Decision Problems in Invertible Automata

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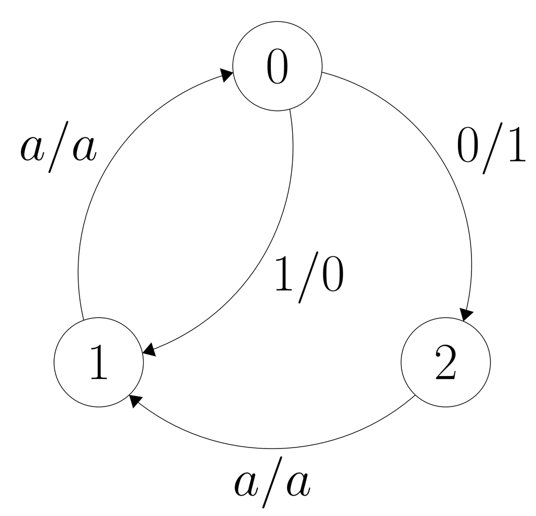
Abstract

We consider a variety of decision problems in groups and semigroups induced by invertible Mealy machines. Notably, we present proof that, in the Abelian case, the automorphism membership problem is decidable in these semigroups. In addition, we prove the undecidability of a Knapsack variant. Partial work toward the decidability of the IsGroup problem is discussed.

Automaton Semigroups

Below is an example *invertible automaton*. It's quite similar to a DFA, but instead of just reading in a string, it also outputs a string (and so induces a relation on strings).



Each state induces a length-preserving functions on strings. For instance, if $\underline{0}$ is the function induced by starting at state 0, we have $\underline{0}(0000) = 1001$.

These functions form a *semigroup* $\mathcal{S}(A)$ under composition (associative and closed).

Interpreting the binary alphabet as an infinite binary tree, these automata can be see as level-preserving, adjacency-preserving maps on the tree (and so are *automorphisms* on the tree).

So then letting Q^+ be the set of all nonempty strings over Q, there is a natural homomorphism $\phi : Q^+ \to \mathrm{Aut}\, \mathbf{2}^*$. We denote the image of ϕ as $\Sigma(A)$.

Many natural classes of semigroups arise as automaton semigroups. For instance, all free semigroups of rank ≥ 2 arise as automaton semigroups.

MEMBERSHIP decidable when Abelian.

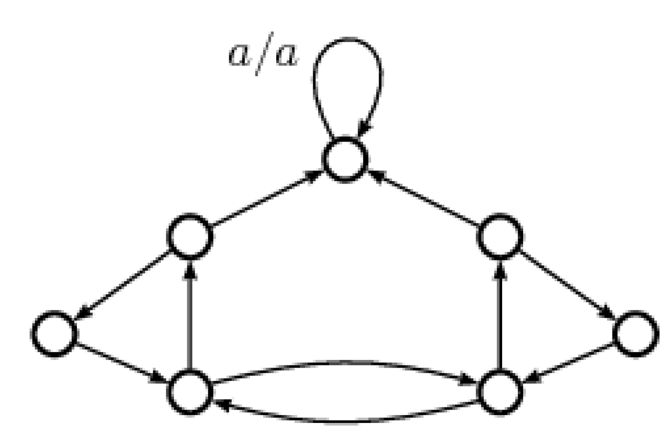
Definition 1. *The automorphism* MEMBERSHIP *problem takes as input two automata, $\mathcal{A}$ and $\mathcal{B}$, a distinguished automorphism $f \in S(\mathcal{A})$ corresponding to some state p, and $\mathcal{B}$'s residual matrix and vector A and r, and ouputs whether $f \in S(\mathcal{B})$.*

Theorem 1. *Automorphism* MEMBERSHIP *in $S(\mathcal{B})$ is decidable for an Abelian automaton $\mathcal{B}$.*

ISGROUP is decidable when Abelian.

Definition 2. *The* ISGROUP *decision problem takes as input an automaton $\mathcal{A}$ and answers the question "is $S(\mathcal{A}) = G(\mathcal{A})$?"*

Proof Sketch: We can represent elements of the semigroup as vectors over $\mathbb{N}^{|Q|}$. Residuation becomes an affine operator, reducing the problem to matrix algebra. □



KNAPSACK is undecidable.

Definition 3. *The* KNAPSACK *problem for automaton semigroups is as follows: given generators $s_1 \dots s_n \in S(A)$ and some element $s \in S(A)$, do there exist $a_1 \dots a_n \in \mathbb{N}$ such that*

$$s_1^{a_1} \dots s_n^{a_n} = s$$

Theorem 2. KNAPSACK *is undecidable.*

Proof Sketch: Reduce from Hilbert's 10 problem. □

A monoid with undecidable ISGROUP

The monoid presented here serves a sort of bound; optimistically suggesting that perhaps the class of automaton semigroups is not so big as to have an undecidable ISGROUP.

Proof Sketch: Reduce from the Halting problem. Given an input TM T, define a group whose generators are configurations of T. Set the halting state to the identity. If one configuration transitions to another, their corresponding group elements are equal.

Then consider the submonoid generated by the start state $\langle s \rangle$. $\langle s \rangle$ is a group iff T halts.

In order to keep the word problem decidable, we need to modify T to be a self-verifying Turing machine. □

Open Questions

- Is MEMBERSHIP decidable in the non-abelian case?
- Is ISGROUP decidable in the non-abelian case?
- A wide variety of other related semigroup decision problems.

Acknowledgements

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