Decision Problems for Automaton Semigroups

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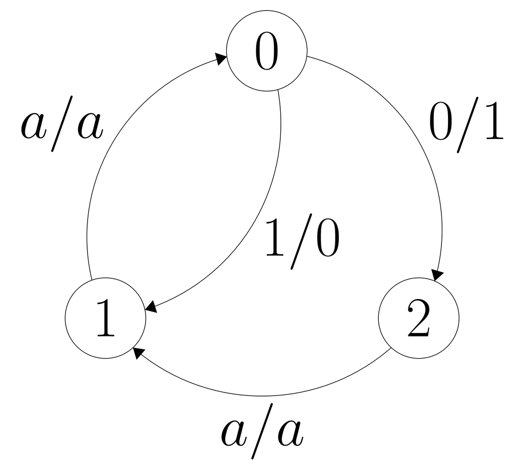
Abstract

The word problem is a classic group-theoretic decision problem. It's known to be undecidable in surprisingly small subclasses of groups. We consider a class of semigroups generated by finite automata for which this problem is decidable. We consider several related decision problems for this subclass of semigroups.

Automaton Semigroups

Below is an example *invertible transducer*. It's quite similar to a DFA, but instead of just reading in a string, it also outputs a string (and so induces a relation on strings).



There's a natural correspondence between states in the machine and length-preserving functions from string to string. For instance, if $\underline{0}$ is the function induced by starting at state 0, we have $\underline{0}(0000) = 1001$.

These functions form a semigroup $\mathcal{S}(A)$ under composition (associative and closed).

More formally, let **2** be the binary alphabet, Q be the state set of the automaton A, Q^+ be all nonempty strings over Q, and $End(\mathbf{2}^*)$ be the semigroup of endomorphisms on the tree $\mathbf{2}^*$.

Then there's a natural homomorphism $\phi : Q^+ \to End(\mathbf{2}^*)$. We denote the image of ϕ as $\Sigma(A)$.

Definition 1. *A semigroup S is called an **automaton semigroup** if there exists an automaton A such that $S \simeq \Sigma(A)$.*

Many natural classes of semigroups arise as automaton semigroups. For instance, all free semigroups of rank ≥ 2 arise as automaton semigroups.

Decidable Word Problem

Definition 2. *The word problem for finitely generated semigroups is as follows: given a f.g. semigroup S with generators $\Sigma = s_1 \dots s_n$, and two words w_1, w_2 over Σ^*, does $w_1 = w_2$ in S?*

Proposition 1. *Automaton semigroups has a decidable word problem.*

Proof Sketch: We're given as input w_1, w_2.

The class of automaton semigroups is closed under direct products, so construct an automata whose state set denotes all words of length $|w_1|$, $|w_2|$ over the generators. Then choose states w_1 and w_2 in these automata as start states, and turn the transducers into DFAs. Minimize and check for equality. □

Primary Results

Most of our current work focuses on the IsGroup decision problem, whether a given automaton semigroup is already a group. Our work toward this goal is summarized below.

IsGroup is decidable in the Abelian case

Proof Sketch: We can represent elements of the semigroup as vectors over $\mathbb{N}^{|Q|}$. Residuation becomes an affine operator, reducing the problem to matrix algebra. □

Knapsack is undecidable

Definition 3. *The* Knapsack *problem for automaton semigroups is as follows: given generators $s_1 \dots s_n \in S(A)$ and some element $s \in S(A)$, do there exist $a_1 \dots a_n \in \mathbb{N}$ such that*

$$s_1^{a_1} \dots s_n^{a_n} = s$$

Theorem 1. Knapsack *is undecidable.*

Proof Sketch: Reduce from Hilbert's 10 problem. □

A Monoid with undecidable IsGroup

While the monoid is not an automaton monoid, it has a decidable word problem.

Proof Sketch: Reduce from the Halting problem. Given an input TM T, define a group whose generators are configurations of T. Set the halting state to the identity. If one configuration transitions to another, their corresponding group elements are equal.

Then consider the submonoid generated by the start state $\langle s \rangle$. $\langle s \rangle$ is a group iff T halts.

In order to keep the word problem decidable, we need to modify T to be a self-verifying Turing machine. □

Open Questions

- Is IsGroup decidable in the non-abelian case?
- Is it decidable, given two automaton, whether they generate isomorphic semigroups?
- Automorphism membership problem: Given an endomorphism $\phi \in End(\mathbf{2}^*)$ and an automata A, is $\phi \in S(A)$? Is this problem decidable?
- A wide variety of other related semigroup decision problems.

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