# Notes on Non-Commuting 1-Toggle Automata

v.3

## 1 Sausage-Type Automata

Assume $\mathcal{A}$ has a single toggle state 0, copy states (some possibly forking) $\{1,\ldots,m\}$ plus an identity state $I$. By the state set of $\mathcal{A}$ we mean $Q = \{0,1,\ldots,m\}$, and $Q$ is required to be strongly connected. We write $Q^\star$ for the free monoid generated by $Q$ and indicate the canonical map $Q^\star \to \mathsf{Aut}(2^\star)$ by an underscore; thus, $\underline{0}$ denotes the map induced by choosing $0 \in Q$ as the initial state. $\mathcal{S}$ denotes the image of the canonical map, the monoid generated by $\mathcal{A}$.

For $\boldsymbol{p} \in Q^\star$ we write $|\boldsymbol{p}|$ for the length of $\boldsymbol{p}$. $\#_k\, \boldsymbol{p}$ is the number of terms $k$ in $\boldsymbol{p}$, the $k$th component of the Parikh vector of $\boldsymbol{p}$. We write the Parikh vector as $\#\,\boldsymbol{p} = (\#_0\,\boldsymbol{p}, \#_+\,\boldsymbol{p})$ We will occasionally confuse $\boldsymbol{p}$ and $f = \underline{\boldsymbol{p}}$ and may write, e.g., $|f|$.

Residuation typically reduces the length of $\boldsymbol{p}$.

**Proposition 1.1**

$$|\partial_s \boldsymbol{p}| = \begin{cases} |\boldsymbol{p}| - \lceil a/2 \rceil & \text{if } s = 0, \\ |\boldsymbol{p}| - \lfloor a/2 \rfloor & \text{otherwise.} \end{cases}$$

In fact, $|\partial_s \boldsymbol{p}| = |\boldsymbol{p}|$ only when $a = 0$ or $a = 1 \wedge s = 1$. For $\boldsymbol{p} \neq 0$ there is some word $x$ such that $|\partial_s \boldsymbol{p}| < |\boldsymbol{p}|$.

This residual length property makes it look like the situation is similar to CCC transducers: in the quotient operations, only the first component needs to be finite. Since the transduction monoids are generally non-commutative, one has to consider both variants

$$f_a f_{\overline{a}} \qquad f_{\overline{a}} f_a$$

when $f$ is odd, i.e., in the case where $\#_0\, f$ is odd.

We refer to monoid elements of the form $\underline{k}^{\,n}$, $n \geq 1$, as pure, all others as mixed. Note that pure elements are closed under residuation. Also, for some $x \in 2^\star$, $\partial_x \underline{k}^{\,n} = I$.

**Proposition 1.2** *$f \in \mathcal{S}$ is orbit rational iff $f$ is pure.*

*Sketch of proof.*

The first component of all quotients of pure functions $\underline{k}^{\,n2^j}$, $n$ odd, is of the form $\underline{\ell}^{\,n2^i}$ for $i < j$.

Now suppose $f$ is mixed, odd. Then

$$|\partial_0 f_a \, \partial_1 f_{\overline{a}}| = \#_0 \, f + 2 \, \#_+ \, f > |f|.$$

Since $Q$ is strongly connected this shows that we can pump the first component check

$\square$

**Proposition 1.3** *$\mathcal{S}$ fails to be a group.*

*Sketch of proof.*

Suppose $\boldsymbol{p} = I$ for some non-trivial vector $\boldsymbol{p}$. We may assume that $|\boldsymbol{p}|$ is minimal.

If $\#_0 \, \boldsymbol{p} \geq 2$ we have $|\partial_s \boldsymbol{p}| < |\boldsymbol{p}|$, contradicting minimality.

If $\#_0 \, \boldsymbol{p} = 0$ choose a state $k$ and a minimal word $x$ such that $\#_0 \, \partial_x \boldsymbol{p} > 0$ ($x$ exists by the SCC condition). But then $|\partial_{x0} \boldsymbol{p}| < |\boldsymbol{p}|$, contradicting minimality. check

$\square$

# 2 5-State Machines

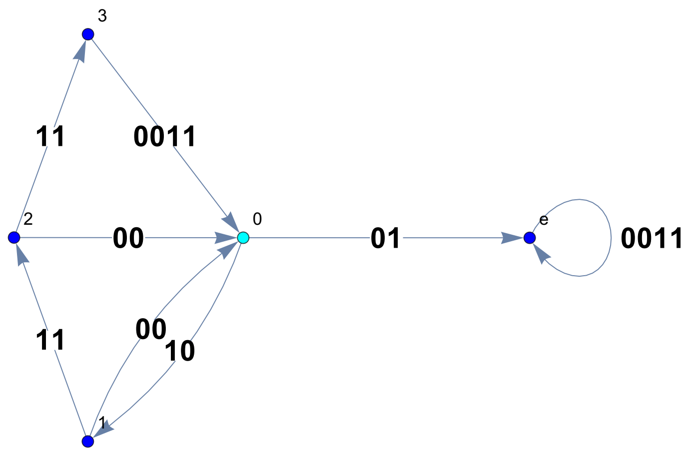
Up to isomorphism, there are 1033 5-state machines of the type under consideration here. check Of these, 371 generate a non-free monoid. 46 have an identity of the form $\underline{0}^{\,2} = \underline{k}$. The other identities appear all to be commutativity or skew-commutativity.

Nota bene: with current resources, the brute-force search for identities breaks down at length 6 or so: the product automata get to be too large. It is far from clear that no larger identities exist.

Is there are better way to search for identities? Or is there some kind of hardness result?

## 2.1 Example 1: Free Monoid

Consider the following 5-state automaton $\mathcal{A}$.




**Proposition 2.1** *$\mathcal{S}$ is the free monoid on 4 generators.*

*Sketch of proof.*

Suppose we have a non-trivial identity $\boldsymbol{p} = \boldsymbol{q}$. We must have $\partial_s \boldsymbol{p} = \partial_s \boldsymbol{q}$ for $s = 0, 1$. The generic situation looks like this:

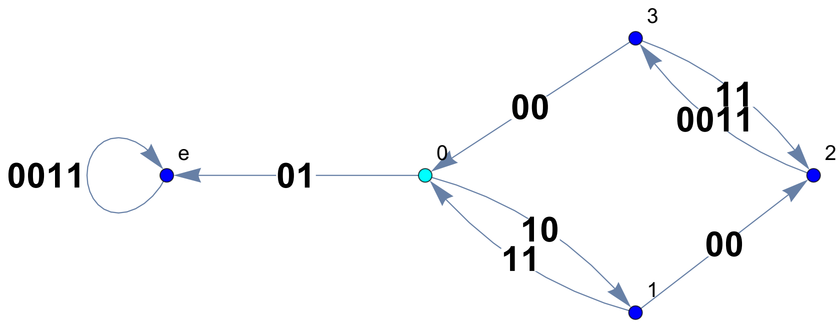
| | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 |
|---|---|---|---|---|---|---|---|---|
| $\partial_0$ | 0 | 0 | 0 | $I$ | 2 | 3 | 0 | 1 |
| $\partial_1$ | 2 | 3 | 0 | 1 | 0 | 0 | 0 | $I$ |

It is not hard to see that $\boldsymbol{p}$ can be reconstructed from the two residuals.

$\square$

## 2.2 Example: Non-Free Monoid

Consider the following 5-state automaton $\mathcal{A}$.




The only non-trivial identities seem to be

$$\underline{0}\,\underline{2} = \underline{2}\,\underline{0} \qquad \underline{1}\,\underline{3} = \underline{3}\,\underline{1}$$

Reconstruction does not work here since

| | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 |
|---|---|---|---|---|---|---|---|---|
| $\partial_0$ | 2 | 3 | 0 | $I$ | 0 | 3 | 2 | 1 |
| $\partial_1$ | 0 | 3 | 2 | 1 | 2 | 3 | 0 | $I$ |

so that

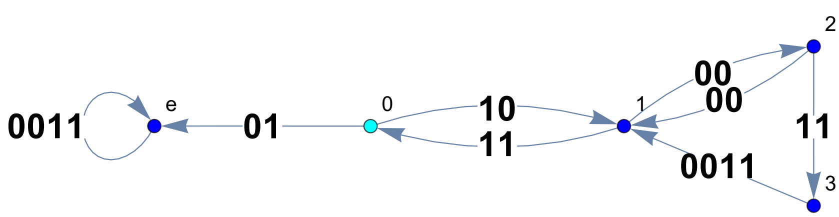
| | 0 | 2 | 2 | 0 | 1 | 3 | 3 | 1 |
|---|---|---|---|---|---|---|---|---|
| $\partial_0$ | $I$ | 3 | 3 | $I$ | 2 | 0 | 0 | 2 |
| $\partial_1$ | 1 | 3 | 3 | 1 | 0 | 2 | 2 | 0 |

and we have a recursive set of identities.

How does this work in general? check

## 2.3 Example: Non-Free Monoid

Consider the following 5-state automaton $\mathcal{A}$.




The only non-trivial identities seem to be

$$\underline{0}^{\,2} = \underline{3} \qquad \underline{1}\,\underline{2} = \underline{2}\,\underline{1}$$