

# DECISION PROBLEMS IN INVERTIBLE AUTOMATA

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### **Abstract**

We consider a variety of decision problems in groups and semigroups induced by invertible Mealy machines. Notably, we present proof that, in the Abelian case, the automorphism membership problem is decidable in these semigroups. In addition, we prove the undecidability of a Knapsack variant. A discussion of iteration and orbit rationality follows.

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## 1 Introduction

The word problem is a classic group-theoretic decision problem. Given a finitely generated group  $G$ , and a word  $w$  over the generators (and their inverses), the word problem asks “is  $w \in G$ .” The word problem is known to be undecidable in surprisingly small classes of groups - see [2] and [3] for background.

The invertible Mealy machines we consider here give rise to a class of semigroups (and sometimes groups) for which the word problem is decidable. The computability picture here is rather nuanced,

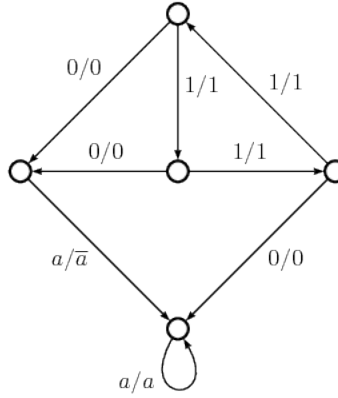


Figure 1: Grigorchuk's 5 state machine

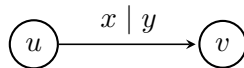
however. Similarly important decision problems, among them the conjugacy problem, and the isomorphism problem are known to be undecidable - see [20] and TODO for details.

In this paper, we present proof that, for the Abelian case, automorphism membership testing is decidable in this class of semigroups.

Serre first suggested the study of subgroups of the full automorphism group  $\text{Aut } 2^*$  of the infinite binary tree  $2^*$  in [18]. This notion has been usefully applied across group theory; a classic result here is Grigorchuk's group of intermediate growth, generated by the 5 state invertible machine shown in figure 1.

## 2 Background

An *automaton* is formally a triple  $(Q, \Sigma, \delta)$ , where  $Q$  is some finite state set,  $\Sigma$  is a finite alphabet of *symbols*, and  $\delta$  is a transformation on  $Q \times \Sigma$ . Automata are typically viewed as directed graphs with vertex set  $Q$  and an edge labeled  $x \mid y$  between  $u, v$  if  $(u, x)\delta = (v, y)$ .



One interprets this as if  $\mathcal{A}$  is in state  $u$  and reads symbol  $x$ , then  $\mathcal{A}$  transitions to state  $v$  and outputs symbol  $y$ . A computation within  $\mathcal{A}$  may then start at some state  $q_0$ , and on input  $\alpha_0\alpha_1 \dots \alpha_k$ , output  $\beta_0\beta_1 \dots \beta_k$ , where  $(q_i, \beta_i) = (q_{i-1}, \alpha_i)\delta$  for all  $i = 0 \dots k$ .

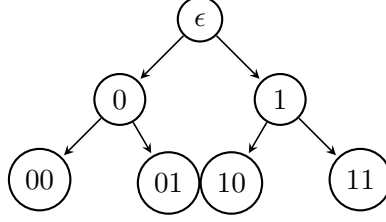


Figure 2:  $\mathbf{2}^*$  interpreted as the infinite binary tree

As in the above case, where  $\delta$  outputs exactly one character for every transition, we call the automaton  $\mathcal{A}$  *alphabetic*. An automaton is called *invertible* when every state in  $Q$  has some bijection  $\pi$  on  $\Sigma$  such that  $(u, x)\delta = (v, \pi(x))$ . A state in  $\mathcal{A}$  is a *copy state* if  $\pi$  is the identity permutation and is a *toggle state* otherwise. The present paper is concerned only with invertible automata.

## 2.1 Actions on the infinite tree

We may identify the set  $\Sigma^*$  with an infinite, regular tree of degree  $|\Sigma|$ . The root is labelled with the empty string  $\epsilon$ , and a vertex labelled  $w$  has the child  $wa$  for each  $a \in \Sigma$ . Out of convenience, we will frequently conflate a vertex with its label.

Each state  $q \in Q$  acts on the corresponding tree, sending vertex  $w$  to  $wq$ . Moreover, if  $\alpha\alpha'q = \beta\beta'$ , then  $\alpha q = \beta$ , for any  $\alpha, \alpha', \beta, \beta' \in \Sigma^*$ . Which is to say,  $q$ 's action on the tree is an adjacency-preserving map and is thus an endomorphism on the tree. Additionally,  $q$  is length-preserving, and thus preserves levels of the tree (and thus is an automorphism of the tree).

We extend the action of  $Q$  on  $\Sigma^*$  to words  $q = q_1 \dots q_n$  over  $Q^+$  by

$$wq = (\dots((wq_1)q_2)\dots q_n).$$

This computation corresponds with running  $\mathcal{A}$  starting at state  $q_1$ , then taking that output and running it through the machine starting at state  $q_2$ , and so on. We adopt the convention of applying functions from the right here. In this way, function composition corresponds naturally with string concatenation.

So there is a natural homomorphism  $\phi : Q^+ \rightarrow \text{Aut } \mathbf{2}^*$ , where  $\text{Aut } \mathbf{2}^*$  denotes the semigroup of automorphisms of the tree  $\mathbf{2}^*$ . We denote the image of  $\phi$  by  $\Sigma(\mathcal{A})$ .

## Semigroup theory

A *semigroup* is a set  $S$  paired with a binary operation  $f : S \times S \rightarrow S$  such that  $S$  is closed under  $f$  and  $f$  is associative over  $S$ . Any set of endofunctions forms a semigroup under composition.

A semigroup is called *Abelian* when its corresponding binary operation is commutative.

For an automaton  $\mathcal{A}$ , we denote by  $S(\mathcal{A})$  the semigroup generated by  $Q$  under composition.  $\mathcal{A}$  is said to be *commutative* or *Abelian* when  $S(\mathcal{A})$  is Abelian. We write  $G(\mathcal{A})$  for the group generated by the elements of  $Q$  and their inverses.

One may also speak about  $S(\mathcal{A})$  and  $G(\mathcal{A})$  without explicit reference to an automaton  $\mathcal{A}$ , yielding the corresponding definition:

**Definition 1.** *We call a semigroup  $S$  an automaton semigroup if there is some automaton  $\mathcal{A}$  with  $S \simeq \Sigma(\mathcal{A})$ . Similarly, a group  $G$  is called an automaton group if  $G \simeq \Sigma(\mathcal{A})$  for some automaton  $\mathcal{A}$ .*

## Wreath Recursions

Any automorphism  $f$  of  $\Sigma^*$  can be written in the recursive form:

$$f = (f_{\alpha_1}, f_{\alpha_2}, \dots, f_{\alpha_n})\tau$$

where  $n = |\Sigma|$  and each  $f_{\alpha}$  is an automorphism of a subtree of the root. Here,  $\tau$  is some permutation on  $\Sigma$ . In the case where  $\Sigma = \{0, 1\}$ , we have  $f = (f_0, f_1)\sigma$  where  $\sigma$  denotes transposition. If  $f = (f_0, f_1)\sigma$ ,  $f$  is said to be *odd*. If  $f = (f_0, f_1)$ ,  $f$  is said to be *even*. That is to say, automorphisms may be classified as even or odd depending on their action on the first level of the tree.

The set of even automorphisms form a subgroup  $H$  of index 2 in  $G(\mathcal{A})$ . Moreover, the residuation maps are group homomorphisms when restricted to  $H$ .

The automorphism semigroup of  $\Sigma^*$  decomposes into a recursive wreath product

$$\text{Aut } \mathbf{2}^* = \text{Aut } \mathbf{2}^* \wr \tau_\Sigma$$

where  $\tau_\Sigma$  is the tranformation semigroup on  $\Sigma$ . Which is to say,

$$\text{Aut } \mathbf{2}^* = \underbrace{(\text{Aut } \mathbf{2}^* \times \dots \times \text{Aut } \mathbf{2}^*)}_{n \text{ times}} \rtimes \tau_\Sigma.$$

**Definition 2.** Define residuation maps  $\partial_a : S(\mathcal{A}) \rightarrow S(\mathcal{A})$  that map  $f = (f_0, f_1)\sigma$  to  $f_a$ , and a parity map  $\text{par}$  such that  $\text{par}(f) = \sigma$ .

Note that a subgroup  $G$  of  $\text{Aut}(\mathbf{2}^*)$  need not be closed under residuation; if it is, we call it *self-similar* or *state-closed*. In this case, the wreath characterization in the full automorphism group carriers over and we have  $G \cong (G \times G) \rtimes \tau_{\mathbf{2}}$ .

### 3 Decision Problems

Automaton semigroups exhibit many interesting and nuanced computability properties. While it is an easy result that the WORD PROBLEM is solvable in such semigroups, similar group-theoretic problems such as the CONJUGACY PROBLEM and FINITENESS PROBLEM have been shown to be undecidable (see [20], and [6], respectively).

Various other semigroup theoretic decision problems have recently been considered for small classes of semigroups by Cain in [3]. We consider a subset of his distinguished properties in the automaton semigroup case here.

#### 3.1 IsAbelian is polynomial time

For a binary invertible automaton  $\mathcal{A}$ , define the *gap* of an automorphism  $f \in G(\mathcal{A})$  to be  $\gamma_f = (\partial_0 f)(\partial_1 f)^{-1}$ .

The following result is adapted from [16].

**Lemma 1.**  $\mathcal{A}$  is Abelian if and only if all even automorphisms in  $S(\mathcal{A})$  have gap  $I$  and odd automorphisms have constant gap.

*Proof.* Suppose  $\mathcal{A}$  is Abelian; so  $fg = gf$  for all  $f, g$  in  $S(\mathcal{A})$ . If  $f$  and  $g$  are both odd, simply residuate both sides to get

$$(\partial_a f)(\partial_{\bar{a}} g) = \partial_a(fg) = (\partial_a gf) = (\partial_a g)(\partial_{\bar{a}} f)$$

which yields  $\gamma_f = \gamma_g$ . If  $f$  is even and  $g$  odd, without loss of generality, we have

$$(\partial_0 f)(\partial_0 g) = \partial_0(fg) = \partial_0(gf) = (\partial_0 g)(\partial_1 f)$$

which, with algebraic manipulation, yields  $\gamma_f = I$ .

Conversely, first suppose  $f$  and  $g$  are both odd. Then  $fg = (\partial_0 f \partial_1 g, \partial_1 f \partial_0 g)$  and  $gf = (\partial_0 g \partial_1 f, \partial_1 g \partial_0 f)$ . Since  $\gamma_f = \gamma_g$ , these wreath recursions are the same. If  $f$  is even and  $g$  odd,  $fg = (\partial_0 f \partial_0 g, \partial_1 f \partial_1 g)\sigma$  and  $gf = (\partial_0 g \partial_1 f, \partial_1 g \partial_r f)\sigma$ .

If  $f$  and  $g$  are both even, the claim follows by induction.  $\square$

**Definition 3.** For an automaton  $\mathcal{A}$  with states  $s_1 \dots s_n$ , the inverse automaton of  $\mathcal{A}$ , denoted  $\mathcal{A}^{-1}$ , has state set  $t_1, \dots, t_n$  and transitions  $\partial_a t_i = \partial_{\bar{a}} s_i$ , with  $t_i$  a toggle state if and only if  $s_i$  is as well.

It is easy to verify by induction that  $t_i = s_i^{-1}$  for all  $i$ .

**Definition 4.** For an automaton  $\mathcal{A} = (Q, \Sigma, \delta)$ , the acceptor of  $\mathcal{A}$  at  $t$ , denoted  $\mathcal{A}(t)$ , is a partial DFA with state set  $Q$ , input alphabet  $Q \times Q$ , and transitions  $s \xrightarrow{a \times b} s'$  for each transition  $t \xrightarrow{a|b} t'$  in  $\mathcal{A}$ . Every state is accepting.

**Lemma 2.** The language of the acceptor  $\mathcal{A}(t)$  is

$$\{(x_1, y_1)(x_2, y_2), \dots, (x_n, y_n) \mid y_1 \dots y_n = (x_1, \dots, x_n)t\}$$

*Proof.* By induction on the length of the input string.  $\square$

**Definition 5.** For an automaton  $\mathcal{A} = (Q, \Sigma, \delta)$ , the product automaton  $\mathcal{A} \times \mathcal{A}$  is a machine with state set  $Q \times Q$  and transition function defined by  $\partial_a(s_1, s_2) = (\partial_a s_1, \partial_{as_1} s_2)$ .

We can see by induction that each state  $(s_1, s_2)$  in the product automaton corresponds to the word  $s_1 s_2 \in S(\mathcal{A})$ .

**Theorem 1.** There is a polynomial time algorithm to check if an automaton  $\mathcal{A}$  is Abelian.

*Proof.* On input automaton  $\mathcal{A}$ , build the inverse automaton  $\mathcal{A}^{-1}$ . Construct the product automaton  $\mathcal{A} \times \mathcal{A}^{-1}$ . Then for each toggle state  $t_i$  of  $\mathcal{A}$ , for the state  $s_i = (\partial_1 t_i, \partial_1 t_i^{-1})$  in  $\mathcal{A} \times \mathcal{A}^{-1}$ , construct the acceptor DFA  $(\mathcal{A} \times \mathcal{A}^{-1})(s_i)$ . Verify all the constructed DFAs are equivalent.  $\square$



The reader may be interested to note that this product automaton construction also provides proof that the word problem for automaton semigroups is decidable.

### 3.2 Automorphism Membership

This section considers the subsemigroup  $S(\mathcal{A})$  of  $\text{Aut } \mathbf{2}^*$  generated by the associated automorphisms of an invertible binary transducer. We assume minimality throughout this section.

We provide proof that the automorphism membership question is decidable in the Abelian case, and discuss partial work toward the general case. Some necessary background from [15] is outlined below.

#### 3.2.1 Linear algebraic background

**Theorem 2.** *If  $\mathcal{A}$  is Abelian, then  $G(\mathcal{A})$  is isomorphic to either a finite Boolean group or to  $\mathbb{Z}^m$  for some  $m \geq 1$ . In the latter case, there is an isomorphism  $\phi : G(\mathcal{A}) \rightarrow \mathbb{Z}^m$  satisfying the following recursion*

$$\phi^{-1}(v) = \begin{cases} (\phi^{-1}(A \cdot v), \phi^{-1}(A \cdot v)) & \text{if } \phi^{-1} \text{ is even} \\ (\phi^{-1}(A \cdot v - r), \phi^{-1}(A \cdot v + r)) & \text{otherwise} \end{cases}$$

where  $A \in GL(m, \mathbb{Q})$  and  $v \in \mathbb{Q}^m$ . Additionally, for all  $v \in \mathbb{Z}^m$ ,  $A \cdot v \in \mathbb{Z}^m$  or  $A \cdot v \pm r \in \mathbb{Z}^m$ .

We call the matrix  $A$  above the *residual matrix* of  $\mathcal{A}$ . The vector  $r$  is referred to as the *residual vector*. Put differently, this theorem specifies that when  $\mathcal{A}$  is Abelian, residuation is an affine map.

We have the following properties of  $A$ :

**Theorem 3.** *If  $G(\mathcal{A}) \cong \mathbb{Z}^m$  and  $A$  is its associated residual matrix,  $A$  satisfies the following properties:*

1.  $A$  is contracting; its spectral radius is less than 1
2.  $A$  is 1/2-integral, meaning that  $A^{-1}$  is a subgroup of index 2 in  $\mathbb{Z}^m$ . Therefore  $A$  be represented

as

$$\begin{bmatrix} \frac{a_{1,1}}{2} & a_{1,2} & \cdots & a_{1,m} \\ \frac{a_{2,1}}{2} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{m,1}}{2} & a_{m,2} & \cdots & a_{m,m} \end{bmatrix}$$

where all  $a_{i,j}$  are integers.

3. The characteristic polynomial  $\chi_A(x)$  is irreducible over  $\mathbb{Q}$  and has the form

$$\chi_A(x) = x^m + \frac{1}{2}g(x)$$

for some  $g \in \mathbb{Z}[x]$  of degree  $m - 1$ . In particular, the constant term is  $+\frac{1}{2}$ .

4.  $A$  is invertible and the characteristic polynomial  $\chi_{A^{-1}}(x)$  is integral and irreducible over  $\mathbb{Q}$ .

From property 2, Laplace expansion yields that  $A^{-1}$  is an integral matrix that is similar to the companion matrix of  $\chi_{A^{-1}}(x)$  over  $\mathbb{Q}$ .

The Latimer and MacDuffee theorem states that:

**Theorem 4.** *If  $p(x) \in \mathbb{Z}[x]$  is monic and irreducible, the  $GL(m, \mathbb{Z})$  similarity classes of integral matrices whose characteristic polynomial coincides with  $\bar{p}(x)$  is in one-to-one correspondance with ideal classes of the ring  $\mathbb{Z}[\theta]$ , where  $\theta$  is any root of  $p(x)$ .*

Property 1 of theorem 3 provides a bound on the coefficients of  $\chi_A(x)$ . When combined with property 4 and the above theorem, it can be shown that, for fixed  $m$ , there exist only finitely many possibilities of  $A$ , up to  $GL(m, \mathbb{Z})$  similarity.

**Definition 6.** *Take  $G$  to be some self-similar group. We may construct the complete group automaton (occasionally abbreviated as the complete automaton) for  $G$ , written  $\mathcal{C}_G$ , as follows: the automaton has  $G$ 's carrier set as state set with transitions  $f \xrightarrow{a|af} \partial_a f$ .*

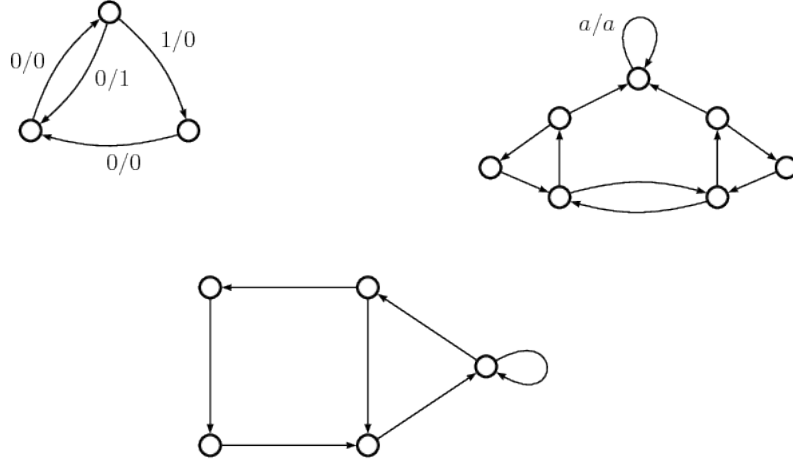
Of course in general, this invertible automaton will be infinite, but certainly  $\mathcal{S}(\mathcal{C}_G)$  is a group and isomorphic to  $G$ . The more interesting case is when  $G$  may be represented in terms of a finite automaton. Toward this end, call  $G$  *finite-state* if for all  $f \in G$ , the number of residuals  $\partial_w f$  is finite. If  $G$  is self-similar, finite-state, and finitely generated, we can construct the *group automaton*  $\mathcal{A}_G$ , a binary Mealy automaton, just like the complete group automaton, but with state set restricted to the collection of all residuals of the generators of  $G$ . Of course, the group generated by  $\mathcal{A}_G$  is isomorphic to  $G$ . One need to be careful, however; the semigroup may be different. Pleasantly,  $\mathcal{A}_G$  is minimal by construction.

**Lemma 3.** *Any complete automaton defined by  $(A, r)$  has only finitely many subautomata, each of which have finitely many states.*

**Example 1.** *The following automata represent the subautomata of the complete automaton generated with residual matrix*

$$A = \begin{bmatrix} -1 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

*and residual vector  $r = [-1, -\frac{3}{2}]^T$ .*



### 3.2.2 Membership is decidable in the Abelian case

**Definition 7.** *For a residual matrix  $A$  and residual vector  $r = A \cdot e_1$ , the principal automaton is the automaton generated from closure of  $e_1$  under residuation defined by the pair  $(A, r)$ .*

Given an invertible automaton  $\mathcal{A}$  and a principal Abelian automaton  $\mathcal{B}$ , we determine if  $f = \mathcal{A}(p)$  is in the semigroup generated by  $\mathcal{B}$ .

Thus one needs to check if there is some product automaton

$$D = \mathcal{B}_{p_1} \times \mathcal{B}_{p_2} \times \dots \times \mathcal{B}_{p_n}$$

that implements  $f$ . We have no computable bound on  $n$ , so a priori this is merely semidecidable.

Now consider the complete automaton  $C$  for  $\mathcal{B}$ , and let  $g$  be the automorphism defined by  $D$ . After minimization,  $D$  produces a subautomaton of  $C$  that consists of a “transient part” and a copy of  $\mathcal{B}$  (there may be SCCs in the transient part, but they are not subautomata). Hence, there is some word  $w$  such that  $\partial_w g$  is just a single state in the copy of  $\mathcal{B}$ ; also,  $w$  can be found effectively (TODO how?). In fact, for all  $u$ , there is a  $w$  such that  $\partial_{uw} g$  is atomic. Essentially this just means that  $g$  is strongly tame.

We may safely assume that  $\mathcal{A}$  is minimal. Then it looks like  $\mathcal{A}$  has to have  $\mathcal{B}$  as a subautomaton to satisfy this ultimate atomicity condition, plus a transient part sitting on top of  $\mathcal{B}$ . It should be decidable if things match up.

If  $\mathcal{B}$  is not principal, there are multiple subautomata of  $C$  to contend with, but that should not make a major difference. Ditto if  $\mathcal{B}$  is just a random subautomaton of  $C$ .

### 3.2.3 Membership is open in the general case

The decidability of MEMBERSHIP is arguably the most important open problem relevant to this thesis. Its decidability would imply the decidability of ISGROUP, as one could simply check that the inverse of each generator is contained in  $\mathcal{S}(\mathcal{A})$ .

## 3.3 IsGroup

### 3.3.1 IsGroup is decidable in the Abelian case

### 3.3.2 IsGroup is open in the general case

## 3.4 Knapsack is undecidable for automaton semigroups

We follow a proof strategy similar to [11].

### 3.4.1 Exponential Equations

Suppose  $\mathcal{X}$  is a countably infinite set of variables.

**Definition 8.** An exponential equation  $E$  over a semigroup  $S$  is an equation of formal products of the form

$$s_1^{x_1} s_2^{x_2} \cdots s_l^{x_l} = t_1^{y_1} t_2^{y_2} \cdots t_l^{y_l}$$

where each  $x_i, y_i$  is in  $\mathcal{X}$  and each  $s_i, t_i$  is in  $S$ . Note that we do not require the  $x_i$ 's and  $y_i$ 's to be distinct.

Denote by  $\text{Var}(E)$  the set of all variables appearing in  $E$ .

**Definition 9.** For a finite set  $X$  and semigroup element  $s$ , the set of  $X$ -solutions of the equation  $E$  is the collection of maps

$$S_X(E = s) = \{v : X \rightarrow \mathbb{N} \mid s_1^{v(x_1)} s_2^{v(x_2)} \dots s_l^{v(x_l)} = s \text{ in } S\}$$

The upcoming reduction involves taking direct products of semigroups, and so we will need the following strategy to transform exponential expressions over a semigroup  $S$  to equivalent exponential expressions over a direct product containing  $S$ .

To this end, suppose we have exponential expressions  $E_i$  for  $i = 1 \dots n$ , each with corresponding semigroups  $S_i$ . In each  $E_i$ , replace the base element  $s_i$  with the element

$$(\underbrace{1, \dots, 1}_{i-1}, s_i, \underbrace{1, \dots, 1}_{n-i}) \in \prod_i S_i.$$

Then take the formal product of each new  $E_i$ .

### 3.4.2 Undecidability of Knapsack

**Definition 10.** We define the KNAPSACK PROBLEM as follows: given as input generators  $g_1 \dots g_k$  and a target semigroup element  $g$ , do there exist natural numbers  $a_1 \dots a_k$  such that

$$g_1^{a_1} \dots g_k^{a_k} = g$$

**Definition 11.** The GENERALIZED KNAPSACK PROBLEM has as input generators  $g_1 \dots g_k, h_1, \dots, h_l$ , and has as output whether there exist natural numbers  $a_1 \dots a_k, b_1, \dots, b_l$  such that

$$g_1^{a_1} \dots g_k^{a_k} = h_1^{b_1} \dots h_l^{b_l}$$

We demonstrate that the GENERALIZED KNAPSACK PROBLEM is undecidable in the class of

automaton semigroups by reducing from Hilbert's tenth problem. The undecidability of the KNAPSACK PROBLEM easily follows.

**Definition 12.** *We define the decision problem HILBERT as following: “given a polynomial over the integers and an integer  $a$ , do there exist values of the arguments to the polynomial such that the polynomial evaluated at this point is equal to  $a$ ?”*

It is well-known that there exist polynomials for which HILBERT is undecidable, see [12] for details.

Recall that the Heisenberg semigroup

$$H_3(\mathbb{N}) = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} ; a, b, c \in \mathbb{N} \right\}$$

is an automaton semigroup [9]. Moreover, the class of automaton semigroups is closed under direct products, proven by Cain in [2]. We denote elements of  $H_3(\mathbb{N})$  as

$$H_{x,y,z} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

**Proposition 1.** *There exist fixed constants  $d, e \in \mathbb{N}$  and an exponential equation  $E$  of the form*

$$s_1^{x_1} s_2^{x_2} \dots s_n^{x_n} = t_1^{y_1} t_2^{y_2} \dots t_n^{y_n}$$

*with  $s_i, t_i \in G = H_3(\mathbb{N})^d \times \mathbb{Z}^e$  for which the GENERALIZED KNAPSACK problem is undecidable.*

*Proof.* From the input polynomial,  $P(x_1, \dots, x_n)$  and target value  $a$ , we separate the positive and negative terms of  $P$  to obtain an equation of the form

$$P_+(x_1, \dots, x_n) = P_-(x_1, \dots, x_n, a)$$

where every coefficient in  $P_+$  and  $P_-$  is positive. From this equation, we construct a system  $S$  of

equations, where each equation has one of the following forms:  $x \cdot y = z$ ,  $x + y = z$ ,  $x = c$  (for  $c \in \mathbb{Z}$ ). We will have that the equation  $P(x_1, \dots, x_n) = a$  has a solution in  $\mathbb{N}$  if and only if the system of equations  $S_a = S \cup \{x_0 = a\}$  has a solution in  $\mathbb{N}$ . Let  $X$  be the set of variables that occur in  $S_a$ .

Take a natural number  $a$  (the input of the reduction). Assume that  $S_a$  contains  $d$  equations of the form  $x \cdot y$ , and  $e$  many equations of the form  $x + y = z$  or  $x = c$ . We enumerate these equations as  $E_1, \dots, E_{d+e}$ , where the first  $d$  equations are of the form  $x \cdot y = z$ . Then set  $G_i = H_3(\mathbb{N})$  for each  $i \leq d$  and set  $G_i = \mathbb{N}$  for each  $i > d$ . For every  $i$ , we define an element  $g_i$  and an exponential expression  $E_i$  over  $G_i$  as follows:

*Case 1:*  $E_i = (x \cdot y = z)$ . Thus we have  $G_i = H_3(\mathbb{N})$ . Set  $g_i$  to be the identity matrix in  $H_3(\mathbb{N})$  and consider the following equation:

$$H_{1,0,0}^x = \begin{bmatrix} 1 & x & xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = H_{1,0,0}^z H_{0,0,1}^y H_{1,0,0}^x$$

A mapping  $v : X \rightarrow \mathbb{N}$  is a solution if and only if  $v(x)v(y) = v(z)$ .

*Case 2:*  $E_i = (x + y = z)$  and so  $G_i = \mathbb{Z}$ . Set  $g_i = 0$  and consider the equation (written additively in the group  $\mathbb{Z}$ )

$$x + y - z = 0$$

Then a mapping  $v : X \rightarrow \mathbb{Z}$  is a solution if and only if  $v(x) + v(y) = v(z)$ .

*Case 3:*  $E_i = (x = c)$ . (This includes our distinguished equation  $x_0 = a$ ). We have  $G_i = \mathbb{Z}$ . Then set  $g_i = c$  and  $E_i = x$ . Then as usual, a mapping  $v$  is a solution if and only if  $v(x) = c$ .

Finally, define  $E$  to be the direct product of the  $E_i$ 's  $\prod_{i=1}^d E_i$  and define  $g = (g_1, \dots, g_{d+e})$ . Then a mapping  $v$  is a solution to the equation  $E = g$  if and only if  $v$  is a solution to the system of equations  $S$ .  $\square$

**Theorem 5.** GENERALIZED KNAPSACK is undecidable in the class of automaton semigroups.

*Proof.* For all  $d, e \in \mathbb{N}$ ,  $H_3(\mathbb{N})^d \times \mathbb{Z}^e$  is an automaton semigroup. It follows that GENERALIZED KNAPSACK is undecidable for automaton semigroups.

□

### 3.5 A monoid with decidable Word Problem and undecidable IsGroup

We establish the existence of a monoid with decidable WORD PROBLEM, but undecidable ISGROUP. We may take this result as an intermediate step toward the decidability of the ISGROUP problem for automaton semigroups.

#### Preliminaries

Here we take a *Turing machine* to be a 6-tuple,  $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$ , where  $Q, \Sigma, \Gamma$  are all finite sets.  $\delta : Q \times \Sigma \rightarrow Q \times \Gamma \times \{L, S, R\}$  is the transition function,  $Q$  is the state set,  $\Sigma$  is the input alphabet,  $\Gamma \supseteq \Sigma$  is the tape alphabet,  $b$  is some blank symbol, with  $b \in \Gamma - \Sigma$ ,  $q_{\text{accept}}$  is the unique accepting final state, and  $q_{\text{reject}}$  the single rejecting final state.

We further define a Turing machine *configuration* to be a triple  $(u, q, v) \in \Gamma^* \times Q \times \Gamma^*$ . Here,  $u$  denotes the tape contents to the left of the tapehead,  $q$  is the current state, and  $v$  begins at the tapehead and extends to the right.

A configuration  $C$  for a TM  $M$  is said to *yield* configuration  $C'$  if  $M$  can step directly from  $C$  to  $C'$ .

For a Turing machine  $M$ , take  $C_M$  to be the set of all valid configurations of  $M$ . Then define  $CG_M$  to be the graph  $(C_M, E)$ , where  $(u, v) \in E$  if and only if  $u$  yields  $v$  in  $M$ .

**Definition 13.** *Define the canonical computation of  $M$  on  $w$ .*

$$\text{canon}(M, w) : \mathcal{T} \times \Sigma^* \rightarrow C_M^* \cup C_M^\omega$$

*to be the function that maps input  $w$  to the sequence of configurations  $M$  takes on while computing over  $w$ . Note that  $\text{canon}(M, w)$  will be a finite sequence if and only if  $M$  halts on  $w$ .*

Certainly, not every configuration in  $C_M$  will be along the sequence  $\text{canon}(M, w)$ . Which is to say, there are unreachable configurations.

Informally, a *self-verifying Turing machine*  $S$  is one that, at every step, verifies that the current configuration lies upon the canonical computation. If  $S$  finds that this is not the case,  $S$  immediately rejects. Otherwise, the computation steps forward a single step.



In the configuration graph  $CG_S$ , there is a path extending from each valid starting configuration  $(\epsilon, q_0, w)$  for  $w \in \Sigma^*$ . The remaining states form an infinite star graph with  $q_{reject}$  as the center.

**Proposition 2.** *There is a computable<sup>1</sup> function  $sv$  that maps Turing machines to equivalent self-verifying Turing machines.*

Speaking informally, as the canonical computation proceeds, a program counter is kept - perhaps to the left of the tapehead. After every step, the Turing machine will examine what “time step” the computation is currently sitting in. It will perform the canonical computation for the first  $n$  steps. If it does not wind up where it’s configuration says it is, it transitions to the death state. Otherwise, it continues.

In the interest of reader intuition, we expound upon a couple of implementation details here. For further reading, see [4], [5], and [19].

TODO rough sketch of implementation details.

### The submonoid in question

Define the Turing machine  $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$  to operate only on the blank tape; for all  $s \in \Sigma$ ,  $\delta(q, s) = (q_{reject}, b, S)$

Then take the ambient Abelian group  $G_M = (C_M, \cdot)$  whose carrier set is all configurations of  $M$ . For  $c, c'$  in  $G_M$ , we have  $c = c'$  if and only if  $c$  yields  $c'$ .

**Proposition 3.**  $G_{sv(M)}$  has a decidable word problem.

*Proof.* For every word  $w$  in  $G_{sv(M)}$ , there exist nonnegative integers  $a, r$  such that  $w = q_{accept}^a q_{reject}^r$ . Further, we may compute  $a$  and  $b$ . Recall that  $sv(M)$  maintains a program counter  $p$  to the left of the input. So we may simply run  $M$  for the first  $p$  steps and then check for configuration equality.

So then given two words  $w_1, w_2$  in  $G_{sv(M)}$ , simply compute  $a_1, r_1, a_2, r_2$ . Then  $w$  is in  $G_{sv(M)}$  if and only if  $a_1 = a_2$  and  $r_1 = r_2$ .  $\square$

**Proposition 4.** *If  $s$  is the start configuration of the Turing machine, it is undecidable whether  $\langle s \rangle$  is a group.*

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<sup>1</sup>The reader may be interested to find that  $sv$  is in fact primitive recursive - see TODO for details.

*Proof.* It is well known that the following language is undecidable

$$\text{HALTS} = \{\langle M \rangle \mid \text{TM } M \text{ halts on } \epsilon\}$$

and so we reduce from HALTS. Given as input a TM  $M$ , we use an oracle for ISGROUP as follows: first, compute  $\text{sv}(M)$ , and then consider  $G_{\text{sv}(M)}$ . Let  $s$  be the starting configuration for  $M$  on  $\epsilon$ .

If  $M$  halts, then the submonoid generated by  $s$  is the trivial group. If  $M$  hangs, then  $\langle s \rangle$  is the free monoid of rank one. So then  $\langle s \rangle$  is a group if and only if  $M$  halts. Since  $\text{sv}(M)$  and  $M$  are equivalent, we are done.  $\square$

## 4 Open Questions

Much partial work has been attempted toward a proof of the solvability or unsolvability of the ISGROUP problem in the general case. The monoid presented here serves a sort of bound for this problem; optimistically suggesting that perhaps the class of automaton semigroups is not so big as to have an undecidable ISGROUP.

There is also a rich area of smaller subclasses of automata to consider. Godin proved the decidability of KNAPSACK for the class of bounded automata in [7]. It is natural to then consider the the decidability of other decision problems, such as ISGROUP, ISOMORPHISM, and others.

One may also consider decision problems from a group presentation angle; all automaton semigroups are recursively presented. If we restrict our considerations to automaton semigroups whose presentations are regular, or context-free, does this affect the decidability of various decision problems? One suspects these questions are probably quite hard.

Additionally, most of the semigroup-theoretic decision properties listed in [3] remain open in the class of automaton semigroups. Notably, determining the decidability of Markov properties would be of great help to future work.

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