Solutions for *Trigonometry* by Gelfand & Saul

Git commit `43c184a`. Published June 19, 2020.

Introduction

Trigonometry by Gelfand and Saul is often recommended as a precalculus text for self-study. However, those who are learning without the help of a teacher can struggle with the lack of solutions to exercises in the text. A partial set of solutions for *Trigonometry* (odd numbered exercises only) has been published by John Beach[1]. It is hoped that this document will eventually contain a complete set of solutions. Contributions are welcome. These can take the form of pull requests or issues submitted to the project's GitHub repository[2].

Chapter 0: Trigonometry

Page 8

1. Statement I applies:

$$\begin{aligned} c^2 &= a^2 + b^2 = 10^2 + 24^2 = 100 + 576 = 676 \\ c &= \sqrt{676} = 26 \end{aligned}$$

2. Statement I applies:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 9^2 &= 41^2 \\ a^2 + 81 &= 1681 \\ a^2 &= 1600 \\ a &= \sqrt{1600} = 40 \end{aligned}$$

[1] https://jbeach50.weebly.com/gelfand–saul-trig-solutions.html
[2] https://github.com/philip-healy/gelfand-trigonometry-solutions

3. $5^2 + 12^2 = 25 + 144 = 169 = 13^2$. By Statement II, a right triangle exists with legs of length 5 and 12, and hypotenuse of length 13.

4. Statement I applies:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 1^2 &= 3^2 \\ a^2 + 1 &= 9 \\ a^2 &= 8 \\ a &= \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \end{aligned}$$

5. Statement I applies, where $a = b$:

$$\begin{aligned} a^2 + a^2 &= c^2 \\ a^2 + a^2 &= 1^2 \\ 2a^2 &= 1 \\ a^2 &= \frac{1}{2} \\ a &= \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \end{aligned}$$

6. From the diagram at the bottom of Page 11, we can see the shorter leg is half the length of the hypotenuse. So in this instance the shorter leg has length 1/2. We can use Statement 1 to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + \left(\frac{1}{2}\right)^2 &= 1^2 \\ a^2 + \frac{1}{4} &= 1 \\ a^2 &= \frac{3}{4} \\ a &= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} \end{aligned}$$

7. For any point Y, we can draw a triangle with sides AY, BY and AB. Let a be the length of side AY, b be the length of side BY and c be the length of side AB. According to Statement II, the subset of these triangles where $a^2 + b^2 = c^2$ are right triangles with legs of length a and b and hypotenuse c. Let X be the subset of Y that are vertices of these right triangles. This set of points describes a circle with its centre at the midpoint of AB, and radius $AB/2$.

8.

Page 9

1. $6^2 + 8^2 = 36 + 64 = 100 = 10^2$. By Statement II on Page 7 (converse of the Pythagorean Theorem), this is a right triangle.

2. 10-24-26 (Exercise 1), 9-40-41 (Exercise 2), 5-12-13 (Exercise 3)

3. Using the Pythagorean Theorem:

$$\begin{aligned} c^2 &= a^2 + b^2 = 8^2 + 15^2 = 64 + 225 = 289 \\ c &= \sqrt{289} = 17 \end{aligned}$$

4. The first column in the table increases by 3, the second increases by 4 and the third increases by 5. Continuing to add rows yields triangles 12-16-20, 15-20-25 and 18-24-30.

5. Shortest side with length 10: 10-24-26. Shortest side with length 15: 15-36-39.

6. Multiplying all sides by the common denominator (5), we get a similar triangle with sides $15/5 = 3$, $20/5 = 4$ and 5. We know that this is a right triangle from the table in Question 4.

7. To find a similar triangle with shorter leg 1, divide all sides by 3, resulting in sides 1-4/3-5/3. To find a similar triangle with longer leg 1, divide all sides by 4, resulting in sides 3/4-1-5/4.

8. To find a similar triangle with hypotenuse 1, divide all sides by 13, resulting in sides 5/13-12/13-1. To find a similar triangle with shorter leg 1, divide all sides by 5, resulting in sides 1-12/5-13/5. To find a similar triangle with longer leg 1, divide all sides by 12, resulting in sides 5/12-1-13/12.

9. To formula for the area of a triangle is $\frac{1}{2}bh$ where b is the length of the base and h is the height. For right triangles, finding the area is easy: one leg is the base and the other leg is the height. For other triangles, finding the height is more difficult: we need to find the length of the altitude drawn from the base. The triangles with sides 5-12-13 and 9-12-15 are both right triangles: see Exercise 3 on Page 8 and Exercise 4 on Page 9. The triangle with sides 13-14-15 is not a right triangle. We can confirm this using Statement I: $a^2 + b^2 = 13^2 + 14^2 = 365$, $c^2 = 15^2 = 225, a^2 + b^2 \neq c^2$. However, if we join the 5-12-13 and 9-12-15 triangles using their equal legs, the resulting triangle has the dimensions we are looking for: 13-14-15. The base of this combined triangle has length $5 + 9 = 14$. We also know the length of the altitude from the base of the combined triangle: 12. So, the area of the 13-14-15 triangle is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ units squared.

10. (a)

    (b)

Page 11

1. $\frac{1}{\sqrt{2}}$ (see the solution for Question 5 on page 8).
   Challenge: $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ (multiplying above and below by $\sqrt{2}$). $\sqrt{2}$ is given to 9 decimal places in the diagram on the top of page 11: 1.4141213562373. Dividing this decimal representation by 2 (using long division if necessary) yields a figure of 0.707060678.

2. $c^2 = a^2 + b^2 = 3^2 + 3^2 = 9 + 9 = 18$. $c = \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$.

3. The hypotenuse of a 30° right triangle is double the length of the shorter leg. In this instance the hypotenuse is 10 units long. We can use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 5^2 &= 10^2 \\ a^2 + 25 &= 100 \\ a^2 &= 75 \\ a &= \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3} \end{aligned}$$

4. We can solve these by finding similar triangles to the 30° right triangle with sides 1-$\sqrt{3}$-2, or the 45° right triangle with sides 1-1-$\sqrt{2}$.

   (a) $x = \sqrt{3}$, $y = 2$
   (b) $x = \sqrt{3}$, $y = 2\sqrt{3}$
   (c) $x = 1/2$, $y = \sqrt{3}/2$
   (d) $x = 4\sqrt{3}$, $y = 8$
   (e) $x = y = 2\sqrt{2}$
   (f) $x = 5$, $y = 5\sqrt{2}$

Page 14 (Examples)

1. Why didn't we need to compare 3^2 with $2^2 + 4^2$, or 2^2 with $3^2 + 4^2$?
   The obtuse angle will always be opposite the longest side.

2. This conclusion is *incorrect*. Why?
   From the footnote at the beginning of Chapter 0: *"Given three arbitrary lengths... they form a triangle if and only if the sum of any two of them is greater than the third."* In this case $1 + 2 = 3$ which is equal to (not greater than) the third side.

Page 14 (Exercise)

1. (a) $c^2 = 8^2 = 64$. $a^2 + b^2 = 6^2 + 7^2 = 36 + 49 = 85$. $c^2 < a^2 + b^2$, so the triangle is acute.
   (b) $c^2 = 10^2 = 100$. $a^2 + b^2 = 6^2 + 8^2 = 36 + 64 = 100$. $c^2 = a^2 + b^2$, so the triangle is a right triangle.
   (c) a and b are the same as in question b), but c is smaller, so the triangle is acute.
   (d) a and b are the same as in question b), but c is larger, so the triangle is obtuse.
   (e) $c^2 = 12^2 = 144$. $a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$. $c^2 < a^2 + b^2$, so the triangle is acute.
   (f) $c^2 = 14^2 = 196$. $a^2 + b^2 = 169$, as above. $c^2 > a^2 + b^2$, so the triangle is obtuse.
   (g) Here a and b are the same as above, but c is larger. So, this triangle is also obtuse.

Chapter 1: Trigonometric Ratios in a Triangle

Page 23

1. (a) $\sin\alpha = 5/13$
   (b) $\sin\alpha = 4/5$
   (c) $\sin\alpha = 5/13$
   (d) $c = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$. $\sin\alpha = 8/10$.
   (e) $\sin\alpha = 3/5$
   (f) $\sin\alpha = 12/13$
   (g) $\sin\alpha = 3/5$
   (h) $c = \sqrt{7^2 + 3^2} = \sqrt{58}$. $\sin\alpha = 7/\sqrt{58}$.

2. (a) $\sin\beta = 12/13$
   (b) $\sin\beta = 3/5$
   (c) $\sin\beta = 12/13$
   (d) $\sin\beta = 6/10$
   (e) $\sin\beta = 4/5$
   (f) $\sin\beta = 5/13$
   (g) $\sin\beta = 4/5$
   (h) $\sin\beta = 3/\sqrt{58}$

3. The example 30-60-90 triangle given on page 11 has sides 1, $\sqrt{3}$, 2. Let β represent the 60°angle. The opposite leg b has length $\sqrt{3}$. The hypotenuse c has length 2. So, $\sin\beta = b/c = \sqrt{3}/2 \approx 1.732/2 = 0.866$.
   Crossing off the numbers listed:
   ~~0.1~~ ~~0.2~~ ~~0.3~~ ~~0.4~~ ~~0.5~~ ~~0.6~~ ~~0.7~~ ~~0.8~~ 0.9

Page 25

1. The Altitude-on-Hypotenuse Theorem tells us that when an altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other. Therefore, the triangles with sides a-b-c, a-p-d and d-b-q are similar, and the ratio for $\sin\alpha$ appears in all of them:
   (a) b/c
   (b) d/a
   (c) q/b
2. (a) $\sin\alpha = h/b$
   (b) Multiplying both sides of formula above by b: $h = b\sin\alpha$
   (c) Substituting $b\sin\alpha$ for h, the formula for the area of ABC can be rewritten as: $bc\sin\alpha/2$.
   (d) $\sin\beta = h/a$. Rewriting this in terms of h: $h = a\sin\beta$. Substituting this for h in the area formula: $ac\sin\beta/2$.
   (e) Let h_2 represent the altitude from A to BC. $sin\beta = h_2/c$. Rewriting in terms of h_2, we get $h_2 = c\sin\beta$.
3. (a) Expressing h in terms of $\sin\alpha$ and b:

$$\sin\alpha = \frac{h}{b}$$
$$h = b\sin\alpha$$

   Expressing h in terms of $\sin\beta$ and a:

$$\sin\beta = \frac{h}{a}$$
$$h = a\sin\beta$$

   (b) Both expresssions are equal to h:

$$a\sin\beta = h = b\sin\alpha$$

   (c) Expressing h_2 in terms of $\sin\beta$ and c:

$$\sin\beta = \frac{h_2}{c}$$
$$h_2 = c\sin\beta$$

Expressing h_2 in terms of $\sin\gamma$ and b:

$$\begin{aligned} \sin\gamma &= \frac{h_2}{b} \\ h_2 &= b\sin\gamma \end{aligned}$$

Both expressions are equal to h_2:

$$b\sin\alpha = h_2 = c\sin\gamma$$

(d) i. We can rewrite the result from part (b) so that the expressions on each side are fractions with sine denominators:

$$\begin{aligned} a\sin\beta &= b\sin\alpha \\ \frac{a\sin\beta}{\sin\alpha\sin\beta} &= \frac{b\sin\alpha}{\sin\alpha\sin\beta} \\ \frac{a}{\sin\alpha} &= \frac{b}{\sin\beta} \end{aligned}$$

ii. We can rewrite the result from part (c) similarly:

$$\begin{aligned} c\sin\beta &= b\sin\gamma \\ \frac{c\sin\beta}{\sin\beta\sin\gamma} &= \frac{b\sin\gamma}{\sin\beta\sin\gamma} \\ \frac{c}{\sin\gamma} &= \frac{b}{\sin\beta} \end{aligned}$$

We can derive the Law of Sines by combining results i. and ii. using the common expression $b/\sin\beta$:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

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1. (a) $\cos\alpha = 12/13$. $\cos\beta = 5/13$.
   (b) $\cos\alpha = 3/5$. $\cos\beta = 4/5$.
   (c) $\cos\alpha = 12/13$. $\cos\beta = 5/13$.
   (d) $\cos\alpha = 6/10$. $\cos\beta = 8/10$.
   (e) $\cos\alpha = 4/5$. $\cos\beta = 3/5$.
   (f) $\cos\alpha = 5/13$. $\cos\beta = 12/13$.
   (g) $\cos\alpha = 4/5$. $\cos\beta = 3/5$.
   (h) $\cos\alpha = 3/\sqrt{58}$. $\cos\beta = 7/\sqrt{58}$.
2. (a) $c = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$. $\cos\alpha = 8/10$. $\cos\beta = 6/10$.

(b) $c = \sqrt{5^2 + 12^2} = \sqrt{25 + 155} = \sqrt{169} = 13$. $\cos\alpha = 12/13$. $\cos\beta = 5/13$.

(c) Scaling up the 1-$\sqrt{3}$-2 30° triangle gives us a value of 20 units for the length of c. Next, we will use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 10^2 + b^2 &= 20^2 \\ b^2 &= 400 - 100 = 300 \\ b &= \sqrt{300} = \sqrt{100}\sqrt{3} = 10\sqrt{3} \end{aligned}$$

We can now find $\cos\alpha$ and $\cos\beta$:

$$\begin{aligned} \cos\alpha &= \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2} \\ \cos\beta &= \frac{10}{20} = \frac{1}{2} \end{aligned}$$

(d) The triangle is congruent to the one above, so the solution is the same.

(e) Consider the 45° right triangle with legs of length 1 and hypotenuse $\sqrt{2}$. $\cos\alpha = \cos\beta = 1/\sqrt{2}$.

(f) $c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. $\cos\alpha = 3/5$. $\cos\beta = 4/5$.

(g) $b = x\sqrt{3}$. $\cos\alpha = x\sqrt{3}/2x = \sqrt{3}/2$. $\cos\beta = x/2x = 1/2$.

3. The Altitude-on-Hypotenuse Theorem tells us that when an altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other. Therefore, the triangles with sides a-b-c, a-p-d and d-b-q are similar, and the ratio for $\cos\alpha$ appears in all of them:

   (a) a/c

   (b) p/a

   (c) d/b

Page 28

1. In this instance, $\alpha = 29°$, $\beta = 61°$, and $\alpha + \beta = 90°$. According to the theorem above, if $\alpha + \beta = 90°$, then $\sin\alpha = \cos\beta$.

2. $x = 90 - 35 = 55°$

3. If $\alpha + \beta = 90°$, then $\beta = 90° - \alpha$. According to the theorem above, $\sin\alpha = \cos\beta$. Substituting $(90 - \alpha)$ for β: $\sin\alpha = \cos(90 - \alpha)$.

Page 29

First, we need to find the length of the hypotenuse: $c = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

1. $\sin^2 \alpha = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$
2. $\sin^2 \beta = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$
3. $\cos^2 \alpha = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$ (same as $\sin^2 \beta$)
4. $\cos^2 \beta = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$ (same as $\sin^2 \alpha$)
5. $\sin^2 \alpha + \cos^2 \alpha = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$
6. $\sin^2 \alpha + \cos^2 \beta = \frac{16}{25} + \frac{16}{25} = \frac{32}{25}$
7. $\cos^2 \alpha + \sin^2 \beta = \frac{9}{25} + \frac{9}{25} = \frac{18}{25}$

Page 30

1. $\sin^2 \alpha + \cos^2 \alpha = \left(\dfrac{4}{5}\right)^2 + \left(\dfrac{3}{5}\right)^2 = \dfrac{16}{25} + \dfrac{9}{25} = \dfrac{25}{25} = 1$
2. It's not an error. According to the corollary of the Pythagoream Theorem, this a right triangle: $a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 = c^2$.
3. $\sin^2 \beta + \cos^2 \beta = \left(\dfrac{3}{5}\right)^2 + \left(\dfrac{4}{5}\right)^2 = \dfrac{9}{25} + \dfrac{16}{25} = \dfrac{25}{25} = 1$
4. $\cos^2 \alpha + \sin^2 \alpha = 1$
$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169} = \frac{144}{169}$$
$$\cos \alpha = \sqrt{\frac{144}{169}} = \frac{12}{13}$$
5. $\cos^2 \alpha + \sin^2 \alpha = 1$
$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{5}{7}\right)^2 = 1 - \frac{25}{49} = \frac{24}{49}$$
$$\cos \alpha = \sqrt{\frac{24}{49}} = \frac{\sqrt{4}\sqrt{6}}{\sqrt{49}} = \frac{2\sqrt{6}}{7}$$

6. We will follow the proof at the bottom of Page 29:

$$\begin{aligned} sin^2\alpha + \sin^2\beta &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\ &= \frac{a^2}{c^2} + \frac{b^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{a^2+b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

7. Again, we will follow the proof at the bottom of Page 29:

$$\begin{aligned} cos^2\alpha + \cos^2\beta &= \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 \\ &= \frac{b^2}{c^2} + \frac{a^2}{c^2} \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{a^2+b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

Page 31

1.

angle x	$\sin x$	$\cos x$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
α	$\frac{4}{5}$	$\frac{3}{5}$
β	$\frac{3}{5}$	$\frac{4}{5}$

2. $\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°$

3. $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

4. We can observe from the table that $\sin x$ increases with the size of an acute angle ($\sin 30° < \sin 45° < \sin 60°$), while $\cos x$ decreases with the size of an acute angle. You can compare the fractions or convert to decimal make sure. We know that $\sin\alpha = \frac{4}{5}$. We also know that α is an acute angle.
Is it larger or smaller than $30°$*?* Larger, $\frac{4}{5} > \frac{1}{2}$ so $\sin\alpha > \sin 30°$.
Than $45°$*?* Larger, $\frac{4}{5} > \frac{1}{\sqrt{2}}$ so $\sin\alpha > \sin 45°$.
Than $60°$*?* Smaller, $\frac{4}{5} < \frac{\sqrt{3}}{2}$ so $\sin\alpha < \sin 60°$.

Page 33 (First)

1. As the angle α get smaller, the ratio of the opposite side to the hypotenuse approaches 0.
2. Recall from the theorem on page 28 that if $\alpha + \beta = 90°$, then $\sin\alpha = \cos\beta$ and $\cos\alpha = \sin\beta$. So, if $\sin 90° = 1$, then $\cos 0° = 1$.
3. $\sin^2 0° + \cos^2 0° = 0^2 + 1^2 = 0 + 1 = 1$
4. $\sin^2 90° + \cos^2 90° = 1^2 + 0^2 = 1 + 0 = 1$
5. Our friend is mistaken; the sine of an angle can never be greater than 1.

Page 33 (Second)

1.

$\sin 0° + \cos 0°$	$0 + 1$	1
$\sin 30° + \cos 30°$	$\frac{1}{2} + \frac{\sqrt{3}}{2}$	1.366 (approx.)
$\sin 45° + \cos 45°$	$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$	1.414 (approx.)
$\sin 60° + \cos 60°$	$\frac{\sqrt{3}}{2} + \frac{1}{2}$	1.366 (approx.)
$\sin\alpha + \cos\alpha$, where α is the smaller...	$\frac{3}{5} + \frac{4}{5}$	1.4
$\sin\alpha + \cos\alpha$, where α is the larger...	$\frac{4}{5} + \frac{3}{5}$	1.4

2. If $\sin\alpha = 1$, then $\cos\alpha = 0$ and $\sin\alpha + \cos\alpha = 1$. If $\cos\alpha = 1$, then $\sin\alpha = 0$ and $\sin\alpha + \cos\alpha = 1$. Otherwise, $\sin\alpha < 1$ and $\cos\alpha < 1$, so $\sin\alpha + \cos\alpha < 2$.
3. First we will expand and simplify $(\sin\alpha + \cos\alpha)^2$:

$$\begin{aligned}
(\sin\alpha + \cos\alpha)^2 &= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha \\
&= (\sin^2\alpha + \cos^2\alpha) + 2\sin\alpha\cos\alpha \\
&= 1 + 2\sin\alpha\cos\alpha
\end{aligned}$$

We know that $0 \le \sin\alpha \le 1$ and $0 \le \cos\alpha \le 1$ because α is acute. So $2\sin\alpha\cos\alpha$ is the product of three nonnegative numbers, and is itself a nonnegative number. A nonnegative number added to 1 results in a number ≥ 1. Therefore, $1{+}2\sin\alpha\cos\alpha \ge 1$. The square root of a number ≥ 1 is itself ≥ 1. Therefore, $\sqrt{1+2\sin\alpha\cos\alpha} \ge 1$. Rewriting the expression on the left: $\sqrt{1+2\sin\alpha\cos\alpha} = \sqrt{(\sin\alpha+\cos\alpha)^2} = \sin\alpha+\cos\alpha$. So, $\sin\alpha+\cos\alpha \ge 1$.

4. $\sin 45° + \cos 45° = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

5. You should notice that the values for $\sin\alpha+\cos\alpha$ increases with larger *alpha* when $0° \le \alpha < 45°$, reaches a maximum value when $\alpha = 45°$, then decreases with larger α when $45° < \alpha \le 90°$.

Page 35

1.

$(\sin 0°)(\cos 0°)$	$0 \cdot 1$	0
$(\sin 30°)(\cos 30°)$	$\frac{1}{2} \cdot \frac{\sqrt{3}}{2}$	0.433 (approx.)
$(\sin 45°)(\cos 45°)$	$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$	0.5
$(\sin 60°)(\cos 60°)$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$	0.433 (approx.)
$(\sin\alpha)(\cos\alpha)$, where α is the smaller...	$\frac{3}{5} \cdot \frac{4}{5}$	0.48
$(\sin\alpha)(\cos\alpha)$, where α is the larger...	$\frac{4}{5} \cdot \frac{3}{5}$	0.48

How large can the product $(\sin\alpha)(\cos\alpha)$ *get?* We can see from the table that the maximum value of the product appears to be when $\alpha = 45°$.

Page 37

1. $\cos\alpha = 3/5$, $\cos\beta = 4/5$, $\sin\alpha = 4/5$, $\sin\beta = 3/5$, $\tan\alpha = 4/3$, $\tan\beta = 3/4$, $\cot\alpha = 3/4$, $\cot\beta = 4/3$.

2. We can show that this assumption is correct using the corollary of the Pythagorean Theorem: $a^2 + b^2 = 3^2 + 4^2 = 25 = c^2$.

3. $\cos\alpha = a/c$, $\cos\beta = b/c$, $\sin\alpha = b/c$, $\sin\beta = a/c$, $\tan\alpha = b/a$, $\tan\beta = a/b$, $\cot\alpha = a/b$, $\cot\beta = b/a$.

4. $c = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$. $\cos\alpha = 12/13$. $\cos\beta = 5/13$. $\cot\alpha = 12/5$. $\cot\beta = 5/12$.

5. First, we will use the Pythagorean Theorem to find the length of the longer leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 7^2 &= 25^2 \\ a^2 + 49 &= 625 \\ a^2 &= 576 \\ a &= 24 \end{aligned}$$

   We can now find the numerical values that were asked for: $\cos\alpha = 24/25$, $\cos\beta = 7/25$, $\cot\alpha = 24/7$, $\cot\beta = 7/24$.

6. $\dfrac{a}{c} = \sin\alpha = \cos\beta$
   $\dfrac{b}{c} = \cos\alpha = \sin\beta$
   $\dfrac{a}{b} = \tan\alpha = \cot\beta$
   $\dfrac{b}{a} = \cot\alpha = \tan\beta$

7. First, we will use the Pythagorean Theorem to find the length of the other leg:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 3^2 &= 5^2 \\ a^2 + 9 &= 25 \\ a^2 &= 16 \\ a &= 4 \end{aligned}$$

   We can now find the numerical values that were asked for: $\cos\alpha = 4/5$, $\cot\alpha = 4/3$.

8. If $\tan\alpha = 1$, then $a/b = 1$, implying that $a = b$ and $\alpha = 45°$. $\cos\alpha = \cos 45° = 1/\sqrt{2}$. $\cot\alpha = 1/1 = 1$.

9. $\tan 45° = 1/1 = 1$.

10. $\tan 30° = 1/\sqrt{3} \approx 0.57735$.

11. $\tan 45° + \sin 30° = 1 + \frac{1}{2} = \frac{3}{2}$. We don't need a calculator because both numbers are rational.

Chapter 2: Relations among Trigonometric Ratios

Page 43

1. $\cos\alpha = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$

   $\tan\alpha = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}$

   $\cot\alpha = \frac{15}{8}$

2. Let the length of the adjacent leg a be $\frac{3}{7}$ and the length of the hypotenuse be 1 (see the first triangle diagram on page 44).

   $$\sin\alpha = \sqrt{1-a^2} = \sqrt{1-\left(\frac{3}{7}\right)^2} = \sqrt{1-\frac{9}{49}} = \sqrt{\frac{40}{49}} \quad = \frac{\sqrt{4}\sqrt{10}}{\sqrt{49}} = \frac{2\sqrt{10}}{7}$$

   $$\tan\alpha = \frac{\sqrt{1-a^2}}{a} = \frac{\frac{2\sqrt{10}}{7}}{\frac{3}{7}} = \frac{2\sqrt{10}}{3}$$

   $$\cot\alpha = \frac{a}{\sqrt{1-a^2}} = \frac{3}{2\sqrt{10}}$$

3. $\sin\alpha = \sqrt{1-b^2},\ \tan\alpha = \frac{\sqrt{1-b^2}}{b},\ \cot\alpha = \frac{b}{\sqrt{1-b^2}}$

4. $\sin\alpha = \frac{d}{\sqrt{1+d^2}},\ \tan\alpha = \frac{1}{\sqrt{1+d^2}},\ \cot\alpha = \frac{1}{d}$

5.

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	a	$\sqrt{1-a^2}$	$\frac{a}{\sqrt{1-a^2}}$	$\frac{\sqrt{1-a^2}}{a}$
$\cos\alpha$	$\sqrt{1-a^2}$	a	$\frac{\sqrt{1-a^2}}{a}$	$\frac{a}{\sqrt{1-a^2}}$
$\tan\alpha$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{\sqrt{1+a^2}}$	a	$\frac{1}{a}$
$\cot\alpha$	$\frac{1}{\sqrt{1+a^2}}$	$\frac{a}{\sqrt{1+a^2}}$	$\frac{1}{a}$	a

Page 45 (First)

1. Given in text
2. $\sin^2 45^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
3.

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\sin\alpha$	$\sin\alpha$	$\sqrt{1-\sin^2\alpha}$	$\frac{a}{\sqrt{1-\sin^2\alpha}}$	$\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$
$\cos\alpha$	$\sqrt{1-\cos^2\alpha}$	$\cos\alpha$	$\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$	$\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}$
$\tan\alpha$	$\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$	$\frac{1}{\sqrt{1+\tan^2\alpha}}$	$\tan\alpha$	$\frac{1}{\tan\alpha}$
$\cot\alpha$	$\frac{1}{\sqrt{1+\cot^2\alpha}}$	$\frac{a}{\sqrt{1+\cot^2\alpha}}$	$\frac{1}{\cot\alpha}$	$\cot\alpha$

Page 45 (Second)

1. $\tan\alpha = \frac{a}{b} = \cot\beta$
2. $\cot\alpha = \frac{b}{a} = \tan\beta$
3. $\sec\alpha = \frac{c}{a} = \csc\beta$
4. $\csc\alpha = \frac{c}{b} = \sec\beta$

Page 47

1. (a) $\sin^2 30^\circ + \cos^2 30^\circ = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$

   (b) $\sin^2 45^\circ + \cos^2 45^\circ = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

   (c) $\sin^2 60^\circ + \cos^2 60^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

2. $$\sin^2\alpha + \cos^2\alpha = 1$$

$$\left(\frac{\sqrt{5}}{4}\right)^2 + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \left(\frac{\sqrt{5}}{4}\right)^2 = 1 - \frac{5}{16} = \frac{11}{16}$$

$$\cos\alpha = \sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4}$$

3. $$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha + \left(\frac{2}{3}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\sin\alpha = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

4. $$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{3}$$

$$\frac{\sin^2\alpha}{1-\sin^2\alpha} = \frac{1}{3}$$

$$3\sin^2\alpha = 1 - \sin^2\alpha$$

$$4\sin^2\alpha = 1$$

$$\sin^2\alpha = \frac{1}{4}$$

$$\sin\alpha = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

5. (a) $\cot x \sin x = \left(\frac{1}{\tan x}\right)\sin x = \frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}} = \frac{\sin x \cos x}{\sin x} = \cos x$

(b) $\frac{\tan x}{\sin x} = \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{\frac{\sin x}{\cos x}\cdot\frac{1}{\sin x}}{\sin x\cdot\frac{1}{\sin x}} = \frac{\frac{\sin x}{\sin x\cos x}}{1} = \frac{\sin x}{\sin x\cos x} = \frac{1}{\cos x}$

(c) $\cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha) = \cos^2\alpha - 1 + \cos^2\alpha = 2\cos^2\alpha - 1$

(d) This one is tricky. You might need to try a few different approaches (squaring above and below, multiplying above and below by $\cos\alpha\sin\alpha$). Eventually it becomes clear that you need to multiply above and below by $(1-\cos\alpha)$ and find a way to cancel out the $\sin\alpha$ factor in the numerator:

$$\frac{\sin\alpha}{1+\cos\alpha} = \frac{\sin\alpha(1-\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)} = \frac{\sin\alpha(1-\cos\alpha)}{1-\cos\alpha+\cos\alpha-\cos^2\alpha}$$
$$= \frac{\sin\alpha(1-\cos\alpha)}{1-\cos^2\alpha} = \frac{\sin\alpha(1-\cos\alpha)}{1-(1-\sin^2\alpha)} = \frac{\sin\alpha(1-\cos\alpha)}{1-1+\sin^2\alpha}$$
$$= \frac{\sin\alpha(1-\cos\alpha)}{\sin^2\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

(e) $$\frac{\sin^2\alpha+2\cos^2\alpha-1}{\cot^2\alpha} = \frac{1-\cos^2\alpha+2\cos^2\alpha-1}{\cot^2\alpha} = \frac{\cos^2\alpha}{\left(\frac{\cos\alpha}{\sin\alpha}\right)^2}$$
$$= \frac{\cos^2\alpha}{\frac{\cos^2\alpha}{\sin^2\alpha}} = \frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha}$$
$$= \sin^2\alpha$$

(f) $$\cos^2\alpha = \frac{\cos^2\alpha}{1} = \frac{\cos^2\alpha}{\cos^2\alpha+\sin^2\alpha}$$
$$= \frac{\frac{\cos^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha+\sin^\alpha}{\cos^2\alpha}} = \frac{1}{\frac{\cos^2\alpha}{\cos^2\alpha}+\frac{\sin^2\alpha}{\cos^2\alpha}}$$
$$= \frac{1}{1+\tan^2\alpha}$$

(g) $$\sin^2\alpha = \frac{\sin^2\alpha}{1} = \frac{\cos^2\alpha}{\cos^2\alpha+\sin^2\alpha}$$
$$= \frac{\frac{\sin^2\alpha}{\sin^2\alpha}}{\frac{\cos^2\alpha+\sin^\alpha}{\sin^2\alpha}} = \frac{1}{\frac{\cos^2\alpha}{\sin^2\alpha}+\frac{\sin^2\alpha}{\sin^2\alpha}}$$
$$= \frac{1}{\cot^2\alpha+1}$$

(h) $$\frac{1-\cos\alpha}{1+\cos\alpha} = \frac{(1-\cos\alpha)(1+\cos\alpha)}{(1+\cos\alpha)(1+\cos\alpha)} = \frac{1+\cos\alpha-\cos\alpha-\cos^2\alpha}{(1+\cos\alpha)^2}$$
$$= \frac{1-\cos^2\alpha}{(1+\cos\alpha)^2} = \frac{\sin^2\alpha}{(1+\cos\alpha)^2}$$
$$= \left(\frac{\sin\alpha}{1+\cos\alpha}\right)^2$$

(i) The key to solving this one is the formula for factoring a difference of cubes: $a^3-b^3=(a-b)(a^2+ab+b^2)$.

$$\frac{\sin^3\alpha-\cos^3\alpha}{\sin\alpha-\cos\alpha} = \frac{(\sin\alpha-\cos\alpha)(\sin^2\alpha+\sin\alpha\cos\alpha+\cos^2\alpha)}{\sin\alpha-\cos\alpha}$$
$$= \sin^2\alpha+\sin\alpha\cos\alpha+\cos^2\alpha$$
$$= 1+\sin\alpha\cos\alpha$$

6. (a) We can rewrite the LHS to show that $\sin^4\alpha - \cos^4\alpha = \cos^2\alpha - \sin^2\alpha$:

$$\begin{aligned}\sin^4\alpha - \cos^4\alpha &= (\sin^2\alpha + \cos^2\alpha)(\sin^2\alpha - \cos^2\alpha) = 1(\sin^2\alpha - \cos^2\alpha)\\ &= \sin^2\alpha - \cos^2\alpha\end{aligned}$$

Answer: There are no angles α for which $\sin^4\alpha - \cos^4\alpha > \cos^2\alpha - \sin^2\alpha$ because the expressions on either side of the inequality are equivalent.

(b) $\sin^4\alpha - \cos^4\alpha >= \cos^2\alpha - \sin^2\alpha$ for all angles α because the expressions on either side of the inequality are equivalent.

7. If we rewrite $2\sin\alpha\cos\alpha$ as a fraction, we can divide above and below by $\cos\alpha$ to convert the numerator and denominator into expressions in terms of $\tan\alpha$:

$$\begin{aligned}2\sin\alpha\cos\alpha &= \frac{2\sin\alpha\cos\alpha}{1} = \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha + \cos^2\alpha}\\ &= \frac{\frac{2\sin\alpha\cos\alpha}{\cos^2\alpha}}{\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}} = \frac{\frac{2\sin\alpha}{\cos\alpha}}{\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha}}\\ &= \frac{2\tan\alpha}{\tan^2\alpha + 1}\end{aligned}$$

Now we can plug in the given value for $\tan\alpha$ to find the value of $2\sin\alpha\cos\alpha$ in this instance:

$$2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{\tan^2\alpha+1} = \frac{2(\frac{2}{5})}{(\frac{2}{5})^2+1} = \frac{\frac{4}{5}}{\frac{4}{25}+1} = \frac{\frac{4}{5}}{\frac{4}{25}+\frac{25}{25}} = \frac{\frac{20}{25}}{\frac{29}{25}} = \frac{20}{29}$$

8. First, we will rewrite the expresssion $\cos^2\alpha - \sin^2\alpha$ in terms of $\tan\alpha$:

$$\begin{aligned}\cos^2\alpha - \sin^2\alpha &= \frac{\cos^2\alpha - \sin^2\alpha}{1} = \frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha + \sin^2\alpha} = \frac{\frac{\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}}\\ &= \frac{1-\tan^2\alpha}{1+\tan^2\alpha}\end{aligned}$$

(a) To find the numerical value of $\cos^2\alpha - \sin^2\alpha$ when $\tan\alpha = \frac{2}{5}$ we can substitute $\frac{2}{5}$ for $\tan\alpha$ in the formula above:

$$\cos^2\alpha - \sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-\left(\frac{2}{5}\right)^2}{1+\left(\frac{2}{5}\right)^2} = \frac{1-\frac{4}{25}}{1+\frac{4}{25}} = \frac{\frac{21}{25}}{\frac{29}{25}} = \frac{21}{29}$$

(b) Substituting r for $\tan\alpha$ in the formula above:

$$\cos^2\alpha - \sin^2\alpha = \frac{1-r^2}{1+r^2}$$

9. First, we will rewrite the expresssion in terms of $\tan\alpha$:

$$\frac{\sin\alpha - 2\cos\alpha}{\cos\alpha - 3\sin\alpha} = \frac{\frac{\sin\alpha-2\cos\alpha}{\cos\alpha}}{\frac{\cos\alpha-3\sin\alpha}{\cos\alpha}} = \frac{\frac{\sin\alpha}{\cos\alpha} - \frac{2\cos\alpha}{\cos\alpha}}{\frac{\cos\alpha}{\cos\alpha} - \frac{3\sin\alpha}{\cos\alpha}} = \frac{\tan\alpha - 2}{1 - 3\tan\alpha}$$

Next, we substitute $\frac{2}{5}$ for $\tan\alpha$:

$$\frac{\tan\alpha - 2}{1 - 3\tan\alpha} = \frac{\frac{2}{5} - 2}{1 - 3\left(\frac{2}{5}\right)} = \frac{\frac{2}{5} - \frac{10}{5}}{\frac{5}{5} - \frac{6}{5}} = \frac{-\frac{8}{5}}{-\frac{1}{5}} = 8$$

10. First, we will rewrite the expresssion in terms of $\tan\alpha$:

$$\frac{a\sin\alpha + b\cos\alpha}{c\cos\alpha + d\sin\alpha} = \frac{\frac{a\sin\alpha}{\cos\alpha} + \frac{b\cos\alpha}{\cos\alpha}}{\frac{c\cos\alpha}{\cos\alpha} + \frac{d\cos\alpha}{\cos\alpha}} = \frac{a\tan\alpha + b}{c + d\tan\alpha}$$

Next, we substitute $\frac{2}{5}$ for $\tan\alpha$ and simplify:

$$\frac{a\tan\alpha + b}{c + d\tan\alpha} = \frac{a\left(\frac{2}{5}\right) + b\left(\frac{5}{5}\right)}{c\left(\frac{5}{5}\right) + d\left(\frac{2}{5}\right)} = \frac{\frac{2a+5b}{5}}{\frac{5c+2d}{5}} = \frac{2a + 5b}{5c + 2d}$$

Now we can see why the problem included the restriction that $5c + 2d \neq 0$; the value of the expression is undefined if the denominator is zero. The sum of two rational numbers is a rational number. Therefore the numerator and denominator in the expression are both rational numbers. The quotient of two rational numbers is a rational number. Therefore, the entire expression evaluates to a rational number for arbitratrary rational values of a, b, c and d.

11. We can expand and simplify the expression:

$$\begin{aligned}
&(\sin\alpha + \cos\alpha)^2 + (\sin\alpha - \cos\alpha)^2 \\
&= \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha \\
&= 2\sin^2\alpha + 2\cos^2\alpha \\
&= 2(\sin^2\alpha + \cos^2\alpha) \\
&= 2(1) \\
&= 2
\end{aligned}$$

As the expression evaluates to a constant, it is as large as possible for all values of α.

Page 49

1. Rewriting any instances of $\sec\alpha$ or $\csc\alpha$ on either side of the identities:

(a) $\tan\alpha \csc\alpha = \sec\alpha$

$$\tan\alpha \frac{1}{\sin\alpha} = \frac{1}{\cos\alpha}$$

$$\frac{\tan\alpha}{\sin\alpha} = \frac{1}{\cos\alpha}$$

(b) $\cot\alpha \csc\alpha = \sec\alpha$

$$\cot\alpha \frac{1}{\cos\alpha} = \frac{1}{\sin\alpha}$$

$$\frac{\cot\alpha}{\cos\alpha} = \frac{1}{\sin\alpha}$$

(c) $\dfrac{1}{\sec\alpha}\csc\alpha = \cot\alpha$

$$\frac{1}{\frac{1}{\cos\alpha}} \cdot \frac{1}{\sin\alpha} = \cot\alpha$$

$$\cos\alpha \frac{1}{\sin\alpha} = \cot\alpha$$

$$\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$$

(d) $\tan^2\alpha = (\sec\alpha + 1)(\sec\alpha - 1)$

$$\tan^2\alpha = \sec^2\alpha - 1$$

$$\tan^2\alpha = \frac{1}{\cos^2\alpha} - 1$$

(e) $\csc^2\alpha = 1 + \cot^2\alpha$

$$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$$

2. Rewriting any instances of $\sin\alpha$ or $\cos\alpha$ on either side of the identities, and eliminating fractions:

(a) $\dfrac{\tan\alpha}{\sin\alpha} = \dfrac{1}{\cos\alpha}$

$$\tan\alpha \frac{1}{\sin\alpha} = \sec\alpha$$

$$\tan\alpha \csc\alpha = \sec\alpha$$

(b) $\dfrac{1}{\sin\alpha}\cos\alpha = \cot\alpha$

$$\frac{\cos\alpha}{\sin\alpha} = \cot\alpha$$

$$\cot\alpha = \cot\alpha$$

(c) $\tan^2\alpha + 1 = \dfrac{1}{\cos^2\alpha}$

$$\tan^2\alpha + 1 = \sec^2\alpha$$

(d) $\dfrac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$
$\csc^2\alpha = 1 + \cot^2\alpha$

Page 50

1. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= (\cos^2\alpha - \sin^2\alpha)^2 + (2\sin\alpha\cos\alpha)^2 \\ &= \cos^4\alpha - 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha + 4\sin^2\alpha\cos^2\alpha \\ &= \cos^4\alpha + 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha \\ &= (\cos^2\alpha + \sin^2\alpha)^2 \\ &= (1)^2 \\ &= 1 \end{aligned}$$

According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

2. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= \left(\sqrt{\frac{1+\cos\alpha}{2}}\right)^2 + \left(\sqrt{\frac{1-\cos\alpha}{2}}\right)^2 \\ &= \frac{1+\cos\alpha}{2} + \frac{1-\cos\alpha}{2} \\ &= \frac{1+\cos\alpha+1-\cos\alpha}{2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

3. First, we will rewrite a and b to eliminate the cube exponents:

$$\begin{aligned} a &= 4\cos^3\alpha - 3\cos\alpha \\ &= 4\cos\alpha\cos^2\alpha - 3\cos\alpha \\ &= 4\cos\alpha(1-\sin^2\alpha) - 3\cos\alpha \\ &= 4\cos\alpha - 4\sin^2\alpha\cos\alpha - 3\cos\alpha \\ &= \cos\alpha - 4\sin^2\alpha\cos\alpha \end{aligned}$$

$$\begin{aligned} b &= 3\sin\alpha - 4\sin^3\alpha \\ &= 3\sin\alpha - 4\sin\alpha\sin^2\alpha \\ &= 3\sin\alpha - 4\sin\alpha(1-\cos^2\alpha) \\ &= -\sin\alpha + 4\sin\alpha\cos^2\alpha \end{aligned}$$

Next, we will expand a^2 and b^2:

$$\begin{aligned} a^2 &= (\cos\alpha - 4\sin^2\alpha\cos\alpha)^2 \\ &= \cos^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha \end{aligned}$$

$$\begin{aligned} b^2 &= (-\sin\alpha + 4\sin\alpha\cos^2\alpha)^2 \\ &= \sin^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^2\alpha\cos^4\alpha \end{aligned}$$

Next, we add the expressions for a^2 and b^2 and simplify to 1:

$$\begin{aligned} a^2 + b^2 &= \cos^2\alpha - 8\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha + \sin^2\alpha - 8\sin^2\alpha\cos^2\alpha + \\ &\qquad 16\sin^2\alpha\cos^4\alpha \\ &= \cos^2\alpha + \sin^2\alpha - 16\sin^2\alpha\cos^2\alpha + 16\sin^4\alpha\cos^2\alpha + 16\sin^2\alpha\cos^4\alpha \\ &= \cos^2\alpha + \sin^2\alpha + 16\sin^2\alpha\cos^2\alpha(-1 + \sin^2\alpha + \cos^2\alpha) \\ &= 1 + 16\sin^2\alpha\cos^2\alpha(0) \\ &= 1 \end{aligned}$$

According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

4. First, we find the value of $a^2 + b^2$:

$$\begin{aligned} a^2 + b^2 &= \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 \\ &= \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2} \\ &= \frac{(1-t^2)^2 + (2t)^2}{(1+t^2)^2} \\ &= \frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)(1+t^2)} \\ &= \frac{(1+t^2)(1+t^2)}{(1+t^2)(1+t^2)} \\ &= 1 \end{aligned}$$

According to the lemma on Page 50, as $a^2 + b^2 = 1$, an angle θ exists such that $a = \cos\theta$ and $b = \sin\theta$.

5. We expand $(p^2 - q^2)^2 + (2pq)^2$ and use the fact that $p^2 + q^2 = 1$ to simplify

to 1:

$$\begin{aligned}(p^2-q^2)^2+(2pq)^2 &= p^4-2p^2q^2+q^4+4p^2q^2\\ &= p^4+2p^2q^2+q^4\\ &= (p^2+q^2)^2\\ &= (1)^2\\ &= 1\end{aligned}$$

This is similar to Exercise 1 above.

Page 51

1. $\sin\alpha < 1$ when α is acute, therefore $1-\sin\alpha > 0$ when α is acute. $1-\sin\alpha = 0$ when $\sin\alpha = 1$, i.e., $\alpha = 90°$.

2. $\cos\alpha < 1$ when α is acute, therefore $1-\cos\alpha > 0$ when α is acute. $1-\cos\alpha = 0$ when $\cos\alpha = 1$, i.e., $\alpha = 0°$.

3. Statement a) is always true. Statements b) and c) both include the case that $\sin^2\alpha + \cos^2\alpha = 1$, which is always true.

4. Let x be the maximum cost of the items in a supermarket. In Supermarket A, $x \leq \$1$. In Supermarket B, $x < \$1$. In Supermarket C, $x \leq \$1$. In Supermarket D, $x > \$1$. We can see that Supermarkets A and C are offering the same terms.

5. Inequality a) is correct. For b) to be correct, an angle α would have to exist such that $\sin\alpha + \cos\alpha = 2$. We know that this is not the case. When $\alpha = 90°$, $\sin\alpha = 1$ and $\cos\alpha = 0$. When $\alpha = 0°$, $\sin\alpha = 0$ and $\cos\alpha = 1$. When $0° < \alpha < 90°$, $\sin\alpha < 1$ and $\cos\alpha < 1$. In all cases, $\sin\alpha + \cos\alpha < 2$.

6. The largest possible value of $\sin\alpha$ is 1, and occurs when $\alpha = 90°$. The largest possible value of $\cos\alpha$ is 1, and occurs when $\alpha = 0°$. See Page 32.

Page 52

1. $\sin 30° = 0.5$, $\sin 45° = 0.707$, $\sin 60° = 0.866$.

2. By using the `tan` button to calculate $\tan 60°$, and the `sqrt` button to calculate $\sqrt{3}$, Betty can compare the results: both are 1.732.

3. Press `tan`, then enter the angle degree measure, then press `1/x`

4.

in radical or rational form				
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$

in decimal form, from calculator				
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$30°$	0.5	0.866	0.577	1.732
$45°$	0.707	0.707	1	1
$60°$	0.866	0.5	1.732	0.577

Page 53

1. The sine of the larger angle is $4/5 = .8$. We can use the inverse sine function to find the angle: $\arcsin .8 = 53.1301°$. The sum of the three angles in the triangles is: $\arcsin .6 + \arcsin .8 + 90° = 36.8699° + 53.1301° + 90° = 180°$.

2. (a) $\arcsin 1 = 90°$

   (b) $\arccos 0.7071067811865 = 45°$

3. $\arccos 0.8 = 36.8699°$

4. $\arcsin 0.6 = 36.8699°$

5. Half of $\sin 30°$ (0.25) seems like a reasonable estimate. The actual value is 0.2588.

6.

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14.

Page 55

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Page 56

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Page 59

1.

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10.

11.

Page 62

1. The degree measure of a semicircle is $180°$. The degree measure of a quarter circle is $90°$.

2. The measure of arc cut off by one side of regular pentagon inscribed in a circle is $360°/5 = 72°$. For a regular hexagon: $360°/6 = 60°$. For a regular octagon: $360°/8 = 45°$.