Orbit Checking for Invertible Transducers

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Abstract

We study iterated transductions defined by a class of invertible transducers over the binary alphabet. We present polynomial time orbit checking algorithms for a subclass of automata associated with Abelien free groups of finite rank.

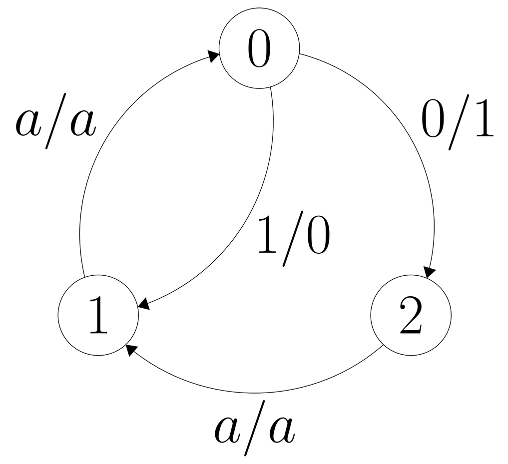
Flipping Pebbles

Suppose we're given a sequence of pebbles, black on one side, white on the other. Starting at the left, flip the current pebble. If the resulting color is black, skip ahead two pebbles. Else, skip ahead one. Repeat until the end of the string.

For example, starting at state 0 with the string 0000, we have

$$0000 \to 1001 \to 0011 \to 1010 \to 0000$$

This can be modeled with the following finite state machine A:



When we have two states p and q with the transition $p^{a/b} \to q$, this means "read character a and output character b". The algorithm previously described starts on state 0.

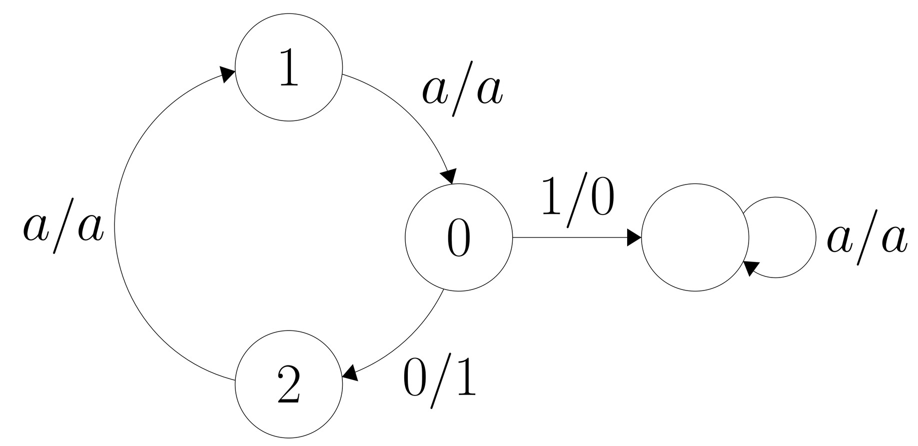
Starting at the 1st state ignores the first character, then proceeds as normal. We can view these pebble flipping algorithm as a function from string to string. These functions form a semigroup $\mathcal{S}(A)$ under composition (associative and closed).

Definition 1. *Given some function $f \in \mathcal{S}(A)$ and some string x, the set of all strings reachable from x by repeated application of f is called the* orbit of x under f.

The central question: given f and two strings x and y, is x in the orbit of y under f?

1-Tree Transducers

Definition 2. *Take a directed acyclic graph with exactly one directed cycle (excluding a copy state self-loop) and exactly one vertex v of outdegree two. Let v be the sole toggle state and every other state be a copy state. The resulting automaton is a 1-tree transducer.*



Proposition 1. *For a 1TT with a cycle of length n, on input x, state i adds (or subtracts) one to the number $\{x_j\}_{j \equiv i \mod n}$ interpreted in reverse binary.*

Theorem 1. *Orbit checking for 1-tree transducers can be done in polynomial time.*

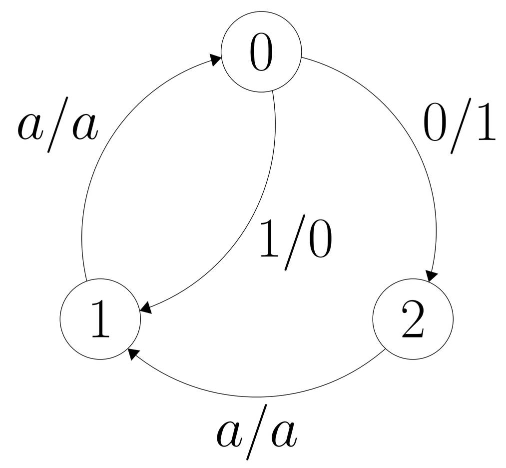
Theorem 2. *The semigroup of a 1TT with cycle length n is the free semigroup of rank n.*

Definition 3. *A relation is* rational *if there is some Mealy automaton recognizing the relation.*

Theorem 3. *The orbit relation of a 1TT is rational.*

A Small CCC

Definition 4. *A Cycle-Cum-Chord automaton (CCC)*



Theorem 4. *Orbit checking for A_2^3 can be done in polynomial time.*

Theorem 5. *The orbit relation for A_2^3 is rational.*

Theorem 6. *$\mathcal{S}(A_2^3)$ is already a group, ismorphic to $\mathbb{Z}^2$.*

Cycle-Cum-Chord Transducers

Theorem 7. *Orbit checking for A_m^n can be done in polynomial time.*

Question 1. *For which n, m is the orbit relation of A_m^n rational?*

Theorem 8. *$\mathcal{S}(A_m^n)$ is already a group, ismorphic to $\mathbb{Z}^{n-gcd(n,m)}$.*

Acknowledgements

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