Practice Proofs — Week 2 (Linear Transformations)

We like vector spaces. They're really useful, and we know a lot about how the operations defined on them work. If we're applying some function to our vectors, we'd like for it not to change how the vectors act so that the outputs act in the same way. This is the idea of a linear transformation: not only is it a function between two sets, but it's a function that preserves the "structure" on the sets which make them vector spaces (the "structure" in this case being the operations of vector addition and scalar multiplication). We can state this formally as follows.

Let V and W be arbitrary vector spaces, $\mathbf{v}$ and $\mathbf{u}$ be vectors in V, and c be a real number.[1] Then the map $T : V \to W$ is a linear transformation if:

1. $T(c\mathbf{v}) = cT(\mathbf{v})$
2. $T(\mathbf{v} + \mathbf{u}) = T(\mathbf{v}) + T(\mathbf{u})$

Keep in mind that $T(\mathbf{v})$ and $T(\mathbf{u})$ are themselves vectors in W, and what we're trying to show is that when we transform two vectors from V they can still be added and scalar multiplied in W while retaining whatever property we use to define T.

Exercise 2 is derived from Ch. 8.1 of Gilbert Strang's *Introduction to Linear Algebra* (5th ed.). Exercises 3 and 4 are from G. Allen at Texas A&M University, and can be found here.

Exercises

1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{v}$ be any vector in $\mathbb{R}^n$, and $\mathbf{u}$ be a fixed vector in $\mathbb{R}^n$. Prove that the following transformations are linear, or show why they're not.

a. $T(\mathbf{v}) = \mathbf{v}^T\mathbf{u}$

$T(cv) = (cv)^T u = c(v^T u) = cT(v)$ ✓

$T(v+w) = (v+w)^T u = (v^T + w^T)u = v^T u + w^T u = T(v) + T(w)$ ✓

b. $T(\mathbf{v}) = \mathbf{v} \times \mathbf{u}$

$T(cv) = (cv) \times u = c(v \times u) = cT(v)$ ✓

$T(v+w) = (v+w) \times u = v \times u + w \times u = T(v) + T(w)$ ✓

c. $T(\mathbf{v}) = \|\mathbf{v}\|$

TRI INEQ

$T(v+w) = \|v+w\| \underline{\leq} \|v\| + \|w\| = T(v) + T(w)$ ✗

NOT =

d. $T(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

$T(cv) = \frac{(cv)\cdot u}{u \cdot u} u = c\left(\frac{v \cdot u}{u \cdot u}\right) u = cT(v)$ ✓

$T(v+w) = \frac{(v+w)\cdot u}{u\cdot u} u = \frac{v\cdot u + w \cdot u}{u \cdot u} u = \left(\frac{v\cdot u}{u\cdot u} + \frac{w \cdot u}{u \cdot u}\right) u = \frac{v\cdot u}{u\cdot u} u + \frac{w\cdot u}{u\cdot u} u = T(v) + T(w)$ ✓

[1] We assume in this class that all our vector spaces are defined over the real numbers. In truth, vector spaces can be defined over any *field*, such as the complex numbers or the integers modulo a prime.

2. Let $\mathcal{P}_2$ be the vector space of polynomials of degree 2 or less.

a. Let $p(x) = ax^2 + bx + c$ be any polynomial in $\mathcal{P}_2$. Is there some choice of a, b, and c that satisfies $p(x) = 0$, other than the trivial one (i.e. other than $a = b = c = 0$)? If not, what does this mean for x^2, x, and 1?

NO! $0x^2+0x+0=0$ HAS INF ROOTS (EVERY x IS ONE) IF $\underset{=p(x)}{ax^2+bx+c}$ IS NOT EQUAL TO 0, THEN $p(x)=0 \Rightarrow ax^2+bx+c$ HAS INF. ROOTS, WHICH CONTRADICTS THE F.T. ALG

ONLY L.C. OF $x^2, x, 1 = 0$ IS TRIVIAL $\Rightarrow$ INDEP
ANY $p \in P_2$ CAN BE WRITTEN $ax^2+bx+c \Rightarrow$ SPAN
$\Rightarrow$ BASIS (OF P_2)

b. Show that the differential operator[2] D is linear on $\mathcal{P}_2$ by differentiating p.[3] $\left[D(p) = \dfrac{dp}{dx}\right]$

$$Dp = \tfrac{d}{dx}(ax^2+bx+c) \overset{\#2}{=} \tfrac{d}{dx}(ax^2) + \tfrac{d}{dx}(bx) + \tfrac{d}{dx}(c)$$
$$\overset{\#1}{=} a\left(\tfrac{d}{dx}x^2\right) + b\left(\tfrac{d}{dx}x\right) + c\left(\tfrac{d}{dx}1\right)$$
$$= 2ax + b$$

NORMALLY, BASIS IS LISTED AS $\{1, x, x^2\}$, NOT $\{x^2, x, 1\}$

c. Let $\mathbf{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $[D]\,\mathbf{p} = \begin{bmatrix} 2a \\ b \end{bmatrix}$. What is $[D]$?

$p = 3\times1$, $[D]p = 2\times1 \Rightarrow [D] = 2\times3$

$$[D] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a \\ b \end{bmatrix} \checkmark$$

$\Rightarrow p = \begin{bmatrix} c \\ b \\ a \end{bmatrix}, [D]p = \begin{bmatrix} b \\ 2a \end{bmatrix}$
$\Downarrow$
$[D] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

d. Show that the indefinite integral operator D^+ is linear on $\mathcal{P}_2$ by integrating $D(p)$. $\left[D^+(D(p)) = \displaystyle\int \frac{dp}{dx}\,dx\right]$

$$D^+(D(p)) = \int (2ax+b)\,dx = \int 2ax\,dx + \int b\,dx$$
$$\overset{\#2}{=} 2a\int x\,dx + b\int dx$$
$$\overset{\#1}{=} ax^2 + bx + c' \leftarrow \text{NOT EQUAL TO } c!$$

e. What is $[D^+]$? Does $D^+ = D^{-1}$ (that is, does $[D^+]\,[D]\,\mathbf{p} = \mathbf{p}$)?[4]

$$[D^+] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow [D^+][D] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$+ \begin{bmatrix} 0 \\ 0 \\ c' \end{bmatrix}$

$$\Rightarrow [D^+][D]p = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \neq p$$

AFTER USING D, WE CANT GET C BACK

NEED TO ADD $\begin{bmatrix} 0 \\ 0 \\ c' \end{bmatrix}$

[2]an *operator* is a transformation on a space of functions. You say that the operator acts "on" the space if the transformation is from a space to itself.
[3]p is still $p(x)$ here–when it's unambiguous what variable you're differentiating with respect to, the variable is dropped to save on parentheses.
[4]D^+ is called the *pseudoinverse* of D, which might give you a hint about the answer.

3. Let $\mathcal{M}_{22}$ be the vector space of 2×2 matrices. Prove that the following transformations on $\mathcal{M}_{22}$ are linear, or show why they're not.

a. $T(A) = A^T$

$T(cA) = (cA)^T = c(A^T) = cT(A)$ ✓

$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$ ✓

b. $T(A) = A + A^T$

$T(cA) = (cA) + (cA)^T = cA + c(A^T) = c(A + A^T) = T(A)$ ✓

$T(A+B) = (A+B) + (A+B)^T = (A + A^T) + (B + B^T) = T(A) + T(B)$ ✓

c. $T(A) = A + I$

$T([0]) = [0] + I = I \neq [0]$ ✗

(IF T IS LINEAR, THEN $T(\vec{0}) = \vec{0}$)

d. $T(A) = \det(A)$

$T(cA) = \det(cA) = c^2 \det(A) = c^2 T(A) \neq cT(A)$ ✗

MULTILINEAR

e. $T(A) = \mathrm{tr}(A)$ ($\mathrm{tr}(A) = \underbrace{\text{trace of } A}_{= \sum_{i=1}^{n} a_{ii}} =$ sum of the diagonal entries of A)

$T(cA) = \sum_{i=1}^{n} (c a_{ii}) = c\left(\sum_{i=1}^{n} a_{ii}\right) = cT(A)$ ✓

$T(A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \left(\sum_{i=1}^{n} a_{ii}\right) + \left(\sum_{i=1}^{n} b_{ii}\right) = T(A) + T(B)$ ✓

4. Let $T : V \to W$ be a linear transformation. $P(n) =$ PROPOSITION THAT $T(nv) = nT(v)$

a. Prove that $T(n\mathbf{v}) = nT(\mathbf{v})$ for some positive integer n using condition $\boxed{2}$.

LOOK UP "MATHEMATICAL INDUCTION" FOR MORE INFO ON METHOD

INDUCTION: IF $P(1)$ IS TRUE AND $P(K) \Rightarrow P(K+1)$, THEN $P(n)$

$P(1)$: $T(1v) = 1T(v) \Rightarrow T(v) = T(v)$

ASSUME $P(K)$ IS TRUE. THEN $T(Kv) = KT(v)$. THEN...

... $T(Kv) + T(v) = KT(v) + T(v) \Rightarrow T(Kv + v) = T((K+1)v) = (K+1)T(v)$

THUS SINCE $P(1)$ IS TRUE AND $P(K) \Rightarrow P(K+1)$, $P(n)$ IS TRUE

b. Prove that $T((p/q)\mathbf{v}) = (p/q)T(\mathbf{v})$ for some positive integers p and q using ~~$\boxed{2}$ and your work~~ from $\boxed{\text{a.}}$ [Hint: If you've proven $\boxed{\text{a.}}$, then you can state $pT(\mathbf{v}) = T(p\mathbf{v}) = T(\cancel{p}(\cancel{q}/\cancel{p})\mathbf{v})$]

q p q

NOT NEEDED

$(p = p \cdot 1 = p \cdot \frac{q}{q} = \frac{pq}{q} = q\frac{p}{q})$

$pT(v) \overset{(a.)}{=} T(pv) = T\left(p\frac{q}{q}v\right) = T\left(q\frac{p}{q}v\right) \overset{(a.)}{=} qT(p/q\, v)$

$pT(v) = qT(p/q\, v) \xRightarrow{\div q} p/q\, T(v) = T(p/q\, v)$