Week 1 — Practice Proofs (Dot Products & Norms)

I've so lovingly lifted this series of exercises from both Ch. I, §4 and Ch. IV, §1 of Serge Lang's *Introduction to Linear Algebra* (2nd edition). I've made some minor adjustments to fit the notation used in this course, but they're largely the same as in Lang, so try to resist looking them up and just play around with them instead (this is not for credit, after all).

As we saw in the notes, the following properties define the dot product:

1. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$
2. $(c\mathbf{v})\cdot\mathbf{w} = c(\mathbf{v}\cdot\mathbf{w})$
3. $\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$
4. $\mathbf{v}\cdot\mathbf{v} \geq 0$ (and $\mathbf{v}\cdot\mathbf{v} > 0$ if $\mathbf{v} \neq \mathbf{0}$)

#'S ABOVE EQUALS SIGNS = WHICH PROP IS USED

The dot product is a special case of the more general inner product, which has the same flavor but can be applied to things like continuous functions. It defines a sense of how close vectors are to each other, but not their length–for that, we need the separate notion of a norm $\| \, \|$, which luckily an inner product induces anyways. It gives another property:

5. $\sqrt{\mathbf{v}\cdot\mathbf{v}} = \|\mathbf{v}\| \geq 0$ (and $\|\mathbf{v}\| > 0$ if $\mathbf{v} \neq \mathbf{0}$)

With these 5 properties we can now answer the following 10 problems.

Problems

1. Prove that $\mathbf{v}\cdot(c\mathbf{w}) = c(\mathbf{v}\cdot\mathbf{w})$, $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$, and $\mathbf{v}\cdot\mathbf{0} = 0$.

THIS TELLS US THAT $c(v\cdot w) = (cv)\cdot w = v\cdot(cw)$

$$v\cdot(cw) \overset{3}{=} (cw)\cdot v \overset{2}{=} c(w\cdot v) \overset{3}{=} c(v\cdot w)$$

$$u\cdot(v+w) \overset{3}{=} (v+w)\cdot u \overset{1}{=} v\cdot u + w\cdot u \overset{3}{=} u\cdot v + u\cdot w$$

$$v\cdot\vec{0} = v\cdot(0v) = 0(v\cdot v) = 0$$

$0\cdot\vec{v} = \vec{0}$ FOR ALL $\vec{v}$

THIS TELLS US THAT

$$(v+w)\cdot(v+w) = (v+w)\cdot v + (v+w)\cdot w$$
$$v\cdot v + w\cdot v + v\cdot w + w\cdot w$$

I.E. FOIL!

2. Prove that $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$ and $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$.

$$v\cdot v = \sqrt{v\cdot v}^{\,2} \overset{5}{=} \|v\|^2$$

$$\|cv\| \overset{5}{=} \sqrt{(cv)\cdot(cv)} = \sqrt{c^2(v\cdot v)} = \sqrt{c^2}\sqrt{v\cdot v} = |c|\,\|v\|$$

$$(v+w)\cdot(v+w) = v\cdot v + v\cdot w + w\cdot v + w\cdot w \Rightarrow \underbrace{\|v+w\|^2 = \|v\|^2 + 2(v\cdot w) + \|w\|^2}_{\text{WILL USE LATER}}$$

3. Prove that $\|\mathbf{v}-\mathbf{w}\| = \|\mathbf{w}-\mathbf{v}\|$.

$$\begin{aligned}\|v-w\|^2 &= (v-w)\cdot(v-w)\\ &= v\cdot v - w\cdot v - v\cdot w + w\cdot w\\ &= w\cdot w - v\cdot w - w\cdot v + v\cdot v\\ &= (w-v)\cdot(w-v)\\ &= \|w-v\|^2\end{aligned}$$

$\left(\|*\| \geq 0 \iff \|*\|^2 \geq 0 \right.$ NORM IS NON-NEG SO SQUARE IS TOO $\left.\right)$

$$\|v-w\|^2 = \|w-v\|^2 \iff \underline{\|v-w\| = \|w-v\|}$$

4. Prove that $\|\mathbf{v}-\mathbf{w}\| = \|\mathbf{v}+\mathbf{w}\|$ if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

"IFF"

(NEED TO SHOW BOTH $\|v+w\| = \|v-w\| \Rightarrow v\cdot w = \vec{0}$ & $v\cdot w = \vec{0} \Rightarrow \|v+w\| = \|v-w\|$)

$$\|v-w\| = \|v+w\| \Rightarrow \|v-w\|^2 = \|v+w\|^2$$

$$\Rightarrow \|v\|^2 - 2(v\cdot w) + \|w\|^2 = \|v\|^2 + 2(v\cdot w) + \|w\|^2$$

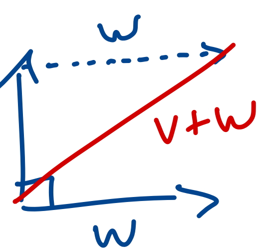
(CANCEL $\|v\|^2$'s & $\|w\|^2$'s) $\Rightarrow -2(v\cdot w) = 2(v\cdot w) \Rightarrow \underline{-(v\cdot w) = (v\cdot w)} \Rightarrow \underline{v\cdot w = 0}$

FOR $\mathbb{R}$, $-x = x$ IFF $x = 0$

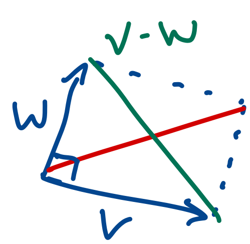
DO ALL OF THIS BACKWARDS

$$v\cdot w = 0 \Rightarrow v\cdot w + v\cdot w = 0 \Rightarrow v\cdot w = -(v\cdot w) \Rightarrow \underline{\|v-w\| = \|v+w\|}$$

5. (*the Pythagorean theorem*) Show that if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then $\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.



RIGHT TRI ⇒ PYTHAG

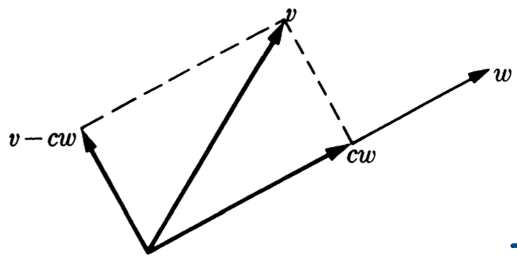


DIAG'S OF A RECT. ARE OF EQ. LENGTH

$$\|v+w\|^2 = \|v\|^2 + \underline{2(v\cdot w)} + \|w\|^2 \quad \text{(FROM BEFORE)}$$

$$v\cdot w = 0 \Rightarrow 2(v\cdot w) = 0 \Rightarrow \underline{\|v+w\|^2 = \|v\|^2 + \|w\|^2}$$

6. Let c be the component of $\mathbf{v}$ along $\mathbf{w}$. Show that if $c = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$, then $\mathbf{v} - c\mathbf{w}$ is orthogonal to $\mathbf{w}$ (the following graphic from Lang might help).



$$c = \frac{V\cdot W}{W\cdot W} \Rightarrow c(W\cdot W) = V\cdot W$$

$$\Rightarrow V\cdot W - c(W\cdot W) = 0$$

$$\Rightarrow V\cdot W - (cW)\cdot W = 0$$

$$\Rightarrow (V - cW)\cdot W = 0$$

$\Rightarrow$ $V - cW$ & W ARE ORTH

ALTHOUGH IF $|c| \leq \|v\|$ AND $c = v\cdot e$ THEN $|v\cdot e| \leq \|v\| = \|v\|\|e\|$ (WHICH IS #8)

TRUE, BUT NOT NEEDED TO PROVE

7. (*the Bessel inequality*) The component of $\mathbf{v}$ along a unit vector $\mathbf{e}$ is $c = \dfrac{\mathbf{v}\cdot\mathbf{e}}{\mathbf{e}\cdot\mathbf{e}} = \dfrac{\mathbf{v}\cdot\mathbf{e}}{1} = \mathbf{v}\cdot\mathbf{e}$. Using ~~this~~ and your work from exercises 5 and 6, show that $|c| \leq \|\mathbf{v}\|$.

$V - ce$ & e ARE ORTH (#6) $\Rightarrow$ $V - ce$ & ce ARE ORTH

#5

$$\Rightarrow \|(V - ce) + (ce)\|^2 = \|V - ce\|^2 + \|ce\|^2$$

$$\|V - ce + ce\|^2 = \|V\|^2 = \|V - ce\|^2 + \|ce\|^2 \geq \|ce\|^2$$

$$\Rightarrow \|V\|^2 \geq \|ce\|^2 = |c|^2\|e\|^2 = |c|^2$$

$$\Rightarrow \|V\|^2 \geq |c|^2 \Longrightarrow |c| \leq \|V\|$$

NEITHER $\|V\| < 0$ NOR $|c| < 0$

8. (*the Schwarz inequality*) Show that $|\mathbf{v}\cdot\mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$ for any two vectors $\mathbf{v}$ and $\mathbf{w}$ by extending your work from exercises 5, 6, and 7.

V-cW & W ARE ORTH $\Rightarrow$ V-cW & cW ARE ORTH

$$\Rightarrow \|v-cw+cw\|^2 = \|v-cw\|^2 + \|cw\|^2 \geq \|cw\|^2$$

$$\Rightarrow \|v\|^2 \geq \|cw\|^2 = |c|^2\|w\|^2 \Rightarrow \frac{\|v\|^2}{\|w\|^2} \geq |c|^2$$

$$\Rightarrow \frac{\|v\|}{\|w\|} \geq |c| = \left|\frac{v\cdot w}{w\cdot w}\right| = \frac{|v\cdot w|}{\|w\|^2}$$

$\|v\|, \|w\|, |c| > 0$

$$\Rightarrow \frac{\|v\|}{\|w\|} \geq \frac{|v\cdot w|}{\|w\|^2} \Rightarrow |v\cdot w| \leq \|v\|\,\|w\|$$

9. Show using your work from exercise 8 that $\cos(\theta) = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$.

$$|v\cdot w| \leq \|v\|\,\|w\| \Rightarrow \frac{|v\cdot w|}{\|v\|\,\|w\|} \leq 1 \Rightarrow \frac{v\cdot w}{\|v\|\,\|w\|} \leq \pm 1$$

RANGE OF $\cos(\theta)$ FOR $0^\circ \leq \theta \leq 180^\circ$, USE LAW OF COSINES TO DERIVE AN EXPLICIT RELATIONSHIP

10. (*the triangle inequality*) Prove that $\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

$$\|v+w\|^2 = \|v\|^2 + 2v\cdot w + \|w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$$

$$\left(v\cdot w \overset{\text{ALWAYS TRUE}}{\leq} |v\cdot w| \overset{\#8}{\leq} \|v\|\,\|w\| \Rightarrow 2v\cdot w \leq 2\|v\|\|w\| \right)$$

$$\Rightarrow \|v+w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2$$

$$\Rightarrow \|v+w\|^2 \leq (\|v\| + \|w\|)^2 \Rightarrow \|v+w\| \leq \|v\| + \|w\|$$