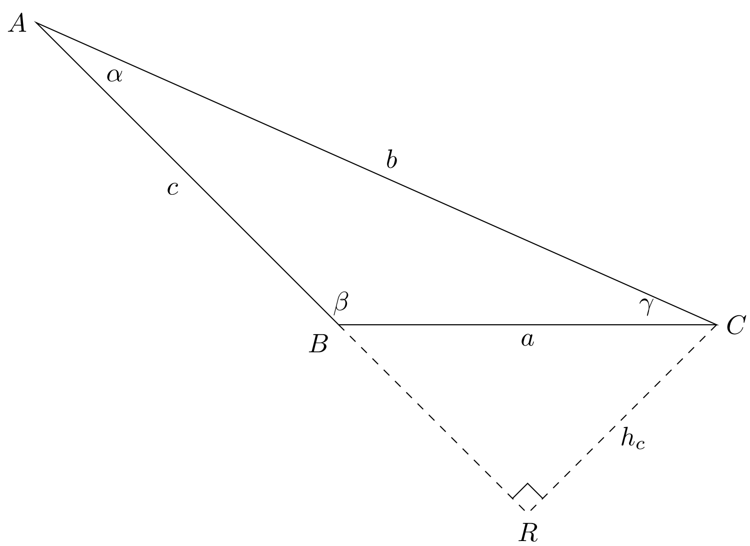
1.



For the h_b formulas, there is nothing new to check as α and γ are both acute angles.

From the above diagram, since $\triangle ARC$ is a right triangle, we can see that $h_c = b\sin\alpha$. Also, since $\triangle BRC$ is a right triangle, we have that $h_c = a\sin(180^\circ - \beta)$. Since we have asserted that the sine of an angle should be equal to the sine of its supplement, the previous formula is equivalent to $h_c = a\sin\beta$.

Page 73

1. We first substitute a with b, b with c, c with a, α with β, β with γ, γ with α.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \to \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

We now substitute a with c, c with b, b with a, α with γ, γ with β, β with α.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \to \frac{c}{\sin\gamma} = \frac{a}{\sin\alpha}$$

Combining these equalities together gives the Law of Sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

2. Let $A = 30°$, $B = 60°$, $C = 90°$. We know that $b = a\sqrt{3}$ and $c = 2a$ by the properties of 30-60-90 triangles.

$$\frac{a}{\sin\alpha} = \frac{a}{1/2} = 2a$$

$$\frac{b}{\sin\beta} = \frac{a\sqrt{3}}{\sqrt{3}/2} = 2a$$

$$\frac{c}{\sin\gamma} = \frac{2a}{1} = 2a$$

Since all of the above ratios are equal to $2a$, we have shown that the Law of Sines holds for 30-60-90 triangles.

3. In the triangle on the left, the angle opposite the side of length 10 measures $60°$. If x is the side length opposite the $50°$ angle and y is the side length opposite the $70°$ angle, then

$$\frac{x}{\sin 50°} = \frac{10}{\sin 60°} \implies x = \frac{10\sin 50°}{\sin 60°} \approx 8.85$$

$$\frac{y}{\sin 70°} = \frac{10}{\sin 60°} \implies y = \frac{10\sin 70°}{\sin 60°} \approx 10.85$$

In the triangle on the right, the missing angle measures $60°$. If x is the side length opposite the $60°$ angle and y is the side length opposite the $65°$ angle, then

$$\frac{x}{\sin 60°} = \frac{12}{\sin 55°} \implies x = \frac{12\sin 60°}{\sin 55°} \approx 12.69$$

$$\frac{y}{\sin 65°} = \frac{12}{\sin 55°} \implies y = \frac{12\sin 65°}{\sin 55°} \approx 13.28$$

4. As discussed in Section 4 of this chapter, the formulas for the altitudes using the sine function remain unchanged for obtuse triangles if we assert that the sine of an angle is equal to the sine of its supplement. Therefore, the proof of the Law of Sines for obtuse triangles is unchanged from the proof presented in the beginning of this section.

5. In the triangle on the left, the angle opposite the side of length 6 measures $45°$. If x is the side length opposite the $15°$ angle and y is the side length opposite the $120°$ angle, then

$$\frac{x}{\sin 15°} = \frac{6}{\sin 45°} \implies x = \frac{6 \sin 15°}{\sin 45°} \approx 2.20$$
$$\frac{y}{\sin 120°} = \frac{6}{\sin 45°} \implies y = \frac{6 \sin 120°}{\sin 45°} \approx 7.35$$

In the triangle on the right, the missing angle measures $115°$. If x is the side length opposite the $40°$ angle and y is the side length opposite the $115°$ angle, then

$$\frac{x}{\sin 40°} = \frac{14}{\sin 25°} \implies x = \frac{14 \sin 40°}{\sin 25°} \approx 21.29$$
$$\frac{y}{\sin 115°} = \frac{14}{\sin 25°} \implies y = \frac{14 \sin 115°}{\sin 25°} \approx 30.02$$

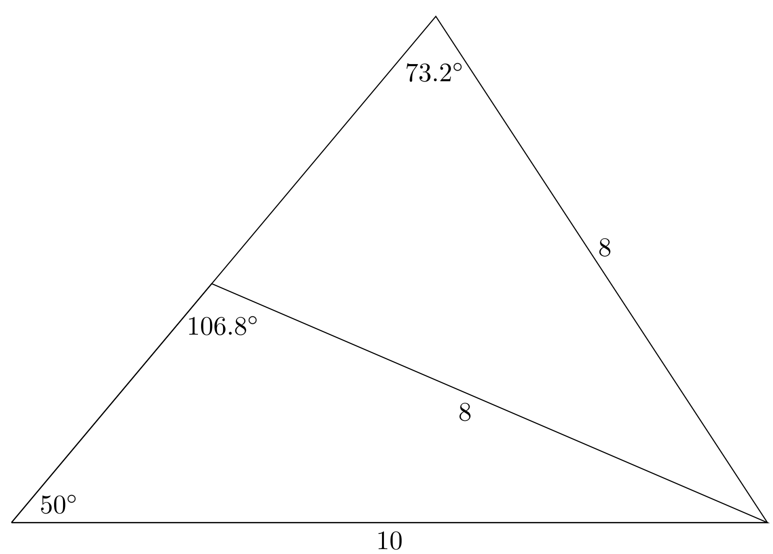
6. We first solve for the angle opposite the side of length 10.

$$\frac{8}{\sin 50°} = \frac{10}{\sin \alpha} \implies \sin \alpha = \frac{10}{8} \sin 50° = 0.9576 \implies \alpha = 73.2°$$

We now solve for the third angle by subtracting the other two angles from $180°$.

$$\beta = 180° - 50° - 73.2° = 56.8°$$

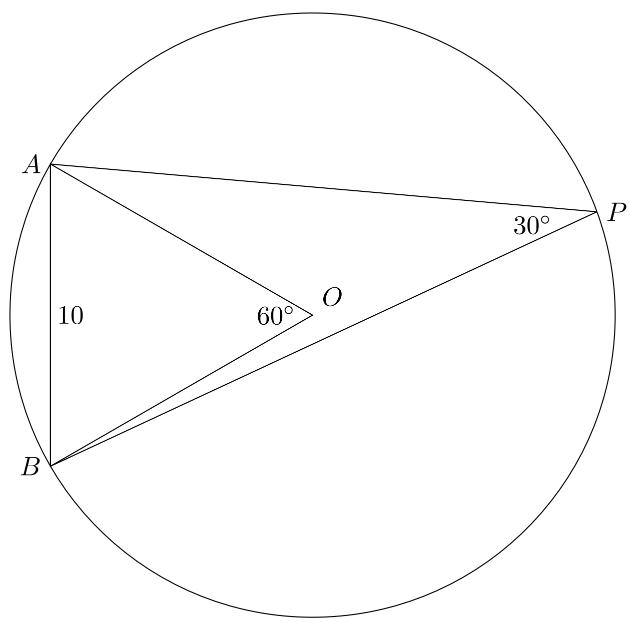
7. Since the sine of an angle is equal to the sine of its supplement, based on our calculations in the previous exercise, α could equal $73.2°$ or $106.8°$. This implies that β can equal $56.8°$ or $23.2°$. The below diagram shows the two triangles that satisfy the given constraints overlaid on top of each other.



8. This result follows immediately from the theorem proven in Section 12 of Chapter 2.

Page 75 (First)

1.
$$2R = \frac{10}{\sin 30^\circ} \implies R = \frac{10}{2\sin 30^\circ} = \frac{10}{2(1/2)} = 10$$



Alternatively, we can find the radius from first principles. Let APB be a triangle inscribed in a circle O with $AB = 10$ and $m\angle APB = 30^\circ$. Since $\angle APB$ is an inscribed angle, the corresponding central angle $\angle AOB$ has a measure of 60°. We also know that segments OA and OB are congruent because they are both radii of the circle O. Therefore, $\triangle AOB$ is isosceles. This combined with the fact that the measure of $\angle AOB$ is 60° implies that $\triangle AOB$ is equilateral. Because AB is equal to 10, we know that the the radius of the circle is also 10 because the other two sides of the equilateral triangle are radii.

2.
$$2R = \frac{8}{\sin 90^\circ} \implies R = \frac{8}{2\sin 90^\circ} = \frac{8}{2(1)} = 4$$

Alternatively, we can apply Thales's theorem (see the appendix of Chapter 2) to show that the hypotenuse of an inscribed triangle is a diameter of the circumscribing circle. If the diameter of the circle is 8, then the radius would be 4.

Page 75 (Second)

1. $$A = \frac{1}{2}(8)(11)\sin 40^\circ = 44\sin 40^\circ \approx 28.3$$

2. (a) $$A = \frac{1}{2}(10)(9)\sin 23^\circ = 45\sin 23^\circ \approx 17.6$$

   (b) $$A = \frac{1}{2}(3)(7)\sin 130^\circ = 10.5\sin 130^\circ \approx 8.0$$

   (c) $$A = \frac{1}{2}(3)(7)\sin 90^\circ = 10.5\sin 90^\circ = 10.5$$

   Since this is a right triangle, the sides adjacent to the 90° angle are altitudes, so we can use the more elementary formula for the area of a triangle: $A = \frac{1}{2}bh$.

3. Let b be the length of AC.
$$40 = \frac{1}{2}b(6)\sin 40^\circ \implies b = \frac{40}{3\sin 40^\circ} \approx 20.7$$

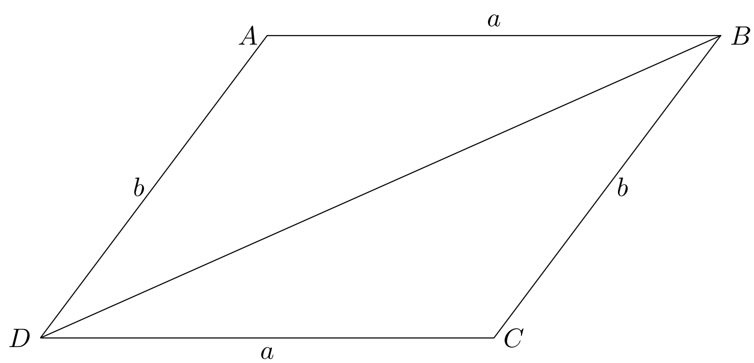
4. Let α be the measure of angle P.
$$9 = \frac{1}{2}(5)(6)\sin\alpha \implies \sin\alpha = \frac{9}{15} \implies \alpha \approx 36.9^\circ \text{ or } \alpha \approx 143.1^\circ$$

5. The area of this triangle is $\frac{1}{2}ab\sin\gamma$, where γ is the angle included between the sides of length a and b. Because $\sin\gamma$ has a maximum value of 1 when γ is 90°, the area of the triangle is maximized when γ is 90°. This gives an area of $\frac{1}{2}ab$. Since $\gamma = 90^\circ$, the triangle is a right triangle.

   In this triangle, a and b are both legs of the right triangle. We can let one of a or b be the hypotenuse of a right triangle to get another non-congruent right triangle with side lengths of a and b. Without loss of generality, we can choose b to be the hypotenuse. By the Pythagorean theorem, the other leg of the triangle has length $\sqrt{b^2 - a^2}$. Therefore, the area of this triangle is
$$\frac{1}{2}a\sqrt{b^2 - a^2}.$$

6. As in the previous exercise, for two given side lengths of a triangle, the area of the triangle is maximized when the angle included between these two sides is equal to 90°. Thus, the maximum area of an isosceles triangle with a leg length of x is
$$\frac{1}{2}(x)(x)\sin 90^\circ = \frac{1}{2}x^2.$$

7.

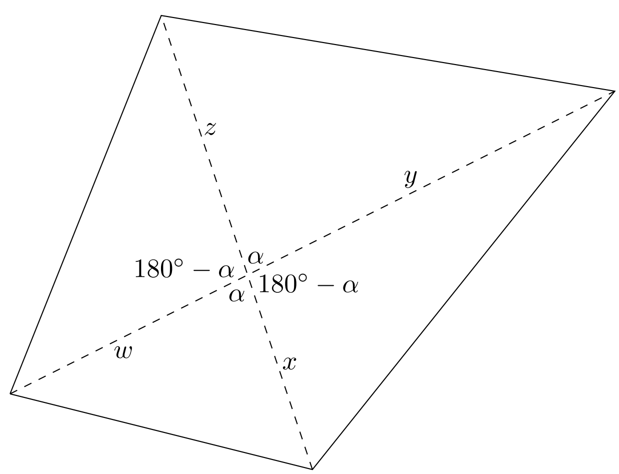


Because opposite angles of a parallelogram are congruent, the area of the above parallelogram is given by

$$\frac{1}{2}ab\sin A + \frac{1}{2}ab\sin C = \frac{1}{2}ab\sin C + \frac{1}{2}ab\sin C = ab\sin C.$$

Notice that A and C are congruent, B and C are supplements, and C and D are supplements by the properties of parallelograms. This means that the above area formula can be used with any angle of the parallelogram since $\sin A = \sin B = \sin C = \sin D$.

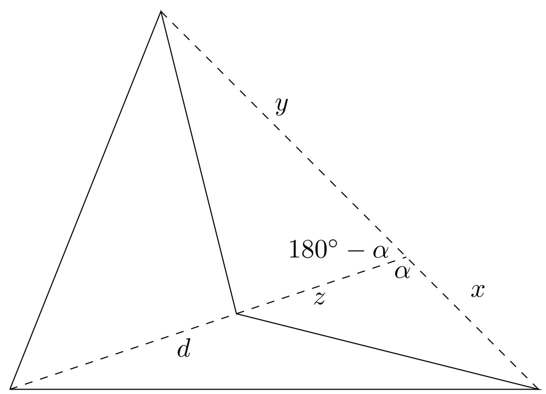
8.



$$
\begin{aligned}
A &= \frac{1}{2}wx\sin\alpha + \frac{1}{2}xy\sin\left(180^\circ - \alpha\right) + \frac{1}{2}yz\sin\alpha + \frac{1}{2}wz\sin\left(180^\circ - \alpha\right) \\
&= \frac{1}{2}wx\sin\alpha + \frac{1}{2}xy\sin\alpha + \frac{1}{2}yz\sin\alpha + \frac{1}{2}wz\sin\alpha \\
&= \frac{1}{2}\left(wx + xy + yz + wz\right)\sin\alpha \\
&= \frac{1}{2}\left(w\left(x+z\right) + y\left(x+z\right)\right)\sin\alpha \\
&= \frac{1}{2}\left(w+y\right)\left(x+z\right)\sin\alpha
\end{aligned}
$$

$w+y$ and $x+z$ are the lengths of the diagonals in the quadrilateral in the diagram above. The above derivation shows that the choice of α does not matter since $\sin\alpha = \sin\left(180^\circ - \alpha\right)$.

9.



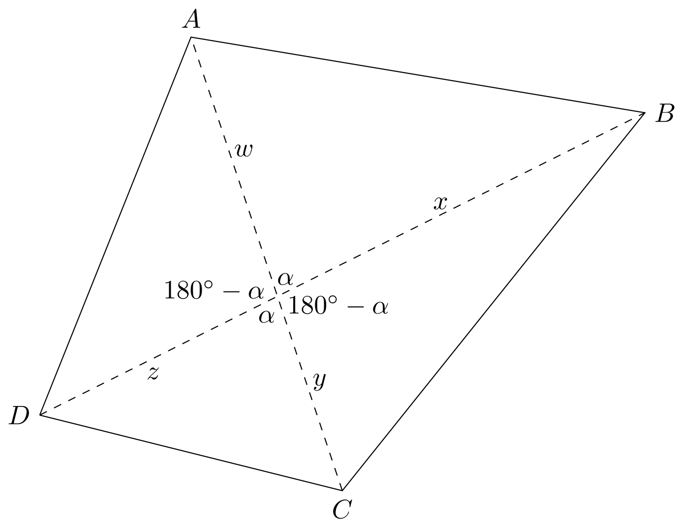
$$
\begin{aligned}
A &= \frac{1}{2}\left(d+z\right)x\sin\alpha + \frac{1}{2}\left(d+z\right)y\sin\left(180^\circ - \alpha\right) - \frac{1}{2}xz\sin\alpha - \frac{1}{2}yz\sin\left(180^\circ - \alpha\right) \\
&= \frac{1}{2}\left(d+z\right)x\sin\alpha + \frac{1}{2}\left(d+z\right)y\sin\alpha - \frac{1}{2}xz\sin\alpha - \frac{1}{2}yz\sin\alpha \\
&= \frac{1}{2}\left[\left(d+z\right)x + \left(d+z\right)y - xz - yz\right]\sin\alpha \\
&= \frac{1}{2}\left[\left(d+z\right)\left(x+y\right) - z\left(x+y\right)\right]\sin\alpha \\
&= \frac{1}{2}d\left(x+y\right)\sin\alpha
\end{aligned}
$$

10.

$$
\frac{\frac{1}{2}\left(4\right)\left(6\right)\sin\alpha}{\frac{1}{2}\left(8\right)\left(10\right)\sin\alpha} = \frac{4\cdot 6}{8\cdot 10} = \frac{3}{10}
$$

11. See solution provided in the textbook

12.



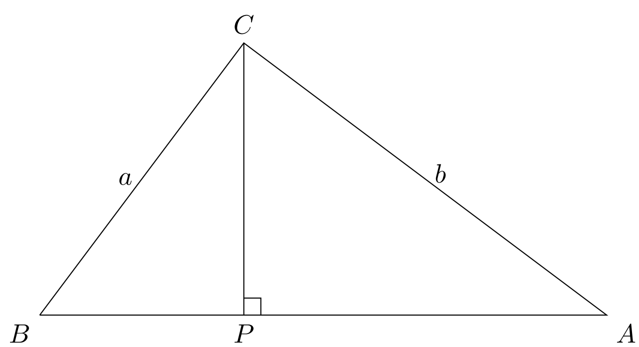
$$|APB| \times |CPD| = \left(\frac{1}{2}wx\sin\alpha\right)\left(\frac{1}{2}yz\sin\alpha\right) = \frac{1}{4}wxyz\sin^2\alpha$$

$$|BPC|\times|DPA| = \left(\frac{1}{2}xy\sin\left(180^\circ - \alpha\right)\right)\left(\frac{1}{2}wz\sin\left(180^\circ - \alpha\right)\right) = \frac{1}{4}wxyz\sin^2\alpha$$

This identity is true even if the diagonals intersect outside of the quadrilateral. Using the diagram above in Exercise 9, we can show that $|APB| \times |CPD|$ and $|BPC| \times |DPA|$ are both equal to

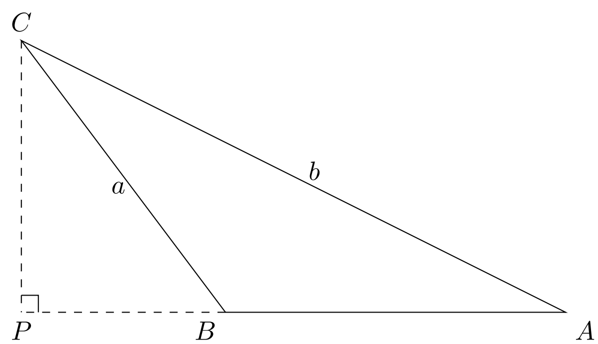
$$\frac{1}{4}xyz\left(d+z\right)\sin^2\alpha.$$

13.



$$c = AB = AP + PB = b\cos A + a\cos B$$

Let's suppose angle A is obtuse. Then we need to subtract ...



$$c = AB = AP - BP = b\cos A - a\cos(180^\circ - B) = b\cos A + a\cos B$$

As shown above, no changes need to be made to this result for obtuse angles.

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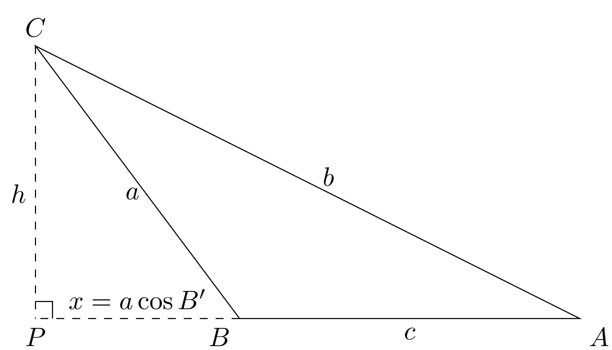
1.



$$\begin{aligned} b^2 &= h^2 + (c+x)^2 \\ &= a^2 - x^2 + c^2 + 2cx + x^2 \\ &= a^2 + c^2 + 2cx \\ &= a^2 + c^2 + 2ac\cos B' \end{aligned}$$

Page 80

1.

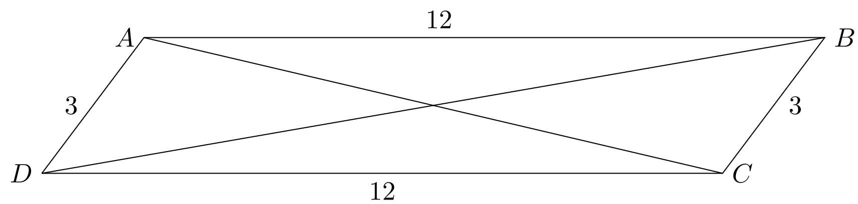
2. (a)
$$x^2 = y^2 + z^2 - 2yz\cos X$$
   (b)
$$x^2 = y^2 + z^2 - 2yz\cos X$$
   (c)
$$a^2 = b^2 + c^2 - 2bc\cos A$$
   (d)
$$b^2 = a^2 + c^2 - 2ac\cos B$$
   (e)
$$c^2 = a^2 + b^2 - 2ac\cos C$$

3. Given two sides and the angle between them, the Law of Cosines can be used to find the length of the third side. Another application of the Law of Cosines can be used to solve for one of the unknown angles since the three sides of the triangle are now known. The final angle of the triangle can be determined by subtracting the measures of the two known angles from 180°.

4. (a)
$$x = \sqrt{12^2 + 15^2 - 2\,(12)\,(15)\cos 50^\circ} \approx 11.7$$
   (b)
$$x = \sqrt{10^2 + 16^2 - 2\,(10)\,(16)\cos 110^\circ} \approx 21.6$$
   (c)
$$\cos x^\circ = \frac{6^2 + 8^2 - 9^2}{2\,(6)\,(8)} = \frac{19}{96} \implies x \approx 78.6$$
   (d)
$$\cos x^\circ = \frac{10^2 + 5^2 - 12^2}{2\,(10)\,(5)} = -\frac{19}{100} \implies x \approx 101.0$$

5.
$$\cos A = \frac{10^2 + 7^2 - 6^2}{2\,(10)\,(7)} = \frac{113}{140} \implies A \approx 36.2^\circ$$
$$\cos B = \frac{10^2 + 6^2 - 7^2}{2\,(10)\,(6)} = \frac{29}{40} \implies B \approx 43.5^\circ$$
$$\cos C = \frac{7^2 + 6^2 - 10^2}{2\,(7)\,(6)} = -\frac{5}{28} \implies C \approx 100.3^\circ$$

6.



In the above diagram of a parallelogram with side lengths of 3 and 12, let d_1 be the length of the diagonal AC and let d_2 be the length of the diagonal BD. By the Law of Cosines, we know that

$$d_1^2 = 3^2 + 12^2 - 2(3)(12)\cos B$$

and

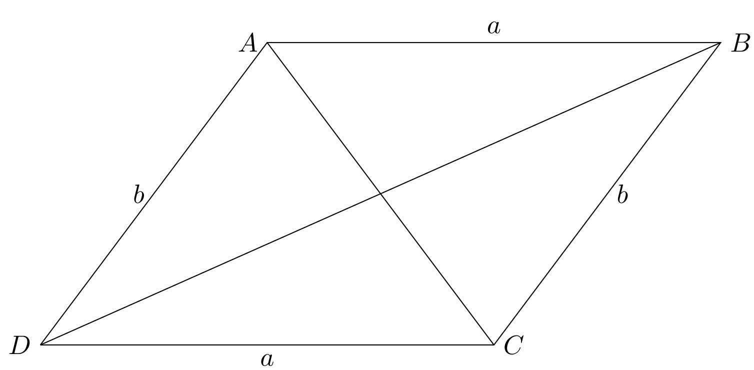
$$d_2^2 = 3^2 + 12^2 - 2(3)(12)\cos A.$$

However, since adjacent angles in a parallelogram are supplementary, $\cos B = -\cos A$. Therefore,

$$\begin{aligned} d_1^2 + d_2^2 &= \big(3^2 + 12^2 - 2(3)(12)\cos B\big) + \big(3^2 + 12^2 - 2(3)(12)\cos A\big) \\ &= \big(3^2 + 12^2 + 2(3)(12)\cos A\big) + \big(3^2 + 12^2 - 2(3)(12)\cos A\big) \\ &= 3^2 + 12^2 + 3^2 + 12^2 \\ &= 306 \end{aligned}$$

Thus, the answer does not depend on the specific shape of the parallelogram. Only the side lengths are pertinent.

7.

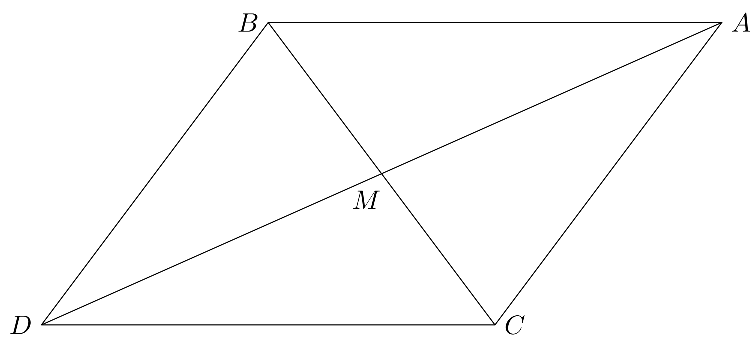


We can generalize the argument from the previous exercise to an arbitrary parallelogram. As before, let d_1 be the length of the diagonal AC and let d_2 be the length of the diagonal BD. Then,

$$\begin{aligned} d_1^2 + d_2^2 &= a^2 + b^2 - 2ab\cos B + a^2 + b^2 - 2ab\cos A \\ &= a^2 + b^2 + 2ab\cos A + a^2 + b^2 - 2ab\cos A \\ &= a^2 + b^2 + a^2 + b^2, \end{aligned}$$

which shows that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the four sides.

8.



We can create a parallelogram from triangle ABC by creating a copy of the triangle and reflecting it through the point M. This allows us to apply the result from the previous exercise about the sum of the squares of the diagonals.

$$\begin{aligned} AD^2 + BC^2 = AB^2 + AC^2 + BD^2 + CD^2 &\implies (2AM)^2 + BC^2 = AB^2 + AC^2 + AC^2 + AB^2 \\ &\implies 4AM^2 + BC^2 = 2AB^2 + 2AC^2 \\ &\implies 4AM^2 = 2AB^2 + 2AC^2 - BC^2 \end{aligned}$$

9. We can get a formula for the length of each median in a triangle by applying cyclic substitutions to the result we derived in the previous exercise.

$$4AM^2 = 2AB^2 + 2AC^2 - BC^2$$

$$4BM^2 = 2BC^2 + 2AB^2 - AC^2$$

$$4CM^2 = 2AC^2 + 2BC^2 - AB^2$$

Adding these three equations together gives

$$4AM^2 + 4BM^2 + 4CM^2 = 3AB^2 + 3AC^2 + 3BC^2.$$

Factoring out 3 on the right-hand side of the equation and dividing both sides by 4 gives the result:

$$AM^2 + BM^2 + CM^2 = \frac{3}{4}\left(AB^2 + AC^2 + BC^2\right).$$

10.

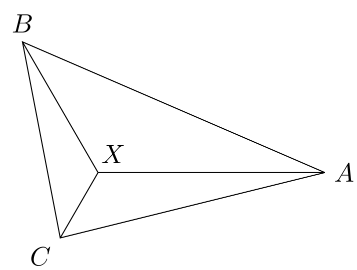
11.

$$c^2 = 1^2 + 4^2 - 2\,(1)\,(4)\cos 60^\circ = 17 - 8\left(\frac{1}{2}\right) = 13 \implies c = \sqrt{13}$$

12.
$$c^2 = a^2 + b^2 - 2ab\cos 60^\circ = a^2 + b^2 - 2ab\left(\frac{1}{2}\right) = a^2 + b^2 - ab$$

Since 120° is the supplement of 60°, the analogous result is that $c^2 = a^2 + b^2 + ab$.

13.



Let A, B, and C be the positions of the three riders at some time t, where t is the number of elapsed hours since departure. Then $AX = 60t$, $BX = 40t$, and $CX = 20t$. Applying the result from the previous exercise concerning the Law of Cosines for triangles with a 120° angle,

$$AB = \sqrt{(60t)^2 + (40t)^2 + (60t)(40t)} \approx 87.2t$$

$$BC = \sqrt{(40t)^2 + (20t)^2 + (40t)(20t)} \approx 52.9t$$

$$AC = \sqrt{(60t)^2 + (20t)^2 + (60t)(20t)} \approx 72.1t$$

Therefore, after 1 hour, riders A and B are 87.2 miles apart, riders B and C are 52.9 miles apart, and riders A and C are 72.1 miles apart. After 2 hours, riders A and B are 174.4 miles apart, riders B and C are 105.8 miles apart, and riders A and C are 144.2 miles apart.

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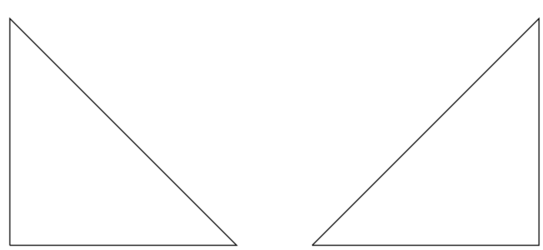
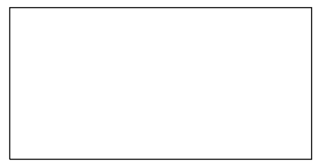
1. 45-45-90 triangle:

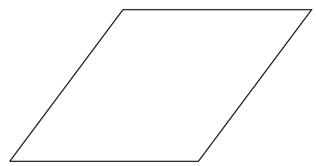


2. The triangles which can be placed on their mirror images without reflection are the isosceles triangles.

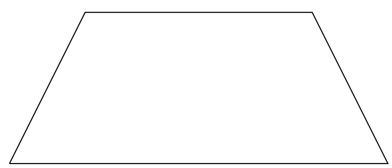
3. Rectangle:



Rhombus:



Isosceles Trapezoid:



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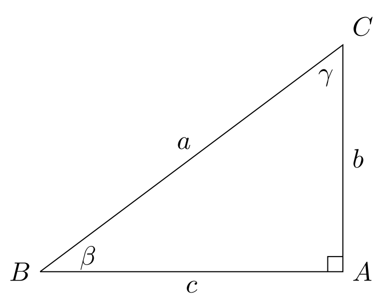
1.



$$\sin\beta = \frac{2S}{ac} = \frac{bc}{ac} = \frac{b}{a}$$
$$\sin\gamma = \frac{2S}{ab} = \frac{bc}{ab} = \frac{c}{a}$$

2.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ &= a^2 + b^2 - 2\left(\frac{2S}{\sin\gamma}\right)\cos\gamma \\ &= a^2 + b^2 - 4S\cot\gamma \end{aligned}$$

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1.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

2. These three quantities are all equal to twice the circumradius of the triangle.

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1. (a) The dimensions of the left-hand side and right-hand side of the equation are both $[L]/[\emptyset]$, where we use $\emptyset$ to represent a dimensionless quantity.
   (b) The dimensions of the left-hand side are $[L]/[L]$, which is a dimensionless quantity. The dimensions of the right-hand side are $[\emptyset]/[\emptyset]$, which is also a dimensionless quantity. Thus, the dimensions in this equation are consistent.
   (c) The dimensions of the left-hand side are $[L]^2$, and the dimensions of the right-hand side are $[L] \cdot [L] \cdot [\emptyset]$, which is equivalent to the dimensions of $[L]^2$.
   (d) The dimensions of the left-hand side are $[L]^2$. The dimensions of each term on the right-hand side are $[L]^2$ (recall that $\cos\gamma$ is dimensionless), so the overall dimensions of the right-hand side are also $[L]^2$.

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1. The semiperimeter of this triangle is $(3 + 4 + 5)/2 = 6$.

$$S = \sqrt{6\,(6-3)\,(6-4)\,(6-5)} = 6$$

2. The semiperimeter of this triangle is $(5 + 12 + 13)/2 = 15$.

$$S = \sqrt{15\,(15-5)\,(15-12)\,(15-13)} = 30$$

3. The semiperimeter of an equilateral triangle with side length l is $3l/2$

$$S = \sqrt{\frac{3l}{2} \cdot \frac{l}{2} \cdot \frac{l}{2} \cdot \frac{l}{2}} = \sqrt{\frac{3}{16} l^4} = l^2 \frac{\sqrt{3}}{4}$$

4. Each angle in an equilateral triangle is $60°$.

$$S = \frac{1}{2} l^2 \sin 60° = l^2 \frac{\sqrt{3}}{4}$$

5. The semiperimeter of a 13-14-15 triangle is $(13 + 14 + 15)/2 = 21$.

$$S = \sqrt{21\,(21 - 13)\,(21 - 14)\,(21 - 15)} = 84$$

The semiperimeter of a 25-39-56 triangle is $(25 + 39 + 56)/2 = 60$.

$$S = \sqrt{60\,(60 - 25)\,(60 - 39)\,(60 - 56)} = 420$$

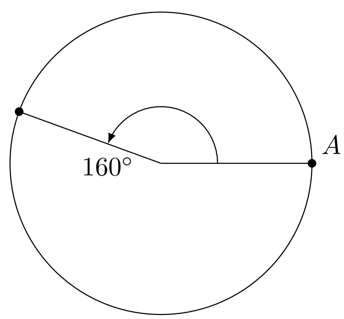
The semiperimeter of a 25-39-16 triangle is $(25 + 39 + 16)/2 = 40$.

$$S = \sqrt{40\,(40 - 25)\,(40 - 39)\,(40 - 16)} = 120$$

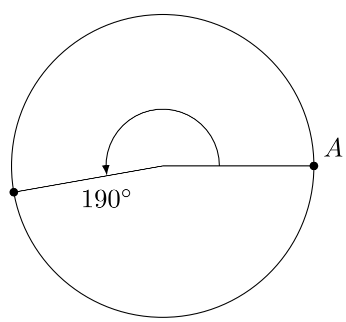
Chapter 4: Angles and Rotations

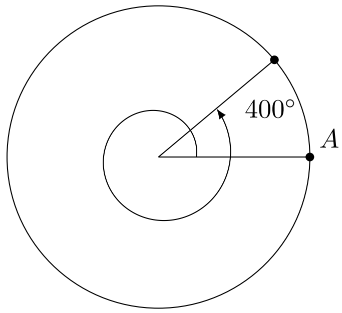
Page 93

1. (a)

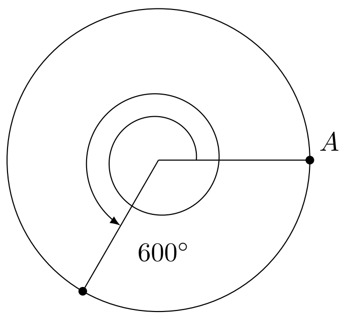


(b)

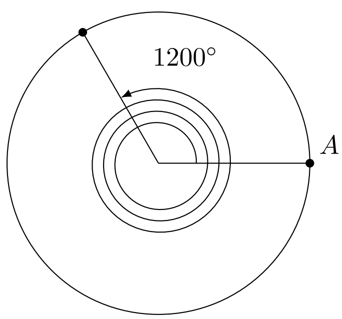




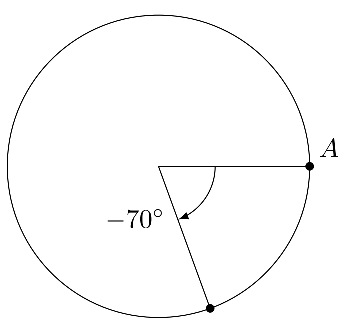
(c)



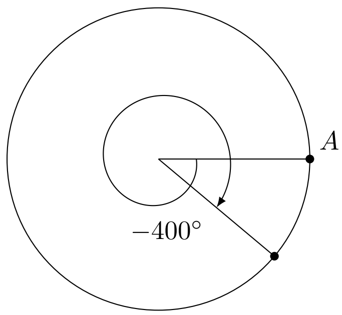
(d)

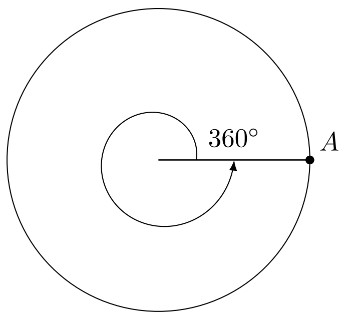


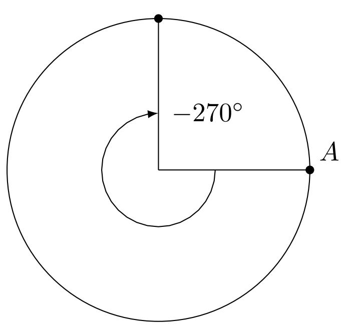
(e)



(f)

(g) 

(h) 

(i) 

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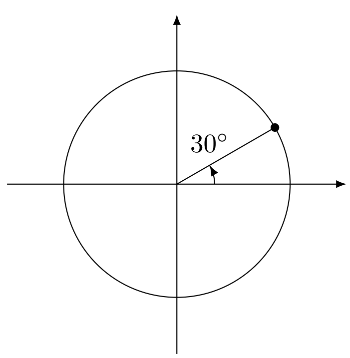
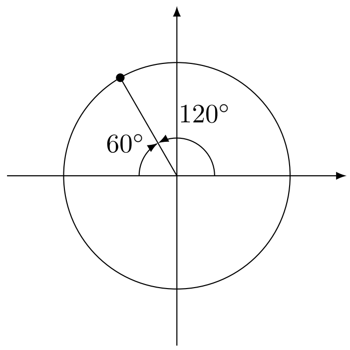
1. (a) 

Because $390° = 360° + 30°$, a point rotated through an angle of 390° would end up at the same location as a point rotated through an angle of 30°. Therefore,

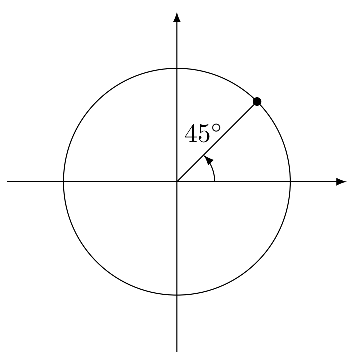
$$\sin 390° = \sin 30° = \frac{1}{2}.$$



(b)

Because $3720° = 10 \cdot 360° + 120°$, a point rotated through an angle of 3720° would end up at the same location as a point rotated through an angle of 120°. Therefore,

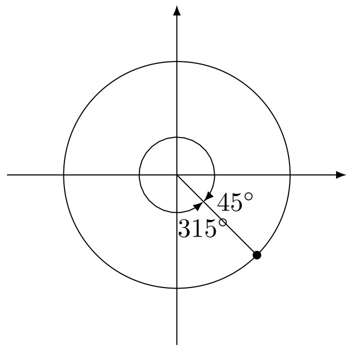
$$\cos 3720° = \cos 120° = -\cos 60° = -\frac{1}{2}.$$



(c)

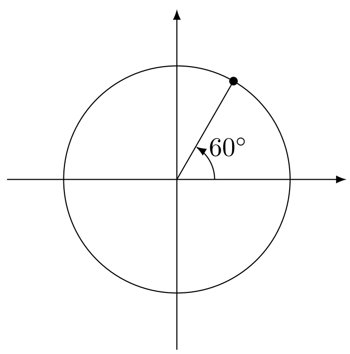
Because $1845° = 5 \cdot 360° + 45°$, a point rotated through an angle of 1845° would end up at the same location as a point rotated through an angle of 45°. Therefore,

$$\tan 1845° = \tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$$

(d) 

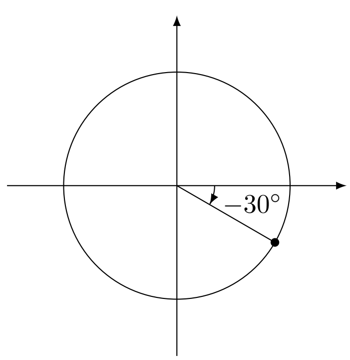
Because $315° = 360° - 45°$, a point rotated through an angle of $315°$ would end up at the same location as a point rotated through an angle of $-45°$. Therefore,

$$\sin 315° = \sin(-45°) = -\sin 45° = -\frac{\sqrt{2}}{2}$$

(e) 

Because $420° = 360° + 60°$, a point rotated through an angle of $420°$ would end up at the same location as a point rotated through an angle of $60°$. Therefore,

$$\cot 420° = \cot 60° = \frac{\cos 60°}{\sin 60°} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}.$$

(f) 

$$\tan(-30°) = \frac{\sin(-30°)}{\cos(-30°)} = \frac{-\sin 30°}{\cos 30°} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

2. (a)

$$\tan 360° = \tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0$$

(b)

$$\sin 180° = \sin 0° = 0$$

(c)

$$\cos 180° = -\cos 0° = -1$$

(d)

$$\cot 90° = \frac{\cos 90°}{\sin 90°} = \frac{0}{1} = 0$$

(e)

$$\cot 360° = \cot 0° = \frac{\cos 0°}{\sin 0°} = \frac{1}{0} \implies \cot 360° \text{ is undefined.}$$

(f)

$$\tan(-270°) = \tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0} \implies \tan(-270°) \text{ is undefined.}$$

Page 100

1. $400° = 360° + 40°$, so P will lie in the first quadrant.

   $3600° = 10 \cdot 360° + 0°$, so P will lie in the first quadrant (on the positive x-axis).

   $1845° = 5 \cdot 360° + 45°$, so P will lie in the first quadrant.

   $-30° = -360° + 330°$, so P will lie in the fourth quadrant.

   $-359° = -360° + 1°$, so P will lie in the first quadrant.

2.

$\sin 30°$	$1/2$	$\sin(-30°)$	$-1/2$
$\sin 135°$	$\sqrt{2}/2$	$\sin(-135°)$	$-\sqrt{2}/2$
$\sin 210°$	$-1/2$	$\sin(-210°)$	$1/2$
$\sin 300°$	$-\sqrt{3}/2$	$\sin(-300°)$	$\sqrt{3}/2$
$\sin 390°$	$1/2$	$\sin(-390°)$	$-1/2$
$\sin 480°$	$\sqrt{3}/2$	$\sin(-480°)$	$-\sqrt{3}/2$

From the table, we can see that $\sin(-\alpha) = -\sin\alpha$.

3. (a)
$$\sin\alpha = 0 \implies \alpha = 180°$$

(b)
$$\cos\alpha = 0 \implies \alpha = 90°, 270°$$

(c)
$$\sin\alpha = 1 \implies \alpha = 90°$$

(d) $\cos\alpha = 1$ is true for $\alpha = 0°$ and $\alpha = 360°$, but these are not in the interval $0 < \alpha < 360°$.

(e)
$$\sin\alpha = -1 \implies \alpha = 270°$$

(f)
$$\cos\alpha = \frac{1}{2} \implies \alpha = 60°, 300°$$

(g)
$$\sin\alpha = -\frac{1}{2} \implies \alpha = 210°, 330°$$

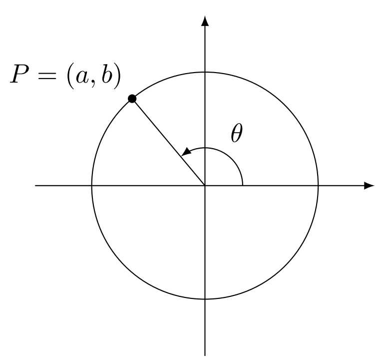
(h)
$$\sin^2\alpha = \frac{1}{2} \implies \sin\alpha = \pm\frac{\sqrt{2}}{2} \implies \alpha = 45°, 135°, 225°, 315°$$

(i) This equation has no solutions because the square of a real number cannot be negative.

4. (a) If $\sin\alpha = 5/13$, then α can lie in either the first or second quadrant. $\cos\alpha = 12/13$ when α is in the first quadrant, and $\cos\alpha = -12/13$ when α is in the second quadrant.

(b) If $\sin\alpha = -5/13$, then α can lie in either the third or fourth quadrant. $\cos\alpha = 12/13$ when α is in the fourth quadrant, and $\cos\alpha = -12/13$ when α is in the third quadrant.

5.



Let $P = (a, b)$ be a point on the coordinate plane. Since $a^2 + b^2 = 1$, P lies on a circle of radius 1 centered at the origin (*the unit circle*). Consider an angle θ between the positive x-axis and the line segment connecting P to the origin. By the extended definitions of the sine and cosine functions, $\sin\theta = b$ and $\cos\theta = a$.

Page 102

1. (a) Even
$$f(-x) = (-x)^6 - (-x)^2 + 7 = x^6 - x^2 + 7 = f(x)$$
(b) Odd
$$f(-x) = (-x)^3 - \sin(-x) = -x^3 + \sin x = -f(x)$$
(c) Neither
$$f(-x) = \frac{1}{-x+1}$$
(d) Even
$$f(-x) = \sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x = f(x)$$
(e) Odd
$$f(-x) = \csc(-x) = \frac{1}{\sin(-x)} = -\frac{1}{\sin(x)} = -\csc x = -f(x)$$
(f) Odd
$$f(-x) = 2\sin(-x)\cos(-x) = -2\sin(x)\cos(x) = -f(x)$$
(g) Even
$$f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$$
(h) Even
$$f(-x) = \cos^2(-x) = (\cos x)^2 = \cos^2 x = f(x)$$
(i) Even
$$f(-x) = \sin^2(-x) + \cos^2(-x) = \sin^2 x + \cos^2 x = f(x)$$

2.
$$g(-x) = \frac{1}{2}[f(-x) + f(-(-x))] = \frac{1}{2}[f(-x) + f(x)] = g(x),$$
so $g(x)$ is even.
$$h(-x) = \frac{1}{2}[f(-x) - f(-(-x))] = \frac{1}{2}[f(-x) - f(x)] = -h(x),$$

so $h(x)$ is odd.

For any function $f(x)$,

$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right] = g(x) + h(x),$$

so $f(x)$ can be written as the sum of an even function and an odd function.

3. Following the notation of the previous question, we denote the even part of $f(x)$ as $g(x)$ and the odd part of $f(x)$ as $h(x)$.

   (a) Since we know $\cos x$ is an even function and $\sin x$ is an odd function,

   $$g(x) = \cos x, h(x) = \sin x$$

   (b) Since a polynomial is an even function when all variables are raised to even exponents and an odd function when all variables are raised to odd exponents,

   $$g(x) = x^2 + 1, h(x) = x^3 + x$$

   (c) Applying the result from the previous exercise,

   $$g(x) = \frac{1}{2}\left(2^x + 2^{-x}\right), h(x) = \frac{1}{2}\left(2^x - 2^{-x}\right)$$

   (d) Applying the result from the previous exercise,

   $$g(x) = \frac{1}{2}\left(\frac{1-\sin x}{1+\sin x} + \frac{1+\sin x}{1-\sin x}\right) = \frac{1}{2}\left(\frac{2+2\sin^2 x}{1-\sin^2 x}\right) = \frac{1+\sin^2 x}{1-\sin^2 x}$$

   $$h(x) = \frac{1}{2}\left(\frac{1-\sin x}{1+\sin x} - \frac{1+\sin x}{1-\sin x}\right) = \frac{1}{2}\left(\frac{-4\sin x}{1-\sin^2 x}\right) = \frac{-2\sin x}{1-\sin^2 x}$$

   (e) Applying the result from the previous exercise,

   $$g(x) = \frac{1}{2}\left(\frac{1}{x+2} + \frac{1}{-x+2}\right) = \frac{1}{2}\left(\frac{4}{4-x^2}\right) = \frac{2}{4-x^2}$$

   $$h(x) = \frac{1}{2}\left(\frac{1}{x+2} - \frac{1}{-x+2}\right) = \frac{1}{2}\left(\frac{-2x}{4-x^2}\right) = -\frac{x}{4-x^2}$$

Chapter 5: Radian Measure

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1. 180 degrees is equal to π radians. 90 degrees is equal to $\pi/2$ radians.

2.

$$\pi^r = 180^\circ \implies 1^r = (180/\pi)^\circ \implies 2^r = (360/\pi)^\circ \approx 114.6^\circ$$

3. Since a full rotation is 2π radians, 1/4 of a full rotation will be $\pi/2$ radians.
4. Since 45 degrees is 1/8 of a full rotation, 1/8 of a full rotation will be $\pi/4$ radians.
5. Filled table below

Degree Measure	Radian Measure
90	$\pi/\mathbf{2}$
180	π
270	$\mathbf{3}\pi/\mathbf{2}$
360	$\mathbf{2}\pi$
90	$\pi/2$
180	π
270	$3\pi/2$
360	2π

6. Filled tables below

Degree Measure	Radian Measure
0	**0**
30	$\pi/\mathbf{6}$
72	$\mathbf{2}\pi/\mathbf{5}$
120	$\mathbf{2}\pi/\mathbf{3}$
135	$\mathbf{3}\pi/\mathbf{4}$
30	$\pi/6$
36	$\pi/5$
45	$\pi/4$
60	$\pi/3$
120	$2\pi/3$
126	$7\pi/10$

Degree Measure	Radian Measure
198	$\mathbf{11}\pi/\mathbf{10}$
210	$\mathbf{7}\pi/\mathbf{6}$
216	$\mathbf{6}\pi/\mathbf{5}$
225	$\mathbf{5}\pi/\mathbf{4}$
240	$\mathbf{4}\pi/\mathbf{3}$
198	$11\pi/10$
200	$10\pi/9$
210	$7\pi/6$
216	$6\pi/5$
225	$5\pi/4$
240	$4\pi/3$

7. Since 360 degrees is equal to 2π radians, then 1 degree is equal to $2\pi/360$, or $\pi/180$, radians.

8. (a) $\sin(1^r) \approx 0.8415$

   (b) $\sin(1^\circ) \approx 0.0175$

9. Filled table below

α (in radian)	$\sin\alpha$	$\cos\alpha$
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0
$2\pi/3$	$\sqrt{3}/2$	$-1/2$
$7\pi/6$	$-1/2$	$-\sqrt{3}/2$
$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$
$3\pi/2$	-1	0
$11\pi/6$	$-1/2$	$\sqrt{3}/2$

10. With an angle of 2 radians and a radius of 1, the length of the arc is $2\times 1 = 2$.

    With an angle of 3 radians and a radius of 1, the length of the arc is $3\times 1 = 3$.

    With an angle of π radians and a radius of 1, the length of the arc is $\pi\times 1 = \pi$.

11. With an angle of 2 radians and a radius of 3, the length of the arc is $2\times 3 = 6$.

    With an angle of 3 radians and a radius of 3, the length of the arc is $3\times 3 = 9$.

    With an angle of π radians and a radius of 3, the length of the arc is $\pi \times 3 = 3\pi$.

12. Using the fact that sine and cosine are cofunctions (see Section 4 of Chapter 1),

$$\sin\frac{\pi}{9} = \cos\left(\frac{\pi}{2} - \frac{\pi}{9}\right) = \cos\frac{7\pi}{18},$$

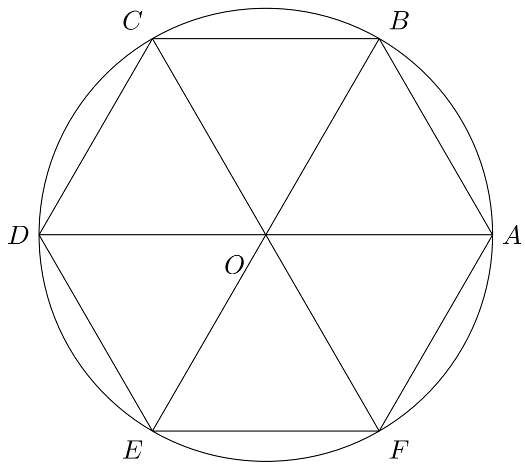
    so $\alpha = 7\pi/18$.

13. Again, applying the properties of cofunctions, we know that $\sin\alpha = \cos(\pi/2 - \alpha)$.

14. A complete rotation around a circle corresponds to an angle of 2π radians, so each of the six sectors in the diagram is an angle of $2\pi/6$, which is a bit more than 1 radian (since $2\pi > 6$).

    However, $2\pi/6$ radians is equal to 60 degrees, so if $2\pi/6$ is a bit more than 1, then 1 radian is less than 60 degrees

    Geometric solution:



In the diagram above, we place the six points A through F so that they are equally spaced across the circle O. This means that adjacent points on the circle are separated by $360^\circ/6 = 60^\circ$. Focusing on $\triangle AOB$, we notice that this triangle is isosceles because AO and BO are both equal to the radius of the circle. This implies that $\angle OAB$ is congruent to $\angle OBA$. Because the three angles in a triangle total to 180°, we have that

$$m\angle AOB = m\angle OAB = m\angle OBA = 60^\circ.$$

In other words, $\triangle AOB$ is an equilateral triangle. This means that the side AB is equal to the radius of O. Therefore, the arc that is intercepted by the chord AB must be longer than the radius of O because the shortest path between two points is a straight line. This implies that the central angle $\angle AOB$ is greater than 1 radian. Since $\angle AOB$ is a 60° angle, we have shown that 60° is greater than 1 radian.

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1. Since a circle of radius 1 traveling 1 foot corresponds to a rotation of 1 radian, a circle of radius 1 traveling 5 feet down a road corresponds to a rotation of 5 radians.

2. From the previous question, the circle has rotated 5 radians, which is equal to $5 \cdot 180^\circ/\pi \approx 286.48^\circ$ (two decimal places).

3. Since a circle of radius 1 rotates 1 radian to travel 1 foot, if it rotates 4 radians, then it has traveled 4 feet.

4. Since $120^\circ = 2\pi/3$ radians, the wheel has traveled $2\pi/3$ feet down the road.

5. A sector of radius 1 with angle α radians has an arc length of $1 \times \alpha$, so:

   An angle of $1/2$ radian has an arc length of $1/2$.

An angle of $\pi/2$ radian has an arc length of $\pi/2$.

An angle of α radian has an arc length of α.

6. Since 2π radians equal to $360°$ then

   $720° = 4\pi$ radians

   $1440° = 8\pi$ radians

   $3600° = 20\pi$ radians

   $15120° = 84\pi$ radians

   12π radians $= 2160°$

   12π radians $= 2160°$

   15π radians $= 2700°$

   100π radians $= 18000°$

7. A sector of radius 3 with angle α radians has an arc length of 3α, so:

   An angle of $1/2$ radian has an arc length of $3/2$.

   An angle of $\pi/2$ radian has an arc length of $3\pi/2$.

   An angle of α radian has an arc length of 3α.

8. A sector of radius 3 and angle α has arc length of 3α.

   An angle of 1.5 radians has an arc length of 4.5.

9. A sector of radius 5 and angle α has arc length of 5α.

   An angle of 80 degrees is equal to $4\pi/9$ radians. Thereforem its arc length is $5 \times 4\pi/9 = 20\pi/9$.

10. A sector of radius 2 and angle α has an arc length of 2α. This means that for an arc length of α, its angle would be $\alpha/2$. If the arc length is 3, then its central angle is $3/2$ radians.

11. A sector of radius 6 and angle α has an arc length of 6α. This means that for arc length of α, its angle would be $\alpha/6$. If the arc length is 2, then its central angle is $1/3$ radians, which is equal to $(60/\pi)^\circ \approx 19.1°$.

12. A circle of radius 7 units will travel 7 units with a rotation of 1 radian. That is, each unit of travel requires a rotation of $1/7$ radians. Therefore, 20 units of travel require $20/7$ radians of rotation.

13. A circle of radius 8 units will travel 8 units with a rotation of 1 radian. Since 150 degrees is equal to $5\pi/6$ radians, the circle has rolled $8 \times 5\pi/6 = 20\pi/3$ units.

14. In twelve hours, the hour hand makes a complete rotation around the watch face (e.g., 12:00 AM to 12:00 PM), so in one hour, the hour hand makes $1/12$ of a full rotation. Because the hands of a clock rotate in a clockwise

sense, the angle of rotation will be negative. Thus, the angle through which the hour hand rotates in one hour is

$$-\frac{1}{12} \cdot 2\pi = -\frac{\pi}{6}.$$

15. In one hour, the minute hand makes one complete rotation around the watch face, so it rotates through an angle of -2π. In the same time, the second hand makes 60 complete rotations around the watch face since it completes one rotation in one minute. Therefore, the second hand rotates through an angle of $-60 \cdot 2\pi = -120\pi$.

16. Because the hands of a watch travel clockwise, Joe's angle should be negative since counterclockwise rotations are positive by convention.

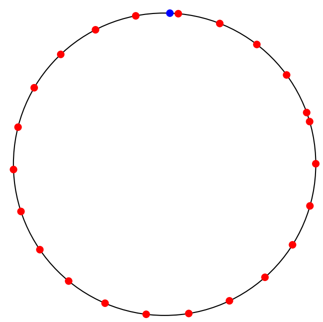
17.
$$\frac{1000 \text{ in.}}{2\pi \text{ in.}} \approx 159.2$$
This trip consists of about 159 full rotations of the hour hand. Multiplying by 12 gives the number of hours the trip takes since a complete rotation of the hour hand takes 12 hours.
$$\frac{1000}{2\pi} \cdot 12 \text{ hours} \approx 1910 \text{ hours}$$

18. The angular speed of the hour hand on a pocket watch and the hour hand on Big Ben should be the same because in one hour, the hour hand travels the same angular distance on both clocks. Since in both this exercise and the previous exercise the clocks are traveling through an angle of 1000 radians, both trips should take the same length of time: 1910 hours.

19. The wheel turns 2π radians before the spoke returns to the same position. After the wheel turns π radians, the spoke will go from pointing vertically downwards to pointing vertically upwards.

20. (a) A wheel of radius one meter will roll 2π meters each revolution, so the blue marks will be 2π meters apart.
    (b) The wheel will make a full revolution in this time, so it has rolled through an angle of 2π radians.
    (c) Not more than once. Suppose we have a red mark and a blue mark coinciding. When they next coincide, the distance between the two coincidences will be equal to $3n$ (measuring using the gaps between red dots) or $2\pi m$ (measuring using the gaps between blue dots), where n and m are positive integers. Because these are two equivalent ways to measure the same distance, we can say that $3n = 2\pi m$. Solving for π, we find that $\pi = \frac{3n}{2m}$. However, this is not possible as π is irrational, so it cannot be expressed as the ratio of integers. This contradiction means that if a red dot and blue dot do coincide, then they cannot coincide again.

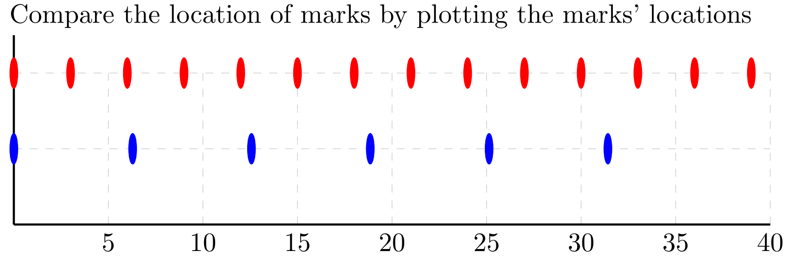
(d)



Let's consider how the red marks hit the wheel as it rolls. Hitting three red marks in a row corresponds to traveling 6 meters (count the space in between the marks). Because the radius of the wheel is 1 meter, when the wheel hits a triple of red marks, the last mark will hit the wheel in a position that is 6 radians ahead of the position where the first mark was hit. Alternatively, we can say that the last mark hits the wheel in a position that is $2\pi - 6$ radians *behind* the position of the first mark. In the above diagram, the red marks are separated by $2\pi - 6$ radians, so they represent the positions of every other red mark that hits the wheel in some period of time. Notice that the position of the red marks on the wheel implies that the blue mark cannot be more than $2\pi - 6$ radians away from a red mark. This means that no matter how far the wheel travels, the blue mark will eventually be placed within $2\pi - 6$ meters of a red mark.

Another perspective:

Compare the location of marks by plotting the marks' locations



Note the distance between blue and red markers will be different but the closest they come together are

Red	Blue	Difference (approx)
6	2π	0.283
12	4π	-0.566
18	6π	-0.850
24	8π	-1.133
30	10π	-1.416

(e) We assume that the blue dot and red dot coincide to begin with. 100 rotations corresponds to the wheel travelling 200π meters. Notice that

$200\pi \approx 209 \cdot 3 + 1.3$. Because $\frac{1}{4} \cdot 3 < 1.3 < \frac{1}{2} \cdot 3$, the blue dot will be between two pink dots: one of the pink dots will be at the midpoint of two red dots, and the second pink dot will be directly behind the first.

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1. The calculator was in radian mode, since $\sin(1^r) \approx 0.8415$, while $\sin(1^\circ) \approx 0.0175$.

2.

x (in radians)	$\sin x$	Difference (approx.)
0.2	0.19867	1.33×10^{-3}
0.15	0.14944	5.62×10^{-4}
0.05	0.04998	2.08×10^{-5}

In all cases, $x > \sin x$.

3. In the below table, the difference being calculated is $x - x^3/6 - \sin x$.

x (in radians)	$\sin x$	$x - x^3/6$	Difference (approx.)
0.2	0.1986693	0.1986667	-2.66×10^{-6}
0.15	0.1494381	0.1494375	-6.32×10^{-7}
0.05	0.0499792	0.0499792	-2.60×10^{-9}

4. $\sin 10^\circ \approx 0.174$, and $10/60 \approx 0.167$, so the error in the approximation is 0.007.

5. (a)
$$\sin 0.1 \approx 0.1 - \frac{0.1^3}{6} = \frac{1}{10} - \frac{1}{6000} = \frac{599}{6000}$$
Error: $\sin 0.1$ is greater than $599/6000$ by 8.33×10^{-8}.

   (b)
$$\sin 0.1^\circ = \sin \frac{\pi}{1800} \approx \frac{\pi}{1800} \approx \frac{3.14}{1800} = \frac{157}{90000}$$
Error: $\sin 0.1^\circ$ is greater than $157/90000$ by 8.84×10^{-7}.

6. (a) $\sin 1000^\circ \approx -0.9848$
   (b) $\sin 1000^r \approx 0.8269$

7. (a) $\sin(\sin 1000^r) \approx 0.7358$
   (b) $\sin 3.14^r \approx 0.00159$

8. Let $\varepsilon = \pi/2 - 1.5707 \approx 9.6 \times 10^{-5}$.
$$\cos 1.5707 = \cos\left(\frac{\pi}{2} - \varepsilon\right) = \sin \varepsilon < \varepsilon < 0.0001$$

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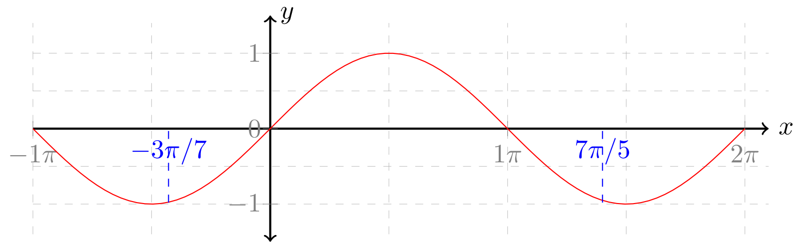
1. Using the graph below



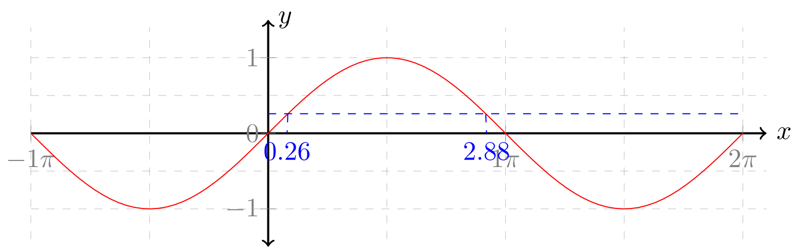
(a) $\sin 7\pi/5$ is negative because $\pi < 7\pi/5 < 2\pi$.
Estimated $x \approx -1$ (actual is -0.9511)

(b) $\sin -3\pi/7$ is negative because $-\pi < -3\pi/7 < 0$.
Estimated $x \approx -1$ (actual is -0.9749)

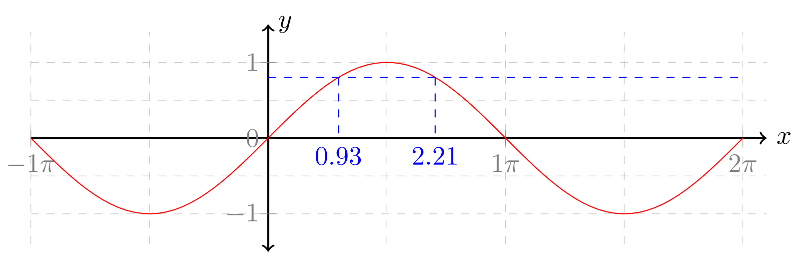
(c) By symmetry, $\sin(\pi - \pi/6) = \sin 5\pi/6 = 1/2$. By periodicity, $\sin(2\pi + \pi/6) = \sin 13\pi/6 = 1/2$. In general, $\sin x = 1/2$ when $x = \pi/6 + 2\pi n$ or $x = 5\pi/6 + 2\pi n$, where n is an integer.

(d) Draw a line at $y = \sin(\pi/12) \approx 0.26$ and then estimate the location of the intersections with the sine wave.

$$\begin{aligned} x &= \pi/12, 11\pi/12, 25\pi/12, \ldots \\ &\approx 0.26, 2.88, 6.54, \ldots \end{aligned}$$



(e) Draw a line at $y = 0.8$ and then estimate the location of the intersections with the sine wave. $x \approx 0.93, 2.21, 7.21, \ldots$



Alternatively, notice that $\sin \pi/4 = \sqrt{2}/2 \approx 0.70$ and $\sin \pi/3 = \sqrt{3}/2 \approx 0.85$. Taking a weighted average, we can estimate that the sine of

$$\frac{1}{3} \cdot \frac{\pi}{4} + \frac{2}{3} \cdot \frac{\pi}{3} = \frac{11\pi}{36}$$

is approximately 0.8. Then, based on the properties of the sine function, $25\pi/36$ and $83\pi/36$ should also have sines of approximately 0.8.

Note: For parts (a) and (b) of the above exercise, we can use the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ to get better estimates. This approximation can be derived from the approximation $\sin x \approx x$ using the identity $\sin^2 x + \cos^2 x = 1$. Let us suppose that x is a positive angle close to zero. Replacing $\sin x$ with x in the aforementioned identity, we get $\cos x \approx \sqrt{1 - x^2}$. Using the binomial approximation, $(1+z)^\alpha \approx 1 + \alpha z$, we arrive at the desired approximation for $\cos x$:

$$\cos x \approx \sqrt{1 - x^2} = \left[1 + \left(-x^2\right)\right]^{1/2} \approx 1 - \frac{1}{2}x^2.$$

For part (a), we get the estimate

$$\sin \frac{7\pi}{5} = -\sin \frac{2\pi}{5} = -\sin\left(\frac{\pi}{2} - \frac{\pi}{10}\right) = -\cos \frac{\pi}{10} \approx \frac{1}{2}\left(\frac{\pi}{10}\right)^2 - 1 \approx \frac{1}{20} - 1 = -0.95.$$

For part (b), we get the estimate

$$\sin \frac{-3\pi}{7} = -\sin \frac{3\pi}{7} = -\sin\left(\frac{\pi}{2} - \frac{\pi}{14}\right) = -\cos \frac{\pi}{14} \approx \frac{1}{2}\left(\frac{\pi}{14}\right)^2 - 1 \approx \frac{1}{40} - 1 = -0.975.$$

Both of these estimates improve upon the first-order approximation of -1.

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1. Results are written to 5 decimal places.

α (radians)	α (degrees)	$\sin\alpha$
1	57.29578	0.84147
0.5	28.64789	0.47943
0.2	11.45916	0.19867
0.1	5.72958	0.09983
0.01	0.57296	0.01000
0.02	1.14592	0.02000
0.001	0.05730	0.00100
0.002	0.11459	0.00200
0.005	0.28648	0.00500

2. Since $\sin x \approx x$ for angles close to zero (when x is in radians), $\sin 0.00123456 \approx 0.00123456$. Since $x > \sin x$ for positive angles, we know this is an overestimate. The calculator result is 0.0012345597, so our estimate is accurate to 7 decimal places.

3. (a) Results are written to 5 decimal places.

α	$\alpha - \alpha^3/6$	$\sin\alpha$
1	0.83333	0.84147
0.5	0.47917	0.47943
0.2	0.19867	0.19867
0.1	0.09983	0.09983
0.05	0.04998	0.04998
0.01	0.00999	0.00999
0.001	0.00099	0.00099

(b) Recall that we can multiply the degree measure of an angle by $\pi/180$ to get the radian measure, so we simply replace α by $\pi D/180$ in our approximation. This gives

$$\sin D \approx \frac{\pi D}{180} - \frac{1}{6}\left(\frac{\pi D}{180}\right)^3 = \frac{\pi D}{180} - \frac{\pi^3 D^3}{34992000}$$

For $D = 1^\circ$, the above approximation gives $\sin 1^\circ \approx 0.017421$, while the calculator result is 0.017452. This is an underestimate by 3.1×10^{-5}.

4. Using the fact that $\alpha = \pi D/180$ from the previous part, the largest possible error for an angle measured in degrees is given by

$$\frac{1}{120}\left(\frac{\pi D}{180}\right)^5 = \frac{\pi^5 D^5}{22674816000000}.$$

5.

$$\begin{aligned} x^5/120 &< 0.001 \\ x^5 &< 0.12 \\ x &< \sqrt[5]{0.12} \\ x &< 0.65438 \end{aligned}$$

6. The signs are alternating between terms, so the third term should be positive.

The exponent of x increases by two in each successive term, so the third term should have an exponent of 5.

The denominator of first term is 1! and the denominator of second term is 3!. Therefore the denominator of the fifth term should be $5! = 120$.

Therefore, the third term is

$$+\frac{x^5}{120}.$$

Hint: This is the Taylor Series for $\sin x$.

Note: **!** is used to denote the factorial function, where the factorial of a positive integer n is the product of all positive integers less than or equal to n.

7. The aliens appear to take clockwise angles to be positive, which is the opposite of the convention we typically use. Furthermore, it is not clear that the aliens even use counterclockwise angles since all of the examples only feature clockwise angles. This could mean that the aliens only work with non-negative (or in our system, non-positive) angles.

   The aliens do consider angles corresponding to rotations greater than a full rotation, as we do. φ seems to be used to denote one complete rotation. Comparing to our system, $\varphi = -2\pi$ radians.

8. It is easier to rewrite each of the radian angle measurements as multiples of π since the first quadrant contains angles between 0 and 0.5π, the second quadrant contains angles between 0.5π and π, etc.

Angle in radians	Angle in terms of π	Quadrant
1	0.318π	1
2	0.636π	2
3	0.954π	2
4	1.273π	3
5	1.591π	4
6	1.909π	4

   1000 radians is approximately equal to 318.31π, which represents 159 full rotations with a remainder of 0.31π radians. Thus an angle of 1000 radians lies in the first quadrant.

   1000 degrees corresponds to $1000/360 \approx 2.78$ rotations, which represents 2 whole rotations and a further 0.78 of a rotation, which is slightly more than a 3/4 rotation. Thus, an angle of 1000 degrees lies in the fourth quadrant.

9. Approximately 1/4 of the angles lie in each quadrant. See the solution below the question in the textbook.

Chapter 6: The Addition Formulas

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1.

α	β	$\sin\alpha$	$\sin\beta$	$\sin\alpha+\sin\beta$	$\sin(\alpha+\beta)$
60°	30°	$\sqrt{3}/2$	$1/2$	$\left(\sqrt{3}+1\right)/2$	$\sin 90^\circ = 1$
$\pi/4$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\left(\sqrt{2}+\sqrt{2}\right)/2=\sqrt{2}$	$\sin\pi/2 = 1$
$\pi/6$	$\pi/3$	$1/2$	$\sqrt{3}/2$	$\left(1+\sqrt{3}\right)/2$	$\sin\pi/2 = 1$

2. For these values of α and β, $\sin\alpha$ and $\sin\beta$ are both at least $1/2$. Furthermore, at least one of $\sin\alpha$ and $\sin\beta$ is strictly greater than $1/2$. Therefore,

$$\sin\alpha+\sin\beta > \frac{1}{2}+\frac{1}{2} = 1 = \sin(\alpha+\beta).$$

3. (a)

$$\sin 60^\circ + \sin 30^\circ = \frac{\sqrt{3}}{2}+\frac{1}{2}$$
$$\sin(60^\circ+30^\circ) = \sin 90^\circ = 1$$

This identity is not correct.

(b)

$$\sin(60^\circ-30^\circ) = \sin 30^\circ = \frac{1}{2}$$
$$\sin 60^\circ - \sin 30^\circ = \frac{\sqrt{3}}{2}-\frac{1}{2}$$

This identity is not correct.

(c)

$$\sin^2 60^\circ - \sin^2 30^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}-\frac{1}{4} = \frac{1}{2}$$
$$\sin(60^\circ+30^\circ)\sin(60^\circ-30^\circ) = \sin 90^\circ\cdot\sin 30^\circ = 1\cdot\frac{1}{2} = \frac{1}{2}$$

This identity is correct for the given angles.

4. See Chapter 2, Section 12 and the Appendix of Chapter 2 to review some of the geometry used in this solution.

(a) Because $\angle ABC$ is subtended by the diameter $\overline{AC}$, $\angle ABC$ is a right angle and $\triangle ABC$ is a right triangle (this fact is known as *Thales's Theorem*). Therefore, $\sin\alpha$ is equal to the length of the opposite side (BC) divided by the length of the hypotenuse (AC). $\overline{AC}$ is a diameter of the circle, so it has length 1. Thus, we have that $\sin\alpha$ is simply equal to BC.

A similar argument shows that $\triangle ADC$ is a right triangle with hypotenuse $\overline{AC}$ of length 1, which implies that $\sin\beta = DC$

(b) Recall that chords of congruent circles which subtend equal angles are themselves equal. This implies that BC in the diagram of part (a) is equal to BC in the diagram of part (b) because in both diagrams, the chord $\overline{BC}$ subtends an angle of measure α. Similarly, DC is the same in both diagrams because in both diagrams, the chord $\overline{DC}$ subtends an angle of measure β. Therefore, BC is still equal to $\sin\alpha$, and DC is still equal to $\sin\beta$.

(c) From part (b) above, we can conclude that a chord which subtends an inscribed angle with measure α in a circle with diameter 1 has length $\sin\alpha$. Thus, we draw $\overline{BD}$, the chord which subtends $\angle BAD$ in both figures and which consequently has length $\sin(\alpha+\beta)$.

Note that the above reasoning implies that the sine of an angle with measure less than 180° cannot exceed 1 since the diameter is the longest chord in a circle.

5. Recall that the sine of any angle is at most 1. Therefore,

$$\sin 105^\circ \leq 1 = \frac{1}{2} + \frac{1}{2} < \sin 45^\circ + \sin 60^\circ,$$

which shows that $\sin 105^\circ$ cannot equal $\sin 45^\circ + \sin 60^\circ$.

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1. Addition formula for sine:

$$\begin{aligned}\sin(60^\circ + 30^\circ) &= \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1 \\ &= \sin 90^\circ\end{aligned}$$

Addition formula for cosine:

$$\begin{aligned}\cos(60^\circ + 30^\circ) &= \cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ \\ &= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ &= 0 \\ &= \cos 90^\circ\end{aligned}$$

Difference formula for sine:

$$\begin{aligned}\sin(60^\circ - 30^\circ) &= \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{3}{4} - \frac{1}{4} \\ &= \frac{1}{2} \\ &= \sin 30^\circ\end{aligned}$$

Difference formula for cosine:

$$\begin{aligned}\cos(60^\circ - 30^\circ) &= \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ \\ &= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ &= \frac{\sqrt{3}}{2} \\ &= \cos 30^\circ\end{aligned}$$

2. Addition formula for sine ($\alpha = 0$):

$$\begin{aligned}\sin(0 + \beta) &= \sin 0 \cos\beta + \cos 0 \sin\beta \\ &= 0 \cdot \cos\beta + 1 \cdot \sin\beta \\ &= \sin\beta\end{aligned}$$

Addition formula for cosine ($\alpha = 0$):

$$\begin{aligned}\cos(0 + \beta) &= \cos 0 \cos\beta - \sin 0 \sin\beta \\ &= 1 \cdot \cos\beta - 0 \cdot \sin\beta \\ &= \cos\beta\end{aligned}$$

Difference formula for sine ($\alpha = 0$):

$$\begin{aligned}\sin(0 - \beta) &= \sin 0 \cos\beta - \cos 0 \sin\beta \\ &= 0 \cdot \cos\beta - 1 \cdot \sin\beta \\ &= -\sin\beta\end{aligned}$$

Notice that this demonstrates that the sine function is *odd*.

Difference formula for cosine ($\alpha = 0$):

$$\begin{aligned}\cos(0 - \beta) &= \cos 0 \cos\beta + \sin 0 \sin\beta \\ &= 1 \cdot \cos\beta + 0 \cdot \sin\beta \\ &= \cos\beta\end{aligned}$$

Notice that this demonstrates that the cosine function is *even*.

Addition formula for sine ($\beta = 0$):

$$\begin{aligned}\sin(\alpha+0) &= \sin\alpha\cos 0 + \cos\alpha\sin 0\\ &= \sin\alpha\cdot 1 + \cos\alpha\cdot 0\\ &= \sin\alpha\end{aligned}$$

Addition formula for cosine ($\beta = 0$):

$$\begin{aligned}\cos(\alpha+0) &= \cos\alpha\cos 0 - \sin\alpha\sin 0\\ &= \cos\alpha\cdot 1 - \sin\alpha\cdot 0\\ &= \cos\alpha\end{aligned}$$

Difference formula for sine ($\beta = 0$):

$$\begin{aligned}\sin(\alpha-0) &= \sin\alpha\cos 0 - \cos\alpha\sin 0\\ &= \sin\alpha\cdot 1 - \cos\alpha\cdot 0\\ &= \sin\alpha\end{aligned}$$

Difference formula for cosine ($\beta = 0$):

$$\begin{aligned}\cos(\alpha+0) &= \cos\alpha\cos 0 + \sin\alpha\sin 0\\ &= \cos\alpha\cdot 1 + \sin\alpha\cdot 0\\ &= \cos\alpha\end{aligned}$$

3. Following the hint, we notice that in a right triangle, the side opposite one of the acute angles is the side adjacent to the other acute angle. Thus, if $\alpha + \beta = \pi/2$, then $\sin\alpha = \cos\beta$ and $\sin\beta = \cos\alpha$ (see also Chapter 1, Section 4).

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \sin\alpha\sin\alpha + \cos\alpha\cos\alpha\\ &= \sin^2\alpha + \cos^2\alpha\\ &= 1\end{aligned}$$

4. Addition formula for sine:

$$\begin{aligned}\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \sin\frac{\pi}{4}\cos\frac{\pi}{4} + \cos\frac{\pi}{4}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{1}{2}+\frac{1}{2}\\ &= 1\\ &= \sin\frac{\pi}{2}\end{aligned}$$

Addition formula for cosine:

$$\begin{aligned}
\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \cos\frac{\pi}{4}\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\sin\frac{\pi}{4}\\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\\
&= \frac{1}{2}-\frac{1}{2}\\
&= 0\\
&= \cos\frac{\pi}{2}
\end{aligned}$$

5. Recall that $(A \pm B)^2 = A^2 \pm 2AB + B^2$.

$$\begin{aligned}
&(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2\\
&\quad= \sin^2\alpha\cos^2\beta+2\sin\alpha\cos\beta\cos\alpha\sin\beta+\cos^2\alpha\sin^2\beta+\\
&\qquad\cos^2\alpha\cos^2\beta-2\cos\alpha\cos\beta\sin\alpha\sin\beta+\sin^2\alpha\sin^2\beta\\
&\quad= \sin^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta\\
&\quad= \sin^2\alpha\left(\cos^2\beta+\sin^2\beta\right)+\cos^2\alpha\left(\sin^2\beta+\cos^2\beta\right)\\
&\quad= \sin^2\alpha\cdot 1+\cos^2\alpha\cdot 1\\
&\quad= 1
\end{aligned}$$

6. After expanding using the identity $(A+B)(A-B) = A^2 - B^2$, we cleverly "add by zero" to get the desired result.

$$\begin{aligned}
&(\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&\quad= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&\quad= \sin^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta-\sin^2\alpha\sin^2\beta-\cos^2\alpha\sin^2\beta\\
&\quad= \sin^2\alpha\left(\cos^2\beta+\sin^2\beta\right)-\sin^2\beta\left(\sin^2\alpha+\cos^2\alpha\right)\\
&\quad= \sin^2\alpha\cdot 1-\sin^2\beta\cdot 1\\
&\quad= \sin^2\alpha-\sin^2\beta
\end{aligned}$$

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1.

2.

Page 131

1.

$$
\begin{aligned}
\sin(30^\circ + 30^\circ) &= \sin 30^\circ \cos 30^\circ + \cos 30^\circ \sin 30^\circ \\
&= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\
&= \frac{\sqrt{3}}{2} \\
&= \sin 60^\circ
\end{aligned}
$$

$$
\begin{aligned}
\cos(30^\circ + 30^\circ) &= \cos 30^\circ \cos 30^\circ - \sin 30^\circ \sin 30^\circ \\
&= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} \\
&= \frac{3}{4} - \frac{1}{4} \\
&= \frac{1}{2} \\
&= \cos 60^\circ
\end{aligned}
$$

2. Assuming α and β are acute angles:

$$\sin\alpha = \frac{3}{5} \implies \cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - (3/5)^2} = \frac{4}{5}$$

$$\sin\beta = \frac{5}{13} \implies \cos\beta = \sqrt{1 - \sin^2\beta} = \sqrt{1 - (5/13)^2} = \frac{12}{13}$$

$$
\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
&= \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13} \\
&= \frac{36}{65} + \frac{20}{65} \\
&= \frac{56}{65}
\end{aligned}
$$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13} \\ &= \frac{48}{65} - \frac{15}{65} \\ &= \frac{33}{65}\end{aligned}$$

3.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

4.

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Notice that $75°$ and $15°$ are complementary angles, so we know $\sin 75° = \cos 15°$ and $\sin 15° = \cos 75°$.

5. (a) Yes, let $\alpha = \beta = \pi/4$.

$$\begin{aligned}\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \cos\frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{1}{2}-\frac{1}{2}\\ &= 0\end{aligned}$$

More generally, we could suppose $\alpha+\beta = \pi/2$ and follow the approach in Exercise 3 of Section 2 earlier in this chapter.

(b) If α and β are acute angles, then $0 < \alpha+\beta < \pi$. Using the unit circle, we can see that $\sin(\alpha+\beta)$ must be positive since the angle $\alpha+\beta$ lies in the upper-half of the plane, where the sine function is positive.

(c) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ is positive when α and β are acute angles since the sum and product of positive real numbers is also positive.

$\cos(\alpha+\beta)$ need not be positive. As shown in part (a) of this exercise, $\cos(\alpha+\beta)$ can equal 0. Furthermore, $\cos(\alpha+\beta)$ can be negative. Let $\alpha = \beta = \pi/3$. Then, assuming that we can extend the cosine addition formula to angles α and β such that $\alpha+\beta$ is obtuse,

$$\begin{aligned}\cos\left(\frac{\pi}{3}+\frac{\pi}{3}\right) &= \cos\frac{\pi}{3}\cos\frac{\pi}{3} - \sin\frac{\pi}{3}\sin\frac{\pi}{3}\\ &= \frac{1}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{1}{4}-\frac{3}{4}\\ &= -\frac{1}{2}\end{aligned}$$

6. As we saw in Exercise 1 of Section 1 of this chapter, $\sin\alpha + \sin\beta$ does not equal $\sin(\alpha+\beta)$ in general. A similar table can be used to show that $\sin\alpha - \sin\beta$ does not equal $\sin(\alpha-\beta)$ in general.

7. This is not a coincidence. The identity holds true even when substituting more "arbitrary" values in for α and β. For example, using $\alpha = 37°$ and $\beta = 19°$, we find that both $\sin^2\alpha - \sin^2\beta$ and $\sin(\alpha+\beta)\sin(\alpha-\beta)$ are equal to approximately 0.2562.

8. You may also refer to the proof in Exercise 6 of Section 2 of this chapter.

$$\begin{aligned}
\sin(\alpha+\beta)\sin(\alpha-\beta) &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta-\sin^2\alpha\sin^2\beta-\cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha\left(\cos^2\beta+\sin^2\beta\right)-\sin^2\beta\left(\sin^2\alpha+\cos^2\alpha\right)\\
&= \sin^2\alpha\cdot 1-\sin^2\beta\cdot 1\\
&= \sin^2\alpha-\sin^2\beta
\end{aligned}$$

9. This proof is nearly identical to the one in the previous part. We just make a small modification in how we "add by zero" in order to obtain to the desired result.

$$\begin{aligned}
\sin(\alpha+\beta)\sin(\alpha-\beta) &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha\cos^2\beta+\cos^2\alpha\cos^2\beta-\cos^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&= \cos^2\beta\left(\sin^2\alpha+\cos^2\alpha\right)-\cos^2\alpha\left(\cos^2\beta+\sin^2\beta\right)\\
&= \cos^2\beta\cdot 1-\cos^2\alpha\cdot 1\\
&= \cos^2\beta-\cos^2\alpha
\end{aligned}$$

10. We apply the sine addition formula in reverse.

$$\begin{aligned}
\sin 18^\circ\cos 12^\circ+\cos 18^\circ\sin 12^\circ &= \sin(18^\circ+12^\circ)\\
&= \sin 30^\circ\\
&= \frac{1}{2}
\end{aligned}$$

11. (a) Since we have not proved that the sine addition formula works for all angles α and β, we use properties of the sine and cosine functions to avoid working with angles larger than 90°.

$$\begin{aligned}
&\sin 113^\circ\cos 307^\circ+\cos 113^\circ\sin 307^\circ\\
&\quad= \sin(180^\circ-67^\circ)\cos(360^\circ-53^\circ)+\cos(180^\circ-67^\circ)\sin(360^\circ-53^\circ)\\
&\quad= \sin 67^\circ\cos 53^\circ+(-\cos 67^\circ)(-\sin 53^\circ)\\
&\quad= \sin(67^\circ+53^\circ)\\
&\quad= \sin 120^\circ\\
&\quad= \sin 60^\circ\\
&\quad= \frac{\sqrt{3}}{2}
\end{aligned}$$

(b) Plugging into a calculator,

$$\sin 113^\circ\cos 307^\circ+\cos 113^\circ\sin 307^\circ\approx 0.866\approx\frac{\sqrt{3}}{2}=\sin 60^\circ$$

(c) Assuming that the sine addition formula does work for non-acute angles, we arrive at the same result.

$$\begin{aligned}\sin 113^\circ \cos 307^\circ + \cos 113^\circ \sin 307^\circ &= \sin(113^\circ + 307^\circ)\\ &= \sin 420^\circ\\ &= \sin 60^\circ\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

12. We can use the addition formulas for sine and cosine by rewriting 2α as $\alpha+\alpha$.

$$\begin{aligned}\sin 2\alpha\cos\alpha - \cos 2\alpha \sin\alpha &= \sin(\alpha+\alpha)\cos\alpha - \cos(\alpha+\alpha)\sin\alpha\\ &= (\sin\alpha\cos\alpha + \cos\alpha\sin\alpha)\cos\alpha - (\cos\alpha\cos\alpha - \sin\alpha\sin\alpha)\sin\alpha\\ &= \sin\alpha\cos^2\alpha + \sin\alpha\cos^2\alpha - \sin\alpha\cos^2\alpha + \sin^3\alpha\\ &= \sin\alpha\cos^2\alpha + \sin^3\alpha\\ &= \sin\alpha\left(\cos^2\alpha + \sin^2\alpha\right)\\ &= \sin\alpha\end{aligned}$$

13.

$$\begin{aligned}\sin(\alpha+\beta)\sin\beta + \cos(\alpha+\beta)\cos\beta &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\sin\beta + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\cos\beta\\ &= \sin\alpha\sin\beta\cos\beta + \sin^2\beta\cos\alpha + \cos\alpha\cos^2\beta - \sin\alpha\sin\beta\cos\beta\\ &= \sin^2\beta\cos\alpha + \cos\alpha\cos^2\beta\\ &= \cos\alpha\left(\sin^2\beta + \cos^2\beta\right)\\ &= \cos\alpha\end{aligned}$$

14.

$$\begin{aligned}\frac{\sin(\alpha+\beta) - \cos\alpha\sin\beta}{\cos(\alpha+\beta) + \sin\alpha\sin\beta} &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta + \sin\alpha\sin\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha}{\cos\alpha}\\ &= \tan\alpha\end{aligned}$$

15.

$$\begin{aligned}\sin\left(\alpha + \frac{\pi}{4}\right) &= \sin\alpha\cos\frac{\pi}{4} + \cos\alpha\sin\frac{\pi}{4}\\ &= \sin\alpha\cdot\frac{\sqrt{2}}{2} + \cos\alpha\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{2}(\sin\alpha + \cos\alpha)\end{aligned}$$

16.

$$\begin{aligned}
\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} &= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= 1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= 1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta} \\
&= 1 - \tan\alpha\tan\beta
\end{aligned}$$

17. Applying the law of cosines, we have that $(b_1+b_2)^2 = c_1^2+c_2^2-2c_1c_2\cos(\alpha+\beta)$. Solving for $\cos(\alpha+\beta)$, we get

$$\cos(\alpha+\beta) = \frac{c_1^2+c_2^2-(b_1+b_2)^2}{2c_1c_2}.$$

Before proceeding further, let's establish some relationships between the variables in the diagram. First, by the Pythagorean theorem, we have that $h^2 = c_1^2 - b_1^2 = c_2^2 - b_2^2$. Additionally, we can compute the sines and cosines for the angles α and β:

$$\sin\alpha = \frac{b_1}{c_1}, \sin\beta = \frac{b_2}{c_2}, \cos\alpha = \frac{h}{c_1}, \cos\beta = \frac{h}{c_2}.$$

We can now simplify our expression for $\cos(\alpha+\beta)$.

$$\begin{aligned}
\cos(\alpha+\beta) &= \frac{c_1^2+c_2^2-(b_1+b_2)^2}{2c_1c_2} \\
&= \frac{c_1^2+c_2^2-b_1^2-2b_1b_2-b_2^2}{2c_1c_2} \\
&= \frac{2h^2-2b_1b_2}{2c_1c_2} \\
&= \frac{h^2-b_1b_2}{c_1c_2} \\
&= \frac{h}{c_1}\cdot\frac{h}{c_2} - \frac{b_1}{c_1}\cdot\frac{b_2}{c_2} \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

Chapter 7: Trigonometric Identities

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1. (a) Yes
   (b) No

(c) Yes
(d) Yes
(e) No
(f) No

2. (a)
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

(b)
$$\begin{aligned}(1+\tan\alpha)(1-\tan\alpha) &= 1-\tan^2\alpha \\ &= 1-\frac{\sin^2\alpha}{\cos^2\alpha} \\ &= \frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha}\end{aligned}$$

(c)
$$\begin{aligned}\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\cdot\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\end{aligned}$$

(d)
$$\begin{aligned}\tan^2\alpha+\cot^2\alpha &= \frac{\sin^2\alpha}{\cos^2\alpha}+\frac{\cos^2\alpha}{\sin^2\alpha} \\ &= \frac{\sin^4\alpha+\cos^4\alpha}{\sin^2\alpha\cos^2\alpha}\end{aligned}$$

(e)
$$\begin{aligned}\tan\alpha\cot\alpha &= \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\cos\alpha}{\sin\alpha} \\ &= 1\end{aligned}$$

(f)
$$\begin{aligned}1+\tan^2\alpha &= 1+\frac{\sin^2\alpha}{\cos^2\alpha} \\ &= \frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha} \\ &= \frac{1}{\cos^2\alpha}\end{aligned}$$

3. The Principle of Analytic Continuation does not apply because $\sqrt{1-\sin^2\alpha}$ is not a rational trigonometric function. The identity is incorrect for $\alpha = 2\pi/3$ as $\cos(2\pi/3) = -1/2$, while $\sqrt{1-\sin^2(2\pi/3)} = 1/2$.

4. The Principle of Analytic Continuation does apply because both $\sin^2\alpha + \cos^2\alpha$ and 1 are rational trigonometric functions. The identity is correct for $\alpha = 2\pi/3$.

$$\sin^2\frac{2\pi}{3} + \cos^2\frac{2\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$$

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1. Since α and β are acute angles,

$$\sin\alpha = \frac{3}{5} \implies \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$\sin\beta = \frac{5}{13} \implies \cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}$$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{3}{5}\cdot\frac{12}{13} + \frac{4}{5}\cdot\frac{5}{13} \\ &= \frac{36}{65} + \frac{20}{65} \\ &= \frac{56}{65}\end{aligned}$$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13} \\ &= \frac{48}{65} - \frac{15}{65} \\ &= \frac{33}{65}\end{aligned}$$

$\alpha + \beta$ lies in the first quadrant because $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ are both positive.

2. Since α and β are acute angles,

$$\sin\alpha = \frac{4}{5} \implies \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

$$\sin\beta = \frac{12}{13} \implies \cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \frac{5}{13}$$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{4}{5}\cdot\frac{5}{13} + \frac{3}{5}\cdot\frac{12}{13} \\ &= \frac{20}{65} + \frac{36}{65} \\ &= \frac{56}{65}\end{aligned}$$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\cdot\frac{12}{13} \\ &= \frac{15}{65} - \frac{48}{65} \\ &= -\frac{33}{65}\end{aligned}$$

$\alpha+\beta$ lies in the second quadrant because $\sin(\alpha+\beta)$ is positive and $\cos(\alpha+\beta)$ is negative.

3.

$$\sin\alpha = \frac{3}{5} \implies \cos\alpha = \pm\sqrt{1-\sin^2\alpha} = \pm\sqrt{1-\left(\frac{3}{5}\right)^2} = \pm\frac{4}{5}$$

$$\sin\beta = \frac{5}{13} \implies \cos\beta = \pm\sqrt{1-\sin^2\beta} = \pm\sqrt{1-\left(\frac{5}{13}\right)^2} = \pm\frac{12}{13}$$

$\cos\alpha > 0,\ \cos\beta > 0$:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{3}{5}\cdot\frac{12}{13} + \frac{4}{5}\cdot\frac{5}{13} \\ &= \frac{36}{65} + \frac{20}{65} \\ &= \frac{56}{65}\end{aligned}$$

$\cos\alpha > 0,\ \cos\beta < 0$:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{3}{5}\left(-\frac{12}{13}\right) + \frac{4}{5}\cdot\frac{5}{13} \\ &= -\frac{36}{65} + \frac{20}{65} \\ &= -\frac{16}{65}\end{aligned}$$

$\cos\alpha < 0$, $\cos\beta > 0$:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{3}{5}\cdot\frac{12}{13} + \left(-\frac{4}{5}\right)\cdot\frac{5}{13} \\ &= \frac{36}{65} - \frac{20}{65} \\ &= \frac{16}{65}\end{aligned}$$

$\cos\alpha < 0$, $\cos\beta < 0$:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{3}{5}\left(-\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\cdot\frac{5}{13} \\ &= -\frac{36}{65} - \frac{20}{65} \\ &= -\frac{56}{65}\end{aligned}$$

There are four possible answers for $\sin(\alpha+\beta)$.

4. (a)

$$\begin{aligned}\sin\frac{2\pi}{3}\cos\frac{\pi}{3} - \cos\frac{2\pi}{3}\sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2}\cdot\frac{1}{2} - \left(-\frac{1}{2}\right)\frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ &= \frac{\sqrt{3}}{2} \\ &= \sin\left(\frac{\pi}{3}\right) \\ &= \sin\left(\frac{2\pi}{3} - \frac{\pi}{3}\right)\end{aligned}$$

(b)

$$\begin{aligned}\sin\frac{\pi}{4}\cos\frac{3\pi}{4} - \cos\frac{\pi}{4}\sin\frac{3\pi}{4} &= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} \\ &= -\frac{1}{2} - \frac{1}{2} \\ &= -1 \\ &= \sin\left(-\frac{\pi}{2}\right) \\ &= \sin\left(\frac{\pi}{4} - \frac{3\pi}{4}\right)\end{aligned}$$

(c)

$$\begin{aligned}
\sin\left(-\frac{\pi}{6}\right)\cos\frac{3\pi}{2} - \cos\left(-\frac{\pi}{6}\right)\sin\frac{3\pi}{2} &= -\frac{1}{2}\cdot 0 - \frac{\sqrt{3}}{2}(-1) \\
&= \frac{\sqrt{3}}{2} \\
&= \sin\left(\frac{\pi}{3}\right) \\
&= \sin\left(-\frac{5\pi}{3}\right) \\
&= \sin\left(-\frac{\pi}{6} - \frac{3\pi}{2}\right)
\end{aligned}$$

5. Applying the identity

$$(A-B)^2 + (A+B)^2 = A^2 - 2AB + B^2 + A^2 + 2AB + B^2 = 2A^2 + 2B^2,$$

we have that,

$$\begin{aligned}
\cos^2(\gamma+\delta) + \cos^2(\gamma-\delta) &= (\cos\gamma\cos\delta - \sin\gamma\sin\delta)^2 + (\cos\gamma\cos\delta + \sin\gamma\sin\delta)^2 \\
&= 2\cos^2\gamma\cos^2\delta + 2\sin^2\gamma\sin^2\delta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\cos^2\alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) &= \cos^2\alpha + 2\cos^2\frac{2\pi}{3}\cos^2\alpha + 2\sin^2\frac{2\pi}{3}\sin^2\alpha \\
&= \cos^2\alpha + 2\left(\frac{1}{4}\right)\cos^2\alpha + 2\left(\frac{3}{4}\right)\sin^2\alpha \\
&= \frac{3}{2}\left(\cos^2\alpha + \sin^2\alpha\right) \\
&= \frac{3}{2}
\end{aligned}$$

6.

$$\begin{aligned}
\sin(x+y) + \sin(x-y) &= \sin x\cos y + \cos x\sin y + \sin x\cos y - \cos x\sin y \\
&= \sin x\cos y + \sin x\cos y \\
&= 2\sin x\cos y
\end{aligned}$$

7.

$$\begin{aligned}
\cos(x+y) + \cos(x-y) &= \cos x\cos y - \sin x\sin y + \cos x\cos y + \sin x\sin y \\
&= \cos x\cos y + \cos x\cos y \\
&= 2\cos x\cos y
\end{aligned}$$

8. Since $(A-B)(A+B)=A^2-B^2$,

$$\begin{aligned}\cos(x+y)\cos(x-y) &= (\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)\\ &= \cos^2 x\cos^2 y-\sin^2 x\sin^2 y\end{aligned}$$

9. Since $(A+B)(A-B)=A^2-B^2$,

$$\begin{aligned}\sin(x+y)\sin(x-y) &= (\sin x\cos y+\cos x\sin y)(\sin x\cos y-\cos x\sin y)\\ &= \sin^2 x\cos^2 y-\cos^2 x\sin^2 y\end{aligned}$$

10.

$$\begin{aligned}\cos(x+y)\cos(x-y)-\sin(x+y)\sin(x-y) &= \cos^2 x\cos^2 y-\sin^2 x\sin^2 y-\left(\sin^2 x\cos^2 y-\cos^2 x\sin^2 y\right)\\ &= \cos^2 x\left(\cos^2 y+\sin^2 y\right)-\sin^2 x\left(\sin^2 y+\cos^2 y\right)\\ &= \cos^2 x-\sin^2 x\end{aligned}$$

11.

$$\begin{aligned}\cos 2x &= \cos(x+x)\\ &= \cos x\cos x-\sin x\sin x\\ &= \cos^2 x-\sin^2 x\end{aligned}$$

There is no error, because $\cos 2x=\cos^2 x-\sin^2 x$.

12.

$$\begin{aligned}\cos(\alpha+\beta)\cos\beta+\sin(\alpha+\beta)\sin\beta &= (\cos\alpha\cos\beta-\sin\alpha\sin\beta)\cos\beta+(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\sin\beta\\ &= \cos\alpha\cos^2\beta-\sin\alpha\sin\beta\cos\beta+\sin\alpha\sin\beta\cos\beta+\sin^2\beta\cos\alpha\\ &= \cos\alpha\cos^2\beta+\sin^2\beta\cos\alpha\\ &= \cos\alpha\left(\cos^2\beta+\sin^2\beta\right)\\ &= \cos\alpha\end{aligned}$$

Alternatively, by applying the cosine difference formula in reverse,

$$\cos(\alpha+\beta)\cos\beta+\sin(\alpha+\beta)\sin\beta=\cos(\alpha+\beta-\beta)\;=\cos\alpha$$

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1.

$$\begin{aligned}
\tan\left(\frac{7\pi}{6}+\frac{5\pi}{3}\right) &= \frac{\tan\frac{7\pi}{6}+\tan\frac{5\pi}{3}}{1-\tan\frac{7\pi}{6}\tan\frac{5\pi}{3}} \\
&= \frac{1/\sqrt{3}-\sqrt{3}}{1-\left(1/\sqrt{3}\right)\left(-\sqrt{3}\right)} \\
&= \frac{1/\sqrt{3}-\sqrt{3}}{2} \\
&= \frac{1/\sqrt{3}-\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{1-3}{2\sqrt{3}} \\
&= -\frac{1}{\sqrt{3}} \\
&= \tan\left(-\frac{\pi}{6}\right) \\
&= \tan\left(\frac{17\pi}{6}\right) \\
&= \tan\left(\frac{7\pi}{6}+\frac{5\pi}{3}\right)
\end{aligned}$$

2. Because the tangent function is odd, we know $\tan -\beta = -\tan\beta$.

$$\begin{aligned}
\tan(\alpha-\beta) &= \tan(\alpha+(-\beta)) \\
&= \frac{\tan\alpha+\tan -\beta}{1-\tan\alpha\tan -\beta} \\
&= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

3.

$$\begin{aligned}
\tan\left(\frac{\pi}{4}+\alpha\right) &= \frac{\tan\frac{\pi}{4}+\tan\alpha}{1-\tan\frac{\pi}{4}\tan\alpha} \\
&= \frac{1+\tan\alpha}{1-\tan\alpha}
\end{aligned}$$

4.

$$\tan\left(\frac{\pi}{4}-\alpha\right) = \frac{\tan\frac{\pi}{4}-\tan\alpha}{1+\tan\frac{\pi}{4}\tan\alpha}$$
$$= \frac{1-\tan\alpha}{1+\tan\alpha}$$

5. Since $\beta = \pi/4 - \alpha$,

$$\begin{aligned}
(1+\tan\alpha)(1+\tan\beta) &= (1+\tan\alpha)\left(1+\tan\left(\frac{\pi}{4}-\alpha\right)\right) \\
&= (1+\tan\alpha)\left(1+\frac{1-\tan\alpha}{1+\tan\alpha}\right) \\
&= 1+\tan\alpha+1-\tan\alpha \\
&= 2
\end{aligned}$$

Alternatively, since $\tan(\alpha+\beta) = \tan\pi/4 = 1$,

$$\begin{aligned}
(1+\tan\alpha)(1+\tan\beta) &= 1+\tan\alpha+\tan\beta+\tan\alpha\tan\beta \\
&= 1+(\tan\alpha+\tan\beta)\left(\frac{1-\tan\alpha\tan\beta}{1-\tan\alpha\tan\beta}\right)+\tan\alpha\tan\beta \\
&= 1+(1-\tan\alpha\tan\beta)\left(\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right)+\tan\alpha\tan\beta \\
&= 1+(1-\tan\alpha\tan\beta)\tan(\alpha+\beta)+\tan\alpha\tan\beta \\
&= 1+1-\tan\alpha\tan\beta+\tan\alpha\tan\beta \\
&= 2
\end{aligned}$$

6.

$$\begin{aligned}
\tan(\alpha+\beta+\gamma) &= \frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma} \\
&= \frac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\tan\gamma} \\
&= \frac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\tan\gamma}\cdot\frac{1-\tan\alpha\tan\beta}{1-\tan\alpha\tan\beta} \\
&= \frac{\tan\alpha+\tan\beta+\tan\gamma(1-\tan\alpha\tan\beta)}{1-\tan\alpha\tan\beta-(\tan\alpha+\tan\beta)\tan\gamma} \\
&= \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\alpha\tan\gamma-\tan\beta\tan\gamma}
\end{aligned}$$

7. Since $\tan(\alpha+\beta+\gamma) = \tan\pi = 0$, from the previous part, we have

   $$\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma = 0 \implies \tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma.$$

   This is because if a fraction equals zero, its numerator must be zero.

8. First, applying the tangent addition formula,

   $$\tan 3\alpha = \tan(2\alpha+\alpha) = \frac{\tan 2\alpha + \tan\alpha}{1-\tan 2\alpha\tan\alpha} \implies \tan 2\alpha+\tan\alpha = \tan 3\alpha\,(1-\tan 2\alpha\tan\alpha).$$

   Therefore,

   $$\tan 3\alpha-\tan 2\alpha-\tan\alpha = \tan 3\alpha-\tan 3\alpha\,(1-\tan 2\alpha\tan\alpha) = \tan 3\alpha\tan 2\alpha\tan\alpha.$$

   $\tan\alpha$ is not defined for $\alpha = \frac{\pi}{2}+n\pi$. Therefore, $\tan 2\alpha$ is not defined for $\alpha = \frac{\pi}{4}+n\frac{\pi}{2}$ and $\tan 3\alpha$ is not defined for $\alpha = \frac{\pi}{6}+n\frac{\pi}{3}$.

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1. (a) Since $\sin^2\alpha+\cos^2\alpha = 1$, we know $\cos\alpha = \sqrt{1-(\frac{7}{25})^2}$ (and cannot be the negative version because $\cos\alpha$ is given as positive).
   Thus $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{7}{25}\cdot\sqrt{1-(\frac{7}{25})^2} = \frac{336}{625}$.
   And $\cos 2\alpha = 1-2\sin^2\alpha = 1-2(\frac{7}{25})^2 = \frac{527}{625}$.

   (b) This part is similar except that we use the negative version of cosine, namely $\cos\alpha = -\sqrt{1-(\frac{7}{25})^2}$.
   Thus $\sin 2\alpha = 2\sin\alpha\cos\alpha = -\frac{336}{625}$.
   However, cosine value remains the same: $\cos 2\alpha = 1-2\sin^2\alpha = 1-2(\frac{7}{25})^2 = \frac{527}{625}$.

2. Firstly, $\sin 2\alpha = 2\sin\alpha\cos\alpha$. In other words, the sine value will be a product of rational numbers, so it will also be rational. Similarly, $\cos 2\alpha = 2\cos^2\alpha-1$ will be rational because $\cos\alpha$ is rational. Exercise 1 confirms this result.

3. Let's use the double angle formula $\cos 2\alpha = 2\cos^2\alpha-1$.

   $$\cos 2\alpha = \cos^2\alpha \implies 2\cos^2\alpha-1 = \cos^2\alpha \implies \cos^2\alpha = 1 \implies \cos\alpha = \pm 1$$

   $\cos\alpha$ has a magnitude of 1 precisely when α is an integer multiple of π, so the student's angle must have also been an integer multiple of π.

4. We start with the given equation:

   $$\begin{aligned} \sin\alpha+\cos\alpha &= 0.2 \\ \sin^2\alpha+2\sin\alpha\cos\alpha+\cos^2\alpha &= 0.04 && \text{(Squared both sides.)} \\ 1+2\sin\alpha\cos\alpha &= 0.04 \\ 2\sin\alpha\cos\alpha &= -0.96 \end{aligned}$$

   Note that that is simply $\sin 2\alpha$. Hooray!

5. We can follow a very similar strategy here to find $1 - 2\sin\alpha\cos\alpha = 0.09$. Then $\sin 2\alpha = 2\sin\alpha\cos\alpha = 0.91$.

6.
$$\begin{aligned}
\cos 2\alpha\cos\alpha + \sin 2\alpha\sin\alpha &= \left(2\cos^2\alpha - 1\right)\cos\alpha + 2\sin\alpha\cos\alpha\sin\alpha \\
&= \cos\alpha\left(2\cos^2\alpha - 1 + 2\sin^2\alpha\right) \\
&= \cos\alpha\,(2-1) \\
&= \cos\alpha
\end{aligned}$$

   Alternatively, we may use the cosine difference formula.

$$\cos 2\alpha\cos\alpha + \sin 2\alpha\sin\alpha = \cos\left(2\alpha - \alpha\right) = \cos\alpha$$

7. Applying the sine addition formula,

$$\sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha = \sin\left(2\alpha + \alpha\right) = \sin 3\alpha$$

   Applying the sine subtraction formula,

$$\sin 4\alpha\cos\alpha - \cos 4\alpha\sin\alpha = \sin\left(4\alpha - \alpha\right) = \sin 3\alpha$$

   Since both sides of the identity are equal to $\sin 3\alpha$, the identity is true.

8. Yes, the book asked you to prove something incorrect! For counterexample, consider that $\cos 2\alpha$ can be negative but $\cos^2\alpha$ is never negative. However, we can prove a relationship.
   Recall that $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$. However, $\sin^2\alpha \geq 0$ since it's a square.

$$\begin{aligned}
\sin^2\alpha &\geq 0 \\
-\sin^2\alpha &\leq 0 \\
\cos^2\alpha - \sin^2\alpha &\leq \cos^2\alpha \\
\cos 2\alpha &\leq \cos^2\alpha
\end{aligned}$$

9.
$$\begin{aligned}
\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2 &= \sin^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} \\
&= 1 - \sin\alpha
\end{aligned}$$

10. Following the hint, we compute the value of $\cos 10^\circ \sin 10^\circ \sin 50^\circ \sin 70^\circ$.

$$\begin{aligned}
M \cos 10^\circ &= \cos 10^\circ \sin 10^\circ \sin 50^\circ \sin 70^\circ \\
&= \frac{1}{2} \sin 20^\circ \sin 50^\circ \sin 70^\circ \\
&= \frac{1}{2} \cos 70^\circ \sin 50^\circ \sin 70^\circ \\
&= \frac{1}{4} \sin 140^\circ \sin 50^\circ \\
&= \frac{1}{4} \sin 40^\circ \sin 50^\circ \\
&= \frac{1}{4} \cos 50^\circ \sin 50^\circ \\
&= \frac{1}{8} \sin 100^\circ \\
&= \frac{1}{8} \sin 80^\circ \\
&= \frac{1}{8} \cos 10^\circ
\end{aligned}$$

Since $M \cos 10^\circ = \frac{1}{8} \cos 10^\circ$, we have that the value of the original expression M is equal to $\frac{1}{8}$.

11. We begin by computing $\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ$.

$$\begin{aligned}
\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ &= \frac{1}{2} \sin 40^\circ \cos 40^\circ \cos 80^\circ \\
&= \frac{1}{4} \sin 80^\circ \cos 80^\circ \\
&= \frac{1}{8} \sin 160^\circ \\
&= \frac{1}{8} \sin 20^\circ
\end{aligned}$$

This implies that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = 1/8$.

12. We begin by computing $\cos \pi/10 \sin \pi/10 \sin \pi/5$.

$$\begin{aligned}
\cos \frac{\pi}{10} \sin \frac{\pi}{10} \cos \frac{\pi}{5} &= \frac{1}{2} \sin \frac{\pi}{5} \cos \frac{\pi}{5} \\
&= \frac{1}{4} \sin \frac{2\pi}{5} \\
&= \frac{1}{4} \cos \frac{\pi}{10}
\end{aligned}$$

This implies that $\sin \pi/10 \cos \pi/5 = 1/4$.

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1.

$$\begin{aligned}
\cos 3\alpha &= \cos(2\alpha+\alpha)\\
&= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha\\
&= \left(2\cos^2\alpha - 1\right)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha\\
&= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha\\
&= 2\cos^3\alpha - \cos\alpha - 2\left(1-\cos^2\alpha\right)\cos\alpha\\
&= 4\cos^3\alpha - 3\cos\alpha
\end{aligned}$$

2.

$$\begin{aligned}
\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha\\
&= 3\left(\frac{3}{5}\right) - 4\left(\frac{3}{5}\right)^3\\
&= \frac{9}{5} - 4\cdot\frac{27}{125}\\
&= \frac{117}{125}
\end{aligned}$$

If $\sin\alpha = 3/5$, then $\cos\alpha = \pm 4/5$. Therefore,

$$\begin{aligned}
\cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha\\
&= 4\left(\pm\frac{4}{5}\right)^3 - 3\left(\pm\frac{4}{5}\right)\\
&= \pm\frac{256}{125} \mp \frac{12}{5}\\
&= \mp\frac{44}{125}
\end{aligned}$$

3. If $\cos\alpha = 4/5$, then $\sin\alpha = \pm 3/5$. Therefore,

$$\begin{aligned}
\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha\\
&= 3\left(\pm\frac{3}{5}\right) - 4\left(\pm\frac{3}{5}\right)^3\\
&= \pm\frac{9}{5} \mp 4\cdot\frac{27}{125}\\
&= \pm\frac{117}{125}
\end{aligned}$$

$$
\begin{aligned}
\cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha \\
&= 4\left(\frac{4}{5}\right)^3 - 3\left(\frac{4}{5}\right) \\
&= \frac{256}{125} - \frac{12}{5} \\
&= -\frac{44}{125}
\end{aligned}
$$

4. (a)

$$
\begin{aligned}
\cos 4\alpha &= 2\cos^2 2\alpha - 1 \\
&= 2\left(2\cos^2\alpha - 1\right)^2 - 1 \\
&= 2\left(4\cos^4\alpha - 4\cos^2\alpha + 1\right) - 1 \\
&= 8\cos^4\alpha - 8\cos^2\alpha + 1
\end{aligned}
$$

(b)

$$
\begin{aligned}
\cos 4\alpha &= 1 - 2\sin^2 2\alpha \\
&= 1 - 8\sin^2\alpha\cos^2\alpha \\
&= 1 - 8\sin^2\alpha\left(1 - \sin^2\alpha\right) \\
&= 1 - 8\sin^2\alpha + 8\sin^4\alpha
\end{aligned}
$$

5.

$$
\begin{aligned}
\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha &= \left(3\sin\alpha - 4\sin^3\alpha\right)\cos\alpha - \left(4\cos^3\alpha - 3\cos\alpha\right)\sin\alpha \\
&= 3\sin\alpha\cos\alpha - 4\sin^3\alpha\cos\alpha - 4\cos^3\alpha\sin\alpha + 3\sin\alpha\cos\alpha \\
&= 2\sin\alpha\cos\alpha\left(3 - 2\sin^2\alpha - 2\cos^2\alpha\right) \\
&= 2\sin\alpha\cos\alpha\,(3 - 2) \\
&= \sin 2\alpha
\end{aligned}
$$

Alternatively, applying the sine difference formula,

$$
\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha = \sin(3\alpha - \alpha) = \sin 2\alpha
$$

6.

$$
\begin{aligned}
\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha} &= \frac{3\sin\alpha - 4\sin^3\alpha}{\sin\alpha} - \frac{4\cos^3\alpha - 3\cos\alpha}{\cos\alpha} \\
&= 3 - 4\sin^2\alpha - \left(4\cos^2\alpha - 3\right) \\
&= 6 - 4\sin^2\alpha - 4\cos^2\alpha \\
&= 2
\end{aligned}
$$

Alternatively, using the result from the previous exercise,

$$\begin{aligned}
\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha} &= \frac{\sin 3\alpha \cos \alpha - \cos 3\alpha \sin \alpha}{\sin \alpha \cos \alpha} \\
&= \frac{\sin 2\alpha}{\sin \alpha \cos \alpha} \\
&= \frac{2 \sin \alpha \cos \alpha}{\sin \alpha \cos \alpha} \\
&= 2
\end{aligned}$$

7. (a) Applying the result of Exercise 9 from Section 3 of this chapter,

$$\begin{aligned}
4 \sin \alpha \sin (60^\circ + \alpha) \sin (60^\circ - \alpha) &= 4 \sin \alpha \left(\sin^2 60^\circ \cos^2 \alpha - \cos^2 60^\circ \sin^2 \alpha\right) \\
&= 4 \sin \alpha \left(\frac{3}{4} \cos^2 \alpha - \frac{1}{4} \sin^2 \alpha\right) \\
&= 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha \\
&= 3 \sin \alpha \left(1 - \sin^2 \alpha\right) - \sin^3 \alpha \\
&= 3 \sin \alpha - 4 \sin^3 \alpha \\
&= \sin 3\alpha
\end{aligned}$$

(b) Applying the result of Exercise 8 from Section 3 of this chapter,

$$\begin{aligned}
4 \cos \alpha \cos (60^\circ + \alpha) \cos (60^\circ - \alpha) &= 4 \cos \alpha \left(\cos^2 60^\circ \cos^2 \alpha - \sin^2 60^\circ \sin^2 \alpha\right) \\
&= 4 \cos \alpha \left(\frac{1}{4} \cos^2 \alpha - \frac{3}{4} \sin^2 \alpha\right) \\
&= \cos^3 \alpha - 3 \sin^2 \alpha \cos \alpha \\
&= \cos^3 \alpha - 3 \left(1 - \cos^2 \alpha\right) \cos \alpha \\
&= 4 \cos^3 \alpha - 3 \cos \alpha
\end{aligned}$$

8.

$$\begin{aligned}
\sin 4\alpha &= 2 \sin 2\alpha \cos 2\alpha \\
&= 4 \sin \alpha \cos \alpha \left(2 \cos^2 \alpha - 1\right) \\
&= 8 \sin \alpha \cos^3 \alpha - 4 \sin \alpha \cos \alpha \\
\implies \frac{\sin 4\alpha}{\sin \alpha} &= 8 \cos^3 \alpha - 4 \cos \alpha
\end{aligned}$$

9. In the penultimate step, we apply the following identity:

$$(A - B)^3 = A^3 - 3A^2 B + 3AB^2 - B^3,$$

taking $A = \cos^2 \alpha$ and $B = \sin^2 \alpha$.

$$
\begin{aligned}
\sin 3\alpha \sin^3\alpha + \cos 3\alpha \cos^3\alpha &= \left(3\sin\alpha - 4\sin^3\alpha\right)\sin^3\alpha + \left(4\cos^3\alpha - 3\cos\alpha\right)\cos^3\alpha \\
&= 3\sin^4\alpha - 4\sin^6\alpha + 4\cos^6\alpha - 3\cos^4\alpha \\
&= 3\sin^4\alpha - 4\sin^4\alpha\left(1-\cos^2\alpha\right) + 4\cos^4\alpha\left(1-\sin^2\alpha\right) - 3\cos^4\alpha \\
&= \sin^4\alpha\left(4\cos^2\alpha - 1\right) + \cos^4\alpha\left(1 - 4\sin^2\alpha\right) \\
&= \sin^4\alpha\left(4\cos^2\alpha - \sin^2\alpha - \cos^2\alpha\right) + \cos^4\alpha\left(\sin^2\alpha + \cos^2\alpha - 4\sin^2\alpha\right) \\
&= 3\sin^4\alpha\cos^2\alpha - \sin^6\alpha + \cos^6\alpha - 3\cos^4\alpha\sin^2\alpha \\
&= \left(\cos^2\alpha - \sin^2\alpha\right)^3 \\
&= \cos^3 2\alpha
\end{aligned}
$$

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1.

$$\cos\alpha = 1 \implies \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+1}{2}} = \pm 1$$

$\cos\alpha$ is equal to 1 when α is an integer multiple of 2π. If $\alpha = 4n\pi$ for some integer n (i.e., an even integer multiple of 2π), then $\cos\alpha/2 = \cos 2n\pi = 1$. Otherwise, if $\alpha = (4n+2)\,\pi$ for some integer n (i.e., an odd integer multiple of 2π), then $\cos\alpha/2 = \cos(2n+1)\,\pi = -1$.

For a particular example of each case, we may set $n = 0$.

$\alpha = 4\,(0)\,\pi = 0$:

$$\cos 0 = 1, \cos\frac{0}{2} = \cos 0 = 1$$

$\alpha = (4\,(0) + 2)\,\pi = 2\pi$:

$$\cos 2\pi = 1, \cos\frac{2\pi}{2} = \cos\pi = -1$$

2. (a) We take the positive square root because $60°/2 = 30°$ lies in the first quadrant.

$$
\begin{aligned}
\cos\frac{60°}{2} &= \sqrt{\frac{1+\cos 60°}{2}} \\
&= \sqrt{\frac{1+1/2}{2}} \\
&= \sqrt{\frac{3}{4}} \\
&= \frac{\sqrt{3}}{2}
\end{aligned}
$$

(b) We take the positive square root because $120°/2 = 60°$ lies in the first quadrant.

$$\begin{aligned}
\cos\frac{120°}{2} &= \sqrt{\frac{1+\cos 120°}{2}} \\
&= \sqrt{\frac{1-1/2}{2}} \\
&= \sqrt{\frac{1}{4}} \\
&= \frac{1}{2}
\end{aligned}$$

(c) We take the negative square root because $240°/2 = 120°$ lies in the second quadrant.

$$\begin{aligned}
\cos\frac{240°}{2} &= -\sqrt{\frac{1+\cos 240°}{2}} \\
&= -\sqrt{\frac{1-1/2}{2}} \\
&= -\sqrt{\frac{1}{4}} \\
&= -\frac{1}{2}
\end{aligned}$$

3.

α	Quadrant α?	$\alpha/2$	Quadrant $\alpha/2$?	$\cos\alpha/2$
$780°$	I	$390°$	I	$\sqrt{3}/2$
$1020°$	IV	$510°$	II	$-\sqrt{3}/2$
$1140°$	I	$570°$	III	$-\sqrt{3}/2$
$1380°$	IV	$690°$	IV	$\sqrt{3}/2$
$-60°$	IV	$-30°$	IV	$\sqrt{3}/2$
$-300°$	I	$-150°$	III	$-\sqrt{3}/2$
$-420°$	IV	$-210°$	II	$-\sqrt{3}/2$
$-660°$	I	$-330°$	I	$\sqrt{3}/2$
$-780°$	IV	$-390°$	IV	$\sqrt{3}/2$

4.

$$\begin{aligned}\sin 15° &= \sqrt{\frac{1-\cos 30°}{2}}\\ &= \sqrt{\frac{1-\sqrt{3}/2}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

$$\begin{aligned}\cos 15° &= \sqrt{\frac{1+\cos 30°}{2}}\\ &= \sqrt{\frac{1+\sqrt{3}/2}{2}}\\ &= \sqrt{\frac{2+\sqrt{3}}{4}}\\ &= \frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

5. Because $|\cos\alpha| \le 1$, $1 \pm \cos\alpha \ge 0$, so the expressions under the square roots in the sine and cosine half-angle formulas will not be negative.

6. The square root sign in the half-angle formula prevents it from being a rational trigonometric function, so the Principle of Analytic Continuation does not apply.

7. (a)

$$\begin{aligned}
\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2} &= \tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\left(\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}\right)\\
&= \tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha+\beta}{2}\left(1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)\\
&= \tan\frac{\alpha}{2}\tan\frac{\beta}{2}+1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\\
&= 1
\end{aligned}$$

Alternatively, we can recall the extended tangent addition formula derived in Exercise 6 of Section 4 of this chapter:

$$\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\alpha\tan\gamma-\tan\beta\tan\gamma}$$

Since $\alpha/2+\beta/2+\gamma/2=\pi/2$, $\tan(\alpha/2+\beta/2+\gamma/2)$ is undefined. This implies that the denominator of the tangent addition formula is 0 (assuming that none of $\tan\alpha/2$, $\tan\beta/2$, or $\tan\gamma/2$ are undefined). Therefore, we can conclude

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=1.$$

(b) Following a similar approach as to the previous part, we begin by noting that $\sin\frac{\alpha+\beta}{2}=\cos\frac{\gamma}{2}$.

$$\begin{aligned}
4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} &= 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\alpha+\beta}{2}\\
&= 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\\
&= 4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos^2\frac{\beta}{2}+4\sin\frac{\beta}{2}\cos\frac{\beta}{2}\cos^2\frac{\alpha}{2}\\
&= 2\sin\alpha\cos^2\frac{\beta}{2}+2\sin\beta\cos^2\frac{\alpha}{2}\\
&= 2\sin\alpha\left(\frac{1+\cos\beta}{2}\right)+2\sin\beta\left(\frac{1+\cos\alpha}{2}\right)\\
&= \sin\alpha+\sin\alpha\cos\beta+\sin\beta+\sin\beta\cos\alpha\\
&= \sin\alpha+\sin\beta+\sin(\alpha+\beta)\\
&= \sin\alpha+\sin\beta+\sin(\pi-\alpha-\beta)\\
&= \sin\alpha+\sin\beta+\sin\gamma
\end{aligned}$$

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1. Because $1 + \cos\alpha$ is non-negative, division by $1 + \cos\alpha$ does not change the sign of $\sin\alpha$, which means $\tan(\alpha/2)$ and $\sin\alpha/(1 + \cos\alpha)$ have the same sign.

2.

$$\begin{aligned}
\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\
&= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}\cdot\frac{1-\cos\alpha}{1-\cos\alpha}} \\
&= \pm\sqrt{\frac{(1-\cos\alpha)^2}{1-\cos^2\alpha}} \\
&= \pm\sqrt{\frac{(1-\cos\alpha)^2}{\sin^2\alpha}} \\
&= \pm\frac{1-\cos\alpha}{\sin\alpha}
\end{aligned}$$

For acute angles α, we need to take the positive branch of the square root so that the signs of both sides of the half-angle formula agree. By the Principle of Analytic Continuation, since the positive branch is correct for all acute angles and both sides of the formula are rational trigonometric expressions, it is correct for all angles in general, so we have

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

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1.

$$\begin{aligned}
\frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta) &= \frac{1}{2}(\cos\alpha\cos\beta + \sin\alpha\sin\beta - \cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= \frac{1}{2}(2\sin\alpha\sin\beta) \\
&= \sin\alpha\sin\beta
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta) &= \frac{1}{2}(\sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta) \\
&= \frac{1}{2}(2\sin\alpha\cos\beta) \\
&= \sin\alpha\cos\beta
\end{aligned}$$

2.

$$\begin{aligned}
\sin 75^\circ \sin 15^\circ &= \frac{1}{2}\cos\left(75^\circ - 15^\circ\right) - \frac{1}{2}\cos\left(75^\circ + 15^\circ\right) \\
&= \frac{1}{2}\cos 60^\circ - \frac{1}{2}\cos 90^\circ \\
&= \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot 0 \\
&= \frac{1}{4}
\end{aligned}$$

3.

$$\begin{aligned}
\sin 75^\circ \cos 15^\circ &= \frac{1}{2}\sin\left(75^\circ + 15^\circ\right) + \frac{1}{2}\sin\left(75^\circ - 15^\circ\right) \\
&= \frac{1}{2}\sin 90^\circ + \frac{1}{2}\sin 60^\circ \\
&= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} \\
&= \frac{2+\sqrt{3}}{4}
\end{aligned}$$

4. (a)

$$\begin{aligned}
\cos 75^\circ \cos 15^\circ &= \frac{1}{2}\cos\left(75^\circ + 15^\circ\right) + \frac{1}{2}\cos\left(75^\circ - 15^\circ\right) \\
&= \frac{1}{2}\cos 90^\circ + \frac{1}{2}\cos 60^\circ \\
&= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

Alternatively, by the cosine subtraction formula, $\cos 60^\circ = \cos 75^\circ \cos 15^\circ + \sin 75^\circ \sin 15^\circ$ so

$$\cos 75^\circ \cos 15^\circ = \cos 60^\circ - \sin 75^\circ \sin 15^\circ = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Alternatively, using the idea of sine and cosine being cofunctions,

$$\cos 75^\circ \cos 15^\circ = \sin\left(90^\circ - 75^\circ\right)\sin\left(90^\circ - 15^\circ\right) = \sin 15^\circ \sin 75^\circ = \frac{1}{4}$$

For the above two alternative solutions, we apply the result from Exercise 2 above that $\sin 75^\circ \sin 15^\circ = 1/4$.

(b)

$$
\begin{aligned}
\cos 75^\circ \sin 15^\circ &= \frac{1}{2}\sin(15^\circ + 75^\circ) + \frac{1}{2}\sin(15^\circ - 75^\circ) \\
&= \frac{1}{2}\sin 90^\circ - \frac{1}{2}\sin 60^\circ \\
&= \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} \\
&= \frac{2-\sqrt{3}}{4}
\end{aligned}
$$

Alternatively, by the sine addition formula, $\sin 90^\circ = \sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$ so

$$
\cos 75^\circ \sin 15^\circ = \sin 90^\circ - \sin 75^\circ \cos 15^\circ = 1 - \frac{2+\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4}
$$

For the above alternative solution, we apply the result from Exercise 3 above that $\sin 75^\circ \cos 15^\circ = \left(2+\sqrt{3}\right)/4$.

5.

$$
\begin{aligned}
2\cos\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}-\alpha\right) &= \cos\left(\frac{\pi}{4}+\alpha+\frac{\pi}{4}-\alpha\right) + \cos\left(\frac{\pi}{4}+\alpha-\frac{\pi}{4}+\alpha\right) \\
&= \cos\frac{\pi}{2}\cos 2\alpha \\
&= \cos 2\alpha
\end{aligned}
$$

Alternatively, applying the result of Exercise 8 in Section 3 of this chapter,

$$
\begin{aligned}
2\cos\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}-\alpha\right) &= 2\left(\cos^2\frac{\pi}{4}\cos^2\alpha - \sin^2\frac{\pi}{4}\sin^2\alpha\right) \\
&= 2\left(\frac{1}{2}\cos^2\alpha - \frac{1}{2}\sin^2\alpha\right) \\
&= \cos^2\alpha - \sin^2\alpha \\
&= \cos 2\alpha
\end{aligned}
$$

6.

$$
\begin{aligned}
&\sin(\alpha+\beta)\sin(\alpha-\beta) + \sin(\beta+\gamma)\sin(\beta-\gamma) + \sin(\gamma+\alpha)\sin(\gamma-\alpha) \\
&= \frac{1}{2}\cos 2\beta - \frac{1}{2}\cos 2\alpha + \frac{1}{2}\cos 2\gamma - \frac{1}{2}\cos 2\beta + \frac{1}{2}\cos 2\alpha - \frac{1}{2}\cos 2\gamma \\
&= 0
\end{aligned}
$$

7.

$$
\begin{aligned}
&\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta)\\
&=\frac{1}{2}\cos(\alpha-\beta+\gamma)-\frac{1}{2}\cos(\alpha+\beta-\gamma)+\frac{1}{2}\cos(\beta-\gamma+\alpha)-\frac{1}{2}\cos(\beta+\gamma-\alpha)\\
&+\frac{1}{2}\cos(\gamma-\alpha+\beta)-\frac{1}{2}\cos(\gamma+\alpha-\beta)\\
&=0
\end{aligned}
$$

Page 155

1. Recall from the previous section that $\cos(\gamma+\delta)+\cos(\gamma-\delta)=2\cos\gamma\cos\delta$. Following Example 54, we let $\gamma=(\alpha+\beta)/2$ and $\delta=(\alpha-\beta)/2$. Substituting for γ and δ, we arrive at the first formula,

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Similarly, we recall that $\cos(\gamma-\delta)-\cos(\gamma+\delta)=2\sin\gamma\sin\delta$. Performing the same substitution as above, we obtain

$$\cos\beta-\cos\alpha=2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\iff\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

2.

$$
\begin{aligned}
\cos 70^\circ+\sin 40^\circ&=\sin 20^\circ+\sin 40^\circ\\
&=2\sin\frac{20^\circ+40^\circ}{2}\cos\frac{20^\circ-40^\circ}{2}\\
&=2\sin 30^\circ\cos(-10^\circ)\\
&=2\cdot\frac{1}{2}\cos 10^\circ\\
&=\cos 10^\circ
\end{aligned}
$$

3.

$$
\begin{aligned}
\cos 55^\circ+\cos 65^\circ&=2\cos\frac{55^\circ+65^\circ}{2}\cos\frac{55^\circ-65^\circ}{2}\\
&=2\cos 60^\circ\cos(-5^\circ)\\
&=2\cdot\frac{1}{2}\cos 5^\circ\\
&=\cos 5^\circ
\end{aligned}
$$

4.

$$\begin{aligned}\cos 20^\circ + \cos 100^\circ + \cos 140^\circ &= 2\cos\frac{100^\circ + 20^\circ}{2}\cos\frac{100^\circ - 20^\circ}{2} + \cos 140^\circ\\ &= 2\cos 60^\circ \cos 40^\circ + \cos 140^\circ\\ &= 2\cdot\frac{1}{2}\cos 40^\circ + \cos 140^\circ\\ &= \cos 40^\circ + \cos 140^\circ\\ &= 2\cos\frac{140^\circ + 40^\circ}{2}\cos\frac{140^\circ - 40^\circ}{2}\\ &= 2\cos 90^\circ \cos 50^\circ\\ &= 0\end{aligned}$$

5.

$$\begin{aligned}\sin 78^\circ + \cos 132^\circ &= \sin 78^\circ - \cos 48^\circ\\ &= \sin 78^\circ - \sin 42^\circ\\ &= 2\cos\frac{78^\circ + 42^\circ}{2}\sin\frac{78^\circ - 42^\circ}{2}\\ &= 2\cos 60^\circ \sin 18^\circ\\ &= 2\cdot\frac{1}{2}\sin 18^\circ\\ &= \sin 18^\circ\end{aligned}$$

6.

$$\begin{aligned}\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ} &= \frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ}\\ &= \frac{2\sin 45^\circ \cos 30^\circ}{2\cos 45^\circ \sin 30^\circ}\\ &= \tan 45^\circ \cot 30^\circ\\ &= \sqrt{3}\end{aligned}$$

7. (a)

$$\sin(\alpha+\beta) = \sin(\pi - \alpha - \beta) = \sin\gamma$$

(b)

$$\cos(\alpha+\beta) = -\cos(\pi - \alpha - \beta) = -\cos\gamma$$

(c)

$$\begin{aligned}\sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 2\sin(\alpha+\beta)\cos(\alpha-\beta) + \sin 2\gamma \\ &= 2\sin\gamma\cos(\alpha-\beta) + 2\sin\gamma\cos\gamma \\ &= 2\sin\gamma\left(\cos(\alpha-\beta)+\cos\gamma\right) \\ &= 2\sin\gamma\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right) \\ &= 2\sin\gamma\left(2\sin\alpha\sin\beta\right) \\ &= 4\sin\alpha\sin\beta\sin\gamma\end{aligned}$$

8.

$$\begin{aligned}\sin\alpha + \sin\left(\alpha+\frac{2\pi}{3}\right) + \sin\left(\alpha+\frac{4\pi}{3}\right) &= 2\sin\left(\alpha+\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{3}\right) + \sin\left(\alpha+\frac{4\pi}{3}\right) \\ &= \sin\left(\alpha+\frac{\pi}{3}\right) + \sin\left(\alpha+\frac{4\pi}{3}\right) \\ &= 2\sin\left(\alpha+\frac{5\pi}{6}\right)\cos\left(-\frac{\pi}{2}\right) \\ &= 0\end{aligned}$$

9. We first note that

$$\sin k\alpha + \sin(k+2)\alpha = 2\sin\frac{(k+2)\alpha+k\alpha}{2}\cos\frac{(k+2)\alpha-k\alpha}{2} = 2\sin(k+1)\alpha\cos\alpha.$$

Therefore,

$$\begin{aligned}\sin\alpha + 2\sin 3\alpha + \sin 5\alpha &= \sin\alpha + \sin 3\alpha + \sin 3\alpha + \sin 5\alpha \\ &= 2\sin 2\alpha\cos\alpha + 2\sin 4\alpha\cos\alpha \\ &= 2\cos\alpha\left(\sin 2\alpha + \sin 4\alpha\right) \\ &= 2\cos\alpha\left(2\sin 3\alpha\cos\alpha\right) \\ &= 4\cos^2\alpha\sin 3\alpha.\end{aligned}$$

10.

$$\begin{aligned}&\frac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma} + \frac{\sin(\gamma-\alpha)}{\sin\gamma\sin\alpha} + \frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} \\ &\quad= \frac{\sin\beta\cos\gamma}{\sin\beta\sin\gamma} - \frac{\cos\beta\sin\gamma}{\sin\beta\sin\gamma} + \frac{\sin\gamma\cos\alpha}{\sin\gamma\sin\alpha} - \frac{\cos\gamma\sin\alpha}{\sin\gamma\sin\alpha} + \frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta} - \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} \\ &\quad= \cot\gamma - \cot\beta + \cot\alpha - \cot\gamma + \cot\beta - \cot\alpha \\ &\quad= 0\end{aligned}$$

This result also follows from the identity which was proven in Exercise 7 of the previous section. This can be seen by rewriting the left-hand side of the equation as a single fraction with a common denominator and noting that the numerator of this fraction is zero.

11.

$$\begin{aligned}\sin(\alpha-\beta)+\sin(\alpha-\gamma)+\sin(\beta-\gamma) &= 2\sin\frac{2\alpha-\beta-\gamma}{2}\cos\frac{\gamma-\beta}{2}+\sin(\beta-\gamma)\\ &= 2\sin\frac{2\alpha-\beta-\gamma}{2}\cos\frac{\beta-\gamma}{2}+2\sin\frac{\beta-\gamma}{2}\cos\frac{\beta-\gamma}{2}\\ &= 2\cos\frac{\beta-\gamma}{2}\left(\sin\frac{2\alpha-\beta-\gamma}{2}+\sin\frac{\beta-\gamma}{2}\right)\\ &= 2\cos\frac{\beta-\gamma}{2}\left(2\sin\frac{\alpha-\gamma}{2}\cos\frac{\alpha-\beta}{2}\right)\\ &= 4\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\cos\frac{\beta-\gamma}{2}\end{aligned}$$

12.

$$\begin{aligned}&\sin(\alpha+\beta+\gamma)+\sin(\alpha-\beta-\gamma)+\sin(\alpha+\beta-\gamma)+\sin(\alpha-\beta+\gamma)\\ &\quad= 2\sin\alpha\cos(\beta+\gamma)+\sin(\alpha+\beta-\gamma)+\sin(\alpha-\beta+\gamma)\\ &\quad= 2\sin\alpha\cos(\beta+\gamma)+2\sin\alpha\cos(\beta-\gamma)\\ &\quad= 2\sin\alpha\left(\cos(\beta+\gamma)+\cos(\beta-\gamma)\right)\\ &\quad= 2\sin\alpha\left(2\cos\beta\cos\gamma\right)\\ &\quad= 4\sin\alpha\cos\beta\cos\gamma\end{aligned}$$

Page 158

1. Let $\beta = 2\gamma$.

$$\begin{aligned}\sin^2\beta+\cos^2\beta &= \sin^2 2\gamma+\cos^2 2\gamma\\ &= \left(\frac{2a}{1+a^2}\right)^2+\left(\frac{1-a^2}{1+a^2}\right)^2\\ &= \frac{4a^2}{1+2a^2+a^4}+\frac{1-2a^2+a^4}{1+2a^2+a^4}\\ &= \frac{1+2a^2+a^4}{1+2a^2+a^4}\\ &= 1\end{aligned}$$

2.

$$\begin{aligned}\tan 2\beta &= \frac{2a}{1-a^2}\\ &= \frac{\frac{2a}{1+a^2}}{\frac{1-a^2}{1+a^2}}\\ &= \frac{\sin 2\beta}{\cos 2\beta}\end{aligned}$$

Page 160

1.

$$\sin\alpha = \frac{2\cdot 2\cdot 3}{2^2+3^2} = \frac{12}{13}$$
$$\cos\alpha = \frac{2^2-3^2}{2^2+3^2} = \frac{-5}{13}$$

These values give the Pythagorean triple 5, 12, 13, provided we take the absolute value of $\cos\alpha$. However, because $\cos\alpha$ is negative, α is not an acute angle, so it cannot correspond to an angle in a right triangle.

2.

$$\sin\alpha = \frac{2\cdot 8\cdot 5}{8^2+5^2} = \frac{80}{89}$$
$$\cos\alpha = \frac{8^2-5^2}{8^2+5^2} = \frac{39}{89}$$

This corresponds to the right triangle with legs of 39 and 80 and a hypotenuse of 89.

3.

$$(2pq)^2 + \left(q^2-p^2\right)^2 = 4p^2q^2 + q^4 - 2q^2p^2 + p^4 = q^4 + 2q^2p^2 + p^4 = \left(q^2+p^2\right)^2$$

The above shows that q^2+p^2 is the hypotenuse.

Page 161 (First)

1.
- $\sin 20^\circ \cos 20^\circ \approx 0.3214$
- $\sin 10^\circ \cos 10^\circ \approx 0.1710$
- $\sin 5^\circ \cos 5^\circ \approx 0.0868$
- $\sin 1^\circ \cos 1^\circ \approx 0.0174$
- $\sin 70^\circ \cos 70^\circ \approx 0.3214$
- $\sin 80^\circ \cos 80^\circ \approx 0.1710$
- $\sin 85^\circ \cos 85^\circ \approx 0.0868$
- $\sin 89^\circ \cos 89^\circ \approx 0.0174$

2.

$$\sin 30^\circ \cos 30^\circ = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$
$$\sin 45^\circ \cos 45^\circ = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{2}$$
$$\sin 60^\circ \cos 60^\circ = \frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{4}$$

Page 161 (Second)

1. In the below answers for this question, n represents an arbitrary integer.

   (a)

$$\begin{aligned}\sin x\cos x=\frac{1}{2} &\implies \frac{1}{2}\sin 2x=\frac{1}{2}\\ &\implies \sin 2x=1\\ &\implies 2x=\frac{\pi}{2}+2n\pi\\ &\implies x=\frac{\pi}{4}+n\pi\end{aligned}$$

   (b)

$$\sin x\cos x=\frac{\sqrt{3}}{2} \implies \frac{1}{2}\sin 2x=\frac{\sqrt{3}}{2} \implies \sin 2x=\sqrt{3}$$

   Because $\sqrt{3}>1$, there are no values of x which satisfy the given equation.

   (c)

$$\begin{aligned}\sin x\cos x=\frac{\sqrt{3}}{4} &\implies \frac{1}{2}\sin 2x=\frac{\sqrt{3}}{4}\\ &\implies \sin 2x=\frac{\sqrt{3}}{2}\\ &\implies 2x=\left(\frac{\pi}{2}\pm\frac{\pi}{6}\right)+2n\pi\\ &\implies x=\left(\frac{\pi}{4}\pm\frac{\pi}{12}\right)+n\pi\end{aligned}$$

2. (c) has no solution because $\sin x\cos x\le 0.5<0.6$.

3. $\sin x\cos x=N$ has a solution when $|N|\le 1/2$.

$$\sin x\cos x=N \implies \frac{1}{2}\sin 2x=N \implies \sin 2x=2N$$

   $2x=\arcsin 2N$ gives one solution to the above equation. Another (not necessarily distinct) solution is $2x=\pi-\arcsin 2N$. All solutions to the equation can be generated by adding an integer multiple of 2π to either of the above angles. Therefore, the general solution to $\sin x\cos x=N$ is

$$x=\frac{1}{2}\arcsin 2N+n\pi \text{ or } x=\frac{1}{2}(\pi-\arcsin 2N)+n\pi.$$

   Using the properties of cofunctions, this solution can be written in a slightly more compact manner as follows:

$$x=\left(\frac{\pi}{4}\pm\frac{1}{2}\arccos 2N\right)+n\pi$$

Page 162 (First)

1.

$$\sin 30^\circ + \cos 30^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2} > \frac{1}{2} + \frac{1}{2} = 1$$

2.

$$\sin 0 + \cos 0 = 1$$

3.

$$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

Page 162 (Second)

1. Yes, because $1.414 < \sqrt{2}$. (Note: $\sin x + \cos x$ must attain the value 1.414 for some x because of the Intermediate Value Theorem).

2. No, because $1.415 > \sqrt{2}$.

3. We square both sides of the equation to solve for x.

   $$\sin x + \cos x = \sqrt{2} \implies 1 + \sin 2x = 2 \implies \sin 2x = 1 \implies 2x = \frac{\pi}{2} + 2n\pi \implies x = \frac{\pi}{4} + n\pi$$

4. Since we know the maximum value of $(\sin x + \cos x)^2$ is 2, the minimum value of $\sin x + \cos x$ cannot be less than $-\sqrt{2}$. The value of $-\sqrt{2}$ is attained when $x = \frac{5\pi}{4}$.

   $$\sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$$

Page 163 (First)

1. $\sin x + \cos x$ is maximized when $x + \frac{\pi}{4} = \frac{\pi}{2} + 2n\pi$, where n is an integer. Therefore,

   $$x = \frac{\pi}{4} + 2n\pi$$

2. The minimum value of $\sin x + \cos x$ is $-\sqrt{2}$. This is achieved when

   $$x = -\frac{3\pi}{4} + 2n\pi.$$

Page 163 (Second)

1. Yes. Because $\sin \alpha$ and $\cos \alpha$ are both positive, α must be in the first quadrant.

2. The minimum value of $3 \sin x + 4 \cos x$ is -5. This occurs when

   $$x = -\alpha - \frac{\pi}{2} + 2n\pi.$$

3. Let α be the positive acute angle such that $\sin\alpha = 7/\sqrt{53}$.

$$2\sin x + 7\cos x = \sqrt{53}\left(\frac{2}{\sqrt{53}}\sin x + \frac{7}{\sqrt{53}}\cos x\right) = \sqrt{53}\sin(\alpha + x)$$

The above shows that the maximum and minimum values of $2\sin x + 7\cos x$ are $\sqrt{53}$ and $-\sqrt{53}$, respectively.

Page 164

1.

$$\begin{aligned}
\cos\frac{\pi}{16} &= \sqrt{\frac{1+\cos\frac{\pi}{8}}{2}} \\
&= \sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}} \\
&= \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}} \\
&= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}
\end{aligned}$$

$$\begin{aligned}
\sin\frac{\pi}{16} &= \sqrt{\frac{1-\cos\frac{\pi}{8}}{2}} \\
&= \sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}} \\
&= \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}} \\
&= \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}
\end{aligned}$$

2.

α	$\cos\alpha$	$\sin\alpha$
$\frac{\pi}{16}$	$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$	$\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$
$\frac{\pi}{32}$	$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$	$\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$
$\frac{\pi}{64}$	$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$	$\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$
$\frac{\pi}{128}$	$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}$	$\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}$

Page 166

1.

$$\frac{1}{2}\sqrt{2} \approx 0.7071$$

$$\frac{1}{2}\sqrt{2+\sqrt{2}} \approx 0.9239$$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} \approx 0.9809$$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \approx 0.9952$$

2.

$$2^2\sqrt{2-\sqrt{2}} \approx 3.0615$$

$$2^3\sqrt{2-\sqrt{2+\sqrt{2}}} \approx 3.1214$$

$$2^4\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}} \approx 3.1365$$

$$2^5\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \approx 3.1403$$

3. (a)

$$\begin{aligned}\cos\frac{\pi}{12} &= \sqrt{\frac{1+\cos(\pi/6)}{2}} \\ &= \sqrt{\frac{1+\sqrt{3}/2}{2}} \\ &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\ &\approx 0.9659\end{aligned}$$

(b)

$$\begin{aligned}\cos\frac{\pi}{24} &= \sqrt{\frac{1+\cos(\pi/12)}{2}} \\ &= \sqrt{\frac{1+\sqrt{2+\sqrt{3}}/2}{2}} \\ &= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}} \\ &\approx 0.9914\end{aligned}$$

(c)

$$\begin{aligned}\cos\frac{\pi}{48} &= \sqrt{\frac{1+\cos(\pi/24)}{2}} \\ &= \sqrt{\frac{1+\sqrt{2+\sqrt{2+\sqrt{3}}}/2}{2}} \\ &= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}} \\ &\approx 0.9979\end{aligned}$$

(d)

$$\begin{aligned}\cos\frac{\pi}{96} &= \sqrt{\frac{1+\cos(\pi/48)}{2}} \\ &= \sqrt{\frac{1+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}/2}{2}} \\ &= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} \\ &\approx 0.9995\end{aligned}$$

These values approach 1 because for small angles x, $\cos x$ gets closer to 1. More formally, the limit of $\cos x$ as x approaches zero is 1.

4. (a)

$$12 \sin \frac{\pi}{12} = 6\sqrt{2-\sqrt{3}} \approx 3.1058$$

(b)

$$24 \sin \frac{\pi}{24} = 12\sqrt{2-\sqrt{2+\sqrt{3}}} \approx 3.1326$$

(c)

$$48 \sin \frac{\pi}{48} = 24\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} \approx 3.1394$$

(d)

$$96 \sin \frac{\pi}{96} = 48\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} \approx 3.1410$$

Page 168

1.

$$\begin{aligned}
\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \ldots + \frac{1}{\sqrt{99}+\sqrt{100}} &= \sum_{k=1}^{99} \frac{1}{\sqrt{k}+\sqrt{k+1}} \\
&= \sum_{k=1}^{99} \frac{1}{\sqrt{k}+\sqrt{k+1}} \cdot \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}} \\
&= \sum_{k=1}^{99} \sqrt{k+1}-\sqrt{k} \\
&= \left(\sqrt{2}-\sqrt{1}\right) + \left(\sqrt{3}-\sqrt{2}\right) + \ldots + \left(\sqrt{100}-\sqrt{99}\right) \\
&= \sqrt{100}-\sqrt{1} \\
&= 10-1 \\
&= 9
\end{aligned}$$

2.

$$
\begin{aligned}
1+3+\ldots+(2n+1) &= \sum_{k=0}^{n} 2k+1 \\
&= \sum_{k=0}^{n} (k+1)^2 - k^2 \\
&= \left(1^2-0^2\right) + \left(2^2-1^2\right) + \ldots + \left((n+1)^2-n^2\right) \\
&= (n+1)^2 - 0^2 \\
&= (n+1)^2 \\
&= n^2+2n+1
\end{aligned}
$$

3.

$$
\begin{aligned}
(1-x)\,P &= (1-x)\,(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)\left(1+x^{16}\right) \\
&= \left(1-x^2\right)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)\left(1+x^{16}\right) \\
&= \left(1-x^4\right)\left(1+x^4\right)\left(1+x^8\right)\left(1+x^{16}\right) \\
&= \left(1-x^8\right)\left(1+x^8\right)\left(1+x^{16}\right) \\
&= \left(1-x^{16}\right)\left(1+x^{16}\right) \\
&= 1-x^{32}
\end{aligned}
$$

Thus,

$$
P = \frac{1-x^{32}}{1-x} = 1+x+\ldots+x^{31}
$$

4.

$$
\begin{aligned}
\sin 20^\circ P &= \sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ \\
&= \frac{1}{2}\sin 40^\circ \cos 40^\circ \cos 80^\circ \\
&= \frac{1}{4}\sin 80^\circ \cos 80^\circ \\
&= \frac{1}{8}\sin 160^\circ \\
&= \frac{1}{8}\sin 20^\circ
\end{aligned}
$$

Thus, $P = \frac{1}{8}$.

Page 170

1.

$$
\begin{aligned}
2\sin x\left(\sin x+\sin 3x+\ldots+\sin 99x\right) &= 2\sin x\sum_{k=0}^{49}\sin\left(2k+1\right)x \\
&= \sum_{k=0}^{49} 2\sin x\sin\left(2k+1\right)x \\
&= \sum_{k=0}^{49}\cos 2kx-\cos\left(2k+2\right)x \\
&= \left(\cos 0-\cos 2x\right)+\left(\cos 2x-\cos 4x\right)+\ldots+\left(\cos 98x-\cos 100x\right) \\
&= 1-\cos 100x
\end{aligned}
$$

Thus, the original sum, $\sin x+\ldots+\sin 99x$, is equal to $\left(1-\cos 100x\right)/\left(2\sin x\right)$. Notice that this is equivalent to $\sin^2 50x/\sin x$, which is what would be found using the formula at the bottom of page 169.

2.

$$
\begin{aligned}
2\sin\frac{\pi}{8}\left[\sin x+\ldots+\sin\left(x+\frac{99\pi}{4}\right)\right] &= 2\sin\frac{\pi}{8}\sum_{k=0}^{99}\sin\left(x+k\frac{\pi}{4}\right) \\
&= \sum_{k=0}^{99} 2\sin\frac{\pi}{8}\sin\left(x+k\frac{\pi}{4}\right) \\
&= \sum_{k=0}^{99}\cos\left(x+\left(k-\frac{1}{2}\right)\frac{\pi}{4}\right)-\cos\left(x+\left(k+\frac{1}{2}\right)\frac{\pi}{4}\right) \\
&= \cos\left(x-\frac{\pi}{8}\right)-\cos\left(x+\frac{199\pi}{8}\right) \\
&= \cos\left(x-\frac{\pi}{8}\right)-\cos\left(x+\frac{7\pi}{8}\right) \\
&= \cos\left(x-\frac{\pi}{8}\right)+\cos\left(\pi-x-\frac{7\pi}{8}\right) \\
&= 2\cos\left(x-\frac{\pi}{8}\right) \\
&= 2\cos x\cos\frac{\pi}{8}+2\sin x\sin\frac{\pi}{8}
\end{aligned}
$$

Dividing by $2\sin\frac{\pi}{8}$, we find that the original sum is equal to $\cot\frac{\pi}{8}\cos x+\sin x$. Notice that this is equivalent to $\sin\left(x+99\pi/8\right)/\sin\left(\pi/8\right)$, which is what would be found using the formula at the bottom of page 169.

An alternative way to evaluate this sum is by noticing that the terms of this sum repeat in periods of 8 because $\sin\left(x+\frac{8\pi}{4}\right)=\sin\left(x\right)$. Furthermore, the first 8 terms of the sum total to zero (try verifying this yourself using the sine addition formulas). Therefore, to evaluate the whole sum, all we need to evaluate is the sum of the last four terms:

$$\sin\left(x+\frac{96\pi}{4}\right)+\sin\left(x+\frac{97\pi}{4}\right)+\sin\left(x+\frac{98\pi}{4}\right)+\sin\left(x+\frac{99\pi}{4}\right).$$

By the periodicity in the sum, we can equivalently evaluate the first four terms of the sum. Using the sine addition formulas, we find that the sum is equal to

$$\begin{aligned}
&\sin x+\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x+\frac{\pi}{2}\right)+\sin\left(x+\frac{3\pi}{4}\right)\\
&\quad=\sin x+\frac{\sqrt{2}}{2}\sin x+\frac{\sqrt{2}}{2}\cos x+\cos x-\frac{\sqrt{2}}{2}\sin x+\frac{\sqrt{2}}{2}\cos x\\
&\quad=\sin x+\cos x+\sqrt{2}\cos x.
\end{aligned}$$

Combining this result with our initial evaluation of this sum shows that $\cot\frac{\pi}{8}=1+\sqrt{2}$.

3. We begin by multiplying the sum by $2\sin x$.

$$\begin{aligned}
2\sin x\left(\cos 2x+\cos 4x+\ldots+\cos 2nx\right)&=\sum_{k=1}^{n}2\sin x\cos 2kx\\
&=\sum_{k=1}^{n}\sin(2k+1)x-\sin(2k-1)x\\
&=(\sin 3x-\sin x)+\ldots+(\sin(2n+1)x-\sin(2n+1)x)\\
&=\sin(2n+1)x-\sin x
\end{aligned}$$

Dividing by $2\sin x$ gives us the value of the sum as

$$\frac{\sin(2n+1)x}{2\sin x}-\frac{1}{2}.$$

4. We begin by multiplying the sum by $2\sin\frac{\pi}{2k}$.

$$\begin{aligned}
2\sin\frac{\pi}{2k}\left(\cos\frac{\pi}{k}+\cos\frac{2\pi}{k}+\ldots+\cos\frac{n\pi}{k}\right)&=\sum_{j=1}^{n}2\sin\frac{\pi}{2k}\cos\frac{j\pi}{k}\\
&=\sum_{j=1}^{n}\sin\left(\frac{\pi}{k}\left(j+\frac{1}{2}\right)\right)-\sin\left(\frac{\pi}{k}\left(j-\frac{1}{2}\right)\right)\\
&=\sin\left(\frac{n\pi}{k}+\frac{\pi}{2k}\right)-\sin\frac{\pi}{2k}\\
&=\sin\frac{n\pi}{k}\cos\frac{\pi}{2k}+\cos\frac{n\pi}{k}\sin\frac{\pi}{2k}-\sin\frac{\pi}{2k}
\end{aligned}$$

Dividing by $2\sin\frac{\pi}{2k}$, we find that the value of the sum is

$$\frac{1}{2}\left(\sin\frac{n\pi}{k}\cot\frac{\pi}{2k}+\cos\frac{n\pi}{k}-1\right).$$

5. The heights of the perpendiculars are given by $\sin \frac{k\pi}{12}$, where k ranges from 1 to 11. We consider this sum multiplied by $2\sin\frac{\pi}{24}$.

$$\begin{aligned}
\sum_{k=1}^{11} 2\sin\frac{\pi}{24}\sin\frac{k\pi}{12} &= \sum_{k=1}^{11} \cos\left(\frac{k\pi}{12}-\frac{\pi}{24}\right) - \cos\left(\frac{k\pi}{12}+\frac{\pi}{24}\right) \\
&= \cos\frac{\pi}{24} - \cos\frac{23\pi}{24} \\
&= 2\cos\frac{\pi}{24}
\end{aligned}$$

Dividing this result by $2\sin\frac{\pi}{24}$ gives the value of the sum as $\cot\frac{\pi}{24}$.

Alternatively, we can use the formula for series of sines with angles in arithmetic progression, setting $x = 0$, $n = 11$, $\alpha = \pi/12$ to find that the sum of the altitudes is

$$\frac{\sin\dfrac{6\pi}{12}\sin\dfrac{11\pi}{24}}{\sin\dfrac{\pi}{24}} = \frac{\sin\dfrac{11\pi}{24}}{\sin\dfrac{\pi}{24}} = \frac{\cos\dfrac{\pi}{24}}{\sin\dfrac{\pi}{24}} = \cot\frac{\pi}{24}.$$

Using the half-angle formulas for sine and cosine, we can show that $\cos\pi/12 = \frac{1}{2}\sqrt{2+\sqrt{3}}$ and $\sin\pi/12 = \frac{1}{2}\sqrt{2-\sqrt{3}}$. Therefore,

$$\begin{aligned}
\cot\frac{\pi}{24} &= \frac{1+\cos\dfrac{\pi}{12}}{\sin\dfrac{\pi}{12}} \\
&= \frac{1+\dfrac{1}{2}\sqrt{2+\sqrt{3}}}{\dfrac{1}{2}\sqrt{2-\sqrt{3}}} \\
&= \frac{2+\sqrt{2+\sqrt{3}}}{\sqrt{2-\sqrt{3}}} \\
&= \frac{2}{\sqrt{2-\sqrt{3}}} + \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} \\
&= \frac{2\sqrt{2-\sqrt{3}}}{2-\sqrt{3}} + 2 + \sqrt{3} \\
&= 2\sqrt{2-\sqrt{3}}\left(2+\sqrt{3}\right) + 2 + \sqrt{3} \\
&= 2\sqrt{2+\sqrt{3}} + 2 + \sqrt{3} \\
&= 2\left(\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}}\right) + 2 + \sqrt{3} \\
&= \sqrt{2} + \sqrt{6} + 2 + \sqrt{3}.
\end{aligned}$$

Let's elaborate on how to evaluate $\sqrt{2+\sqrt{3}}$. Suppose $\sqrt{2+\sqrt{3}} = \sqrt{a} + \sqrt{b}$ for some rational numbers a and b. Then, squaring both sides, we get $2+\sqrt{3} = a + 2\sqrt{ab} + b$. Matching the rational and irrational parts, we get two equations relating a and b: $a+b=2$ and $4ab=3$. Solving this system shows that a and b are $1/2$ and $3/2$ (the order doesn't matter because the equations are symmetric). Thus, we have shown that

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}}.$$

Chapter 8: Graphs of Trigonometric Functions

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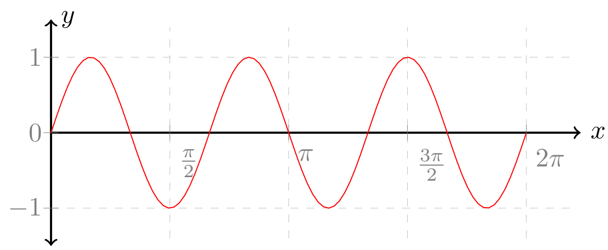
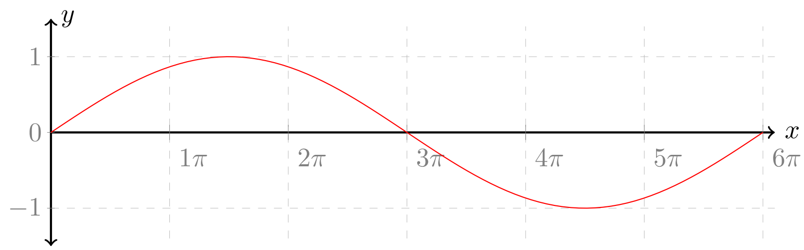
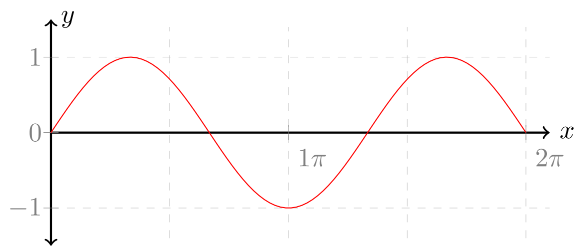
1. Since $k=5$, Period is 5, and Frequency is $\frac{\pi}{5}$
2. Since $k=\frac{1}{4}$, Period is $\frac{1}{4}$, and Frequency is $\frac{\pi}{1/4} = 4\pi$
3. Since $k=\frac{4}{5}$, Period is $\frac{4}{5}$, and Frequency is $\frac{\pi}{4/5} = \frac{5\pi}{4}$
4. Since $k=\frac{5}{4}$, Period is $\frac{5}{4}$, and Frequency is $\frac{\pi}{5/4} = \frac{4\pi}{5}$
5. Period is $\frac{2\pi}{3}$



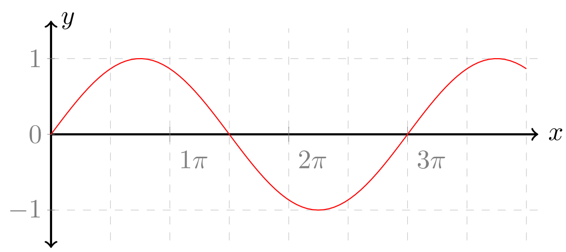
6. Period is $\frac{2\pi}{1/3} = 6\pi$



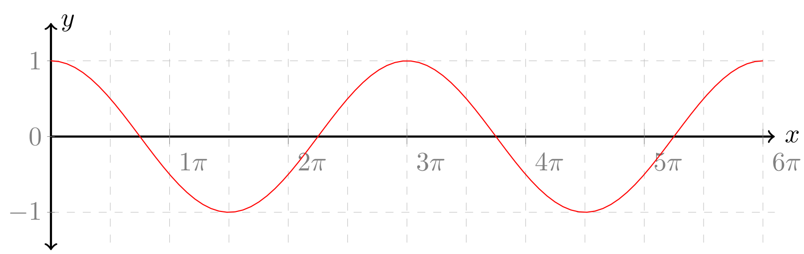
7. Period is $\frac{2\pi}{3/2} = \frac{4\pi}{3}$



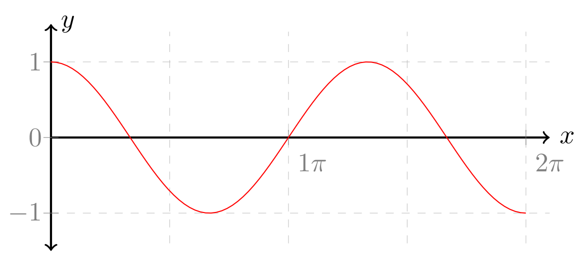
8. Period is $\frac{2\pi}{2/3} = 3\pi$



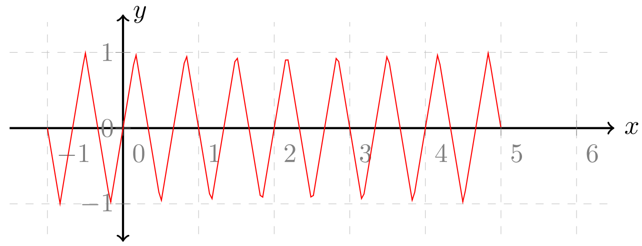
9. Period is $\frac{2\pi}{2/3} = 3\pi$



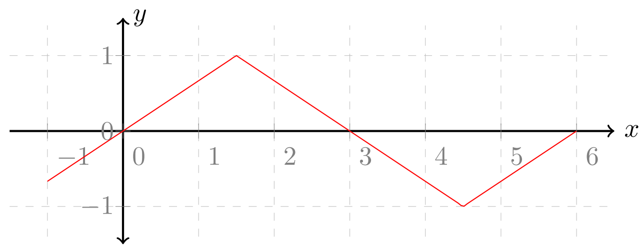
10. Period is $\frac{2\pi}{3/2} = \frac{4\pi}{3}$



11. For $y = f(3x)$

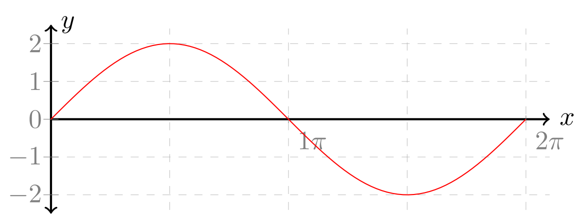


For $y = f(\frac{x}{3})$

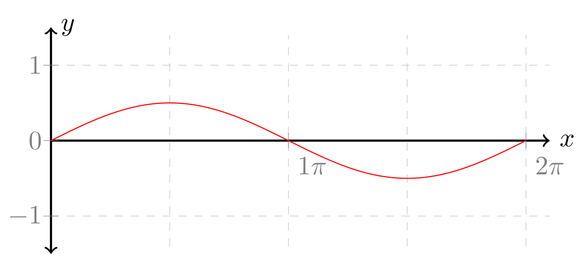


Page 179

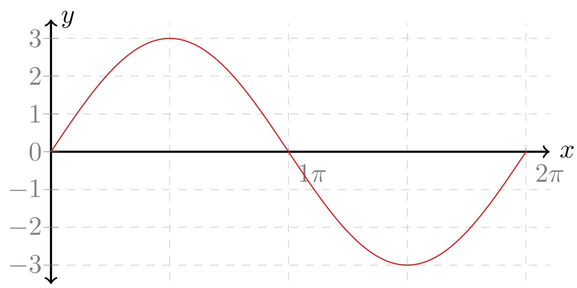
1. $y = 2\sin(x)$



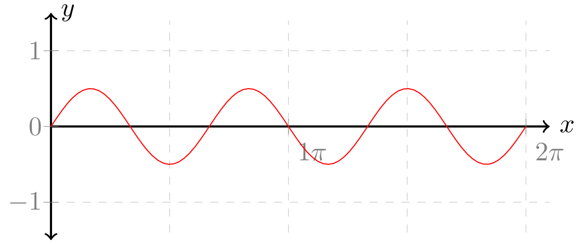
2. $y = \frac{1}{2}\sin(x)$



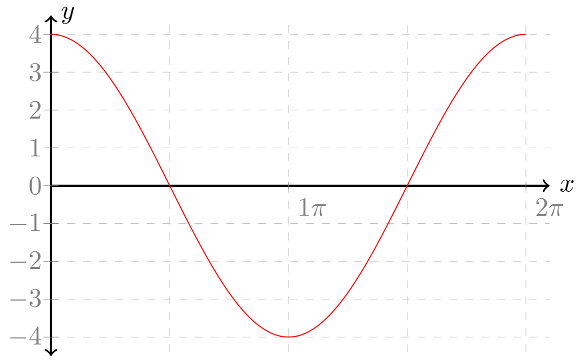
3. $y = 3\sin(2x)$



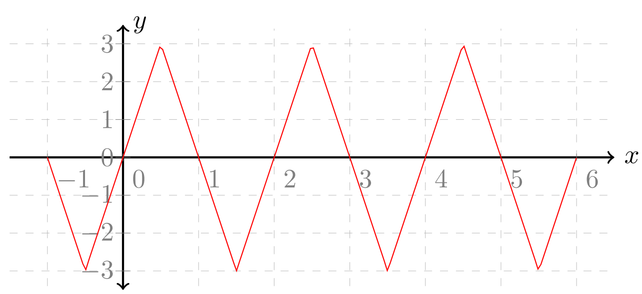
4. $y = \frac{1}{2}\sin(3x)$



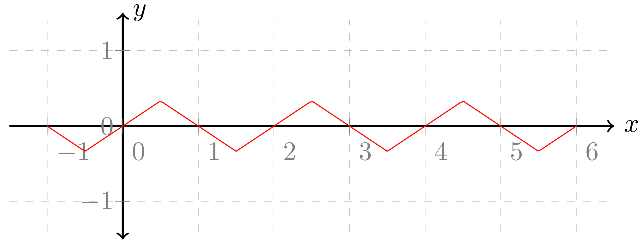
5. $y = 4\cos(x)$



6. For $y = 3f(x)$

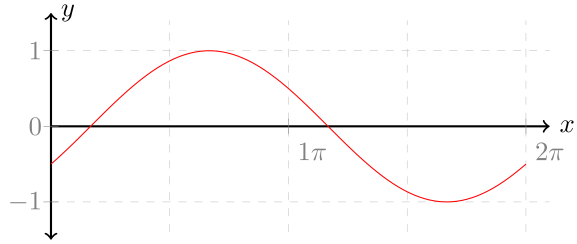


7. For $y = \frac{1}{3}f(x)$

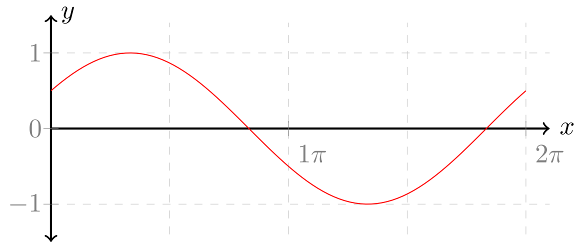


Page 181

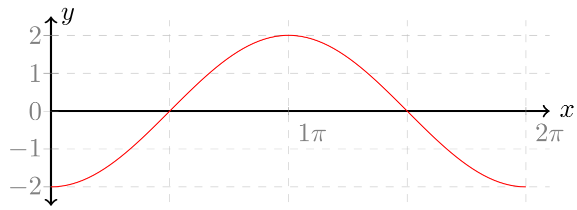
1. $y = \sin\left(x - \frac{\pi}{6}\right)$



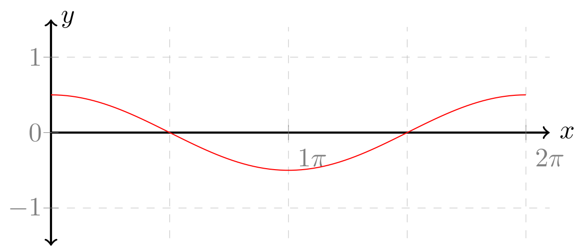
2. $y = \sin\left(x + \frac{\pi}{6}\right)$



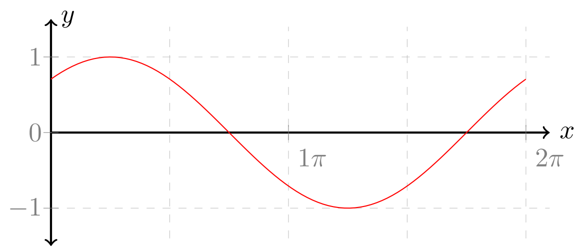
3. $y = 2\sin\left(x - \frac{\pi}{2}\right)$



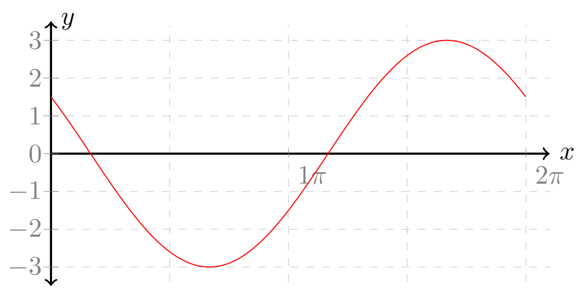
4. $y = \frac{1}{2}\sin\left(x + \frac{\pi}{2}\right)$



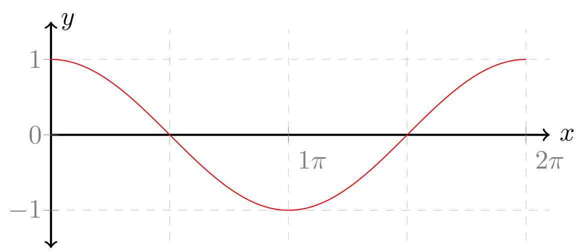
5. $y = \cos\left(x - \frac{\pi}{4}\right)$



6. $y = 3\cos\left(x + \frac{\pi}{3}\right)$



7. $y = \sin\left(x - 2x\right)$



8. Since the sine wave has shifted right by $\frac{\pi}{3}$. The equation is

$$y = \sin\left(x - \frac{\pi}{3}\right)$$

9. Since the sine wave has shifted right by $\frac{2\pi}{3}$. The equation is

$$y = \sin\left(x - \frac{2\pi}{3}\right)$$

10. Since the sine wave has shifted left by $\frac{\pi}{6}$. The equation is

$$y = \sin\left(x - \frac{\pi}{6}\right)$$

11. Note: this looks like the sine wave has shifted right by $\frac{\pi}{4}$, but at that point, the sine wave is on its wave down.

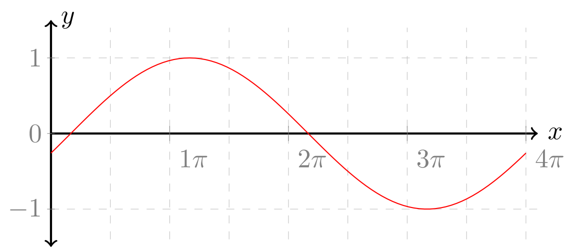
    The actual shift is the x intercept on the far left, calculate its location by $\frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ as it's half a period away, that is, if 2π radians is one period, then π radians is a half period

    Therefore, the sine wave has shifted left by $\frac{3\pi}{4}$. The equation is

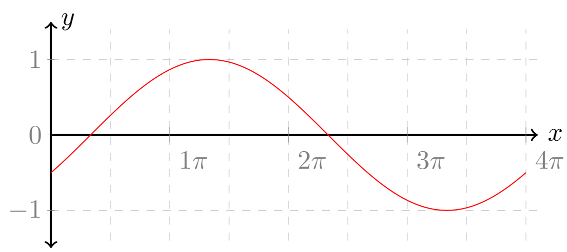
$$y = \sin\left(x + \frac{3\pi}{4}\right)$$

Page 183

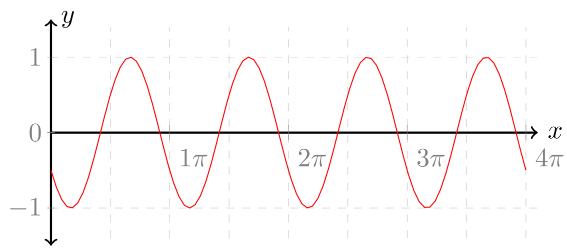
1. $y = \sin\frac{1}{2}\left(x - \frac{\pi}{6}\right)$



2. $y = \sin\left(\frac{1}{3}x - \frac{\pi}{6}\right)$



3. $y = \cos 2\left(x + \frac{\pi}{3}\right)$



4. Since period is π, making $k = \frac{2\pi}{\pi} = 2$, making the equation

$$y = \sin 2(x - \beta)$$

To solve for β, substitute $x = 0, y = -1$, as the sine wave passes point $(0, -1)$, giving

$$\begin{aligned} -1 &= \sin 2(0 - \beta) \\ -1 &= \sin -2\beta \\ -\frac{\pi}{2} &= -2\beta \\ \frac{\pi}{4} &= \beta \\ \beta &= \frac{\pi}{4} \end{aligned}$$

Therefore the equation is

$$y = \sin 2\left(x - \frac{\pi}{4}\right)$$

5. Since half period is 2π, its full period is 4πmaking $k = \frac{2\pi}{4\pi} = \frac{1}{2}$, making the equation

$$y = \sin \frac{1}{2}(x - \beta)$$

To solve for β, substitute $x = \frac{5\pi}{6}, y = 1$, as the sine wave passes point $(\frac{5\pi}{6}, 1)$, giving

$$\begin{aligned} 1 &= \sin \frac{1}{2}\left(\frac{5\pi}{6} - \beta\right) \\ \frac{\pi}{2} &= \frac{1}{2}\left(\frac{5\pi}{6} - \beta\right) \\ \pi &= \frac{5\pi}{6} - \beta \\ \frac{\pi}{6} &= -\beta \\ \beta &= -\frac{\pi}{6} \end{aligned}$$

Therefore the equation is

$$y = \sin \frac{1}{2}\left(x - \frac{\pi}{6}\right)$$

Page 186

1. Answer A) equal to $\sin x$ since

$$\begin{aligned}\sin(x+2\pi) &= \sin(x+2\pi-2\pi)\\ &= \sin(x)\end{aligned}$$

2. Answer C) equal to $-\sin x$ since

$$\begin{aligned}\sin(x+3\pi) &= \sin(x+3\pi-2\pi)\\ &= \sin(x+\pi)\\ &= -\sin(x)\end{aligned}$$

3. Answer B) equal to $\cos x$ since

$$\begin{aligned}\sin\left(x+\frac{9}{2}\pi\right) &= \sin\left(x+\frac{9}{2}\pi-4\pi\right)\\ &= \sin\left(x+\frac{\pi}{2}\right)\\ &= \cos(x)\end{aligned}$$

4. Answer D) equal to $-\cos x$ since

$$\sin\left(x-\frac{\pi}{2}\right) = -\cos(x)$$

5. Answer B) equal to $\cos x$ since

$$\begin{aligned}\sin\left(x-\frac{3}{2}\pi\right) &= \sin\left(-\frac{3}{2}\pi+2\pi\right)\\ &= \sin\left(x+\frac{\pi}{2}\right)\\ &= \cos(x)\end{aligned}$$

6. Answer D) equal to $-\cos x$ since

$$\begin{aligned}\sin\left(x+\frac{19}{2}\pi\right) &= \sin\left(x+\frac{19}{2}\pi-10\pi\right)\\ &= \sin\left(x-\frac{\pi}{2}\right)\\ &= -\cos(x)\end{aligned}$$

7. Answer D) equal to $-\cos x$ since

$$
\begin{aligned}
-\sin\left(x-\frac{19}{2}\pi\right) &= -\sin\left(x+\frac{19}{2}\pi+10\pi\right)\\
&= -\sin\left(x+\frac{\pi}{2}\right)\\
&= -\cos(x)
\end{aligned}
$$

8. Answer B) equal to $\cos x$ since

$$
\begin{aligned}
\sin\left(x+\frac{157}{2}\pi\right) &= \sin\left(x+\frac{157}{2}\pi-78\pi\right)\\
&= \sin\left(x+\frac{\pi}{2}\right)\\
&= \cos(x)
\end{aligned}
$$

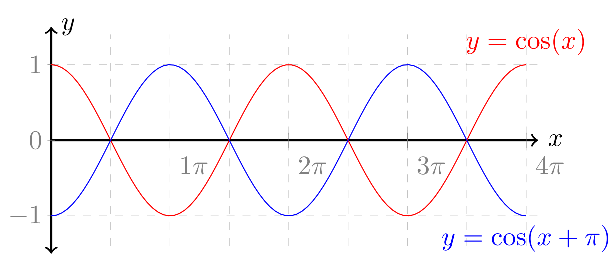
9. Answer D) equal to $-\cos x$ since

$$
\begin{aligned}
\sin\left(x-\frac{157}{2}\pi\right) &= \sin\left(x-\frac{157}{2}\pi+78\pi\right)\\
&= \sin\left(x-\frac{\pi}{2}\right)\\
&= -\cos(x)
\end{aligned}
$$

10. Note: there is no question 10 in textbook (printing error?)

11. Using definition that p is called a half-period of function f is $f(x+p) = -f(x)$, for all values of x which $f(x)$ and $f(x+p)$ are defined

    Since $\cos(x)$ has period of 2π radians, therefore its half-period is π radians, or we can compare diagrams

    Which can be seen that $\cos(x+\pi)$ is the same as $-\cos(x)$, therefore by definition, $\cos x$ has a half-period of π radians

For $y = \tan x$

$$\begin{aligned}\tan(x+\pi) &= \frac{\sin(x+\pi)}{\cos(x+\pi)} \\ &= \frac{-\sin x}{-\cos x} \\ &= \tan x\end{aligned}$$

So π is period of $y = \tan x$

For $y = \cot x$

$$\begin{aligned}\cot(x+\pi) &= \frac{\cos(x+\pi)}{\sin(x+\pi)} \\ &= \frac{-\cos x}{-\sin x} \\ &= \cot x\end{aligned}$$

So π is period of $y = \cot x$

12. Using definition that q is called a half-period of function f is $f(x+q) = -f(x)$, for all values of x which $f(x)$ and $f(x+q)$ are defined

$$\begin{aligned}f(x+2q) &= f\left(x+q+q\right) \\ &= -f\left(x+q\right) \\ &= -\left(-f(x)\right) \\ &= f(x)\end{aligned}$$

Therefore if q is half period of some function f, then $2q$ is a period of f

13. (a)

$$\begin{aligned}\cos\left(x+k\pi/2\right) &= \cos\left(x+\frac{\pi(4n+1)}{2}\right) \\ &= \cos\left(x+2n\pi+\pi/2\right) \\ &= \cos(x+\pi/2) \\ &= -\sin x\end{aligned}$$

(b)

$$\begin{aligned}\cos\left(x+k\pi/2\right) &= \cos\left(x+\frac{\pi(4n+2)}{2}\right) \\ &= \cos\left(x+2n\pi+\pi\right) \\ &= \cos(x+\pi) \\ &= -\cos x\end{aligned}$$

(c)

$$\begin{aligned}
\cos\left(x + k\pi/2\right) &= \cos\left(x + \frac{\pi(4n+3)}{2}\right) \\
&= \cos\left(x + 2n\pi + \frac{3\pi}{2}\right) \\
&= \cos(x + \frac{3\pi}{2}) \\
&= \cos(x - \frac{\pi}{2}) \\
&= \sin x
\end{aligned}$$

(d)

$$\begin{aligned}
\cos\left(x + k\pi/2\right) &= \cos\left(x + \frac{\pi(4n)}{2}\right) \\
&= \cos\left(x + 2n\pi\right) \\
&= \cos x
\end{aligned}$$

14. (a) As the equation has a negative coefficient, we need to shift this by a half-period of π radians

$$\begin{aligned}
y &= -2\sin x \\
&= 2\sin(x + \pi)
\end{aligned}$$

Comparing coefficients, we get $a = 2$ and $k = 1$ and $\beta = \pi$

(b) As the equation has a negative coefficient, we need to shift this by a half-period of π radians

$$\begin{aligned}
y &= 2\sin(x - \pi/3) \\
&= 2\sin(x - \pi/3 + \pi) \\
&= 2\sin(x + 2\pi/3)
\end{aligned}$$

Comparing coefficients, we get $a = 2$ and $k = 1$ and $\beta = -2\pi/3$

(c) As the equation has a negative coefficient, we need to shift this by a half-period of π radians

$$\begin{aligned}
y &= -2\sin(x + \pi/4) \\
&= 2\sin(x + \pi/4 + \pi) \\
&= 2\sin(x + 5\pi/4)
\end{aligned}$$

Comparing coefficients, we get $a = 2$ and $k = 1$ and $\beta = -5\pi/4$

(d) As the equation is a cosine function, we need to re-express as a sine function

$$3\cos x = 3\sin(x + \pi/2)$$

Comparing coefficients, we get $a = 3$ and $k = 1$ and $\beta = \pi/2$

(e) As the equation is a cosine function, we need to re-express as a sine function

$$\begin{aligned} y &= 3\cos(x - \pi/6) \\ &= 3\sin(x - \pi/6 + \pi/2) \\ &= 3\sin(x + \pi/3) \end{aligned}$$

Comparing coefficients, we get $a = 3$ and $k = 1$ and $\beta = -\pi/3$

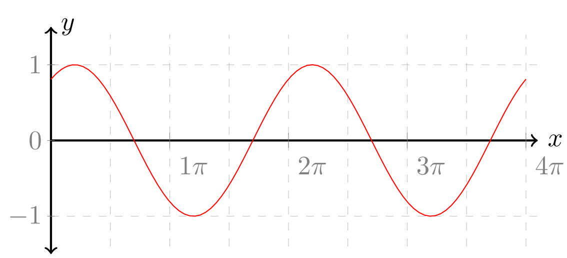
(f) As the equation is a cosine function, we need to re-express as a sine function. Then re-express as a postive amplitude

$$\begin{aligned} y &= -3\cos(x - \pi/6) \\ &= -3\sin(x - \pi/6 + \pi/2) \\ &= 3\sin(x - \pi/6 + \pi/2 + \pi) \\ &= 3\sin(x + 4\pi/3) \end{aligned}$$

Comparing coefficients, we get $a = 3$ and $k = 1$ and $\beta = -4\pi/3$

15.

$$\cos(x - \pi/5)$$



16. We need to shift to the right by $\frac{\pi}{2}$
We need to shift to the left by $\frac{3\pi}{2}$

17. To express k as an odd number, let $k = 2n + 1$ for any integer n

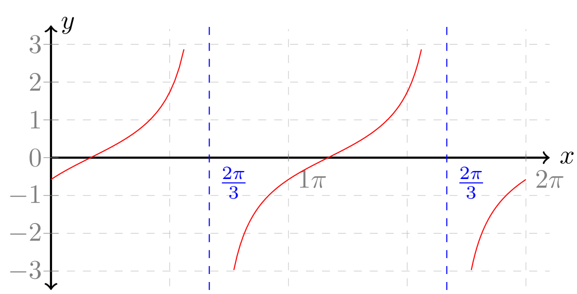
$$\begin{aligned}
\tan(x + k\pi/2) &= \tan\left(x + \frac{\pi(2n+1)}{2}\right) \\
&= \tan(x + 2n\pi + \pi/2) \\
&= \tan(x + \pi/2) \\
&= \frac{\sin(x + \pi/2)}{\cos(x + \pi/2)} \\
&= \frac{-\cos x}{\sin x} \\
&= -\cot x
\end{aligned}$$

To express k as an even number, let $k = 2n$ for any integer n

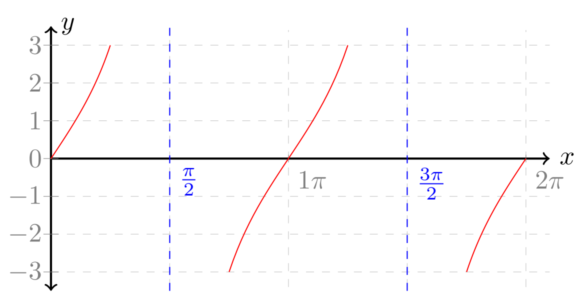
$$\begin{aligned}
\tan(x + k\pi/2) &= \tan\left(x + \frac{\pi(2n)}{2}\right) \\
&= \tan(x + 2n\pi) \\
&= \tan x
\end{aligned}$$

Page 188

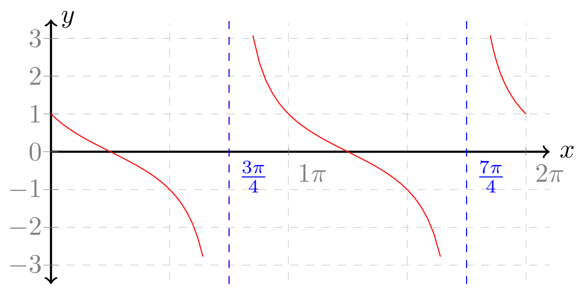
1. a) $y = \tan\left(x - \frac{\pi}{6}\right)$



b) $y = 3\tan(x)$



c) $y = \cot\left(x + \frac{\pi}{4}\right)$



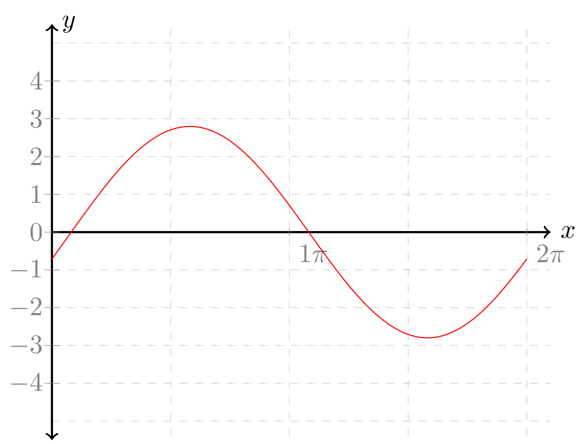
2. Not really, although there is a trigonometric identity where

$$\tan x = \cot\left(\frac{\pi}{2} - x\right)$$

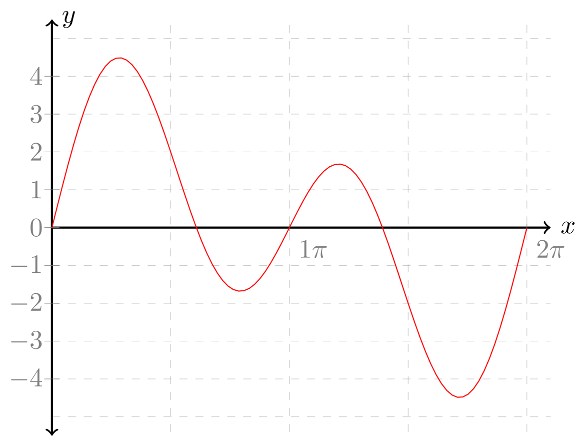
This equation has a negative x inside. Therefore it won't be in form of $y = \cot(x + \phi)$ (note the positive x inside)

Page 189

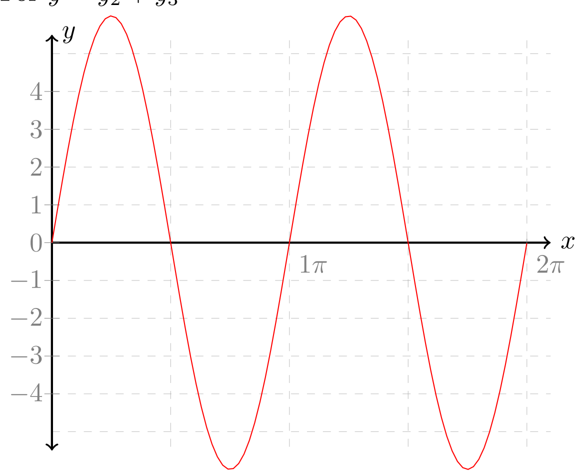
1. (a) For $y = y_1 + y_2$



(b) For $y = y_1 + y_2$



(c) For $y = y_2 + y_3$



2. Graph 1a appear to be a sinusoidal function

Page 191

1. For $y = 2\sin x + 3\cos x$, compare with standard form $y = A\sin kx + B\cos kx$, we get values of $A = 2$, $B = 3$, and $k = 1$

   To convert to the other standard form $y = a\sin k(x + \phi)$, we calculate its amplitude by

$$\begin{aligned} a &= \sqrt{A^2 + B^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

While its phase $\phi = \alpha/k$ is calculated by

$$\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$$
$$\cos\alpha = \frac{2}{\sqrt{13}}$$
$$\alpha = \arccos(2/\sqrt{13})\text{Since} \qquad k = 1$$
$$\phi = \arccos(2/\sqrt{13})$$

Therefore the equation is $y = \sqrt{2}\sin(x+\phi)$ where $\phi = \arccos 2\sqrt{13}$

Since amplitude is $\sqrt{2}$, then its maximum value is $\sqrt{2}$

2. Since amplitude is $\sqrt{13}$, then its maximum value is $\sqrt{13}$

3. For $y = \sin x + \cos x$, compare with standard form $y = A\sin kx + B\cos kx$, we get values of $A = 1$, $B = 1$, and $k = 1$

   To convert to the other standard form $y = a\sin k(x+\phi)$, we calculate its amplitude a by

$$a = \sqrt{A^2+B^2}$$
$$= \sqrt{1^2+1^2}$$
$$= \sqrt{2}$$

   While its phase $\phi = \alpha/k$ is calculated by

$$\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$$
$$\cos\alpha = \frac{1}{\sqrt{2}}$$
$$\alpha = \pi/4$$
$$\text{Since } k = 1$$
$$\phi = \pi/4$$

   Therefore the equation is $y = \sqrt{2}sin\,(x+\pi/4)$

   Since amplitude is $\sqrt{2}$, then its maximum value is $\sqrt{2}$

4. For $y = \sin x - \cos x$, compare with standard form $y = A\sin kx + B\cos kx$, we get values of $A = 1$, $B = 1$, and $k = 1$

   To convert to the other standard form $y = a\sin k(x-\beta)$, we calculate its amplitude by

$$a = \sqrt{A^2+B^2}$$
$$= \sqrt{1^2+1^2}$$
$$= \sqrt{2}$$

While its phase $\beta = \alpha/k$ is calculated by

$$\begin{aligned}
\cos\alpha &= \frac{A}{\sqrt{A^2+B^2}} \\
\cos\alpha &= \frac{1}{\sqrt{2}} \\
\alpha &= \pi/4 \\
\text{Since } k &= 1 \\
\beta &= \pi/4
\end{aligned}$$

Therefore the equation is $y = \sqrt{2}\sin(x - \pi/4)$

Since amplitude is $\sqrt{2}$, then its maximum value is $\sqrt{2}$

5. For $y = 4\sin x + 3\cos x$, compare with standard form $y = A\sin kx + B\cos kx$, we get values of $A = 4$, $B = 3$, and $k = 1$

   To convert to the other standard form $y = a\sin k(x + \phi)$, we calculate its amplitude by

$$\begin{aligned}
a &= \sqrt{A^2+B^2} \\
&= \sqrt{4^2+3^2} \\
&= \sqrt{25} \\
&= 5
\end{aligned}$$

   While its phase $\phi = \alpha/k$ is calculated by

$$\begin{aligned}
\cos\alpha &= \frac{A}{\sqrt{A^2+B^2}} \\
\cos\alpha &= \frac{4}{5} \\
\alpha &= \arccos(\frac{4}{5}) \\
\text{Since } k &= 1 \\
\phi &= \arccos(\frac{4}{5})
\end{aligned}$$

   Therefore the equation is $y = 5\sin(x + \phi)$ where $\phi = \arccos(4/5)$

   Since amplitude is 5, then its maximum value is 5

6. For $y = \sin 2x + 3\cos 2x$, compare with standard form $y = A\sin kx + B\cos kx$, we get values of $A = 1$, $B = 3$, and $k = 2$

To convert to the other standard form $y = a \sin k(x + \phi)$, we calculate its amplitude by

$$\begin{aligned} a &= \sqrt{A^2 + B^2} \\ &= \sqrt{1^2 + 3^2} \\ &= \sqrt{10} \end{aligned}$$

While its phase $\phi = \alpha/k$ is calculated by

$$\begin{aligned} \cos\alpha &= \frac{A}{\sqrt{A^2 + B^2}} \\ \cos\alpha &= \frac{1}{\sqrt{10}} \\ \alpha &= \arccos(1/\sqrt{10}) \\ \text{Since } k &= 2 \\ \phi &= \frac{1}{2}\arccos(\frac{1}{\sqrt{10}}) \end{aligned}$$

Therefore the equation is $y = \sqrt{10}\sin(x + \phi)$ where $\phi = \frac{1}{2}\arccos(\frac{1}{\sqrt{10}})$

7. For $y = \sin(x - \pi/4)$, compare with standard form $y = a \sin k(x - \beta)$, we get values of $a = 1$, $\beta = \pi/4$, and $k = 1$

   To convert to the other standard form $y = A \sin kx - B \cos kx$, we need to solve for A and B using the definition $A = a \cos k\beta$ and $B = a \sin k\beta$

$$\begin{aligned} A &= 1\cos(\pi/4) \\ &= \frac{1}{\sqrt{2}} \\ B &= 1\sin(\pi/4) \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

   Therefore the equation is $y = \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}i}\cos x$

8. For $y = 4\sin(x + \pi/6)$, compare with standard form $y = a \sin k(x + \phi)$, we get values of $a = 4$, $\phi = \pi/6$, and $k = 2$

   To convert to the other standard form $y = A \sin kx + B \cos kx$, we need to solve for A and B using the definition $A = a\cos k\beta$ and $B = a\sin k\beta$

$$\begin{aligned}
A &= 4\cos(2\pi/6)\\
&= 4\times 1/2\\
&= 2\\
B &= 4\sin(2\pi/6)\\
&= 4\times\sqrt{3}/2\\
&= 2\sqrt{3}
\end{aligned}$$

Therefore the equation is $y = 2\sin 2x + 2\sqrt{3}\cos 2x$

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1. (a)

$$\begin{aligned}
y &= 2\sin(x+\pi/6) + \cos(x+\pi/6)\\
y &= 2\left(\sin x\cos\pi/6 + \cos x\sin\pi/6\right) + \left(\cos x\cos\pi/6 - \sin x\sin\pi/6\right)\\
y &= 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right) + \left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)\\
y &= \sqrt{3}\sin x + \cos x + \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\\
y &= \left(\sqrt{3} - \frac{1}{2}\right)\sin x + \left(1 + \frac{\sqrt{3}}{2}\right)\cos x
\end{aligned}$$

   (b)

$$\begin{aligned}
y &= 2\sin 2(x+\pi/4) - \cos 2(x+\pi/4)\\
y &= 2\sin(2x+\pi/2) - \cos(2x+\pi/2)\\
y &= 2\left(\sin 2x\cos\pi/2 + \cos 2x\sin\pi/2\right) + \left(\cos 2x\cos\pi/2 - \sin 2x\sin\pi/2\right)\\
y &= 2\left(\sin 2x\times 0 + \cos 2x\times 1\right) + \left(\cos 2x\times 0 - \sin 2x\times 1\right)\\
y &= 2\cos 2x - \sin 2x\\
y &= -\sin 2x + 2\cos 2x
\end{aligned}$$

2. (a)

$$\begin{aligned}
y_1 + y_1 &= 2\sin x + \sin(x-\pi/4)\\
&= 2\sin x + \left(\sin x\cos\pi/4 - \cos x\sin\pi/4\right)\\
&= 2\sin x + \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\\
&= \left(2 + \frac{1}{\sqrt{2}}\right)\sin x - \frac{1}{\sqrt{2}}\cos x
\end{aligned}$$

   (b) The functions have different frequencies, which would need advanced mathematics topics such as Fourier Series

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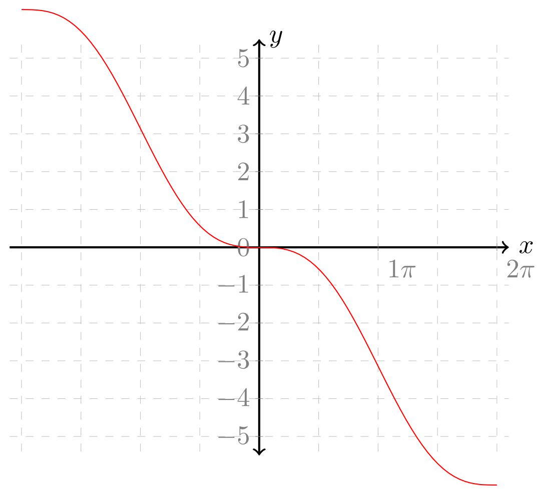
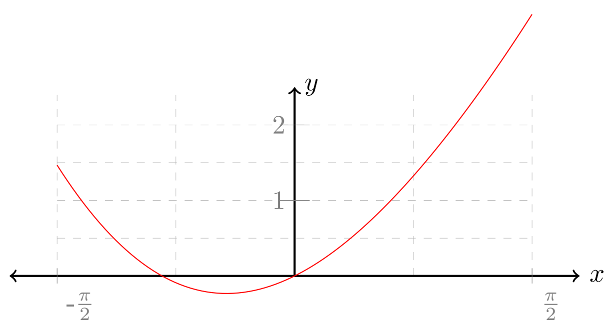
1.

$$y = -x + \sin x$$



2.

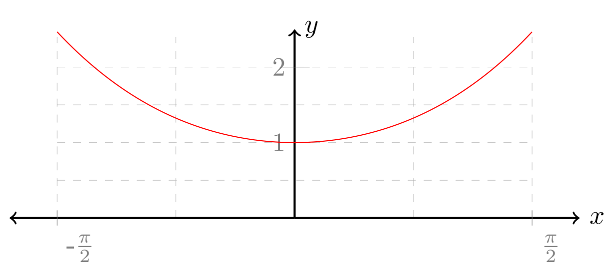
$$y = x^2 + \sin x$$



3.

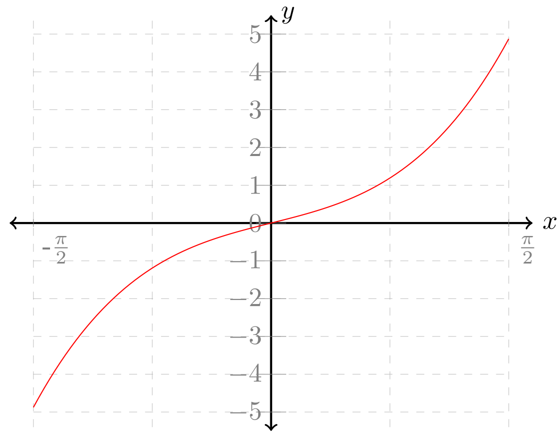
$$y = x^2 + \cos x$$

Note this is an even function since $f(-x) = f(x)$



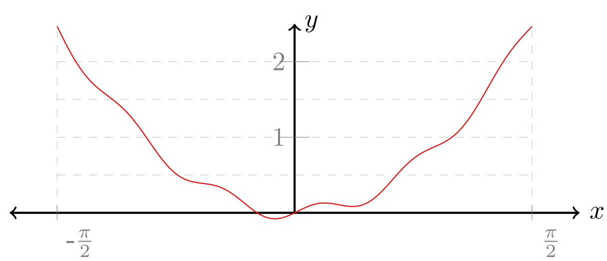
4.

$$y = x^3 + \sin x$$



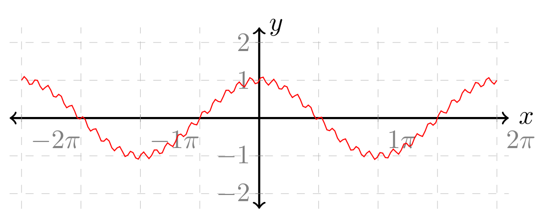
5.

$$y = x^2 + \frac{1}{10}\sin x$$



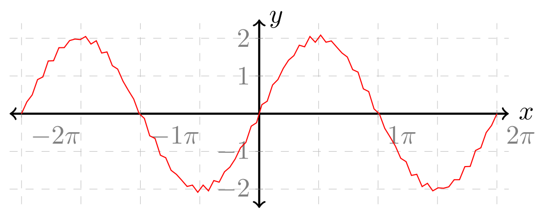
6.

$$y = \cos x + \frac{1}{10}\sin 20x$$



7.

$$y = 2\sin x + \frac{1}{10}\sin 20x$$



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1. In $y = \sin 2x + \sin 3x$. The period of $\sin 2x$ has form of $m(2\pi/2)$, while the period of $\sin 3x$ has form of $n(2\pi/3)$.

   To be a period of both parts

   $$m \times \frac{2\pi}{2} = n \times \frac{2\pi}{3}$$
   $$3m = 2n$$

   Using $m = 2$ and $n = 3$. the period of each part will be 2π radians

2. In $y = \sin 3x + \sin 6x$. The period of $\sin 3x$ has form of $m(2\pi/3)$, while the period of $\sin 6x$ has form of $n(2\pi/6)$.

   To be a period of both parts

   $$m \times \frac{2\pi}{3} = n \times \frac{2\pi}{6}$$
   $$6m = 3n$$
   $$2m = 1n$$

   Using $m = 1$ and $n = 2$. the period of each part will be $2\pi/3$ radians

3. In $y = \sin 4x + \sin 6x$. The period of $\sin 4x$ has form of $m(2\pi/4)$, while the period of $\sin 6x$ has form of $n(2\pi/6)$.

   To be a period of both parts

   $$m \times \frac{2\pi}{4} = n \times \frac{2\pi}{6}$$
   $$6m = 4n$$
   $$3m = 2n$$

   Using $m = 2$ and $n = 3$. the period of each part will be π radians

4.

5. In $y = \sin \sqrt{2}x + \sin 3\sqrt{2}x$. The period of $\sin \sqrt{2}x$ has form of $m(2\pi/\sqrt{2})$, while the period of $\sin 3\sqrt{2}x$ has form of $n(2\pi/3\sqrt{2})$.

To be a period of both parts

$$m \times \frac{2\pi}{\sqrt{2}} = n \times \frac{2\pi}{3\sqrt{2}}$$
$$3m = 1n$$

Using $m = 1$ and $n = 3$. the period of each part will be $2\pi/\sqrt{2} = \sqrt{2}\pi$ radians

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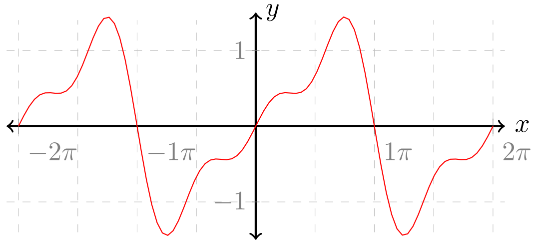
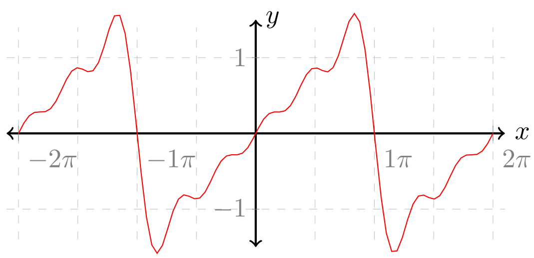
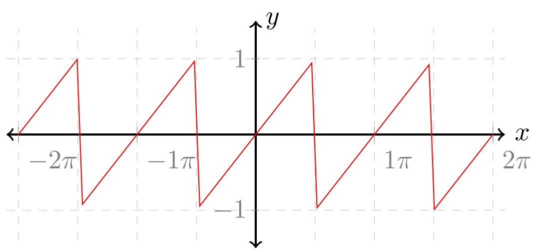
1.
$$y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$$



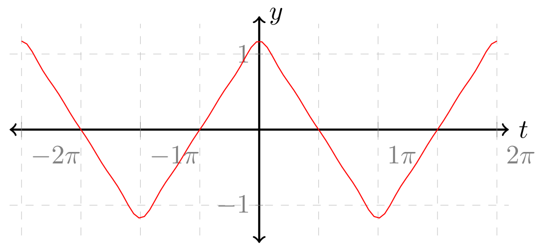
2.
$$y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x$$



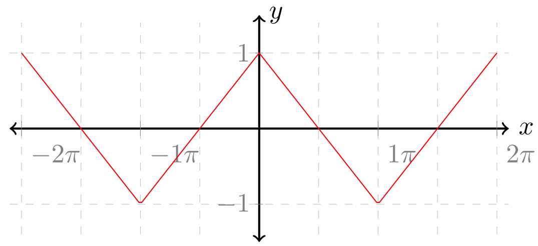
3. The limit will be a sawtooth wave



4. $$y = \cos x - \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x - \frac{1}{49}\cos 7x$$



Therefore, the limiting function is the triangular function



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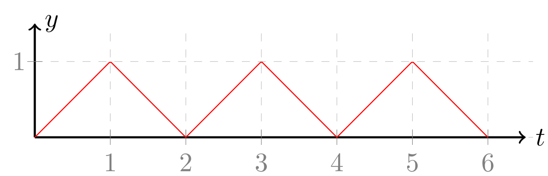
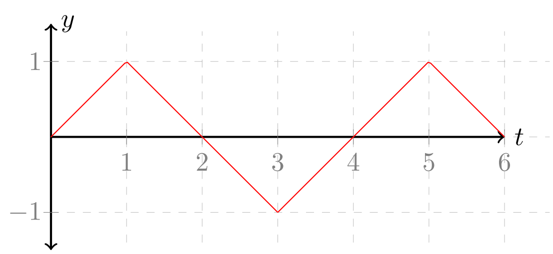
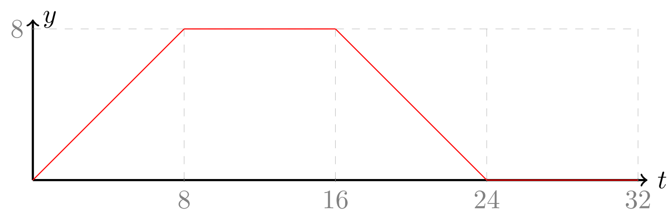
1. The graph will be



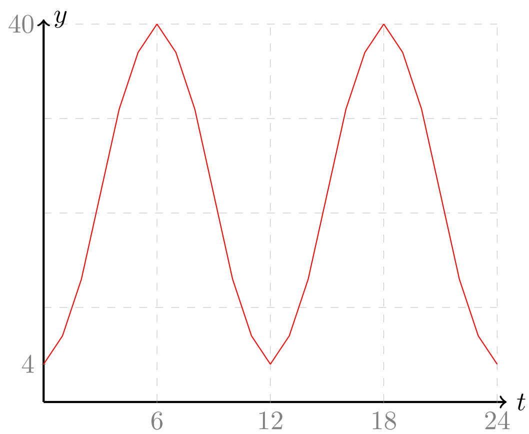
2. The graph will be



3. The graph will be



4. The graph will be



Tide is rising fastest at 3rd, 4th, 15th, and 16th hour

Tide is receding fastest at 9th, 10th, 21st, and 22nd hour

Tide is rising slowest at 1st, 5th, 13th and 17th hour

Tide is receding slowest at 7th, 11th, 19th and 23rd hour

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1. Substituing x values
   - $\sin \pi = 0$
   - $\sin 2\pi = 0$
   - $\sin 3\pi = 0$
   - $\sin 4\pi = 0$
2. For $y = \sin 4\pi x$, the value of k is $k = 4\pi$, therefore the period is $\frac{2\pi}{4\pi}$, so the period is $\frac{1}{2}$
3. Since period is $\frac{2\pi}{k}$, if period is 3

$$3 = \frac{2\pi}{k}$$
$$k = \frac{2\pi}{3}$$

Therefore the equation is $y = \sin \frac{2\pi}{3} x$

4. Since period is $\frac{2\pi}{k}$, if period is 2

$$2 = \frac{2\pi}{k}$$
$$k = \frac{2\pi}{2}$$

Therefore the equation is $y = \sin \pi x$

5. Since period is $\frac{2\pi}{k}$, if period is n

$$n = \frac{2\pi}{k}$$
$$k = \frac{2\pi}{n}$$

Therefore the equation is $y = \sin \frac{2\pi}{n} x$

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1. The period is estimated to be 12 months by distance between peaks at $t = 8$ and $t = 20$. Therefore the value for k is $k = 2\pi/20$

   The amplitude is estimated to be mid-way between peak (15) and trough (9.5), which is $(15 - 9.5)/2 = 2.75$

   Therefore the equation is $y = 2.75 \sin \frac{\pi}{10}(x - \pi)$

2. If (a) is in northern hemisphere, (b) and (c) would be in northern hemisphere too as the peaks occur at similar month. (d) would be in southern hemispehere as the wave is shifted by half-period

3. For all graphs, the 'average' number of daylight hours occur at month 4, month 10, month 16, and month 22

Chapter 9: Inverse Functions and Trigonometric Equations

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1. (a) $\arcsin 0.5 = \pi/6$
   (b) $\arccos 0.5 = \pi/3$
   (c) $\arctan 0.5 = \pi/4$
   (d) $\arcsin(-\sqrt{3}/2) = -\pi/3$
   (e) $\arccos(-\sqrt{3}/2) = 5\pi/6$
   (f) $\arctan(-\sqrt{3}) = -\pi/3$

(g) Note: not valid as the domain of $\arcsin x$ is $-1 \leq x \leq 1$

2. (a)

$$\begin{aligned}&\sin(\arcsin 0.5)\\&= \sin(\pi/6)\\&= -0.5\end{aligned}$$

(b)

$$\begin{aligned}&\cos(\arccos 0.5)\\&= \cos(\pi/3)\\&= 0.5\end{aligned}$$

(c)

$$\begin{aligned}&\tan(\arctan(-1))\\&= \tan(\pi/4)\\&= -\pi/4\end{aligned}$$

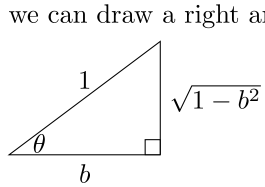
(d)

$$\begin{aligned}&\arcsin(\sin(\pi/3))\\&= \arcsin(-\sqrt{3}/2)\\&= \pi/3\end{aligned}$$

(e)

$$\begin{aligned}&\arccos(\cos 11\pi/6)\\&= \arccos(-\sqrt{3}/2)\\&= \pi/6\end{aligned}$$

3. Let $\theta = \arccos b$ which mean $\cos\theta = b$. If we assume angle θ is acute, then we can draw a right angle triangle



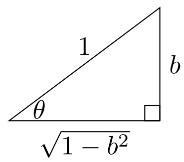
Giving $\sin\theta = \sqrt{1-b^2}$ when $0 \leq \theta \leq \pi/2$

For $\pi/2 \leq \theta\pi$, $\sin\theta = \sin(\pi-\theta)$, making the sign positive

For $\pi \leq \theta 3\pi/2$, $\sin\theta = -\sin(\theta-\pi)$, making the sign negative

For $3\pi/2 \leq \theta 2\pi$, $\sin\theta = -\sin(2\theta-\theta)$, making the sign negative

4. Let $\theta = \arcsin b$ which mean $\sin\theta = b$. If we assume angle θ is acute, then we can draw a right angle triangle



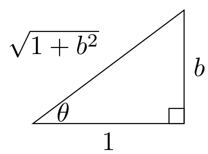
Giving $\tan\theta = \frac{b}{\sqrt{1-b^2}}$ when $0 \le \theta \le \pi/2$

For $\pi/2 \le \theta\pi$, $\tan\theta = -\tan(\pi - \theta)$, making the sign negative

For $\pi \le \theta 3\pi/2$, $\tan\theta = \tan(\theta - \pi)$, making the sign positive

For $3\pi/2 \le \theta 2\pi$, $\tan\theta = -\tan(2\theta - \theta)$, making the sign negative

5. Let $\theta = \arctan b$ which mean $\tan\theta = b$. If we assume angle θ is acute, then we can draw a right angle triangle



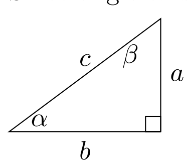
Giving $\cos\theta = \frac{1}{\sqrt{1+b^2}}$ when $0 \le \theta \le \pi/2$

For $\pi/2 \le \theta\pi$, $\cos\theta = -\cos(\pi - \theta)$, making the sign negative

For $\pi \le \theta 3\pi/2$, $\cos\theta = -\cos(\theta - \pi)$, making the sign negative

For $3\pi/2 \le \theta 2\pi$, $\cos\theta = \cos(2\theta - \theta)$, making the sign positive

6. Since angle α is acute, then we can draw a right angle triangle



Note that $\sin\alpha = a/b$ and $\cos\beta = a/b$, and $\beta = \pi/2 - \alpha$

$$\begin{aligned} \sin\alpha &= \cos\beta \\ \arccos(\sin\alpha) &= \beta \\ \arccos(\sin\alpha) &= \pi/2 - \alpha \end{aligned}$$

For $\pi/2 \le \alpha \le \pi$, the expression $\sin\alpha$ becomes $\sin(\pi - \alpha)$, therefore making

$$\begin{aligned} \arccos(\sin\alpha) &= \arccos(\sin(\pi - \alpha)) \\ &= \pi/2 - (\pi/2 - \alpha) \\ &= \alpha \end{aligned}$$

For $\pi \leq \alpha \leq 3\pi/2$, the expression $\sin\alpha$ becomes $-\sin(\alpha - \pi)$, therefore making

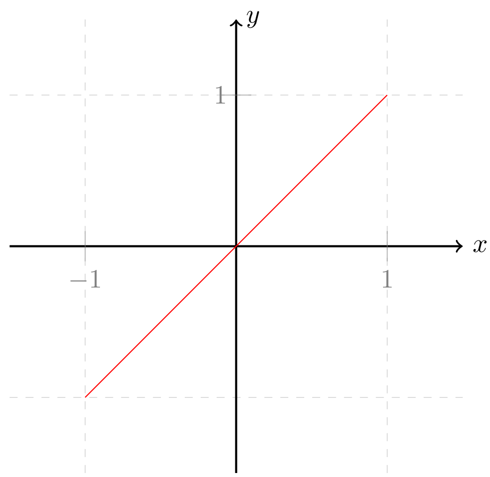
$$\begin{aligned}\arccos(\sin\alpha) &= \arccos(-\sin(\alpha-\pi)) \\ &= \arccos(\sin(-\alpha+\pi)) \\ &= \pi/2 - (-\alpha+\pi) \\ &= \alpha - \pi/2\end{aligned}$$

For $3\pi/2 \leq \alpha \leq 2\pi$, the expression $\sin\alpha$ becomes $-\sin(2\pi - \alpha)$, therefore making

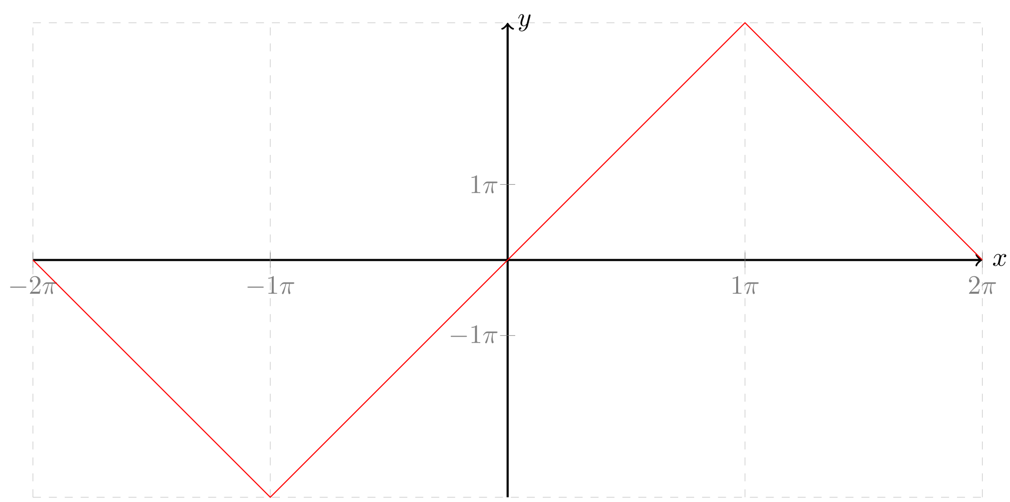
$$\begin{aligned}\arccos(\sin\alpha) &= \arccos(-\sin(2\pi-\alpha)) \\ &= \arccos(\sin(-2\pi+\alpha)) \\ &= \arccos(\sin(\alpha)) \\ &= \pi/2 - \alpha\end{aligned}$$

7. Note that the angles for parts (a) to (e) are less than $\pi/2$, so no change in angle is needed, unlike (f)
   (a) $\arcsin(\sin\pi/11) = \pi/11$
   (b) $\arcsin(\sin 2\pi/11) = 2\pi/11$
   (c) $\arcsin(\sin 3\pi/11) = 3\pi/11$
   (d) $\arcsin(\sin 4\pi/11) = 4\pi/11$
   (e) $\arcsin(\sin 5\pi/11) = 5\pi/11$
   (f)
   $$\begin{aligned}&\arcsin(\sin 6\pi/11) \\ &= \arcsin(\sin(\pi - 6\pi/11)) \\ &= \arcsin(\sin 5\pi/11) \\ &= 5\pi/11\end{aligned}$$

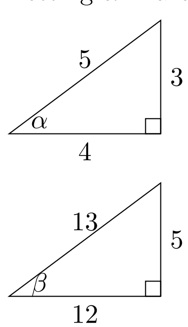
8. Graph of $y = \cos(\arccos x)$



9. Graph of $y = \arccos(\cos x)$



10. Letting $\alpha = \arcsin 3/5$ and $\beta = \arcsin 5/13$. Then drawing the triangles



$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
&= 3/5 \times 12/13 + 4/5 \times 5/13 \\
&= 56/65
\end{aligned}
$$

11. Letting $\alpha = \arctan a$ and $\beta = \arctan b$. Therefore we have $\tan\alpha = a$ amd $\tan\beta = b$

$$
\begin{aligned}
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
&= \frac{a+b}{1-ab}
\end{aligned}
$$

12. Let $\tan\alpha = 1/3$ and $\tan\beta = 1/2$ and $\tan\gamma = 1$

$$
\begin{aligned}
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\tan(\alpha+\beta) &= \frac{1/3 + 1/2}{1 - 1/3 \times 1/2} \\
\tan(\alpha+\beta) &= 1
\end{aligned}
$$

But Since both $\tan(\alpha+\beta) = 1$ and $\tan\gamma = 1$. Therefore $\alpha + \beta = \gamma$

Note: I think the author meant Question 11 instead of 'Problem 8'

13. Using plane geometry, draw another row of squares underneath the diagram and draw lines AH and HC.

Note that the lengths of AH and HC are the same, making triangle AHC an isosceles triangle, therefore $\angle CAH = \angle ACH$

Since triangle AHG and HEC is congruent with triangle BCD, $\angle BAH = \beta$, and $\angle HCE = \beta$

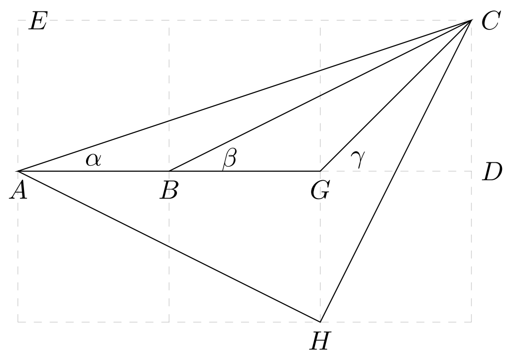
Since triangle FCA is congruent with triangle ACD, $\angle FCA = \alpha$

Therefore

$$
\begin{aligned}
\angle FCA + \angle ACH + \angle HCE &= 90^\circ \\
\alpha + \angle ACH + \beta &= 90^\circ \\
\angle ACH &= 90^\circ - \alpha - \beta
\end{aligned}
$$

Since $\angle\gamma = 45^\circ$, $\angle CAH = \alpha + \beta$ and $\angle CAH = \angle ACH$

$$
\begin{aligned}
\alpha + \beta &= 90^\circ - \alpha - \beta \\
2(\alpha+\beta) &= 90^\circ \\
\alpha + \beta &= 45^\circ \\
\alpha + \beta &= \gamma
\end{aligned}
$$



Note: I think the author meant Question 12 instead of 'Problem 9'

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1. From the graph, $\sin x > 1/2$ when $\pi/6 \le x \le 5\pi/6$ and $13\pi/6 \le x \le 17\pi/6$
2. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

   For $\sin x = -1/2$, the value for α is $\arcsin(-1/2) = -\pi/6$, so the solution is $\pi k + (-1)^k(-\pi/6)$
3. Using the general equation $2n\pi \pm (\alpha)$ for any integer n

   For $\cos x = \sqrt{2}/2$, the value for α is $\arccos(\sqrt{2}/2) = \pi/4$, so the solution is $2\pi n \pm \pi/4$
4. Using the general equation $n\pi + (\alpha)$ for any integer n

   For $\tan x = 1$, the value for α is $\arctan(1) = \pi/4$, so the solution is $\pi n + \pi/4$
5. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

   For $\sin x = -1$, the value for α is $\arcsin(-1) = -\pi/2$, so the solution is $\pi k + (-1)^k(-\pi/2)$

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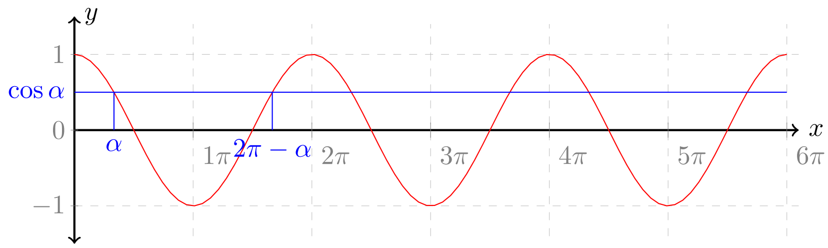
1. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

   For $\sin x = \sin \pi/5$, the value for α is $\pi/5$, so the solution is $\pi k + (-1)^k(\pi/5)$
2. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

   For $\sin x = \sin \pi/2$, the value for α is $\pi/2$, so the solution is $\pi k + (-1)^k(\pi/2)$

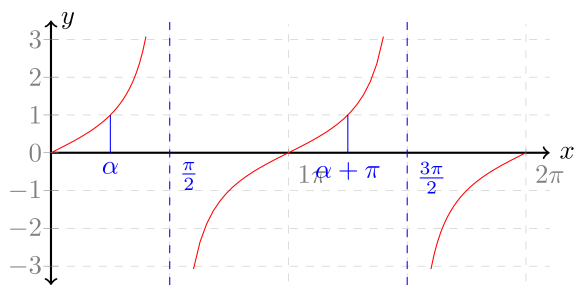
3. 

   Since period of $\cos x$ is 2π, the solutions are $2\pi n + \alpha$ and $2\pi n - \alpha$ for any integer n

4. Using the general equation $2n\pi \pm (\alpha)$ for any integer n

   For $\cos x = \cos \pi/5$, the value for α is $\pi/5$, so the solution is $2\pi n \pm \pi/5$

5. Since period of $\tan x$ is π, then if α is a solution, then the general equation for solutions are $\alpha + \pi n$ for any integer n

6. Using the general equation $n\pi + (\alpha)$ for any integer n

   For $\tan x = \tan \pi/5$, the value for α is $\pi/5$, so the solution is $\pi n + \pi/5$

7. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

$$\begin{aligned}\sin x &= -\sin\alpha \\ \sin x &= \sin(-\alpha) \\ x &= \pi k + (-1)^k(-\alpha)\end{aligned}$$

8. Using the general equation $2n\pi \pm (\alpha)$ for any integer n

$$\begin{aligned}\cos x &= -\cos\alpha \\ \cos x &= \cos(\pi-\alpha) \\ x &= 2n\pi \pm (\pi-\alpha)\end{aligned}$$

9. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

$$\begin{aligned}\sin x &= \sin\alpha \\ \sin x &= \sin(\alpha) \\ x &= \pi k + (-1)^k(\alpha)\end{aligned}$$

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1. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

$$\begin{aligned} \sin 2x &= 1 \\ 2x &= \pi k + (-1)^k \times \arcsin 1 \\ 2x &= \pi k + (-1)^k \times \pi/2 \\ x &= \frac{1}{2}\left(\pi k + (-1)^k \times \pi/2\right) \end{aligned}$$

2. Using the general equation $\pi k + (-1)^k(\alpha)$ for any integer k

$$\begin{aligned} \sin x/2 &= 1/2 \\ x/2 &= \pi k + (-1)^k \times \arcsin 1/2 \\ x/2 &= \pi k + (-1)^k \times \pi/6 \\ x &= 2\left(\pi k + (-1)^k \times \pi/6\right) \end{aligned}$$

3.

$$\begin{aligned} \cos x &= \sin 2x \\ \cos x &= 2\sin x \cos x \\ 0 &= 2\sin x \cos x - \cos x \\ 0 &= \cos x\,(2\sin x - 1) \end{aligned}$$

Therefore there are two set of solutions. Using general solutions $2n\pi \pm (\alpha)$

$$\begin{aligned} \cos x &= 0 \\ x &= 2n\pi \pm (\arccos 0) \\ x &= 2n\pi \pm (\pi/2) \end{aligned}$$

and $\pi k + (-1)^k(\alpha)$

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= 1/2 \\ x &= \pi k + (-1)^k(\arcsin 1/2) \\ x &= \pi k + (-1)^k(\pi/6) \end{aligned}$$

Note: I think the question is supposed to be $\cos x = \sin^2 x$?

4.

$$\begin{aligned}
\sin x &= \sin 3x \\
\sin x &= \sin(2x + x) \\
\sin x &= \sin 2x \cos x + \cos 2x \sin x \\
\sin x &= 2 \sin x \cos x \cos x + (\cos^2 x - \sin^2 x) \sin x \\
\sin x &= 2 \sin x(1 - \sin^2 x) + (1 - \sin^2 x - \sin^2 x) \sin x \\
\sin x &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\
0 &= -4 \sin^3 x + 2 \sin x \\
0 &= -2 \sin x(2 \sin^2 -1)
\end{aligned}$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + (-1)^k(\alpha)$

$$\begin{aligned}
\sin x &= 0 \\
x &= \pi k + (-1)^k(\arcsin 0) \\
x &= \pi k
\end{aligned}$$

The other sets of solutions are

$$\begin{aligned}
2 \sin^2 x - 1 &= 0 \\
\sin^2 x &= 1/2 \\
\sin x &= \pm 1/\sqrt{2} \\
\text{For } \sin x &= 1/\sqrt{2} \\
x &= \pi k + (-1)^k(\arcsin 1/\sqrt{2}) \\
x &= \pi k + (-1)^k \times \pi/4 \\
\text{For } \sin x &= -1/\sqrt{2} \\
x &= \pi k + (-1)^k(\arcsin -1/\sqrt{2}) \\
x &= \pi k + (-1)^k \times -\pi/4
\end{aligned}$$

Note: I think the questions is supposed to be $\sin x = \sin^3 x$?

5.

$$\begin{aligned}
\cos x &= \sin 4x \\
\cos x &= 2\sin 2x \cos 2x \\
\cos x &= 2 \times 2\sin x \cos x \times (\cos^2 x - \sin^2 x) \\
\cos x &= 4\sin x \cos x \times (1 - \sin^2 x - \sin^2 x) \\
0 &= 4\sin x \cos x \times (1 - 2\sin^2 x) - \cos x \\
0 &= \cos x \left(4\sin x(1 - 2\sin^2 x) - 1\right) \\
0 &= \cos x \left(-8\sin^3 x + 4\sin x - 1\right)
\end{aligned}$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + (-1)^k(\alpha)$ and $2n\pi \pm (\alpha)$

$$\begin{aligned}
\cos x &= 0 \\
x &= 2n\pi \pm \arccos 0 \\
x &= 2n\pi \pm \pi/2
\end{aligned}$$

Solving $-8\sin^3 x + 4\sin x - 1 = 0$ requires using cubic functions, which is advanced level and is beyond this book's level

Note: I think the question is supposed to be $\cos x = \sin 4x$?

6.

$$\begin{aligned}
26\sin^2 x + \cos^2 x &= 10 \\
26\sin^2 x + 1 - \sin^2 x - 10 &= 0 \\
25\sin^2 x - 9 &= 0 \\
(5\sin x + 3)(5\sin x - 3) &= 0
\end{aligned}$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + (-1)^k(\alpha)$

For $5\sin x + 3 = 0$

$$\begin{aligned}
5\sin x + 3 &= 0 \\
\sin x &= -3/5 \\
x &= \pi k + (-1)^k(\arcsin -3/5)
\end{aligned}$$

For $5\sin x - 3 = 0$

$$\begin{aligned}
5\sin x - 3 &= 0 \\
\sin x &= 3/5 \\
x &= \pi k + (-1)^k(\arcsin 3/5)
\end{aligned}$$

7.

$$\begin{aligned}
\cos^2 x - \cos x &= \sin^2 x \\
\cos^2 x - \cos x &= 1 - \cos^2 x \\
2\cos^2 x - \cos x - 1 &= 0 \\
(2\cos x + 1)(\cos x - 1) &= 0
\end{aligned}$$

Therefore there are multiple sets of solutions to solve and using the general equation $2\pi n \pm (\alpha)$

For $2\cos x + 1 = 0$

$$\begin{aligned}
2\cos x + 1 &= 0 \\
\cos x &= -1/2 \\
x &= 2\pi n \pm \arccos(-1/2) \\
x &= 2\pi n \pm 2\pi/3
\end{aligned}$$

For $2\cos x - 1 = 0$

$$\begin{aligned}
2\cos x - 1 &= 0 \\
\cos x &= 1/2 \\
x &= 2\pi n \pm \arccos(1/2) \\
x &= 2\pi n \pm \pi/3
\end{aligned}$$

8.

$$\begin{aligned}
3\tan^2 x &= 12 \\
\tan^2 x &= 4 \\
\tan^2 x - 4 &= 0 \\
(\tan x + 2)(\tan x - 2) &= 0
\end{aligned}$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + (\alpha)$

For $\tan x + 2 = 0$

$$\begin{aligned}
\tan x + 2 &= 0 \\
\tan x &= -2 \\
x &= \pi k + \arctan -2
\end{aligned}$$

For $\tan x - 2 = 0$

$$\begin{aligned}
\tan x - 2 &= 0 \\
\tan x &= 2 \\
x &= \pi k + \arctan 2
\end{aligned}$$

9.

$$
\begin{aligned}
\cos 2x &= 2\sin^2 x \\
\cos^2 x - \sin^2 x &= 2\sin^2 x \\
1 - \sin^2 x - \sin^2 x &= 2\sin^2 x \\
1 - 4\sin^2 x &= 0 \\
(1 - 2\sin x)(1 + 2\sin x) &= 0
\end{aligned}
$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + (-1)^k(\alpha)$

For $1 - 2\sin x = 0$

$$
\begin{aligned}
1 - 2\sin x &= 0 \\
\sin x &= 1/2 \\
x &= \pi k + (-1)^k \arcsin 1/2 \\
x &= \pi k + (-1)^k \pi/6
\end{aligned}
$$

For $1 + 2\sin x = 0$

$$
\begin{aligned}
1 + 2\sin x &= 0 \\
\sin x &= -1/2 \\
x &= \pi k + (-1)^k \arcsin -1/2 \\
x &= \pi k + (-1)^k(-\pi/6)
\end{aligned}
$$

10.

$$
\begin{aligned}
\tan^2 x &= \cot x \\
\tan^3 x &= 1 \\
\tan^3 x - 1 &= 0 \\
(\tan x - 1)(\tan^2 x + \tan x + 1) &= 0
\end{aligned}
$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + \alpha$

For $\tan x - 1 = 0$

$$
\begin{aligned}
\tan x - 1 &= 0 \\
\tan x &= 1 \\
x &= \pi k + \arctan 1 \\
x &= \pi k + \pi/4
\end{aligned}
$$

For $\tan^2 x + \tan x + 1 = 0$, apply quadratic formula

$$\tan x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$
$$\tan x = \frac{-1 \pm \sqrt{-3}}{2}$$

Which does not have any solutiosn as the number inside the square root is negative

11.

$$\begin{aligned}\frac{5}{\cos^2 x} &= 7\tan x + 3\\ 5\sec^2 x &= 7\tan x + 3\\ 5(\tan^2 x + 1) &= 7\tan x + 3\\ 5\tan^2 x + 5 &= 7\tan x + 3\end{aligned}$$
$$5\tan^2 x - 7\tan x + 2 = 0$$

Apply quadratic formula

$$\tan x = \frac{7 \pm \sqrt{49 - 4(5)(2)}}{2(5)}$$
$$\tan x = \frac{7 \pm 3}{10}$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + \alpha$

For $\tan x = \frac{7+3}{10}$

$$\begin{aligned}\tan x &= 1\\ x &= \pi k + \arctan 1\\ x &= \pi k + \pi/4\end{aligned}$$

For $\tan x = \frac{7-3}{10}$

$$\begin{aligned}\tan x &= 2/5\\ x &= \pi k + \arctan 2/5\end{aligned}$$

12.

$$\sqrt{3}\tan^2 x + 1 = (1 + \sqrt{3})\tan x$$
$$(\sqrt{3}\tan x + 1)(\tan x + 1) = 0$$

Therefore there are multiple sets of solutions to solve and using the general equation $\pi k + \alpha$

For $\sqrt{3}\tan x + 1 = 0$

$$\begin{aligned}
\sqrt{3}\tan x + 1 &= 0 \\
\sqrt{3}\tan x &= -1 \\
\tan x &= -1/\sqrt{3} \\
x &= \pi n + \arctan -1/\sqrt{3} \\
x &= \pi n + (-\pi/6)
\end{aligned}$$

For $\tan x + 1 = 0$

$$\begin{aligned}
\tan x + 1 &= 0 \\
\tan x &= -1 \\
x &= \pi n + \arctan -1 \\
x &= \pi n + (-\pi/4)
\end{aligned}$$

13. For Solution 1, re-express the equations such that

$$\begin{aligned}
&\text{with n} \\
x &= \pi/6 + 2\pi n/3 \\
&\text{with n+1} \\
&= \pi/6 + 2\pi(n+1)/3 \\
&= \pi/6 + 2\pi n/3 + 2\pi/3 \\
&= 5\pi/6 + 2\pi n/3 \\
&\text{with n+2} \\
&= \pi/6 + 2\pi(n+2)/3 \\
&= \pi/6 + 2\pi n/3 + 4\pi/3 \\
&= 9\pi/6 + 2\pi n/3 \\
&= 3\pi/2 + 2\pi n/3
\end{aligned}$$

Which corresponds to each line of Solution 2, provided that the number n for Solution 1 is multiples of 3

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1. If $\alpha = 1.6$ then

$$\begin{aligned}
x &= \alpha - \tan\alpha \\
&\approx 35.8325
\end{aligned}$$

Since near $x = \pi/2$, the tangent will be very flat and only reaches the x-axis after travelling a long distance

2. If $\alpha = \pi/4$ then

$$\begin{aligned} x &= \alpha - \tan \alpha \\ &\approx -0.2146 \end{aligned}$$

Since the tangent is intersecting x axis to the left

Note, the smaller the α, the closer to it intersecting x-axis at $x = 0$

If $\alpha = 0.1$ then

$$\begin{aligned} x &= \alpha - \tan \alpha \\ &\approx -0.000334 \end{aligned}$$

3. To solve for location of R where $x = 0$ and noting $x \approx \tan x$ for small x value

$$\begin{aligned} x &= \alpha - \tan \alpha \\ 0 &= \alpha - \tan \alpha \\ \tan \alpha &= \alpha \\ \alpha &= 0 \end{aligned}$$

Therefore R will be at origin when we take a tangent at $x = 0$

4. if $x = \pi/2$, then the tangent will be horizontal, therefore not intersecting with x-axis at all

If x is a bit smaller than $\pi/2$, then the tangent will intersect with x-axis to the left (far away)

If x is a bit bigger than $\pi/2$, then the tangent will intersect with x-axis to the right (far away)

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1. Using $A \approx \frac{h}{\sin h/2}$, where $h = \pi/n$

   Using $n = 4$, so $h = \pi/4$ gives $A \approx \frac{\pi/4}{\sin \pi/8}$, which is 2.05234

   Using $n = 8$, so $h = \pi/8$ gives $A \approx \frac{\pi/8}{\sin \pi/16}$, which is 2.01291

2. Since area between $0 \leq x \leq \pi$ is approximately 2, threfore the area under curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ would be around 1

3.

$$
\begin{aligned}
& h\sin x_1 + h\sin x_2 + h\sin x_3 + \cdots + h\sin x_m \\
&= h\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{2\pi}\right) \\
&= h\frac{\sin\frac{n+1}{2}\frac{\pi}{2n}\sin\frac{n}{2}\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} \\
&= h\frac{\sin\frac{n+1}{2}\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\sin\frac{\pi}{4} \\
&= h\frac{\sin\frac{n+1}{n}\frac{\pi}{4}}{\sin\frac{\pi}{2n}}\frac{1}{\sqrt{2}}
\end{aligned}
$$

if n is very large

$$
\begin{aligned}
&\approx h\frac{\sin\pi/4}{\sin\pi/2n}\frac{1}{\sqrt{2}} \\
&\approx \frac{h}{2\sin h/2}
\end{aligned}
$$

letting $n = 8$, giving $h = \pi/8$, gives approximate area of $\frac{\pi/8}{2\sin\pi/16} \approx 1.00645$