Exercise 1. Model Linearization:

$$u = \begin{bmatrix} \delta \\ F \end{bmatrix}; \quad S_1 = \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix}; \quad S_2 = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Write Dynamics in linear form: $\cos\delta = 1 \rightarrow$ S.A.A, $\delta \leq \frac{\pi}{6}$

S_1:

$$f_1 : \dot{y} = \dot{y}$$

$$f_2 : \ddot{y} = -\dot{\psi}\dot{x} + \frac{2C_\alpha}{m}\left((1)\left(\delta - \frac{\dot{y} + l_f\dot{\psi}}{\dot{x}}\right) - \frac{\dot{y} - l_r\dot{\psi}}{\dot{x}} \right)$$

$$f_3 : \dot{\psi} = \dot{\psi}$$

$$f_4 : \ddot{\psi} = \frac{2l_fC_\alpha}{I_z}\left(\delta - \frac{\dot{y} + l_f\dot{\psi}}{\dot{x}}\right) - \frac{2l_rC_\alpha}{I_z}\left(-\frac{\dot{y} - l_r\dot{\psi}}{\dot{x}}\right)$$

Equilibria:

$$\underline{\bar{\dot{y}} = 0}; \quad \underline{\bar{\dot{\psi}} = 0}$$

$$\ddot{y} = 0 = -\bar{\dot{\psi}}\dot{x} + \frac{2C_\alpha}{m}\left((1)\left(\bar{\delta} - \frac{\bar{\dot{y}} + l_f\bar{\dot{\psi}}}{\dot{x}}\right) - \frac{\bar{\dot{y}} - l_r\bar{\dot{\psi}}}{\dot{x}}\right)$$

$$0 = (0)\dot{x} + \frac{2C_\alpha}{m}\left(\left(\bar{\delta} - \frac{0 + l_f(0)}{\cancel{\dot{x}}}\right) - \frac{0 - l_r(0)}{\cancel{\dot{x}}}\right)$$

$$0 = \frac{2C_\alpha}{m}\bar{\delta} \rightarrow \underline{\bar{\delta} = 0}$$

Finding State Jacobian:

$$\frac{\partial f_1}{\partial y} = \underline{0}; \quad \frac{\partial f_1}{\partial \dot{y}} = \underline{1}; \quad \frac{\partial f_1}{\partial \psi} = \underline{0}; \quad \frac{\partial f_1}{\partial \dot{\psi}} = \underline{0}$$

$$\frac{\partial f_2}{\partial y} = 0; \quad \frac{\partial f_2}{\partial \dot{y}} = \frac{2C_\alpha}{m\dot{x}} - \frac{1}{\dot{x}} = \underline{\frac{2C_\alpha/m - 1}{\dot{x}}}; \quad \frac{\partial f_2}{\partial \psi} = \underline{0}$$

$$\frac{\partial f_2}{\partial \dot{\psi}} = -\dot{x} - \frac{2C_\alpha l_f}{m\dot{x}} + \frac{l_r}{\dot{x}} = \underline{-\dot{x} - \frac{2C_\alpha l_f/m + l_r}{\dot{x}}}; \quad \frac{\partial f_3}{\partial y} = \underline{0}$$

$$\frac{\partial f_3}{\partial \dot{y}} = \underline{0}; \quad \frac{\partial f_3}{\partial \psi} = \underline{0} \quad \frac{\partial f_3}{\partial \dot{\psi}} = \underline{1}; \quad \frac{\partial f_4}{\partial y} = \underline{0}$$

$$\frac{\partial f_4}{\partial \dot{y}} = -\underline{\frac{2l_fC_\alpha + 2l_rC_\alpha}{I_z\dot{x}}}, \quad \frac{\partial f_4}{\partial \psi} = \underline{0}; \quad \frac{\partial f_4}{\partial \dot{\psi}} = \frac{-2l_f^2C_\alpha - 2l_r^2C_\alpha}{I_z\dot{x}}$$

continued on next...

Exercise 1. (continued)

s_1 (continued):

Finding input jacobian:

$$\frac{\partial f_1}{\partial \delta} = 0 \; ; \; \frac{\partial f_1}{\partial F} = 0 \; ; \; \frac{\partial f_2}{\partial \delta} = \frac{2 C_\alpha}{m} \; ; \; \frac{\partial f_2}{\partial F} = 0$$

$$\frac{\partial f_3}{\partial \delta} = 0 \; ; \; \frac{\partial f_3}{\partial F} = 0 \; ; \; \frac{\partial f_4}{\partial \delta} = \frac{2 l_f C_\alpha}{I_z}$$

$$\rightarrow A_1 = \left.\frac{\partial f}{\partial s_1}\right|_{\bar{s}_1, \bar{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4C_\alpha}{m\dot{x}} & 0 & -\dot{x} + \frac{2 C_\alpha (l_r + l_f)}{m\dot{x}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-2 l_f C_\alpha + 2 l_r C_\alpha}{I_z \dot{x}} & 0 & \frac{-2 l_f^2 C_\alpha - 2 l_r^2 C_\alpha}{I_z \dot{x}} \end{bmatrix}$$

$$B_1 = \left.\frac{\partial f}{\partial s_1}\right|_{\bar{s}_1, \bar{u}} = \begin{bmatrix} 0 & 0 \\ \frac{2C_\alpha}{m} & 0 \\ 0 & 0 \\ 0 & \frac{2 l_f C_\alpha}{I_z} \end{bmatrix}$$

$$\rightarrow \boxed{\dot{s}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-4C_\alpha/m}{\dot{x}} & 0 & -\dot{x} + \frac{2C_\alpha (l_r + l_f)}{m\dot{x}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-2 l_f C_\alpha + 2 l_r C_\alpha}{I_z \dot{x}} & 0 & \frac{-2 l_f^2 C_\alpha - 2 l_r^2 C_\alpha}{I_z \dot{x}} \end{bmatrix} s_1 + \begin{bmatrix} 0 & 0 \\ \frac{2C_\alpha}{m} & 0 \\ 0 & 0 \\ 0 & \frac{2 l_f C_\alpha}{I_z} \end{bmatrix} u}$$

Exercise 1. (continued)

s_2:

f_1: $\dot{x} = \dot{x}$

f_2: $\ddot{x} = \dot{\psi}\dot{y} + \frac{1}{m}(F - f_{rg})$

Finding state Jacobian

$$\frac{\partial f_1}{\partial x} = 0;\ \frac{\partial f_1}{\partial \dot{x}} = 1;\ \frac{\partial f_2}{\partial x} = 0;\ \frac{\partial f_2}{\partial \dot{x}} = 0$$

Finding input Jacobian

$$\frac{\partial f_1}{\partial \delta} = 0;\ \frac{\partial f_1}{\partial F} = 0;\ \frac{\partial f_2}{\partial \delta} = 0;\ \frac{\partial f_2}{\partial F} = \frac{1}{m}$$

$$\rightarrow A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};\ B_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$$

$$\rightarrow \boxed{s_2' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s_2 + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{bmatrix} u}$$