Practice Proofs — Week 2 (One-to-One & Onto)

Functions, maps, and transformations are all words for the same thing: a rule that relates each element in a set X to a unique element in a set Y. We can split this one condition into two and give a formal definition as follows:

Let f be a relation over sets X and Y. Then f is a **well defined function** if

1. for every x in X there exists a y in Y such that $f(x) = y$. [all of X is sent to Y]
2. if $x_i = x_j$ then $f(x_i) = f(x_j)$. [each element in X is sent to only one element in Y]

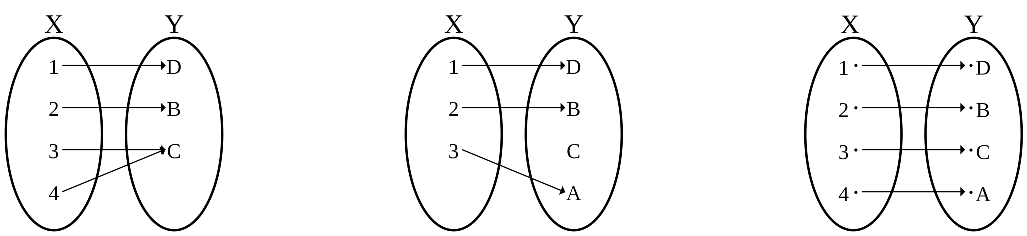
Ideally, our function is strong enough to work the other way around, but to be well defined in the opposite direction it would need to retain these two conditions. If 1 still works from Y to X then we call the function onto, and if 2 still works from Y to X we call it one-to-one. If both still work, then the opposite direction is well defined, and we can invert the function.

Let $f : \text{X} \to \text{Y}$ be a well defined function. Then

(a) f is **onto** (*surjective*) if for every y in Y there exists a x in X such that $f(x) = y$.

(b) f is **one-to-one** (*injective*) if $f(x_i) = f(x_j)$ implies that $x_i = x_j$.

(c) f is **invertible** (*bijective*) if f is both one-to-one and onto.



(a) Onto (b) One-to-One (c) Invertible

Notice in the diagram above that the range of f in (a) is all of Y–this is an equivalent way of stating that f is onto. We'll be able to say a lot more about these properties if f is a linear transformation once we cover subspaces and dimensionality, but for now we just need the following.

Let $T : V \to W$ be a linear transformation between vector spaces V and W. Then

(a) T is **onto** if
- for every w in W there exists a v in V such that $T(v) = w$.
- the range of T is equal to the codomain W.

(b) T is **one-to-one** if $T(v_i) = T(v_j)$ implies that $v_i = v_j$.

(c) T is **invertible** if
- T is onto and one-to-one.
- there exists a linear transformation $S : W \to V$ such that $S \circ T : V \to V$ is equal to I on V, and $T \circ S : W \to W$ is equal to I on W.

There are also more equivalent definitions if we talk about the matrices, solution sets, etc. of these transformations, but for this section we'll just be looking as the transformation in the abstract.

The exercises in this worksheet are adapted from Ch. 3 of Sheldon Axler's *Linear Algebra Done Right* (3rd ed.), although the notation and proofs have been modified to fit this course.

Exercises

1. Prove that T is injective if and only if $T(v) = \mathbf{0}$ has only the trivial solution $v = \mathbf{0}$.

2. Prove that T is invertible if and only if T is injective and surjective.

3. Prove that the inverse of T is unique.

4. Prove that T is injective if and only if there exists a S such that $ST = I$.

5. Prove that T is surjective if and only if there exists a S such that $TS = I$.