Solutions for *Trigonometry* by Gelfand & Saul

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Introduction

Trigonometry by Gelfand and Saul is often recommended as a precalculus text for self-study. However, those who are learning without the help of a teacher can struggle with the lack of solutions to exercises in the text. A partial set of solutions for *Trigonometry* (odd numbered exercises only) has been published by John Beach[1]. It is hoped that this document will eventually contain a complete set of solutions. Contributions are welcome. These can take the form of pull requests or issues submitted to the project's GitHub repository[2].

Chapter 0: Trigonometry

Page 8

1. Statement I applies:

$$\begin{aligned} c^2 &= a^2 + b^2 = 10^2 + 24^2 = 100 + 576 = 676 \\ c &= \sqrt{676} = 26 \end{aligned}$$

2. Statement I applies:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 9^2 &= 41^2 \\ a^2 + 81 &= 1681 \\ a^2 &= 1600 \\ a &= \sqrt{1600} = 40 \end{aligned}$$

[1] https://jbeach50.weebly.com/gelfand–saul-trig-solutions.html
[2] https://github.com/philip-healy/gelfand-trigonometry-solutions

3. $5^2 + 12^2 = 25 + 144 = 169 = 13^2$. By Statement II, a right triangle exists with legs of length 5 and 12, and hypotenuse of length 13.

4. Statement I applies:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 1^2 &= 3^2 \\ a^2 + 1 &= 9 \\ a^2 &= 8 \\ a &= \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \end{aligned}$$

5. Statement I applies, where $a = b$:

$$\begin{aligned} a^2 + a^2 &= c^2 \\ a^2 + a^2 &= 1^2 \\ 2a^2 &= 1 \\ a^2 &= \frac{1}{2} \\ a &= \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \end{aligned}$$

6. From the diagram at the bottom of Page 11, we can see the shorter leg is half the length of the hypotenuse. So in this instance the shorter leg has length 1/2. We can use Statement 1 to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + \left(\frac{1}{2}\right)^2 &= 1^2 \\ a^2 + \frac{1}{4} &= 1 \\ a^2 &= \frac{3}{4} \\ a &= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} \end{aligned}$$

7. For any point Y, we can draw a triangle with sides AY, BY and AB. Let a be the length of side AY, b be the length of side BY and c be the length of side AB. According to Statement II, the subset of these triangles where $a^2 + b^2 = c^2$ are right triangles with legs of length a and b and hypotenuse c. Let X be the subset of Y that are vertices of these right triangles. This set of points describes a circle with its centre at the midpoint of AB, and radius $AB/2$.

8.

Page 9

1. $6^2 + 8^2 = 36 + 64 = 100 = 10^2$. By Statement II on Page 7 (converse of the Pythagorean Theorem), this is a right triangle.

2. 10-24-26 (Exercise 1), 9-40-41 (Exercise 2), 5-12-13 (Exercise 3)

3. Using the Pythagorean Theorem:

$$\begin{aligned} c^2 &= a^2 + b^2 = 8^2 + 15^2 = 64 + 225 = 289 \\ c &= \sqrt{289} = 17 \end{aligned}$$

4. The first column in the table increases by 3, the second increases by 4 and the third increases by 5. Continuing to add rows yields triangles 12-16-20, 15-20-25 and 18-24-30.

5. Shortest side with length 10: 10-24-26. Shortest side with length 15: 15-36-39.

6. Multiplying all sides by the common denominator (5), we get a similar triangle with sides $15/5 = 3$, $20/5 = 4$ and 5. We know that this is a right triangle from the table in Question 4.

7. To find a similar triangle with shorter leg 1, divide all sides by 3, resulting in sides 1-4/3-5/3. To find a similar triangle with longer leg 1, divide all sides by 4, resulting in sides 3/4-1-5/4.

8. To find a similar triangle with hypotenuse 1, divide all sides by 13, resulting in sides 5/13-12/13-1. To find a similar triangle with shorter leg 1, divide all sides by 5, resulting in sides 1-12/5-13/5. To find a similar triangle with longer leg 1, divide all sides by 12, resulting in sides 5/12-1-13/12.

9. To formula for the area of a triangle is $\frac{1}{2}bh$ where b is the length of the base and h is the height. For right triangles, finding the area is easy: one leg is the base and the other leg is the height. For other triangles, finding the height is more difficult: we need to find the length of the altitude drawn from the base. The triangles with sides 5-12-13 and 9-12-15 are both right triangles: see Exercise 3 on Page 8 and Exercise 4 on Page 9. The triangle with sides 13-14-15 is not a right triangle. We can confirm this using Statement I: $a^2 + b^2 = 13^2 + 14^2 = 365$, $c^2 = 15^2 = 225, a^2 + b^2 \neq c^2$. However, if we join the 5-12-13 and 9-12-15 triangles using their equal legs, the resulting triangle has the dimensions we are looking for: 13-14-15. The base of this combined triangle has length $5 + 9 = 14$. We also know the length of the altitude from the base of the combined triangle: 12. So, the area of the 13-14-15 triangle is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ units squared.

10. (a)

    (b)

Page 11

1. $\frac{1}{\sqrt{2}}$ (see the solution for Question 5 on page 8).
   Challenge: $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ (multiplying above and below by $\sqrt{2}$). $\sqrt{2}$ is given to 9 decimal places in the diagram on the top of page 11: 1.4141213562373. Dividing this decimal representation by 2 (using long division if necessary) yields a figure of 0.707060678.

2. $c^2 = a^2 + b^2 = 3^2 + 3^2 = 9 + 9 = 18$. $c = \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$.

3. The hypotenuse of a 30° right triangle is double the length of the shorter leg. In this instance the hypotenuse is 10 units long. We can use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 5^2 &= 10^2 \\ a^2 + 25 &= 100 \\ a^2 &= 75 \\ a &= \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3} \end{aligned}$$

4. We can solve these by finding similar triangles to the 30° right triangle with sides 1-$\sqrt{3}$-2, or the 45° right triangle with sides 1-1-$\sqrt{2}$.
   (a) $x = \sqrt{3}$, $y = 2$
   (b) $x = \frac{1}{\sqrt{3}}$, $y = \frac{2}{\sqrt{3}}$
   (c) $x = 1/2$, $y = \sqrt{3}/2$
   (d) $x = 4\sqrt{3}$, $y = 8$
   (e) $x = y = 2\sqrt{2}$
   (f) $x = 5$, $y = 5\sqrt{2}$

Page 14 (Examples)

1. Why didn't we need to compare 3^2 with $2^2 + 4^2$, or 2^2 with $3^2 + 4^2$?
   The obtuse angle will always be opposite the longest side.

2. This conclusion is *incorrect*. Why?
   From the footnote at the beginning of Chapter 0: *"Given three arbitrary lengths... they form a triangle if and only if the sum of any two of them is greater than the third."* In this case $1 + 2 = 3$ which is equal to (not greater than) the third side.

Page 14 (Exercise)

1. (a) $c^2 = 8^2 = 64$. $a^2 + b^2 = 6^2 + 7^2 = 36 + 49 = 85$. $c^2 < a^2 + b^2$, so the triangle is acute.
   (b) $c^2 = 10^2 = 100$. $a^2 + b^2 = 6^2 + 8^2 = 36 + 64 = 100$. $c^2 = a^2 + b^2$, so the triangle is a right triangle.
   (c) a and b are the same as in question b), but c is smaller, so the triangle is acute.
   (d) a and b are the same as in question b), but c is larger, so the triangle is obtuse.
   (e) $c^2 = 12^2 = 144$. $a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$. $c^2 < a^2 + b^2$, so the triangle is acute.
   (f) $c^2 = 14^2 = 196$. $a^2 + b^2 = 169$, as above. $c^2 > a^2 + b^2$, so the triangle is obtuse.
   (g) The sum of two sides must be larger than the third, but $12 + 5 = 17$ in this case.

Chapter 1: Trigonometric Ratios in a Triangle

Page 23

1. (a) $\sin\alpha = 5/13$
   (b) $\sin\alpha = 4/5$
   (c) $\sin\alpha = 5/13$
   (d) $c = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$. $\sin\alpha = 8/10$.
   (e) $\sin\alpha = 3/5$
   (f) $\sin\alpha = 12/13$
   (g) $\sin\alpha = 3/5$
   (h) $c = \sqrt{7^2 + 3^2} = \sqrt{58}$. $\sin\alpha = 7/\sqrt{58}$.

2. (a) $\sin\beta = 12/13$
   (b) $\sin\beta = 3/5$
   (c) $\sin\beta = 12/13$
   (d) $\sin\beta = 6/10$
   (e) $\sin\beta = 4/5$
   (f) $\sin\beta = 5/13$
   (g) $\sin\beta = 4/5$
   (h) $\sin\beta = 3/\sqrt{58}$

3. The example 30-60-90 triangle given on page 11 has sides 1, $\sqrt{3}$, 2. Let β represent the 60°angle. The opposite leg b has length $\sqrt{3}$. The hypotenuse c has length 2. So, $\sin\beta = b/c = \sqrt{3}/2 \approx 1.732/2 = 0.866$.
   Crossing off the numbers listed:
   ~~0.1~~ ~~0.2~~ ~~0.3~~ ~~0.4~~ ~~0.5~~ ~~0.6~~ ~~0.7~~ ~~0.8~~ 0.9

Page 25

1. The Altitude-on-Hypotenuse Theorem tells us that when an altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other. Therefore, the triangles with sides a-b-c, a-p-d and d-b-q are similar, and the ratio for $\sin\alpha$ appears in all of them:
   (a) b/c
   (b) d/a
   (c) q/b
2. (a) $\sin\alpha = h/b$
   (b) Multiplying both sides of formula above by b: $h = b\sin\alpha$
   (c) Substituting $b\sin\alpha$ for h, the formula for the area of ABC can be rewritten as: $bc\sin\alpha/2$.
   (d) $\sin\beta = h/a$. Rewriting this in terms of h: $h = a\sin\beta$. Substituting this for h in the area formula: $ac\sin\beta/2$.
   (e) Let h_2 represent the altitude from A to BC. $sin\beta = h_2/c$. Rewriting in terms of h_2, we get $h_2 = c\sin\beta$.
3. (a) Expressing h in terms of $\sin\alpha$ and b:

$$\begin{aligned}\sin\alpha &= \frac{h}{b}\\ h &= b\sin\alpha\end{aligned}$$

   Expressing h in terms of $\sin\beta$ and a:

$$\begin{aligned}\sin\beta &= \frac{h}{a}\\ h &= a\sin\beta\end{aligned}$$

   (b) Both expresssions are equal to h:

$$a\sin\beta = h = b\sin\alpha$$

   (c) Expressing h_2 in terms of $\sin\beta$ and c:

$$\begin{aligned}\sin\beta &= \frac{h_2}{c}\\ h_2 &= c\sin\beta\end{aligned}$$

Expressing h_2 in terms of $\sin\gamma$ and b:

$$\sin\gamma = \frac{h_2}{b}$$
$$h_2 = b\sin\gamma$$

Both expressions are equal to h_2:

$$b\sin\alpha = h_2 = c\sin\gamma$$

(d) i. We can rewrite the result from part (b) so that the expressions on each side are fractions with sine denominators:

$$a\sin\beta = b\sin\alpha$$
$$\frac{a\sin\beta}{\sin\alpha\sin\beta} = \frac{b\sin\alpha}{\sin\alpha\sin\beta}$$
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

ii. We can rewrite the result from part (c) similarly:

$$c\sin\beta = b\sin\gamma$$
$$\frac{c\sin\beta}{\sin\beta\sin\gamma} = \frac{b\sin\gamma}{\sin\beta\sin\gamma}$$
$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta}$$

We can derive the Law of Sines by combining results i. and ii. using the common expression $b/\sin\beta$:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Page 26

1. (a) $\cos\alpha = 12/13$. $\cos\beta = 5/13$.
   (b) $\cos\alpha = 3/5$. $\cos\beta = 4/5$.
   (c) $\cos\alpha = 12/13$. $\cos\beta = 5/13$.
   (d) $\cos\alpha = 6/10$. $\cos\beta = 8/10$.
   (e) $\cos\alpha = 4/5$. $\cos\beta = 3/5$.
   (f) $\cos\alpha = 5/13$. $\cos\beta = 12/13$.
   (g) $\cos\alpha = 4/5$. $\cos\beta = 3/5$.
   (h) $\cos\alpha = 3/\sqrt{58}$. $\cos\beta = 7/\sqrt{58}$.
2. (a) $c = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$. $\cos\alpha = 8/10$. $\cos\beta = 6/10$.

(b) $c = \sqrt{5^2 + 12^2} = \sqrt{25 + 155} = \sqrt{169} = 13$. $\cos\alpha = 12/13$. $\cos\beta = 5/13$.

(c) Scaling up the 1-$\sqrt{3}$-2 30° triangle gives us a value of 20 units for the length of c. Next, we will use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 10^2 + b^2 &= 20^2 \\ b^2 &= 400 - 100 = 300 \\ b &= \sqrt{300} = \sqrt{100}\sqrt{3} = 10\sqrt{3} \end{aligned}$$

We can now find $\cos\alpha$ and $\cos\beta$:

$$\begin{aligned} \cos\alpha &= \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2} \\ \cos\beta &= \frac{10}{20} = \frac{1}{2} \end{aligned}$$

(d) The triangle is congruent to the one above, so the solution is the same.

(e) Consider the 45° right triangle with legs of length 1 and hypotenuse $\sqrt{2}$. $\cos\alpha = \cos\beta = 1/\sqrt{2}$.

(f) $c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. $\cos\alpha = 3/5$. $\cos\beta = 4/5$.

(g) $b = x\sqrt{3}$. $\cos\alpha = x\sqrt{3}/2x = \sqrt{3}/2$. $\cos\beta = x/2x = 1/2$.

3. The Altitude-on-Hypotenuse Theorem tells us that when an altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other. Therefore, the triangles with sides a-b-c, a-p-d and d-b-q are similar, and the ratio for $\cos\alpha$ appears in all of them:

   (a) a/c

   (b) p/a

   (c) d/b

Page 28

1. In this instance, $\alpha = 29°$, $\beta = 61°$, and $\alpha + \beta = 90°$. According to the theorem above, if $\alpha + \beta = 90°$, then $\sin\alpha = \cos\beta$.

2. $x = 90 - 35 = 55°$

3. If $\alpha + \beta = 90°$, then $\beta = 90° - \alpha$. According to the theorem above, $\sin\alpha = \cos\beta$. Substituting $(90 - \alpha)$ for β: $\sin\alpha = \cos(90 - \alpha)$.

Page 29

First, we need to find the length of the hypotenuse: $c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

1. $\sin^2 \alpha = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$
2. $\sin^2 \beta = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$
3. $\cos^2 \alpha = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$ (same as $\sin^2 \beta$)
4. $\cos^2 \beta = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$ (same as $\sin^2 \alpha$)
5. $\sin^2 \alpha + \cos^2 \alpha = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$
6. $\sin^2 \alpha + \cos^2 \beta = \frac{16}{25} + \frac{16}{25} = \frac{32}{25}$
7. $\cos^2 \alpha + \sin^2 \beta = \frac{9}{25} + \frac{9}{25} = \frac{18}{25}$

Page 30

1. $\sin^2 \alpha + \cos^2 \alpha = \left(\dfrac{4}{5}\right)^2 + \left(\dfrac{3}{5}\right)^2 = \dfrac{16}{25} + \dfrac{9}{25} = \dfrac{25}{25} = 1$
2. It's not an error. According to the corollary of the Pythagoream Theorem, this a right triangle: $a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 = c^2$.
3. $\sin^2 \beta + \cos^2 \beta = \left(\dfrac{3}{5}\right)^2 + \left(\dfrac{4}{5}\right)^2 = \dfrac{9}{25} + \dfrac{16}{25} = \dfrac{25}{25} = 1$
4. $\cos^2 \alpha + \sin^2 \alpha = 1$
$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169} = \frac{144}{169}$$
$$\cos \alpha = \sqrt{\frac{144}{169}} = \frac{12}{13}$$
5. $\cos^2 \alpha + \sin^2 \alpha = 1$
$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{5}{7}\right)^2 = 1 - \frac{25}{49} = \frac{24}{49}$$
$$\cos \alpha = \sqrt{\frac{24}{49}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{49}} = \frac{2\sqrt{6}}{7}$$

6. We will follow the proof at the bottom of Page 29:

$$\begin{aligned} sin^2\alpha + \sin^2\beta &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\ &= \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{a^2+b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

7. Again, we will follow the proof at the bottom of Page 29:

$$\begin{aligned} cos^2\alpha + \cos^2\beta &= \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 \\ &= \frac{b^2}{c^2} + \frac{a^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{a^2+b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

Page 31

1.

angle x	$\sin x$	$\cos x$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
α	$\frac{4}{5}$	$\frac{3}{5}$
β	$\frac{3}{5}$	$\frac{4}{5}$

2. $\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°$

3. $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

4. We can observe from the table that $\sin x$ increases with the size of an acute angle ($\sin 30° < \sin 45° < \sin 60°$), while $\cos x$ decreases with the size of an acute angle. You can compare the fractions or convert to decimal make sure. We know that $\sin\alpha = \frac{4}{5}$. We also know that α is an acute angle.
Is it larger or smaller than $30°$*?* Larger, $\frac{4}{5} > \frac{1}{2}$ so $\sin\alpha > \sin 30°$.
Than $45°$*?* Larger, $\frac{4}{5} > \frac{1}{\sqrt{2}}$ so $\sin\alpha > \sin 45°$.
Than $60°$*?* Smaller, $\frac{4}{5} < \frac{\sqrt{3}}{2}$ so $\sin\alpha < \sin 60°$.

Page 33 (First)

1. As the angle α get smaller, the ratio of the opposite side to the hypotenuse approaches 0.

2. Recall from the theorem on page 28 that if $\alpha + \beta = 90°$, then $\sin\alpha = \cos\beta$ and $\cos\alpha = \sin\beta$. So, if $\sin 90° = 1$, then $\cos 0° = 1$.

3. $\sin^2 0° + \cos^2 0° = 0^2 + 1^2 = 0 + 1 = 1$

4. $\sin^2 90° + \cos^2 90° = 1^2 + 0^2 = 1 + 0 = 1$

5. Our friend is mistaken; the sine of an angle can never be greater than 1.

Page 33 (Second)

1.

$\sin 0° + \cos 0°$	$0 + 1$	1
$\sin 30° + \cos 30°$	$\frac{1}{2} + \frac{\sqrt{3}}{2}$	1.366 (approx.)
$\sin 45° + \cos 45°$	$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$	1.414 (approx.)
$\sin 60° + \cos 60°$	$\frac{\sqrt{3}}{2} + \frac{1}{2}$	1.366 (approx.)
$\sin 90° + \cos 90°$	$1 + 0$	1
$\sin\alpha + \cos\alpha$, where α is the smaller...	$\frac{3}{5} + \frac{4}{5}$	1.4
$\sin\alpha + \cos\alpha$, where α is the larger...	$\frac{4}{5} + \frac{3}{5}$	1.4

2. If $\sin\alpha = 1$, then $\cos\alpha = 0$ and $\sin\alpha + \cos\alpha = 1$. If $\cos\alpha = 1$, then $\sin\alpha = 0$ and $\sin\alpha + \cos\alpha = 1$. Otherwise, $\sin\alpha < 1$ and $\cos\alpha < 1$, so $\sin\alpha + \cos\alpha < 2$.

3. First we will expand and simplify $(\sin\alpha + \cos\alpha)^2$:

$$\begin{aligned}(\sin\alpha + \cos\alpha)^2 &= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha \\ &= (\sin^2\alpha + \cos^2\alpha) + 2\sin\alpha\cos\alpha \\ &= 1 + 2\sin\alpha\cos\alpha\end{aligned}$$

   We know that $0 \le \sin\alpha \le 1$ and $0 \le \cos\alpha \le 1$ because α is acute. So $2\sin\alpha\cos\alpha$ is the product of three nonnegative numbers, and is itself a nonnegative number. A nonnegative number added to 1 results in a number ≥ 1. Therefore, $1+2\sin\alpha\cos\alpha \ge 1$. The square root of a number ≥ 1 is itself ≥ 1. Therefore, $\sqrt{1+2\sin\alpha\cos\alpha} \ge 1$. Rewriting the expression on the left: $\sqrt{1+2\sin\alpha\cos\alpha} = \sqrt{(\sin\alpha+\cos\alpha)^2} = \sin\alpha + \cos\alpha$. So, $\sin\alpha + \cos\alpha \ge 1$.

4. $\sin 45^\circ + \cos 45^\circ = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

5. You should notice that the values for $\sin\alpha+\cos\alpha$ increases with larger *alpha* when $0^\circ \le \alpha < 45^\circ$, reaches a maximum value when $\alpha = 45^\circ$, then decreases with larger α when $45^\circ < \alpha \le 90^\circ$.

Page 35

1.

$(\sin 0^\circ)(\cos 0^\circ)$	$0 \cdot 1$	0
$(\sin 30^\circ)(\cos 30^\circ)$	$\frac{1}{2} \cdot \frac{\sqrt{3}}{2}$	0.433 (approx.)
$(\sin 45^\circ)(\cos 45^\circ)$	$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$	0.5
$(\sin 60^\circ)(\cos 60^\circ)$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$	0.433 (approx.)
$(\sin\alpha)(\cos\alpha)$, where α is the smaller...	$\frac{3}{5} \cdot \frac{4}{5}$	0.48
$(\sin\alpha)(\cos\alpha)$, where α is the larger...	$\frac{4}{5} \cdot \frac{3}{5}$	0.48

   How large can the product $(\sin\alpha)(\cos\alpha)$ *get?* We can see from the table that the maximum value of the product appears to be when $\alpha = 45^\circ$.

Page 37

1. $\cos\alpha = 3/5$, $\cos\beta = 4/5$, $\sin\alpha = 4/5$, $\sin\beta = 3/5$, $\tan\alpha = 4/3$, $\tan\beta = 3/4$, $\cot\alpha = 3/4$, $\cot\beta = 4/3$.

2. We can show that this assumption is correct using the corollary of the Pythagorean Theorem: $a^2 + b^2 = 3^2 + 4^2 = 25 = c^2$.

3. $\cos\alpha = a/c$, $\cos\beta = b/c$, $\sin\alpha = b/c$, $\sin\beta = a/c$, $\tan\alpha = b/a$, $\tan\beta = a/b$, $\cot\alpha = a/b$, $\cot\beta = b/a$.

4. $c = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$. $\cos\alpha = 12/13$. $\cos\beta = 5/13$. $\cot\alpha = 12/5$. $\cot\beta = 5/12$.

5. First, we will use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 7^2 &= 25^2 \\ a^2 + 49 &= 625 \\ a^2 &= 576 \\ a &= 24 \end{aligned}$$

   We can now find the numerical values that were asked for: $\cos\alpha = 24/25$, $\cos\beta = 7/25$, $\cot\alpha = 24/7$, $\cot\beta = 7/24$.

6. $\frac{a}{c} = \sin\alpha = \cos\beta$
   $\frac{b}{c} = \cos\alpha = \sin\beta$
   $\frac{a}{b} = \tan\alpha = \cot\beta$
   $\frac{b}{a} = \cot\alpha = \tan\beta$

7. First, we will use the Pythagorean Theorem to find the length of the other leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 3^2 &= 5^2 \\ a^2 + 9 &= 25 \\ a^2 &= 16 \\ a &= 4 \end{aligned}$$

   We can now find the numerical values that were asked for: $\cos\alpha = 4/5$, $\cot\alpha = 4/3$.

8. If $\tan\alpha = 1$, then $a/b = 1$, implying that $a = b$ and $\alpha = 45°$. $\cos\alpha = \cos 45° = 1/\sqrt{2}$. $\cot\alpha = 1/1 = 1$.

9. $\tan 45° = 1/1 = 1$.

10. $\tan 30° = 1/\sqrt{3} \approx 0.57735$.

11. $\tan 45° + \sin 30° = 1 + \frac{1}{2} = \frac{3}{2}$. We don't need a calculator because both numbers are rational.

Chapter 2: Relations among Trigonometric Ratios

Page 43

1. $\cos\alpha = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \sqrt{1 - \frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$

   $\tan\alpha = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}$

   $\cot\alpha = \frac{15}{8}$

2. Let the length of the adjacent leg a be $\frac{3}{7}$ and the length of the hypotenuse be 1 (see the first triangle diagram on page 44).

   $$\sin\alpha = \sqrt{1-a^2} = \sqrt{1 - \left(\frac{3}{7}\right)^2} = \sqrt{1 - \frac{9}{49}} = \sqrt{\frac{40}{49}} = \frac{\sqrt{4}\sqrt{10}}{\sqrt{49}} = \frac{2\sqrt{10}}{7}$$

   $$\tan\alpha = \frac{\sqrt{1-a^2}}{a} = \frac{\frac{2\sqrt{10}}{7}}{\frac{3}{7}} = \frac{2\sqrt{10}}{3}$$

   $$\cot\alpha = \frac{a}{\sqrt{1-a^2}} = \frac{3}{2\sqrt{10}}$$

3. $\sin\alpha = \sqrt{1-b^2}$, $\tan\alpha = \dfrac{\sqrt{1-b^2}}{b}$, $\cot\alpha = \dfrac{b}{\sqrt{1-b^2}}$

4. $\sin\alpha = \dfrac{d}{\sqrt{1+d^2}}$, $\cos\alpha = \dfrac{1}{\sqrt{1+d^2}}$, $\cot\alpha = \dfrac{1}{d}$

5.

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	a	$\sqrt{1-a^2}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{\sqrt{1-a^2}}{a}$
$\cos\alpha$	$\sqrt{1-a^2}$	a	$\frac{\sqrt{1-a^2}}{a}$	$\frac{a}{\sqrt{1-a^2}}$
$\tan\alpha$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	a	$\frac{1}{a}$
$\cot\alpha$	$\frac{1}{\sqrt{1+a^2}}$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{a}$	a

Page 45 (First)

1. Given in text

2. $\sin^2 45^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

3.

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	$\sin\alpha$	$\sqrt{1-\sin^2\alpha}$	$\frac{a}{\sqrt{1-\sin^2\alpha}}$	$\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$
$\cos\alpha$	$\sqrt{1-\cos^2\alpha}$	$\cos\alpha$	$\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$	$\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}$
$\tan\alpha$	$\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$	$\frac{1}{\sqrt{1+\tan^2\alpha}}$	$\tan\alpha$	$\frac{1}{\tan\alpha}$
$\cot\alpha$	$\frac{1}{\sqrt{1+\cot^2\alpha}}$	$\frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$	$\frac{1}{\cot\alpha}$	$\cot\alpha$

Page 45 (Second)

1. $\tan\alpha = \frac{a}{b} = \cot\beta$

2. $\cot\alpha = \frac{b}{a} = \tan\beta$

3. $\sec\alpha = \frac{c}{a} = \csc\beta$

4. $\csc\alpha = \frac{c}{b} = \sec\beta$

Page 47

1. (a) $\sin^2 30^\circ + \cos^2 30^\circ = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

   (b) $\sin^2 45^\circ + \cos^2 45^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

   (c) $\sin^2 60^\circ + \cos^2 60^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

2. $\sin^2\alpha + \cos^2\alpha = 1$

$$\left(\frac{\sqrt{5}}{4}\right)^2 + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \left(\frac{\sqrt{5}}{4}\right)^2 = 1 - \frac{5}{16} = \frac{11}{16}$$

$$\cos\alpha = \sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4}$$

3. $\sin^2\alpha + \cos^2\alpha = 1$

$$\sin^2\alpha + \left(\frac{2}{3}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\sin\alpha = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

4. $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{1}{\sqrt{3}}$

$$\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{3}$$

$$\frac{\sin^2\alpha}{1-\sin^2\alpha} = \frac{1}{3}$$

$$3\sin^2\alpha = 1 - \sin^2\alpha$$

$$4\sin^2\alpha = 1$$

$$\sin^2\alpha = \frac{1}{4}$$

$$\sin\alpha = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$
\begin{aligned}
\frac{\sin\alpha}{\cos\alpha} &= \tan\alpha = \frac{1}{\sqrt{3}} \\
\frac{\sin^2\alpha}{\cos^2\alpha} &= \frac{1}{3} \\
\frac{1-\cos^2\alpha}{\cos^2\alpha} &= \frac{1}{3} \\
3 * \frac{1-\cos^2\alpha}{\cos^2\alpha} &= 3 * \frac{1}{3} \\
\frac{3*(1-\cos^2\alpha)}{\cos^2\alpha} &= 1 \\
3*(1-\cos^2\alpha) &= \cos^2\alpha \\
3 - 3\cos^2\alpha &= \cos^2\alpha \\
3 &= 4\cos^2\alpha \\
\frac{3}{4} &= \cos^2\alpha \\
\frac{\sqrt{3}}{2} &= \cos\alpha
\end{aligned}
$$

And then to check our solution we can calculate the fraction we are given $\frac{1}{\sqrt{3}}$ from our $\cos\alpha$ and $\sin\alpha$ fractions.

$$
\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}
$$

$$
\frac{2}{2*\sqrt{3}}
$$

$$
\frac{1}{\sqrt{3}}
$$

5. (a) $\cot x \sin x = \left(\frac{1}{\tan x}\right)\sin x = \frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}} = \frac{\sin x \cos x}{\sin x} = \cos x$

   (b) $\frac{\tan x}{\sin x} = \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{\frac{\sin x}{\cos x}\cdot\frac{1}{\sin x}}{\sin x\cdot\frac{1}{\sin x}} = \frac{\frac{\sin x}{\sin x\cos x}}{1} = \frac{\sin x}{\sin x\cos x} = \frac{1}{\cos x}$

   (c) $\cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha) = \cos^2\alpha - 1 + \cos^2\alpha = 2\cos^2\alpha - 1$

   (d) This one is tricky. You might need to try a few different approaches (squaring above and below, multiplying above and below by $\cos\alpha\sin\alpha$). Eventually it becomes clear that you need to multiply above and below by $(1-\cos\alpha)$ and find a way to cancel out the $\sin\alpha$ factor in the numerator:

$$
\begin{aligned}
\frac{\sin\alpha}{1+\cos\alpha} &= \frac{\sin\alpha(1-\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)} = \frac{\sin\alpha(1-\cos\alpha)}{1-\cos\alpha+\cos\alpha-\cos^2\alpha} \\
&= \frac{\sin\alpha(1-\cos\alpha)}{1-\cos^2\alpha} = \frac{\sin\alpha(1-\cos\alpha)}{1-(1-\sin^2\alpha)} = \frac{\sin\alpha(1-\cos\alpha)}{1-1+\sin^2\alpha} \\
&= \frac{\sin\alpha(1-\cos\alpha)}{\sin^2\alpha} = \frac{1-\cos\alpha}{\sin\alpha}
\end{aligned}
$$

(e) $\dfrac{\sin^2\alpha + 2\cos^2\alpha - 1}{\cot^2\alpha} = \dfrac{1-\cos^2\alpha + 2\cos^2\alpha - 1}{\cot^2\alpha} = \dfrac{\cos^2\alpha}{\left(\frac{\cos\alpha}{\sin\alpha}\right)^2}$

$$= \frac{\cos^2\alpha}{\frac{\cos^2\alpha}{\sin^2\alpha}} = \frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha}$$
$$= \sin^2\alpha$$

(f) $\cos^2\alpha = \dfrac{\cos^2\alpha}{1} = \dfrac{\cos^2\alpha}{\cos^2\alpha + \sin^2\alpha}$

$$= \frac{\frac{\cos^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha + \sin^\alpha}{\cos^2\alpha}} = \frac{1}{\frac{\cos^2\alpha}{\cos^2\alpha} + \frac{\sin^2\alpha}{\cos^2\alpha}}$$
$$= \frac{1}{1+\tan^2\alpha}$$

(g) $\sin^2\alpha = \dfrac{\sin^2\alpha}{1} = \dfrac{\cos^2\alpha}{\cos^2\alpha + \sin^2\alpha}$

$$= \frac{\frac{\sin^2\alpha}{\sin^2\alpha}}{\frac{\cos^2\alpha + \sin^\alpha}{\sin^2\alpha}} = \frac{1}{\frac{\cos^2\alpha}{\sin^2\alpha} + \frac{\sin^2\alpha}{\sin^2\alpha}}$$
$$= \frac{1}{\cot^2\alpha + 1}$$

(h) $\dfrac{1-\cos\alpha}{1+\cos\alpha} = \dfrac{(1-\cos\alpha)(1+\cos\alpha)}{(1+\cos\alpha)(1+\cos\alpha)} = \dfrac{1+\cos\alpha - \cos\alpha - \cos^2\alpha}{(1+\cos\alpha)^2}$

$$= \frac{1-\cos^2\alpha}{(1+\cos\alpha)^2} = \frac{\sin^2\alpha}{(1+\cos\alpha)^2}$$
$$= \left(\frac{\sin\alpha}{1+\cos\alpha}\right)^2$$

(i) The key to solving this one is the formula for factoring a difference of cubes: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$$\frac{\sin^3\alpha - \cos^3\alpha}{\sin\alpha - \cos\alpha} = \frac{(\sin\alpha - \cos\alpha)(\sin^2\alpha + \sin\alpha\cos\alpha + \cos^2\alpha)}{\sin\alpha - \cos\alpha}$$
$$= \sin^2\alpha + \sin\alpha\cos\alpha + \cos^2\alpha$$
$$= 1 + \sin\alpha\cos\alpha$$

6. (a) We can rewrite the LHS to show that $\sin^4\alpha - \cos^4\alpha = \cos^2\alpha - \sin^2\alpha$:

$$\sin^4\alpha - \cos^4\alpha = (\sin^2\alpha + \cos^2\alpha)(\sin^2\alpha - \cos^2\alpha) = 1(\sin^2\alpha - \cos^2\alpha)$$
$$= \sin^2\alpha - \cos^2\alpha$$

Answer: There are no angles α for which $\sin^4\alpha - \cos^4\alpha > \cos^2\alpha - \sin^2\alpha$ because the expressions on either side of the inequality are equivalent.

(b) $\sin^4\alpha - \cos^4\alpha >= \cos^2\alpha - \sin^2\alpha$ for all angles α because the expressions on either side of the inequality are equivalent.

7. If we rewrite $2\sin\alpha\cos\alpha$ as a fraction, we can divide above and below by $\cos\alpha$ to convert the numerator and denominator into expressions in terms of $\tan\alpha$:

$$\begin{aligned}
2\sin\alpha\cos\alpha &= \frac{2\sin\alpha\cos\alpha}{1} = \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha+\cos^2\alpha}\\
&= \frac{\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha}}{\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{\frac{\sin^2\alpha}{\cos^2\alpha}+\frac{\cos^2\alpha}{\cos^2\alpha}}\\
&= \frac{2\tan\alpha}{\tan^2\alpha+1}
\end{aligned}$$

Now we can plug in the given value for $\tan\alpha$ to find the value of $2\sin\alpha\cos\alpha$ in this instance:

$$2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{\tan^2\alpha+1} = \frac{2(\frac{2}{5})}{(\frac{2}{5})^2+1} = \frac{\frac{4}{5}}{\frac{4}{25}+1} = \frac{\frac{4}{5}}{\frac{4}{25}+\frac{25}{25}} = \frac{\frac{20}{25}}{\frac{29}{25}} = \frac{20}{29}$$

8. First, we will rewrite the expresssion $\cos^2\alpha-\sin^2\alpha$ in terms of $\tan\alpha$:

$$\begin{aligned}
\cos^2\alpha-\sin^2\alpha &= \frac{\cos^2\alpha-\sin^2\alpha}{1} = \frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha+\sin^2\alpha} = \frac{\frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}}\\
&= \frac{1-\tan^2\alpha}{1+\tan^2\alpha}
\end{aligned}$$

   (a) To find the numerical value of $\cos^2\alpha-\sin^2\alpha$ when $\tan\alpha=\frac{2}{5}$ we can substitute $\frac{2}{5}$ for $\tan\alpha$ in the formula above:

$$\cos^2\alpha-\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-\left(\frac{2}{5}\right)^2}{1+\left(\frac{2}{5}\right)^2} = \frac{1-\frac{4}{25}}{1+\frac{4}{25}} = \frac{\frac{21}{25}}{\frac{29}{25}} = \frac{21}{29}$$

   (b) Substituting r for $\tan\alpha$ in the formula above:

$$\cos^2\alpha-\sin^2\alpha = \frac{1-r^2}{1+r^2}$$

9. First, we will rewrite the expresssion in terms of $\tan\alpha$:

$$\frac{\sin\alpha-2\cos\alpha}{\cos\alpha-3\sin\alpha} = \frac{\frac{\sin\alpha-2\cos\alpha}{\cos\alpha}}{\frac{\cos\alpha-3\sin\alpha}{\cos\alpha}} = \frac{\frac{\sin\alpha}{\cos\alpha}-\frac{2\cos\alpha}{\cos\alpha}}{\frac{\cos\alpha}{\cos\alpha}-\frac{3\sin\alpha}{\cos\alpha}} = \frac{\tan\alpha-2}{1-3\tan\alpha}$$

Next, we substitute $\frac{2}{5}$ for $\tan\alpha$:

$$\frac{\tan\alpha-2}{1-3\tan\alpha} = \frac{\frac{2}{5}-2}{1-3\left(\frac{2}{5}\right)} = \frac{\frac{2}{5}-\frac{10}{5}}{\frac{5}{5}-\frac{6}{5}} = \frac{-\frac{8}{5}}{-\frac{1}{5}} = 8$$

10. First, we will rewrite the expresssion in terms of $\tan\alpha$:

$$\frac{a\sin\alpha + b\cos\alpha}{c\cos\alpha + d\sin\alpha} = \frac{\frac{a\sin\alpha}{\cos\alpha} + \frac{b\cos\alpha}{\cos\alpha}}{\frac{c\cos\alpha}{\cos\alpha} + \frac{d\cos\alpha}{\cos\alpha}} = \frac{a\tan\alpha + b}{c + d\tan\alpha}$$

Next, we substitute $\frac{2}{5}$ for $\tan\alpha$ and simplify:

$$\frac{a\tan\alpha + b}{c + d\tan\alpha} = \frac{a\left(\frac{2}{5}\right) + b\left(\frac{5}{5}\right)}{c\left(\frac{5}{5}\right) + d\left(\frac{2}{5}\right)} = \frac{\frac{2a+5b}{5}}{\frac{5c+2d}{5}} = \frac{2a+5b}{5c+2d}$$

Now we can see why the problem included the restriction that $5c + 2d \neq 0$; the value of the expression is undefined if the denominator is zero. The sum of two rational numbers is a rational number. Therefore the numerator and denominator in the expression are both rational numbers. The quotient of two rational numbers is a rational number. Therefore, the entire expression evaluates to a rational number for arbitratrary rational values of a, b, c and d.

11. We can expand and simplify the expression:

$$\begin{aligned}
&(\sin\alpha + \cos\alpha)^2 + (\sin\alpha - \cos\alpha)^2 \\
&= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha \\
&= 2\sin^2\alpha + 2\cos^2\alpha \\
&= 2(\sin^2\alpha + \cos^2\alpha) \\
&= 2(1) \\
&= 2
\end{aligned}$$

As the expression evaluates to a constant, it is as large as possible for all values of α.

Page 49

1. Rewriting any instances of $\sec\alpha$ or $\csc\alpha$ on either side of the identities:

   (a) $$\begin{aligned}\tan\alpha\csc\alpha &= \sec\alpha \\ \tan\alpha\frac{1}{\sin\alpha} &= \frac{1}{\cos\alpha} \\ \frac{\tan\alpha}{\sin\alpha} &= \frac{1}{\cos\alpha}\end{aligned}$$

   (b) $$\begin{aligned}\cot\alpha\csc\alpha &= \sec\alpha \\ \cot\alpha\frac{1}{\cos\alpha} &= \frac{1}{\sin\alpha} \\ \frac{\cot\alpha}{\cos\alpha} &= \frac{1}{\sin\alpha}\end{aligned}$$

(c) $$\frac{1}{\sec\alpha}\csc\alpha = \cot\alpha$$
$$\frac{1}{\frac{1}{\cos\alpha}} \cdot \frac{1}{\sin\alpha} = \cot\alpha$$
$$\cos\alpha\frac{1}{\sin\alpha} = \cot\alpha$$
$$\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$$

(d) $$\tan^2\alpha = (\sec\alpha + 1)(\sec\alpha - 1)$$
$$\tan^2\alpha = \sec^2\alpha - 1$$
$$\tan^2\alpha = \frac{1}{\cos^2\alpha} - 1$$

(e) $$\csc^2\alpha = 1 + \cot^2\alpha$$
$$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$$

2. Rewriting any instances of $\sin\alpha$ or $\cos\alpha$ on either side of the identities, and eliminating fractions:

(a) $$\frac{\tan\alpha}{\sin\alpha} = \frac{1}{\cos\alpha}$$
$$\tan\alpha\frac{1}{\sin\alpha} = \sec\alpha$$
$$\tan\alpha\csc\alpha = \sec\alpha$$

(b) $$\frac{1}{\sin\alpha}\cos\alpha = \cot\alpha$$
$$\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$$
$$\cot\alpha = \cot\alpha$$

(c) $$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}$$
$$\tan^2\alpha + 1 = \sec^2\alpha$$

(d) $$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$$
$$\csc^2\alpha = 1 + \cot^2\alpha$$

Page 50

1. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= (\cos^2\alpha - \sin^2\alpha)^2 + (2\sin\alpha\cos\alpha)^2 \\ &= \cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha + 4\sin^2\alpha\cos^2\alpha \\ &= \cos^4\alpha + 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha \\ &= (\cos^2\alpha + \sin^2\alpha)^2 \\ &= (1)^2 \\ &= 1 \end{aligned}$$

   According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

2. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= \left(\sqrt{\frac{1+\cos\alpha}{2}}\right)^2 + \left(\sqrt{\frac{1-\cos\alpha}{2}}\right)^2 \\ &= \frac{1+\cos\alpha}{2} + \frac{1-\cos\alpha}{2} \\ &= \frac{1+\cos\alpha+1-\cos\alpha}{2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

3. First, we will rewrite a and b to eliminate the cube exponents:

$$\begin{aligned} a &= 4\cos^3\alpha - 3\cos\alpha \\ &= 4\cos\alpha\cos^2\alpha - 3\cos\alpha \\ &= 4\cos\alpha(1-\sin^2\alpha) - 3\cos\alpha \\ &= 4\cos\alpha - 4\sin^2\alpha\cos\alpha - 3\cos\alpha \\ &= \cos\alpha - 4\sin^2\alpha\cos\alpha \end{aligned}$$

$$\begin{aligned} b &= 3\sin\alpha - 4\sin^3\alpha \\ &= 3\sin\alpha - 4\sin\alpha\sin^2\alpha \\ &= 3\sin\alpha - 4\sin\alpha(1-\cos^2\alpha) \\ &= -\sin\alpha + 4\sin\alpha\cos^2\alpha \end{aligned}$$

   Next, we will expand a^2 and b^2:

$$\begin{aligned} a^2 &= (\cos\alpha - 4\sin^2\alpha\cos\alpha)^2 \\ &= \cos^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha \end{aligned}$$

$$\begin{aligned} b^2 &= (-\sin\alpha + 4\sin\alpha\cos^2\alpha)^2 \\ &= \sin^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^2\alpha\cos^4\alpha \end{aligned}$$

Next, we add the expressions for a^2 and b^2 and simplify to 1:

$$\begin{aligned} a^2 + b^2 = {} & \cos^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha + \sin^2\alpha - 8\sin^2\alpha\cos^2\alpha + \\ & 16\sin^2\alpha\cos^4\alpha \\ = {} & \cos^2\alpha + \sin^2\alpha - 16\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha + 16\sin^2\alpha\cos^4\alpha \\ = {} & \cos^2\alpha + \sin^2\alpha + 16\sin^2\alpha\cos^2\alpha(-1 + \sin^2\alpha + \cos^2\alpha) \\ = {} & 1 + 16\sin^2\alpha\cos^2\alpha(0) \\ = {} & 1 \end{aligned}$$

According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

4. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 \\ &= \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2} \\ &= \frac{(1-t^2)^2 + (2t)^2}{(1+t^2)^2} \\ &= \frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)(1+t^2)} \\ &= \frac{(1+t^2)(1+t^2)}{(1+t^2)(1+t^2)} \\ &= 1 \end{aligned}$$

According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

5. We expand $(p^2 - q^2)^2 + (2pq)^2$ and use the fact that $p^2 + q^2 = 1$ to simplify to 1:

$$\begin{aligned} (p^2 - q^2)^2 + (2pq)^2 &= p^4 - 2p^2q^2 + q^4 + 4p^2q^2 \\ &= p^4 + 2p^2q^2 + q^4 \\ &= (p^2 + q^2)^2 \\ &= (1)^2 \\ &= 1 \end{aligned}$$

This is similar to Exercise 1 above.

Page 51

1. $\sin \alpha < 1$ when α is acute, therefore $1 - \sin \alpha > 0$ when α is acute. $1 - \sin \alpha = 0$ when $\sin \alpha = 1$, i.e., $\alpha = 90°$.

2. $\cos \alpha < 1$ when α is acute, therefore $1 - \cos \alpha > 0$ when α is acute. $1 - \cos \alpha = 0$ when $\cos \alpha = 1$, i.e., $\alpha = 0°$.

3. Statement a) is always true. Statements b) and c) both include the case that $\sin^2 \alpha + \cos^2 \alpha = 1$, which is always true.

4. Let x be the maximum cost of the items in a supermarket. In Supermarket A, $x \leq \$1$. In Supermarket B, $x < \$1$. In Supermarket C, $x \leq \$1$. In Supermarket D, $x > \$1$. We can see that Supermarkets A and C are offering the same terms.

5. Inequality a) is correct. For b) to be correct, an angle α would have to exist such that $\sin \alpha + \cos \alpha = 2$. We know that this is not the case. When $\alpha = 90°$, $\sin \alpha = 1$ and $\cos \alpha = 0$. When $\alpha = 0°$, $\sin \alpha = 0$ and $\cos \alpha = 1$. When $0° < \alpha < 90°$, $\sin \alpha < 1$ and $\cos \alpha < 1$. In all cases, $\sin \alpha + \cos \alpha < 2$.

6. The largest possible value of $\sin \alpha$ is 1, and occurs when $\alpha = 90°$. The largest possible value of $\cos \alpha$ is 1, and occurs when $\alpha = 0°$. See Page 32.

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1. $\sin 30° = 0.5$, $\sin 45° = 0.707$, $\sin 60° = 0.866$.

2. By using the `tan` button to calculate $\tan 60°$, and the `sqrt` button to calculate $\sqrt{3}$, Betty can compare the results: both are 1.732.

3. Press `tan`, then enter the angle degree measure, then press `1/`x

4.

in radical or rational form				
α	$\sin \alpha$	$\cos \alpha$	$\tan \alpha$	$\cot \alpha$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$

in decimal form, from calculator				
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
30°	0.5	0.866	0.577	1.732
45°	0.707	0.707	1	1
60°	0.866	0.5	1.732	0.577

Page 53

1. The sine of the larger angle is $4/5 = .8$. We can use the inverse sine function to find the angle: $\arcsin .8 = 53.1301^\circ$. The sum of the three angles in the triangles is: $\arcsin .6 + \arcsin .8 + 90^\circ = 36.8699^\circ + 53.1301^\circ + 90^\circ = 180^\circ$.
2. (a) $\arcsin 1 = 90^\circ$
   (b) $\arccos 0.7071067811865 = 45^\circ$
3. $\arccos 0.8 = 36.8699^\circ$
4. $\arcsin 0.6 = 36.8699^\circ$
5. Half of $\sin 30^\circ$ (0.25) seems like a reasonable estimate. The actual value is 0.2588.
6.
7.
8.
9.
10.
11.
12.
13.
14.

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1.
2.
3.
4.

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1.
2.
3.
4.

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1.
2.
3.
4.
5.
6.
7.
8.
9.
10.
11.

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1. The degree measure of a semicircle is 180°. The degree measure of a quarter circle is 90°.

2. The measure of arc cut off by one side of regular pentagon inscribed in a circle is $360^\circ/5 = 72^\circ$. For a regular hexagon: $360^\circ/6 = 60^\circ$. For a regular octagon: $360^\circ/8 = 45^\circ$.

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Chapter 3: Relationships in a Triangle

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Chapter 4: Angles and Rotations

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Chapter 5: Radian Measure

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Chapter 6: The Addition Formulas

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1.

α	β	$\sin\alpha$	$\sin\beta$	$\sin\alpha+\sin\beta$	$\sin(\alpha+\beta)$
60°	30°	$\sqrt{3}/2$	$1/2$	$\left(\sqrt{3}+1\right)/2$	$\sin 90^\circ = 1$
$\pi/4$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\left(\sqrt{2}+\sqrt{2}\right)/2=\sqrt{2}$	$\sin\pi/2=1$
$\pi/6$	$\pi/3$	$1/2$	$\sqrt{3}/2$	$\left(1+\sqrt{3}\right)/2$	$\sin\pi/2=1$

2. For these values of α and β, $\sin\alpha$ and $\sin\beta$ are both at least $1/2$. Furthermore, at least one of $\sin\alpha$ and $\sin\beta$ is strictly greater than $1/2$. Therefore,

$$\sin\alpha+\sin\beta>\frac{1}{2}+\frac{1}{2}=1=\sin(\alpha+\beta).$$

3. (a)

$$\begin{aligned}\sin 60^\circ+\sin 30^\circ &= \frac{\sqrt{3}}{2}+\frac{1}{2}\\ \sin(60^\circ+30^\circ) &= \sin 90^\circ = 1\end{aligned}$$

This identity is not correct.

(b)

$$\begin{aligned}\sin(60^\circ-30^\circ) = \sin 30^\circ &= \frac{1}{2}\\ \sin 60^\circ-\sin 30^\circ &= \frac{\sqrt{3}}{2}-\frac{1}{2}\end{aligned}$$

This identity is not correct.

(c)

$$\begin{aligned}\sin^2 60^\circ-\sin^2 30^\circ &= \left(\frac{\sqrt{3}}{2}\right)^2-\left(\frac{1}{2}\right)^2=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}\\ \sin(60^\circ+30^\circ)\sin(60^\circ-30^\circ) &= \sin 90^\circ\cdot\sin 30^\circ = 1\cdot\frac{1}{2}=\frac{1}{2}\end{aligned}$$

This identity is correct for the given angles.

4. See Chapter 2, Section 12 and the Appendix of Chapter 2 to review some of the geometry used in this solution.

   (a) Because $\angle ABC$ is subtended by the diameter $\overline{AC}$, $\angle ABC$ is a right angle and $\triangle ABC$ is a right triangle (this fact is known as *Thales's Theorem*). Therefore, $\sin\alpha$ is equal to the length of the opposite side (BC) divided by the length of the hypotenuse (AC). $\overline{AC}$ is a diameter of the circle, so it has length 1. Thus, we have that $\sin\alpha$ is simply equal to BC.

   A similar argument shows that $\triangle ADC$ is a right triangle with hypotenuse $\overline{AC}$ of length 1, which implies that $\sin\beta = DC$

(b) Recall that chords of congruent circles which subtend equal angles are themselves equal. This implies that BC in the diagram of part (a) is equal to BC in the diagram of part (b) because in both diagrams, the chord $\overline{BC}$ subtends an angle of measure α. Similarly, DC is the same in both diagrams because in both diagrams, the chord $\overline{DC}$ subtends an angle of measure β. Therefore, BC is still equal to $\sin\alpha$, and DC is still equal to $\sin\beta$.

(c) From part (b) above, we can conclude that a chord which subtends an inscribed angle with measure α in a circle with diameter 1 has length $\sin\alpha$. Thus, we draw $\overline{BD}$, the chord which subtends $\angle BAD$ in both figures and which consequently has length $\sin(\alpha+\beta)$.

Note that the above reasoning implies that the sine of an angle with measure less than 180° cannot exceed 1 since the diameter is the longest chord in a circle.

5. Recall that the sine of any angle is at most 1. Therefore,

$$\sin 105^\circ \leq 1 = \frac{1}{2} + \frac{1}{2} < \sin 45^\circ + \sin 60^\circ,$$

which shows that $\sin 105^\circ$ cannot equal $\sin 45^\circ + \sin 60^\circ$.

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1. Addition formula for sine:

$$\begin{aligned}\sin(60^\circ + 30^\circ) &= \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{1}{2} \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1 \\ &= \sin 90^\circ\end{aligned}$$

Addition formula for cosine:

$$\begin{aligned}\cos(60^\circ + 30^\circ) &= \cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ \\ &= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ &= 0 \\ &= \cos 90^\circ\end{aligned}$$

Difference formula for sine:

$$\begin{aligned}\sin(60^\circ - 30^\circ) &= \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{3}{4} - \frac{1}{4} \\ &= \frac{1}{2} \\ &= \sin 30^\circ\end{aligned}$$

Difference formula for cosine:

$$\begin{aligned}\cos(60^\circ - 30^\circ) &= \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ \\ &= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ &= \frac{\sqrt{3}}{2} \\ &= \cos 30^\circ\end{aligned}$$

2. Addition formula for sine ($\alpha = 0$):

$$\begin{aligned}\sin(0 + \beta) &= \sin 0 \cos\beta + \cos 0 \sin\beta \\ &= 0 \cdot \cos\beta + 1 \cdot \sin\beta \\ &= \sin\beta\end{aligned}$$

Addition formula for cosine ($\alpha = 0$):

$$\begin{aligned}\cos(0 + \beta) &= \cos 0 \cos\beta - \sin 0 \sin\beta \\ &= 1 \cdot \cos\beta - 0 \cdot \sin\beta \\ &= \cos\beta\end{aligned}$$

Difference formula for sine ($\alpha = 0$):

$$\begin{aligned}\sin(0 - \beta) &= \sin 0 \cos\beta - \cos 0 \sin\beta \\ &= 0 \cdot \cos\beta - 1 \cdot \sin\beta \\ &= -\sin\beta\end{aligned}$$

Notice that this demonstrates that the sine function is *odd*.

Difference formula for cosine ($\alpha = 0$):

$$\begin{aligned}\cos(0 - \beta) &= \cos 0 \cos\beta + \sin 0 \sin\beta \\ &= 1 \cdot \cos\beta + 0 \cdot \sin\beta \\ &= \cos\beta\end{aligned}$$

Notice that this demonstrates that the cosine function is *even*.

Addition formula for sine ($\beta = 0$):

$$\begin{aligned}\sin(\alpha + 0) &= \sin\alpha\cos 0 + \cos\alpha\sin 0\\ &= \sin\alpha \cdot 1 + \cos\alpha \cdot 0\\ &= \sin\alpha\end{aligned}$$

Addition formula for cosine ($\beta = 0$):

$$\begin{aligned}\cos(\alpha + 0) &= \cos\alpha\cos 0 - \sin\alpha\sin 0\\ &= \cos\alpha \cdot 1 - \sin\alpha \cdot 0\\ &= \cos\alpha\end{aligned}$$

Difference formula for sine ($\beta = 0$):

$$\begin{aligned}\sin(\alpha - 0) &= \sin\alpha\cos 0 - \cos\alpha\sin 0\\ &= \sin\alpha \cdot 1 - \cos\alpha \cdot 0\\ &= \sin\alpha\end{aligned}$$

Difference formula for cosine ($\beta = 0$):

$$\begin{aligned}\cos(\alpha + 0) &= \cos\alpha\cos 0 + \sin\alpha\sin 0\\ &= \cos\alpha \cdot 1 + \sin\alpha \cdot 0\\ &= \cos\alpha\end{aligned}$$

3. Following the hint, we notice that in a right triangle, the side opposite one of the acute angles is the side adjacent to the other acute angle. Thus, if $\alpha + \beta = \pi/2$, then $\sin\alpha = \cos\beta$ and $\sin\beta = \cos\alpha$ (see also Chapter 1, Section 4).

$$\begin{aligned}\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \sin\alpha\sin\alpha + \cos\alpha\cos\alpha\\ &= \sin^2\alpha + \cos^2\alpha\\ &= 1\end{aligned}$$

4. Addition formula for sine:

$$\begin{aligned}\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) &= \sin\frac{\pi}{4}\cos\frac{\pi}{4} + \cos\frac{\pi}{4}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\\ &= \frac{1}{2} + \frac{1}{2}\\ &= 1\\ &= \sin\frac{\pi}{2}\end{aligned}$$

Addition formula for cosine:

$$\begin{aligned}
\cos\left(\frac{\pi}{4}+\frac{\pi}{4}\right) &= \cos\frac{\pi}{4}\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{1}{2}-\frac{1}{2} \\
&= 0 \\
&= \cos\frac{\pi}{2}
\end{aligned}$$

5. Recall that $(A \pm B)^2 = A^2 \pm 2AB + B^2$.

$$\begin{aligned}
&(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2 \\
&\quad= \sin^2\alpha\cos^2\beta+2\sin\alpha\cos\beta\cos\alpha\sin\beta+\cos^2\alpha\sin^2\beta+ \\
&\quad\quad \cos^2\alpha\cos^2\beta-2\cos\alpha\cos\beta\sin\alpha\sin\beta+\sin^2\alpha\sin^2\beta \\
&\quad= \sin^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta \\
&\quad= \sin^2\alpha\left(\cos^2\beta+\sin^2\beta\right)+\cos^2\alpha\left(\sin^2\beta+\cos^2\beta\right) \\
&\quad= \sin^2\alpha\cdot 1+\cos^2\alpha\cdot 1 \\
&\quad= 1
\end{aligned}$$

6. After expanding using the identity $(A+B)(A-B) = A^2 - B^2$, we cleverly "add by zero" to get the desired result.

$$\begin{aligned}
&(\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta) \\
&\quad= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta \\
&\quad= \sin^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta-\sin^2\alpha\sin^2\beta-\cos^2\alpha\sin^2\beta \\
&\quad= \sin^2\alpha\left(\cos^2\beta+\sin^2\beta\right)-\sin^2\beta\left(\sin^2\alpha+\cos^2\alpha\right) \\
&\quad= \sin^2\alpha\cdot 1-\sin^2\beta\cdot 1 \\
&\quad= \sin^2\alpha-\sin^2\beta
\end{aligned}$$

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1.

$$\begin{aligned}\sin(30^\circ + 30^\circ) &= \sin 30^\circ \cos 30^\circ + \cos 30^\circ \sin 30^\circ\\ &= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\\ &= \frac{\sqrt{3}}{2}\\ &= \sin 60^\circ\end{aligned}$$

$$\begin{aligned}\cos(30^\circ + 30^\circ) &= \cos 30^\circ \cos 30^\circ - \sin 30^\circ \sin 30^\circ\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{2}\\ &= \frac{3}{4} - \frac{1}{4}\\ &= \frac{1}{2}\\ &= \cos 60^\circ\end{aligned}$$

2. Assuming α and β are acute angles:

$$\sin\alpha = \frac{3}{5} \implies \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-(3/5)^2} = \frac{4}{5}$$

$$\sin\beta = \frac{5}{13} \implies \cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-(5/13)^2} = \frac{12}{13}$$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \frac{3}{5}\cdot\frac{12}{13} + \frac{4}{5}\cdot\frac{5}{13}\\ &= \frac{36}{65} + \frac{20}{65}\\ &= \frac{56}{65}\end{aligned}$$

$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&= \frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13} \\
&= \frac{48}{65} - \frac{15}{65} \\
&= \frac{33}{65}
\end{aligned}
$$

3.

$$
\begin{aligned}
\sin 75^\circ &= \sin(45^\circ + 30^\circ) \\
&= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}
$$

$$
\begin{aligned}
\cos 75^\circ &= \cos(45^\circ + 30^\circ) \\
&= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}
$$

4.

$$
\begin{aligned}
\sin 15^\circ &= \sin(45^\circ - 30^\circ) \\
&= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}
$$

$$
\begin{aligned}
\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\
&= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}
$$

Notice that 75° and 15° are complementary angles, so we know $\sin 75^\circ = \cos 15^\circ$ and $\sin 15^\circ = \cos 75^\circ$.

5. (a) Yes, let $\alpha = \beta = \pi/4$.

$$\begin{aligned}
\cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right) &= \cos\frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}$$

More generally, we could suppose $\alpha + \beta = \pi/2$ and follow the approach in Exercise 3 of Section 2 earlier in this chapter.

(b) If α and β are acute angles, then $0 < \alpha + \beta < \pi$. Using the unit circle, we can see that $\sin(\alpha + \beta)$ must be positive since the angle $\alpha + \beta$ lies in the upper-half of the plane, where the sine function is positive.

(c) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ is positive when α and β are acute angles since the sum and product of positive real numbers is also positive.

$\cos(\alpha + \beta)$ need not be positive. As shown in part (a) of this exercise, $\cos(\alpha + \beta)$ can equal 0. Furthermore, $\cos(\alpha + \beta)$ can be negative. Let $\alpha = \beta = \pi/3$. Then, assuming that we can extend the cosine addition formula to angles α and β such that $\alpha + \beta$ is obtuse,

$$\begin{aligned}
\cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right) &= \cos\frac{\pi}{3}\cos\frac{\pi}{3} - \sin\frac{\pi}{3}\sin\frac{\pi}{3} \\
&= \frac{1}{2}\cdot\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} \\
&= \frac{1}{4} - \frac{3}{4} \\
&= -\frac{1}{2}
\end{aligned}$$

6. As we saw in Exercise 1 of Section 1 of this chapter, $\sin\alpha + \sin\beta$ does not equal $\sin(\alpha + \beta)$ in general. A similar table can be used to show that $\sin\alpha - \sin\beta$ does not equal $\sin(\alpha - \beta)$ in general.

7. This is not a coincidence. The identity holds true even when substituting more "arbitrary" values in for α and β. For example, using $\alpha = 37^\circ$ and $\beta = 19^\circ$, we find that both $\sin^2\alpha - \sin^2\beta$ and $\sin(\alpha + \beta)\sin(\alpha - \beta)$ are equal to approximately 0.2562.

8. You may also refer to the proof in Exercise 6 of Section 2 of this chapter.

$$\begin{aligned}
\sin(\alpha+\beta)\sin(\alpha-\beta) &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta-\sin^2\alpha\sin^2\beta-\cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha\left(\cos^2\beta+\sin^2\beta\right)-\sin^2\beta\left(\sin^2\alpha+\cos^2\alpha\right)\\
&= \sin^2\alpha\cdot 1-\sin^2\beta\cdot 1\\
&= \sin^2\alpha-\sin^2\beta
\end{aligned}$$

9. This proof is nearly identical to the one in the previous part. We just make a small modification in how we "add by zero" in order to obtain to the desired result.

$$\begin{aligned}
\sin(\alpha+\beta)\sin(\alpha-\beta) &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&= \sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha\cos^2\beta+\cos^2\alpha\cos^2\beta-\cos^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\
&= \cos^2\beta\left(\sin^2\alpha+\cos^2\alpha\right)-\cos^2\alpha\left(\cos^2\beta+\sin^2\beta\right)\\
&= \cos^2\beta\cdot 1-\cos^2\alpha\cdot 1\\
&= \cos^2\beta-\cos^2\alpha
\end{aligned}$$

10. We apply the sine addition formula in reverse.

$$\begin{aligned}
\sin 18^\circ\cos 12^\circ+\cos 18^\circ\sin 12^\circ &= \sin(18^\circ+12^\circ)\\
&= \sin 30^\circ\\
&= \frac{1}{2}
\end{aligned}$$

11. (a) Since we have not proved that the sine addition formula works for all angles α and β, we use properties of the sine and cosine functions to avoid working with angles larger than 90°.

$$\begin{aligned}
&\sin 113^\circ\cos 307^\circ+\cos 113^\circ\sin 307^\circ\\
&\quad= \sin(180^\circ-67^\circ)\cos(360^\circ-53^\circ)+\cos(180^\circ-67^\circ)\sin(360^\circ-53^\circ)\\
&\quad= \sin 67^\circ\cos 53^\circ+(-\cos 67^\circ)(-\sin 53^\circ)\\
&\quad= \sin(67^\circ+53^\circ)\\
&\quad= \sin 120^\circ\\
&\quad= \sin 60^\circ\\
&\quad= \frac{\sqrt{3}}{2}
\end{aligned}$$

(b) Plugging into a calculator,

$$\sin 113^\circ\cos 307^\circ+\cos 113^\circ\sin 307^\circ\approx 0.866\approx\frac{\sqrt{3}}{2}=\sin 60^\circ$$

(c) Assuming that the sine addition formula does work for non-acute angles, we arrive at the same result.

$$\begin{aligned}\sin 113^\circ \cos 307^\circ + \cos 113^\circ \sin 307^\circ &= \sin(113^\circ + 307^\circ)\\ &= \sin 420^\circ\\ &= \sin 60^\circ\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

12. We can use the addition formulas for sine and cosine by rewriting 2α as $\alpha + \alpha$.

$$\begin{aligned}\sin 2\alpha \cos\alpha - \cos 2\alpha \sin\alpha &= \sin(\alpha+\alpha)\cos\alpha - \cos(\alpha+\alpha)\sin\alpha\\ &= (\sin\alpha\cos\alpha + \cos\alpha\sin\alpha)\cos\alpha - (\cos\alpha\cos\alpha - \sin\alpha\sin\alpha)\sin\alpha\\ &= \sin\alpha\cos^2\alpha + \sin\alpha\cos^2\alpha - \sin\alpha\cos^2\alpha + \sin^3\alpha\\ &= \sin\alpha\cos^2\alpha + \sin^3\alpha\\ &= \sin\alpha\left(\cos^2\alpha + \sin^2\alpha\right)\\ &= \sin\alpha\end{aligned}$$

13.

$$\begin{aligned}\sin(\alpha+\beta)\sin\beta + \cos(\alpha+\beta)\cos\beta &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)\sin\beta + (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\cos\beta\\ &= \sin\alpha\sin\beta\cos\beta + \sin^2\beta\cos\alpha + \cos\alpha\cos^2\beta - \sin\alpha\sin\beta\cos\beta\\ &= \sin^2\beta\cos\alpha + \cos\alpha\cos^2\beta\\ &= \cos\alpha\left(\sin^2\beta + \cos^2\beta\right)\\ &= \cos\alpha\end{aligned}$$

14.

$$\begin{aligned}\frac{\sin(\alpha+\beta) - \cos\alpha\sin\beta}{\cos(\alpha+\beta) + \sin\alpha\sin\beta} &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta + \sin\alpha\sin\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha}{\cos\alpha}\\ &= \tan\alpha\end{aligned}$$

15.

$$\begin{aligned}\sin\left(\alpha + \frac{\pi}{4}\right) &= \sin\alpha\cos\frac{\pi}{4} + \cos\alpha\sin\frac{\pi}{4}\\ &= \sin\alpha\cdot\frac{\sqrt{2}}{2} + \cos\alpha\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{2}(\sin\alpha + \cos\alpha)\end{aligned}$$

16.

$$\begin{aligned}
\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} &= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= 1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= 1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta} \\
&= 1 - \tan\alpha\tan\beta
\end{aligned}$$

17. Applying the law of cosines, we have that $(b_1+b_2)^2 = c_1^2+c_2^2-2c_1c_2\cos(\alpha+\beta)$. Solving for $\cos(\alpha+\beta)$, we get

$$\cos(\alpha+\beta) = \frac{c_1^2+c_2^2-(b_1+b_2)^2}{2c_1c_2}.$$

Before proceeding further, let's establish some relationships between the variables in the diagram. First, by the Pythagorean theorem, we have that $h^2 = c_1^2 - b_1^2 = c_2^2 - b_2^2$. Additionally, we can compute the sines and cosines for the angles α and β:

$$\sin\alpha = \frac{b_1}{c_1}, \sin\beta = \frac{b_2}{c_2}, \cos\alpha = \frac{h}{c_1}, \cos\beta = \frac{h}{c_2}.$$

We can now simplify our expression for $\cos(\alpha+\beta)$.

$$\begin{aligned}
\cos(\alpha+\beta) &= \frac{c_1^2+c_2^2-(b_1+b_2)^2}{2c_1c_2} \\
&= \frac{c_1^2+c_2^2-b_1^2-2b_1b_2-b_2^2}{2c_1c_2} \\
&= \frac{2h^2-2b_1b_2}{2c_1c_2} \\
&= \frac{h^2-b_1b_2}{c_1c_2} \\
&= \frac{h}{c_1}\cdot\frac{h}{c_2} - \frac{b_1}{c_1}\cdot\frac{b_2}{c_2} \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

Chapter 7: Trigonometric Identities

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1. (a) Since $\sin^2\alpha + \cos^2\alpha = 1$, we know $\cos\alpha = \sqrt{1 - (\frac{7}{25})^2}$ (and cannot be the negative version because $\cos\alpha$ is given as positive).
   Thus $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \cdot \frac{7}{25} \cdot \sqrt{1 - (\frac{7}{25})^2} = 0.5376$.
   And $\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - (\frac{7}{25})^2 = 0.9216$.

(b) This part is similar except that we use the negative version of cosine, namely $\cos\alpha = -\sqrt{1 - (\frac{7}{25})^2}$.
Thus $\sin 2\alpha = 2\sin\alpha\cos\alpha = -0.5376$.
However, cosine value remains the same: $\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - (\frac{7}{25})^2 = 0.9216$.

2. Firstly, $\sin 2\alpha = 2\sin\alpha\cos\alpha$. In other words, the sine value will be a product of rational numbers, so it will also be rational. Similarly, $\cos 2\alpha = 2\cos^2\alpha - 1$ will be rational because $\cos\alpha$ is rational. Exercise 1 confirms this result. (The decimal solutions for Exercise 1 are not rounded.)

3. Let's use the double angle formula $\cos 2\alpha = 2\cos^2\alpha - 1$.
$$\begin{aligned}\cos^2\alpha &= \cos 2\alpha = 2\cos^2\alpha - 1\\ -\cos\alpha^2 &= -1 && \text{(Subtracted } 2\cos^2\alpha \text{ from each side)}\\ \cos\alpha^2 &= 1\\ \cos\alpha &= \pm 1\end{aligned}$$
These values of $\cos\alpha$ can happen only at $\alpha = 0, \pi$.

4. We start with the given equation:
$$\begin{aligned}\sin\alpha + \cos\alpha &= 0.2\\ \sin^2\alpha + 2\cos\alpha\sin\alpha + \cos^2\alpha &= 0.04 && \text{(Squared both sides.)}\\ 1 + 2\cos\alpha\sin\alpha &= 0.04\\ 2\cos\alpha\sin\alpha &= -0.96\end{aligned}$$
Note that that is simply $\sin 2\alpha$. Hooray!

5. We can follow a very similar strategy here to find $1 - 2\cos\alpha\sin\alpha = 0.09$. Then $\sin 2\alpha = 2\cos\alpha\sin\alpha = 0.91$.

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9. Yes, the book asked you to prove something incorrect! For counterexample, consider that $\cos 2\alpha$ can be negative but $\cos^2\alpha$ is never negative. However, we can prove a relationship.
Recall that $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$. However, $\sin^2\alpha \geq 0$ since it's a square.
$$\begin{aligned}\sin^2\alpha &\geq 0\\ -\sin^2\alpha &\leq 0\\ \cos^2\alpha - \sin^2\alpha &\leq \cos^2\alpha\\ \cos 2\alpha &\leq \cos^2\alpha\end{aligned}$$

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Chapter 9: Inverse Functions and Trigonometric Equations

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