Practice Proofs — Week 3 (Invertible Matrices & Diff Eq's)

This worksheet's exercises are somewhat of a mixed bag–we've already seen some results about invertible matrices proved algebraically, but it's still good practice to prove results about invertibility without relying on row reduction. Lots of this stuff will come up again when you take differential equations, so I thought I'd include a small glimpse of what you'll be doing in 220 as well.

Exercises 1 to 3 are from Ch. 2.2 of David C. Lay, Steven R. Lay, and Judi J. McDonald's *Linear Algebra and its Applications* (5th ed.). Exercises 4 to 6 are sourced from Ch. 6.3 of Gilbert Strang's *Introduction to Linear Algebra* (5th ed.), as well as from course notes for Math 220 by Jim Kruidenier, SBCC.

Exercises

1. Suppose $(A-B)C = [\,0\,]$, where A and B are $m \times n$ and C is invertible. Prove that $A = B$.

$(A-B)C = AC - BC = [0] \Rightarrow AC = BC$

C^{-1} EXISTS $\Rightarrow (AC)C^{-1} = (BC)C^{-1} \Rightarrow AI = BI \Rightarrow \underline{A = B}$

$\left(C^{-1}(A+X)B^{-1} = I \Rightarrow A + X = CB \Rightarrow X = CB - A \right)$

2. Suppose A, B, and C are invertible square matrices. Does there exist some square matrix X such that $C^{-1}(A+X)B^{-1} = I$? If so, what is X?

YES! WORK BACKWARDS

LET $X = CB - A$. THEN X IS SQ, AND

$C^{-1}(A + (CB - A))B^{-1} = C^{-1}(CB)B^{-1} = (C^{-1}C)(BB^{-1}) = \underline{I}$

3. Suppose A, B, and X are square matrices, and that $(A - AX)^{-1} = X^{-1}B$. If A, X, and $A - AX$ are invertible, then

a. how do you know B^{-1} exists?

BY A PREVIOUS RESULT — IF $C = AB$, AND C, A ARE INVT'BLE, THEN B IS INVT'BLE

b. what is X? [if you invert any other matrices in the process, then also explain why they exist]

$(A - AX)^{-1} = X^{-1}B \Rightarrow I = X^{-1}B(A - AX) = X^{-1}BA - X^{-1}BAX$

$\Rightarrow X = X(X^{-1}BA - X^{-1}BAX) = BA - BAX$

$\Rightarrow X + BAX = (I + BA)X = BA$

$\Rightarrow \underline{X = (I + BA)^{-1}BA}$

BA, X ARE INVT'BLE $\Rightarrow (I + BA)$ IS INVT'BLE

4. The Maclaurin series of the exponential function e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Since $n!$ is just a number, and since we know that if a matrix is square then we can take its powers, it's natural to extend the exponential function to matrices.

Let A be a square matrix. Then $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \ldots$

a. If $AB = BA$, then $e^A e^B = e^{(A+B)}$.[1] Prove that e^A is invertible. [Hint: consider $-A$]

$$e^A e^{-A} = e^{A-A} = e^{[0]} = \underline{\underline{I}} \quad \Big(\text{SINCE } \underbrace{[0]^0 = I}_{\text{WE DEFINE IT THIS WAY}},\ [0]^n = [0] \text{ FOR } n \geq 1\Big)$$

$$\Rightarrow e^{-A} \text{ IS } (e^A)^{-1}$$

b. Prove that $e^{2A} = \left(e^A\right)^2$ using your work from a.

$$e^{2A} = e^{A+A} = e^A e^A = \underline{\underline{(e^A)^2}}$$

5. You might remember from Math 160 that the constant-coefficient linear differential equation $\frac{du}{dt} = \lambda u$ has the general solution $u(t) = Ce^{\lambda t}$. In Math 220 you'll deal with systems of differential equations, where your unknown function $u(t)$ is an unknown vector $\mathbf{u}(t)$, and your constant coefficient λ is a constant matrix A.

a. The system $\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{f}(t)$ is homogeneous if $\mathbf{f}(t) = \mathbf{0}$. If we suppose that $\mathbf{u}(t) = e^{\lambda t}\mathbf{x}$ is a solution to the homogeneous system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ (where $\mathbf{x}$ is a constant vector), then what must be true of A, λ, and $\mathbf{x}$?

$$\partial\vec{u}/\partial t = \partial/\partial t\left(e^{\lambda t}\vec{x}\right) = \lambda e^{\lambda t}\vec{x}$$

$$A\vec{u} = Ae^{\lambda t}\vec{x}$$

$$\partial\vec{u}/\partial t = A\vec{u} \Longrightarrow \lambda e^{\lambda t}\vec{x} = Ae^{\lambda t}\vec{x} \Longrightarrow \underline{\underline{A\vec{x} = \lambda\vec{x}}}$$

b. Using the definition of e^A, show that $\mathbf{u}(t) = e^{At}\mathbf{x}$ is also a solution to the homogeneous system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$. Does this pair with your result for a.?

$$\partial u/\partial t = \partial/\partial t\left(e^{At}\vec{x}\right) = \partial/\partial t\left(\sum_{n=0}^{\infty}\frac{(At)^n}{n!}\vec{x}\right) = \left(\partial/\partial t \sum_{n=0}^{\infty}\frac{A^n t^n}{n!}\right)\vec{x}$$

$$= \left(\sum_{n=0}^{\infty}\frac{A^n(\partial/\partial t\, t^n)}{n!}\right)\vec{x} = \left(\sum_{n=0}^{\infty}\frac{A^n n t^{n-1}}{n!}\right)\vec{x} = A\left(\sum_{n=0}^{\infty}\frac{A^{n-1}t^{n-1}}{(n-1)!}\right)\vec{x}$$

$$= Ae^{At}x = \underline{\underline{Au}}$$

RE-INDEX E.G. $j = n-1$

[1] Proving this is doable but strenuous. If you want to take a whack at it, I'd recommend using the binomial theorem.

6. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

a. Find A^2, A^3, and A^4. What is A^4? What does it tell you about A^5, A^6, A^7, and A^8?

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A^6, A^{10}, \ldots$$

$$A^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A^7, A^{11}, \ldots$$

$$A^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{I}} = A^8, A^{12}, \ldots$$

$$\Rightarrow A^5 = A^4 A = IA = A = A^9, A^{13}, \ldots$$

b. Consider e^{At}. Compute the first four terms in the series, and add them. What do you notice about the sums that appear in each entry?

$$e^{At}\Big|_{n=4} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + 1/2\begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix} + 1/6\begin{pmatrix} 0 & -t^3 \\ t^3 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 - 1/2\,t^2 + \ldots & t - 1/6\,t^3 + \ldots \\ -t + 1/6\,t^3 - \ldots & 1 - 1/2\,t^2 + \ldots \end{pmatrix}}} \leftarrow \sin \,\&\, \cos\,?$$

c. State e^{At} in full, with its entries stated in terms of the appropriate infinite series.

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = 1 - 1/2\,t^2 + 1/24\,t^4 - \ldots$$

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = t - 1/6\,t^3 + 1/120\,t^5 - \ldots$$

$$\Rightarrow \underline{\underline{e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}}}$$