A Cofactor Theorem

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Consider the following:

$$\frac{C^T}{|A|} = A^{-1} \implies C^T = |A|A^{-1} \tag{1}$$

Left multiply (1) by A to get:

$$A(C^T) = A(|A|A^{-1}) \implies AC^T = A|A|A^{-1} = |A|AA^{-1} = |A|I \tag{2}$$

Visually, (2) looks like:

$$\overbrace{\begin{bmatrix} — & a_1 & — \\ — & a_2 & — \\ & \vdots & \\ — & a_n & — \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} | & | & & | \\ c_1^T & c_2^T & \cdots & c_n^T \\ | & | & & | \end{bmatrix}}^{C^T} = \overbrace{\begin{bmatrix} |A| & & & \\ & |A| & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & |A| \end{bmatrix}}^{|A|I}$$

Consider the value of one entry of $|A|I$, which we obtain using the entry-by-entry method of matrix multiplication. Visually, it looks like:

$$\begin{bmatrix} — & a_i & — \end{bmatrix} \begin{bmatrix} | \\ c_j^T \\ | \end{bmatrix} = |A|I_{ij}$$

If $i = j$, then the entry $|A|I_{ij}$ lies on the diagonal of $|A|I$, and so is equal to $|A|$. If $i \neq j$, then the entry lies off the diagonal, and so is equal to 0. The above illustration can then be explicitly stated as:

$$\text{Row}_i(A)\text{Col}_j(C^T) = \begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{3}$$

If we had right multiplied (1) by A instead of left multiplying, we would instead have that $C^T A = |A|I$. We can then use a similar method to derive the same result as (3) but with the placement of

A and C^T switched, since we now have that the multiplication of C^T and A is commutative. This results in the following theorem.

Theorem 1 (Cofactor Theorem) *Let A be an $n \times n$ matrix and C the transpose of its cofactor matrix. Then*

$$\left.\begin{array}{l} Row_i(A)\,Col_j(C^T) \\ Row_i(C^T)\,Col_j(A) \end{array}\right\} = \begin{cases} |A| & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$$

If we'd rather deal with theorem 1 in terms of rows or columns exclusively, or if we don't want to compute the transpose of C, we can instead consider the multiplication of C and A by taking the transpose of C^T in the previous result. This changes the rows and columns of C^T into the columns and rows of C (respectively), which invalidates the multiplication as stated in theorem 1, but we can conserve it by writing the multiplication explicitly as a dot product.

$$\begin{aligned} \mathrm{Row}_i(A)\mathrm{Col}_j(C^T) &= \mathrm{Row}_i(A) \cdot \mathrm{Row}_j(C) \\ \mathrm{Row}_i(C^T)\mathrm{Col}_j(A) &= \mathrm{Col}_i(C) \cdot \mathrm{Col}_j(A) \end{aligned} \tag{4}$$

Since the dot product is commutative, we can change the order and indices in the lower half of (4), and conclude with a corollary of theorem 1.

Corollary 1.1 *Let C be the cofactor matrix of A. Then*

$$\left.\begin{array}{l} Row_i(A) \cdot Row_j(C) \\ Col_i(A) \cdot Col_j(C) \end{array}\right\} = \begin{cases} |A| & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$$