Composite Trapezoidal Rule

Evan Bullock

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In calculus, we discuss the fact that not all integrable functions have anti derivatives so we cannot use the fundamental theorem to evaluate the integral.Recall that the second fundamental theorem of calculus states that the integral of a function f over some interval can be computed by using any one, F , of its infinitely many anti derivatives. This part of the theorem is important because it can find an antiderivative of a function by symbolic integration avoiding numerical integration to compute intervals.

To review the second part of the fundamental theorem, Let f be a function on a closed interval $[a, b]$ and F an anti derivative of f in $[a, b]$:

$$F'(x) = f(x). \tag{1}$$

If f is Riemann integrable on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a). \tag{2}$$

Trapezoidal Rule

Often times there are functions that we simply cannot integrate. The trapezoidal rule is a technique for approximating a definite integral. For moments when we simply cannot use equation (2). The trapezoidal rule approximates the area found under a curve by using a series of trapezoids and summing all the individual trapezoids areas. It can be shown as:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{n-1}+f(x_n)), \tag{3}$$

where $\Delta x = \frac{b-a}{n}$,and $x_i = a + i\Delta x$.

A better approximation which includes the sum of each individual trapezoid, is done by partitioning the integration interval while applying the trapezoidal rule to each subinterval.This allows for a "chained" trapezoidal but is indeed the true definition of the trapezoidal rule.Let $\{x_i\}$be a partition of $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and Δx_i is the length of the i-th subinterval (while $\Delta x_i = x_i - x_{i-1}$), then

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x_i. \tag{4}$$

where n is the number of subintervals of the partition. Note that the approximation becomes more accurate as the subinterval total increases of the partition(larger n means Δx_i decreases).Essentially with more intervals, n, the more accurate the approximation will be.

Errors

Remember that the Trapezoidal Rule is an approximation, which means the approximation can produce errors.The following can occur:

- If the integrand is concave up, the error is negative and the trapezoidal rule overestimates the true value.
- If the integrand is concave down, the rule underestimates because area is unaccounted for under the curve, but none is counted above.

Big O

Keeping in mind with errors, the error in the trapezoidal rule is $O(h^2)$ under the terms of "big O" notation. Big O can be defined as a mathematical notation that describes the limiting behavior of a function when the arguement tends towards a particular value or infinity. Big O is used in computer science as a way to classify certain algorithms with their running time as input size grows, essentially how efficient an algorithm runs in a program in respect to memory.In mathematical theory, big O is often used to express a bound on the difference between an arithmetical function and a better understood approximation.The letter O is used from the growth rate of function, which is also called the "order of the function" . Having the error of $O(h^2)$ for the Composite Trapezoidal rule means that the antiderivates exponentially converge to zero. For example, going from 1 to 1/2, to 1/4, etc. The error in the approximation to the integral goes down proportionally to the square of the inverse of the number of subintervals used in the approximation.

Example

A sample question may be the following: Apply the trapezoidal rule to the function $p(x) = -x^2 + 2x + \frac{1}{2}$ on the interval $[0, 3]$. To solve:

- Determine an appropriate number of subdivisions needed to get a accurate approximation, this will be n. Determine a, b from the interval.

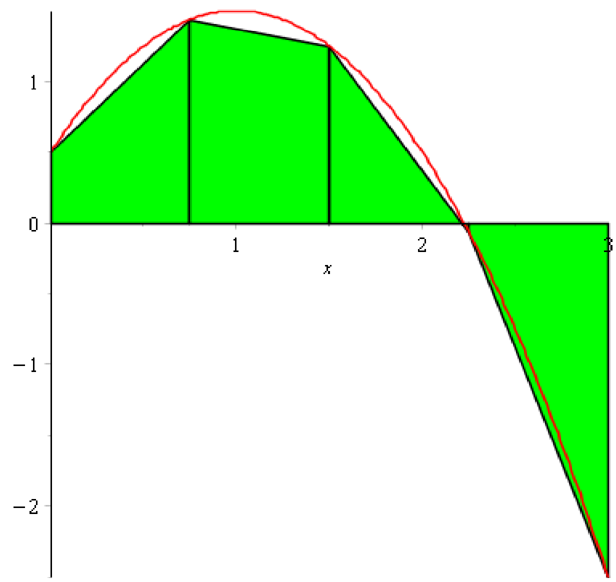

Figure 1: Trapezoidal rule on 4 subintervals

- Recall the formula for the rule,I would use the equation 5 in this case:
(5)
$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{n-1}+f(x_n)).$
- Solve for Δx by $\frac{b-a}{n}$. Plug in values for a,b and n.
- Solve for each $f(0), 2f(x_1), \cdots, 2f(x_{n-1}), f(x_n)$. For visualization of this step, and each individual trapezoid segment see Figure 1 .
- Plug everything back into the original equation. Recall,

$$\frac{\Delta x}{2}(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{n-1}+f(x_n)).$$

For a visualization of the accuracy of each interval of integration see Table 1. Note that Δx in the table represents the uniform spacing of the partition for the interval of integration. Recall that $\Delta x = \frac{b-a}{n}$.

Conclusion

In closing, the composite Trapezoid rule can be used to solve integrals that may seem impossible to solve. The rule simply finds an accurate approximation by finding the area under the curve by summing up multiple trapezoids areas. In

Table 1: Trapezoidal rule for $p(x)$

Δx	Integral	Error
10^{-1}	1.495	5×10^{-3}
10^{-2}	1.49995	5×10^{-5}
10^{-3}	1.4999995	5×10^{-7}
10^{-4}	1.499999995	5×10^{-9}
10^{-5}	1.4999999995	4.9998×10^{-11}

basic terms, this rule can get extremely close to the exact answer simply by "creating more trapezoids" under the curve.

$$\begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}$$

Some things

$$y = a \cos \pi$$
$$x = b \sin \pi$$

References

[1] Fundamental theorem of calculus.(2018) *Fundamental theorem of calculus- Wikipedia, the free encyclopedia.* Retrieved from **https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus#Second_part** ([Online; accessed 3-October-2018)

[2] Trapezoidal rule.(2018) *Trapezoidal rule- Wikipedia, the free encyclopedia.* Retrieved from **https://en.wikipedia.org/wiki/Trapezoidal_rule** ([Online; accessed 3-October-2018)

[3] Big O notation.(2018) *Big O notation- Wikipedia, the free encyclopedia.* Retrieved from **https://en.wikipedia.org/wiki/Big_O_notation** ([Online; accessed 3-October-2018)