Practice Proofs — Week 4 (Determinants)

Once again, the exercises on this worksheet are a mixed bag, since many types of matrices can be characterized by their determinants. We can also use determinants to pose questions about matrices that are difficult to answer with linear algebra in terms of other objects we might know better, such as polynomials. To answer these exercises, really focus in on the properties listed in the beginning of the "Properties of the Determinant" section of the notes.

Exercises 1, 2, 4 and 6 are from Ch. 5 of Gilbert Strang's *Introduction to Linear Algebra* (5th ed.)

Exercises

$$\Longrightarrow \quad Q^TQ = QQ^T = I$$

1. Let Q be a $n \times n$ matrix such that $Q^{-1} = Q^T$.[1]

a. Prove that $\det Q = \pm 1$ using the product and transpose rules (properties 7 and 10 in the notes).

$$Q^TQ = I \Rightarrow \mathrm{DET}(Q^TQ) = \mathrm{DET}(I) \Rightarrow \mathrm{DET}(Q^T)\,\mathrm{DET}(Q) = 1$$

$$\Rightarrow (\mathrm{DET}(Q))^2 = 1 \Rightarrow \underline{\underline{\mathrm{DET}(Q) = \pm 1}}$$

b. Prove that $\det Q = \pm 1$ using only the product rule. [Hint: for any matrix A, if $|\det A| > 1$, then $\lim_{n\to\infty} \det A^n$ diverges. Why can't this happen to Q^n?]

I CANT FIGURE THIS ONE OUT. MAYBE PROOF BY CONTRIDICTION W/ ε-δ INVOLVED? WE MIGHT NEED MORE MATERIAL ABOUT ORTHOGONALITY

$$\mathrm{DET}(A)I = \begin{pmatrix} \mathrm{DET}A & & 0 \\ & \mathrm{DET}A & \\ 0 & & \ddots \end{pmatrix}$$

2. Assume that A is $n \times n$ and that $AC^T = (\det A)I$.[2] Prove that $\det C = (\det A)^{n-1}$.

$$\mathrm{DET}(AC^T) = \mathrm{DET}(\mathrm{DET}(A)I) \Rightarrow \mathrm{DET}(A)\,\mathrm{DET}(C^T) = \mathrm{DET}(A)^n$$

$$\Rightarrow \underset{\substack{\parallel \\ (\mathrm{DET}(C^T))}}{\mathrm{DET}(C)} = \frac{\mathrm{DET}(A)^n}{\mathrm{DET}(A)} = \underline{\underline{\mathrm{DET}(A)^{n-1}}}$$

3. Prove that if A and V are $n \times n$ matrices and V is invertible, then $\det\left(VAV^{-1}\right) = \det A$.

$$\mathrm{DET}(VAV^{-1}) = \mathrm{DET}(V)\,\mathrm{DET}(A)\,\mathrm{DET}(V^{-1}) = \underbrace{\mathrm{DET}(V)\,\mathrm{DET}(V)^{-1}}_{=1}\mathrm{DET}(A) = \underline{\underline{\mathrm{DET}(A)}}$$

[1] Q is called an *orthogonal* matrix. We'll cover these matrices and their properties in a few weeks.
[2] the proof of this is trivial, but you can see my paper on the cofactor theorem for its implications.

4. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and suppose $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$ and some constant λ.

a. Find the matrix B such that $B\mathbf{x} = \cancel{0}$. $\vec{0}$

$$Ax = \lambda x \Rightarrow Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0$$

$$\Rightarrow \underline{\underline{B = A - \lambda I}}$$

b. If $B\mathbf{x} = \cancel{0}$ $\vec{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then B is singular. Find the values of λ such that B is singular.

B IS SINGULAR $\Longleftrightarrow$ DET(B) = 0

$$\mathrm{DET}(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

$$= 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$= \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

$$\Rightarrow \underline{\underline{\lambda = 3, 1}}$$

5. (*Jacobi's theorem*) Let A be a $n \times n$ matrix such that $A^T = -A$.[3]

a. Show that $\det A = (-1)^n \det A$.

$$\mathrm{DET}(A) = \mathrm{DET}(A^T) = \mathrm{DET}(-1 \cdot A) = \underline{\underline{(-1)^n\,\mathrm{DET}(A)}}$$

b. What does this imply if n is odd?

n IS ODD $\Rightarrow (-1)^n = -1$

$\Rightarrow$ DET(A) = − DET(A)

$\Rightarrow$ DET(A) = 0

$\Rightarrow$ <u>A IS SING.</u>

[3] A is called a *skew-symmetric* matrix.

6. (For those who have taken Math 200) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and let $f(a,b,c,d) = \ln(\det A)$. Show that $A^{-1} = \begin{bmatrix} \partial f/\partial a & \partial f/\partial b \\ \partial f/\partial c & \partial f/\partial d \end{bmatrix}$.

$$f = \ln(\det(A)) = \ln(ad - bc)$$

$$\partial f/\partial a = \frac{\left(\frac{\partial(ad-bc)}{\partial a}\right)}{ad-bc} = \frac{d}{ad-bc}$$

$$\partial f/\partial b = \frac{\left(\frac{\partial(ad-bc)}{\partial b}\right)}{ad-bc} = \frac{-c}{ad-bc}$$

$$\left(\partial/\partial x \ln f(x) = \frac{f'(x)}{f(x)} \right)$$

$$\partial f/\partial c = \frac{\left(\frac{\partial(ad-bc)}{\partial c}\right)}{ad-bc} = \frac{-b}{ad-bc}$$

$$\partial f/\partial d = \frac{\left(\frac{\partial(ad-bc)}{\partial d}\right)}{ad-bc} = \frac{a}{ad-bc}$$

$$\begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc} = \underline{\underline{A^{-1}}}$$