Week 1 — Practice Proofs (Symmetric & Invertible Matrices)

The examples labeled "*Strang*" are taken from the problem sets in Ch. 2, §5 & §7 of Gilbert Strang's *Introduction to Linear Algebra*, fifth edition. Otherwise, the full author's name is given, or the problem is so well known that a citation is unwarranted.

Problems

1. (*Strang*) If A and B are $n \times n$ symmetric matrices, then show that the following are also symmetric, or provide a counterexample.

a. $A^2 - B^2$

b. $(A+B)(A-B)$

c. ABA

d. $ABAB$

$(A^2)^T = A^TA^T = (A^T)^2$

SYM ✓

a. $(A^2 - B^2)^T = (A^2)^T - (B^2)^T = (A^T)^2 - (B^T)^2 = A^2 - B^2$

b. $((A+B)(A-B))^T = (A-B)^T(A+B)^T = (A^T - B^T)(A^T + B^T)$

$= (A-B)(A+B)$

$\neq (A+B)(A-B)$

NOT SYM

C.EX. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$(A+B)(A-B) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$

c. $(ABA)^T = A^TB^TA^T = ABA$ SYM

d. $(ABAB)^T = B^TA^TB^TA^T = BABA \neq ABAB$ NOT SYM

C.EX. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \Rightarrow ABAB = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix} \neq \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}$

2. (*Jim Kruidenier, SBCC*) If $A^2 = [0]$, does $(I-A)^{-1}$ exist? Prove or provide a counterexample.

IF $(I-A)^{-1}$ EXISTS, THEN $B(I-A) = (I-A)B = I$ FOR SOME B

$A^2 = [0] \Rightarrow I^2 - A^2 = I \Rightarrow$ TRY A DIF OF SQ'S

$(I-A)(I+A) = I^2 - AI + IA - A^2 = I^2 - A + A - [0] = I$
$(I+A)(I-A) = I^2 + AI - IA - A^2 = I^2 + A - A - [0] = I$

$\Rightarrow$ EXISTS, $(I-A)^{-1} = I + A$

3. (*Strang*) Suppose that A is $m \times n$, S is $m \times m$ and symmetric, and both A and S have real valued entries.

a. Show that A^TA has no negative diagonal entries.

b. Is A^TSA symmetric? What are its dimensions?

a. $A^TA_{ii} = R_i(A^T)^T C_i(A) = C_i(A)^T C_i(A)$
$= \sum_{j=1}^{m} a_{ij}^2$ (THE SUM OF THE SQUARES OF THE iTH COLUMN OF A'S ENTRIES)

$a_{ij}^2 \geq 0$ FOR ALL $j \Rightarrow \sum_{j=1}^{m} a_{ij}^2 \geq 0$

b. $(A^TSA)^T = A^TS^T(A^T)^T = A^TSA$ - YES!

DIM: $\underset{n\times m}{A^T} \; \underset{\cancel{m}\times \cancel{m}}{S} \; \underset{\cancel{m}\times n}{A} \Rightarrow A^TSA$ IS $n \times n$

SIMILAR PROOF, B^{-1} GIVEN BELOW

4. (*Strang*) Let $C = AB$ and $D = ABC$.

a. If C is invertible, then A is invertible. Find A^{-1} in terms of C^{-1} and B.

b. If D is invertible, then B is invertible. Find B^{-1} in terms of D^{-1}, A, and C.

MEANT TO SAY "PROVE THIS"!

IF $C = AB$, C^{-1} AND B^{-1} EXISTS, THEN A^{-1} EXISTS

a. C^{-1} EXISTS $\Rightarrow CC^{-1} = (AB)C^{-1} = A(BC^{-1}) = I$ RIGHT INVS

$C^{-1}C = C^{-1}(AB) = (C^{-1}A)B = I$

B^{-1} EXISTS $\Rightarrow (C^{-1}A)BB^{-1} = IB^{-1} \Rightarrow C^{-1}A = B^{-1}$

$\Rightarrow BC^{-1}A = BB^{-1} = I \Rightarrow (BC^{-1})A = I$ LEFT INVS

$\therefore A^{-1} = BC^{-1}$

$B^{-1} = CD^{-1}A$

$BB^{-1} = BCD^{-1}A = I \Rightarrow D^{-1} = C^{-1}B^{-1}A^{-1}$

$\Rightarrow D = ABC$

$B^{-1}B = CD^{-1}AB = I \Rightarrow D^{-1} = C^{-1}B^{-1}A^{-1}$

5. Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

a. Find the matrix A such that $A\mathbf{b} = \mathbf{a} \times \mathbf{b}$, where $\times$ is the cross product.

b. What is A^T?

a. $a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \begin{matrix} = 0 b_1 + -a_3 b_2 + a_2 b_3 \\ = a_3 b_1 + 0 b_2 - a_1 b_3 \\ = -a_2 b_1 + a_1 b_2 + 0 b_3 \end{matrix}$

$\Rightarrow A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$

b. $A^T = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} = -A$

~~**6.** A matrix is skew-symmetric if $A^T = -A$. Show that if A is skew symmetric, then $(I + A)^{-1}$ exists.~~

WE NEED A LOT MORE THAN WHAT WE HAVE SO FAR TO SHOW THIS (DETERMINANTS, EIGENVALUES, ETC.)

SKIP FOR NOW, GOOD PROBLEM TO RETURN TO LATER THOUGH

7. A little algebra shows that $A = \frac{1}{2}(A - A^T) + \frac{1}{2}(A + A^T)$.

a. Show that $\frac{1}{2}(A - A^T)$ is skew symmetric (I.E. $A^T = -A$)

b. Show that $\frac{1}{2}(A + A^T)$ is symmetric

a. $\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -\left(\frac{1}{2}(A - A^T)\right)$

b. $\left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T)$