Practice Proofs — Week 2 (One-to-One & Onto)

Functions, maps, and transformations are all words for the same thing: a rule that relates each element in a set X to a unique element in a set Y. We can split this one condition into two and give a formal definition as follows:

Let f be a relation over sets X and Y. Then f is a **well defined function** if

1. for every x in X there exists a y in Y such that $f(x) = y$. [all of X is sent to Y]
2. if $x_i = x_j$ then $f(x_i) = f(x_j)$. [each element in X is sent to only one element in Y]

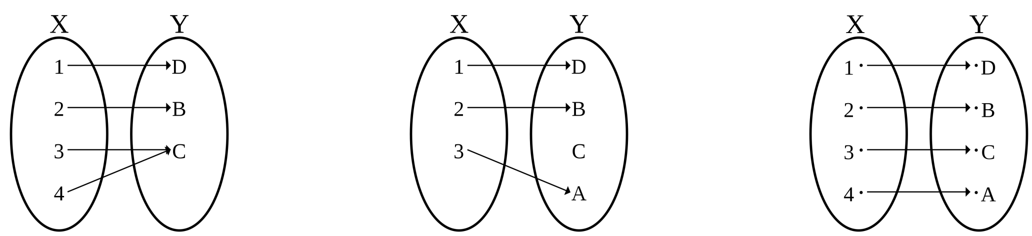
Ideally, our function is strong enough to work the other way around, but to be well defined in the opposite direction it would need to retain these two conditions. If 1 still works from Y to X then we call the function onto, and if 2 still works from Y to X we call it one-to-one. If both still work, then the opposite direction is well defined, and we can invert the function.

Let $f : \mathrm{X} \to \mathrm{Y}$ be a well defined function. Then

(a) f is **onto** (*surjective*) if for every y in Y there exists a x in X such that $f(x) = y$.

(b) f is **one-to-one** (*injective*) if $f(x_i) = f(x_j)$ implies that $x_i = x_j$.

(c) f is **invertible** (*bijective*) if f is both one-to-one and onto.



(a) Onto (b) One-to-One (c) Invertible

Notice in the diagram above that the range of f in (a) is all of Y–this is an equivalent way of stating that f is onto. We'll be able to say a lot more about these properties if f is a linear transformation once we cover subspaces and dimensionality, but for now we just need the following.

Let $T : V \to W$ be a linear transformation between vector spaces V and W. Then

(a) T is **onto** if

- for every w in W there exists a v in V such that $T(v) = w$.
- the range of T is equal to the codomain W.

(b) T is **one-to-one** if $T(v_i) = T(v_j)$ implies that $v_i = v_j$.

(c) T is **invertible** if

- T is onto and one-to-one.
- there exists a linear transformation $S : W \to V$ such that $S \circ T : V \to V$ is equal to I on V, and $T \circ S : W \to W$ is equal to I on W.

There are also more equivalent definitions if we talk about the matrices, solution sets, etc. of these transformations, but for this section we'll just be looking ~~as~~ at the transformation in the abstract.

The exercises in this worksheet are adapted from Ch. 3 of Sheldon Axler's *Linear Algebra Done Right* (3rd ed.), although the notation and proofs have been modified to fit this course.

Exercises

"P IF AND ONLY IF Q" → SHOW $P \Rightarrow Q$ AND $Q \Rightarrow P$

1. Prove that T is injective if and only if $T(v) = \mathbf{0}$ has only the trivial solution $v = \mathbf{0}$.

($\Rightarrow$) ASSUME T IS ONE-TO-ONE. THEN $T(v)=T(w) \Rightarrow v=w$. LET v BE SOME SOLUTION TO $T(v)=\vec{0}$. BY LINEARITY, $T(\vec{0})$ MUST BE $\vec{0}$, SO WE HAVE $T(v)=\vec{0}=T(\vec{0})$. SINCE T IS INJ., $T(v)=T(\vec{0}) \Rightarrow v=\vec{0}$

($\Leftarrow$) ASSUME $v=\vec{0}$ IS THE ONLY SOL. TO $T(v)=\vec{0}$. SUPPOSE THERE EXIST SOME u,w SUCH THAT $T(u)=T(w)$. THEN $T(u)-T(w)=T(u-w)=\vec{0}$, AND SINCE $v=\vec{0}$ IS THE ONLY SOL TO $T(v)=\vec{0}$, $v=u-w \Rightarrow u-w=\vec{0} \Rightarrow u=w$

THUS $T(u)=T(w) \Rightarrow u=w$, SO T IS 1-1

WE'RE TRYING TO PROVE THE DEFINITION GIVEN EARLIER THAT T IS INVT. IFF T IS 1-1 & ONTO, SO WE CAN ONLY USE THE OTHER DEF (THAT $\exists S$ S.T. $S \circ T = T \circ S = I$)

2. Prove that T is invertible if and only if T is injective and surjective.

($\Rightarrow$) ASSUME T IS INVERTIBLE.

SUPPOSE $u,v \in V$ AND $T(u)=T(v)$.

THEN $u=T^{-1}(T(u))=T^{-1}(T(v))=v$, SO $T(u)=T(v) \Rightarrow u=v \Rightarrow$ T IS 1-1

LET $w \in W$. THEN $w=T(T^{-1}(w))$

SO FOR ANY $w \in W$ THERE IS A $T^{-1}(w) \in V$ S.T. $T(T^{-1}(w))=w \Rightarrow$ T IS ONTO

($\Leftarrow$) ASSUME T IS 1-1 AND ONTO

THEN FOR $\underbrace{\text{EACH } w \in W \text{ THERE}}_{\text{ONTO}}$ IS A $\underbrace{\text{UNIQUE}}_{\text{1-1}}$ $Sw \in V$ S.T. $T(S(w))=w$

"$T \circ S$" IS "T COMPOSED WITH S"

NOTATION: $(T \circ S)(v)$ MEANS $T(S(v))$

THUS $(T \circ S)(w)=w \Rightarrow$ $T \circ S = I$ ON W

FUNCTION COMPOSITION IS ASSOCIATIVE $\Rightarrow$ $(T \circ S \circ R)(v)=(T \circ S)(R(v))=T((S \circ R)(v))$

LET $v \in V$. THEN $T((S \circ T)(v))=(T \circ S)(T(v))=I \circ T(v)=T(v)$

SINCE T IS 1-1, $T((S \circ T)(v))=T(v) \Rightarrow (S \circ T)(v)=v \Rightarrow$ $S \circ T = I$ ON V

(WE ALSO NEED TO SHOW THAT S IS LINEAR)

SUPPOSE $w_1, w_2 \in W$. THEN

$T(S(w_1)+S(w_2))=(T \circ S)(w_1)+\underbrace{(T \circ S)}_{=I}(w_2)=w_1+w_2$

$\Rightarrow S(T(S(w_1)+S(w_2)))=S(w_1+w_2)$

$=\underbrace{(S \circ T)}_{=I}(S(w_1)+S(w_2))=S(w_1)+S(w_2)=S(w_1+w_2)$ ✓

SUPPOSE $c \in \mathbb{R}$. THEN

$T(cS(w))=cT(S(w))=c\underbrace{(T \circ S)}_{=I}(w)=cw$

$\Rightarrow S(T(cS(w)))=S(cw) \Rightarrow \underbrace{(S \circ T)}_{=I}(cS(w))=cS(w)=S(cw)$ ✓

3. Prove that the inverse of T is unique.

SUPPOSE S, R ARE TWO INV'S.

THEN $S = S \circ I = S \circ (T \circ R) = (S \circ T) \circ R = I \circ R = R$

THUS $S = R$

CAN ONLY DO PART OF IT

~~**4.**~~ Prove that T is injective if and only if there exists a S such that $ST = I$.

$(\Rightarrow)$ ASSUME T IS INJ. DEFINE S': RANGE OF $T \to V$ BY $S'(T(v)) = v$

THEN $S' \circ T = I$ ON V (CHECKING LINEARITY IS EASY, BUT WE NEED STUFF ABOUT DIMENSIONS/SUBSPACES TO EXTEND S' TO S, WHERE THE DOMAIN OF S IS ALL OF W)

$(\Leftarrow)$ ASSUME THERE EXISTS A S S.T. $ST = I$. SUPPOSE

SUPPOSE $T(u) = T(v)$. THEN $S(T(u)) = S(T(v)) \Rightarrow u = v$,

SO T IS INJ

5. Prove that T is surjective if and only if there exists a S such that $TS = I$.

$\Rightarrow$ ANY $w' \in W$ CAN BE WRITTEN AS $w' = \sum_{i=1}^{n} c_i w_i$

$(\Rightarrow)$ ASSUME T IS SURJ. LET $w_1, w_2, \ldots, w_n$ BE A BASIS OF W.

GOOD PROOF IN CONTEXT OF "b-COORD" SECTION

THEN, SINCE T IS SURJ, FOR EACH $w_i \in$ BASIS OF W THERE EXISTS A $v_i \in V$ S.T. $T(v_i) = w_i$. WE CAN PROVE THAT THERE EXISTS A UNIQUE LINEAR TRANS. S SUCH THAT $S(w_i) = v_i \in$ BASIS OF V. THUS FOR SOME $w' \in W$,

$$T(S(w')) = T\left(S\left(\sum_{i=1}^{n} c_i w_i\right)\right) = T\left(\sum_{i=1}^{n} c_i S(w_i)\right) = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i T(v_i) = \sum_{i=1}^{n} c_i w_i = w'$$

$(\Leftarrow)$ ASSUME THERE EXISTS SOME S S.T. $TS = I$. THEN FOR ANY $w \in W$, $w = TS(w) = T(S(w)) \Rightarrow$ THERE EXISTS A $S(w)$ FOR EACH $w \in W$ S.T. $T(S(w)) = w \Rightarrow T$ IS SURJ