DECISION PROBLEMS IN INVERTIBLE AUTOMATA

EVAN BERGERON & KLAUS SUTNER

May 5, 2017

Abstract

We consider a variety of decision problems in groups and semigroups induced by invertible Mealy machines. Notably, we present proof that the automorphism membership problem in decidable in these semigroups. In addition, we prove undecidability of a Knapsack variant. A discussion of iteration and orbit rationality follows.

CONTENTS

INTRODUCTION

The word problem is a class group-theoretic decision problem. Given a finitely generated group G, and a word w over the generators (and their inverses), the word problem asks "is $w \in G$." The word problem is known to be undecidable in surprisely small classes of groups - see TODO for background.

The invertible Mealy machines we consider here give rise to a class of semigroups (and sometimes groups) for which the word problem is decidable. The computability picture here is rather nuaced, however. Other important decision problems, among them the conjugacy problem, and the isomorphism problem are known to be undecidable. (See TODO and TODO for details).

We present proof that, for the Abelian case, automorphism membership testing is decidable in this class of semigroups.

BACKGROUND

An *automaton* is a formally a triple (Q, Σ, δ), where Q is some finite state set, Σ is a finite alphabet of *symbols*, and δ is a transformation on $Q \times \Sigma$. Automata are typically viewed as directed graphs with vertex set Q and an edge between u, v if $(u, x)\delta = (v, y)$.

An automaton is said to be *synchronous* when δ outputs exactly one character for every transition and is called *invertible* when every state in Q has some bijection π on Σ such that $(u, x)\delta = (v, \pi(x))$. A state is a *copy state* if π is the identity permutation and is a *toggle state* otherwise.

Each state $q \in Q$ acts on Σ^*, the set of finite strings over Σ. We commonly view Σ^* as the infinite $|\Sigma|$-nary tree, so we may view q as a transformation sending vertex w to wq.

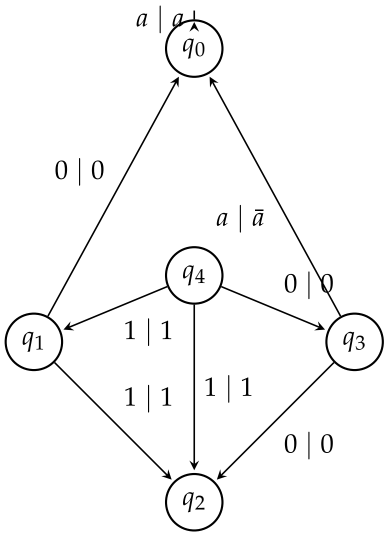

Figure 1: Grigorchuk's machine

We extend the action of Q on Σ^* to words $q = q_1 \cdots q_n$ over Q^+ by

$$wq = (\cdots((wq_1)q_2)\cdots q_n)$$

We adopt the convention of applying functions from the right here. In this way, function composition corresponds naturally with concatenation.

For an automaton A, we denote by $S(A)$ the semigroup generated by Q under composition. A is said to be *commutative* or *Abelian* when $S(A)$ is Abelian. We write $G(A)$ for the group generated by the elements of Q and their inverses.

One may also speak about $S(A)$ and $G(A)$ without explicit reference to an automaton A. As such, we call a semigroup S an *automaton semigroup* if there is some automaton A with $S \simeq \Sigma(A)$. *Automaton groups G* are similarly defined.

Invertible automata have recently been usefully applied the group theory. A classic result here is Grigorchuk's group of intermediate growth, generated by the 5 state invertible machine shown in figure 1.

Decision Problems

Automaton semigroups exhibit many interesting and nuanced computability properties. While it is an easy result that the WORD PROBLEM is solvable in such semigroups, similar group-theoretic problems such as the CONJUGACY PROBLEM and FINITENESS PROBLEM have been shown to be undecidable ([12], and [4], respectively).

Various other semigroup theoretic decision problems have recently been considered for small classes of semigroups by Cain in [3]. We consider a subset of his distinguished properties in the automaton semigroup case here.

DECIDABLE ABELIAN AUTOMORPHISM MEMBERSHIP

KNAPSACK IS UNDECIDABLE FOR AUTOMATON SEMIGROUPS

We follow a proof strategy similar to [7].

We define the *Knapsack Problem* as follows: given as input generators $g_1 \ldots g_k$ and a target group element g, do there exist natural numbers $a_1 \ldots a_k$ such that

$$g_1^{a_1} \cdots g_k^{a_K} = g$$

We prove that this problem is undecidable for automaton semigroups by reducing from Hilbert's tenth problem.

abelian automata

open questions

- All automaton semigroups are recursively presented. If these presentations are regular, or context-free, does that affect the soluability of these questions?
- Having a zero
- Isomorphism problem
- Bounded automata, etc

references

[1] L. Bartholdi and P. V. Silva. Groups defined by automata. *CoRR*, abs/1012.1531, 2010.

[2] A. J. Cain. Automaton semigroups. *TCS*, 410(47–49):5022–5038, 2009.

[3] A. J. Cain and V. Maltcev. Decision problems for finitely presented and one-relation semigroups and monoids. *International Journal of Algebra and Computation*, 19(6):747–770, 2009.

[4] Pierre Gillibert. The finiteness problem for automaton semigroups is undecidable. *CoRR*, abs/1304.2295, 2013.

[5] T. Godin. Knapsack problem for automaton groups. *HAL*, 2016.

[6] O. Kharlampovich, B. Khoussainov, and A. Miasnikov. From automatic structures to automatic groups. *ArXiv e-prints*, July 2011.

[7] D. Konig, M. Lohrey, and G. Zetzsche. Knapsack and subset sum problems in nilpotent, polycyclic, and co-context-free groups. *Contemporary Mathematics*, 2015.

[8] Y. Muntyan. *Automata Groups*. PhD thesis, Texas A&M University, May 2009.

[9] V. Nekrashevych. *Self-Similar Groups*, volume 117 of *Math. Surveys and Monographs*. AMS, 2005.

[10] V. Nekrashevych and S. Sidki. *Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2-endomorphisms*. Cambridge University Press, 2004.

[11] T. Okano. *Invertible Binary Transducers and Automorphisms of the Binary Tree*. PhD thesis, Carnegie Mellon University, May 2015.

[12] Z. Sunic and E. Ventura. The conjugacy problem in automaton groups is not solvable. *Journal of Algebra*, 364(148–154), 2012.

[13] K. Sutner. Invertible transducers, iterations and coordinates. 2013.

[14] K. Sutner and K. Lewi. Iterating inverse binary transducers. *Journal of Automata, Languages, and Combinators*, 17(2–4):293–313, 2012.