Week 1 — Practice Proofs (Symmetric & Invertible Matrices)

The examples labeled "*Strang*" are taken from the problem sets in Ch. 2, §5 & §7 of Gilbert Strang's *Introduction to Linear Algebra*, fifth edition. Otherwise, the full author's name is given, or the problem is so well known that a citation is unwarranted.

Problems

1. (*Strang*) If A and B are $n \times n$ symmetric matrices, then show that the following are also symmetric, or provide a counterexample.

a. $A^2 - B^2$

b. $(A + B)(A - B)$

c. ABA

d. $ABAB$

2. (*Jim Kruidenier, SBCC*) If $A^2 = [\,0\,]$, does $(I - A)^{-1}$ exist? Prove or provide a counterexample.

3. (*Strang*) Suppose that A is $m \times n$, S is $m \times m$ and symmetric, and both A and S have real valued entries.

a. Show that $A^T A$ has no negative diagonal entries.

b. Is $A^T S A$ symmetric? What are its dimensions?

4. (*Strang*) Let $C = AB$ and $D = ABC$.

a. If C is invertible, then A is invertible. Find A^{-1} in terms of C^{-1} and B.

b. If D is invertible, then B is invertible. Find B^{-1} in terms of D^{-1}, A, and C.

5. Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

a. Find the matrix A such that $A\mathbf{b} = \mathbf{a} \times \mathbf{b}$, where $\times$ is the cross product.

b. What is A^T?

6. A matrix is skew-symmetric if $A^T = -A$. Show that if A is skew symmetric, then $(I + A)^{-1}$ exists.

7. A little algebra shows that $A = \frac{1}{2}(A - A^T) + \frac{1}{2}(A + A^T)$.

a. Show that $\frac{1}{2}(A - A^T)$ is skew symmetric

b. Show that $\frac{1}{2}(A + A^T)$ is symmetric