SOME RECENT USES OF ELASTICITY OF SUBSTITUTION— A SURVEY[1]

By Irving Morrissett

The concept of elasticity of substitution was set forth in the early 1930's and was at first used primarily in discussions of production theory under simplified conditions. During the last decade the concept has been used in a number of studies of interrelated consumer demands, without adequate recognition of the original assumptions in production theory nor of the additional problems which arise when the concept is carried over to the theory of demand. An examination of the assumptions which must be made in order to use elasticity of substitution in the theoretical or empirical study of interrelated consumer demands indicates that the concept is of little value for these purposes.

I. INTRODUCTION

A number of writers in recent years have employed the concept of elasticity of substitution in the measurement of relationships between the demands for rival products.[2] In most instances there has been little or no discussion of the significance of an elasticity of substitution between two goods of, say, -2.0. The assumption is usually present that a high value of this measure is associated with close rivalry between the two goods under consideration, but no benchmarks are given by which one may judge whether a particular value is high or low.

Many difficulties arise in connection with the use of the concept of elasticity of substitution, and with attempts to measure it with market data. It is the purpose of this paper to reconsider the various meanings that have been attached to the concept of elasticity of substitution, and to examine some of the efforts made in recent years to apply this concept to empirical measurement of interrelated demands.

II. DEFINITIONS OF ELASTICITY OF SUBSTITUTION

When elasticity of substitution (which will be denoted by E_s) is defined precisely and unambiguously, it is usually defined with respect to movement along an isoquant with the assumption that all other relevant quantities are held constant. Allen [**1,** p. 341] and Lerner [**13,** p.68; **14**], for example, have been quite explicit and consistent in this respect.[3]

[1] I wish to acknowledge the helpful comments and suggestions of Professors Robert Dorfman, George Kuznets, and Jerzy Neyman.

[2] Among them, Brems [**2**, pp. 36–48], Chang [**3**], Clawson [**4**], D. J. Morgan and W. J. Corlett [**16**], James N. Morgan [**17**], Polak [**19**], and Tinbergen [**23**].

[3] Lerner [**14**] is emphatic about the desirability of confining the definition of E_s to measures obtained when movement is confined to an isoquant. He refers to a definition of E_s in which movement is permitted from one isoquant to another as "a loose concept of elasticity of substitution," which Hicks was "guilty" of introducing in his *Theory of Wages* [**7**].

Hicks refused to admit his guilt, and in a reply [**6**] to Lerner he insisted that his

Thus E_s may be defined with respect to movement along a firm's iso-product curve as factors X_1 and X_2 vary in quantity, assuming quantities of all other factors are held constant; with respect to movement along an individual's indifference curve as goods X_1 and X_2 vary in quantity, assuming that quantities of all other goods are held constant; or with respect to movement along a firm's transformation (iso-factor) curve as products X_1 and X_2 vary in quantity, assuming quantities of all other products are held constant. E_s has also been defined, less commonly and not always explicitly, with respect to movement along a community indifference curve and with respect to movement along industry or economy iso-product curves. To complete the classification, filling up the rest of the "boxes," it may also be defined with respect to industry or economy transformation curves, giving eight possible meanings in all.

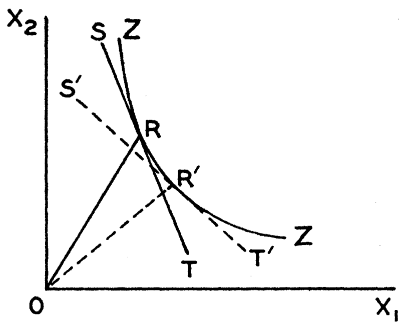

FIGURE 1

Elasticity of substitution is a concept which does not have a great deal of intuitive appeal. Its purpose is to measure how easily the proportion of X_1 to X_2 can change. The "ease of change" is measured by the ratio of the change in the quantity x_1/x_2 (where x_1 and x_2 represent the quantities of X_1 and X_2, respectively) to the change in the marginal rate of substitution of X_1 for X_2. Thus, if it is possible to make a large change in the ratio of x_1 to x_2 and the corresponding change in the marginal rate of substitution of X_1 for X_2 is small, E_s is large. In terms of produc-

definition of E_s was not intended to refer to movement along an isoquant, and only reduces to E_s so defined in the special case of constant returns. In "A Reconsideration of the Theory of Value" [**9**], however, Hicks joined with Allen in using the Lerner definition of E_s rather than his own, earlier definition. In Hicks' *Value and Capital* [**8**], E_s is conspicuous by its absence; this appears to be an implicit refutation of Hicks' prediction in the early article [**9**, p. 58, parentheses added] that "the rate of increase of the marginal rate of substitution (the principal ingredient) of E_s may be expected to play an important role in the development of theory."

tion functions, E_s is large if it is possible to substitute a large amount of factor X_1 for factor X_2 without causing the marginal rate of substitution of the two factors to change very much. Or, to express the same thing in geometric terms: in Figure 1, E_s is the change in the ratio of x_1 to x_2 (which is the change in the slope of OR) divided by the corresponding change in the slope of the iso-product curve (ZZ) (which is the change in the slope of ST). In terms of differentials, this becomes $d(x_1/x_2)/d(dx_2/dx_1)$, which must be multiplied by the factor $(dx_2/dx_1)/(x_1/x_2)$ to make the measure independent of the units of measurement of X_1 and X_2. E_s is, thus, defined as

$$\frac{d(x_1/x_2)}{d(dx_2/dx_1)} \frac{dx_2/dx_1}{x_1/x_2}, \tag{1}$$

or

$$\frac{d \log (x_1/x_2)}{d \log (dx_2/dx_1)}. \tag{2}$$

(For convenience this definition—either (1) or (2)—will sometimes be referred to as the "basic" definition.) It is apparent that a large value of E_s would be associated with an iso-product curve which has little curvature, while a small value of E_s would be associated with an iso-product curve which is sharply curved. In the forms in which it is written above, E_s is normally negative for indifference and iso-product curves and positive for transformation curves.

E_s is also defined sometimes as

$$\frac{d(x_1/x_2)}{d(dx_1/dx_2)} \frac{dx_1/dx_2}{x_1/x_2}, \tag{3}$$

or

$$\frac{d \log (x_1/x_2)}{d \log (dx_1/dx_2)}. \tag{4}$$

Since $\log (dx_2/dx_1) = -\log (dx_1/dx_2)$, (3) and (4) have the same value as (1) and (2) but with changed signs. Further, since $\log (x_1/x_2) = -\log (x_2/x_1)$, it is apparent that

$$\frac{d \log (x_1/x_2)}{d \log (dx_2/dx_1)} = \frac{d \log (x_2/x_1)}{d \log (dx_1/dx_2)}; \tag{5}$$

i.e., E_s of X_1 for X_2 has the same value as E_s of X_2 for X_1. It may be noted that (4) is equivalent to the form in which Allen [**1**, p. 341] has presented E_s. For movement along iso-product curves, the value of (4) is normally positive.

In expressions (1)–(5), changes are confined to movement along an isoquant. If additional factors or goods, X_3, X_4, $\cdots$, X_n, and output or a utility index, Z, are explicitly introduced, then the restriction to movement along an isoquant might be indicated by use of the notation for partial derivatives in (1)–(5).

In competitive equilibrium, the ratio of the marginal productivities of two factors is equal to the ratio of the prices of the two factors; i.e., $(\partial z/\partial x_1)/(\partial z/\partial x_2) = p_1/p_2$ where z is quantity of output and the p's are factor prices. Therefore $\partial x_2/\partial x_1 = -p_1/p_2$, and, if equilibrium exists and movement is confined to an iso-product curve,

$$\frac{d(x_1/x_2)}{d(dx_2/dx_1)} \frac{dx_2/dx_1}{x_1/x_2} = \frac{d(x_1/x_2)}{d(p_1/p_2)} \frac{p_1/p_2}{x_1/x_2} = \frac{d \log (x_1/x_2)}{d \log (p_1/p_2)}, \tag{6}$$

which is another way of defining E_s. [For convenience, (6) will sometimes be referred to as the "empirical" definition.] Similarly, a consumer in equilibrium is said to equate the ratio of marginal utilities of two goods to the ratio of the prices of the two goods; therefore (6) also defines E_s for an individual consumer, with movement restricted to an indifference curve. It we admit community indifference curves, (6) also defines E_s between two goods for an economy, with changes restricted to a single indifference curve. A similar proposition applies to the relation between the marginal rate of transformation and the price ratio of outputs. The "empirical" definition (6), thus, is equivalent to (1) or (2), in competitive equilibrium and with respect to movement along a single isoquant, where the isoquant may be any one of the eight types described above.

III. THE CONDITIONS UNDER WHICH THE "BASIC" AND "EMPIRICAL" DEFINITIONS ARE EQUIVALENT

Under actual conditions of production, there is no a priori reason to expect a producer to remain on the same iso-product curve in the (X_1, X_2) plane—a condition that implies constant output *and* no changes in the amounts of factors X_3, X_4, $\cdots$, X_n employed—when p_1 and/or p_2 changes. The producer's change in the ratio of quantities of X_1 and X_2 used, in response to a change in the ratio p_1/p_2 (assuming only one product and no other price changes), can be indicated by

$$\begin{aligned}\frac{d \log (x_1/x_2)}{d \log (p_1/p_2)} &= \frac{\partial \log (x_1/x_2)}{\partial \log (p_1/p_2)} + \frac{\partial \log (x_1/x_2)}{\partial \log x_3} \frac{d \log x_3}{d \log (p_1/p_2)} + \cdots \\ &\quad + \frac{\partial \log (x_1/x_2)}{\partial \log x_n} \frac{d \log x_n}{d \log (p_1/p_2)} + \frac{\partial \log (x_1/x_2)}{\partial \log z} \frac{d \log z}{d \log (p_1/p_2)}.\end{aligned} \tag{7}$$

In equation (7), $d \log (x_1/x_2)/d \log (p_1/p_2)$ represents the "empirical" definition of E_s, and $\partial \log (x_1/x_2)/\partial \log (p_1/p_2)$ is equivalent to the "basic" definition, the partial derivative sign indicating that output and other inputs are unchanged. If all other members of the right-hand side of (7) are zero, then

$$\text{(8)} \qquad \frac{d \log (x_1/x_2)}{d \log (p_1/p_2)} = \frac{\partial \log (x_1/x_2)}{\partial \log (p_1/p_2)};$$

that is, the "basic" and "empirical" definitions are equivalent. For the case of only two factors, (8) will hold if

$$\text{(9)} \qquad \frac{d \log z}{d \log (p_1/p_2)} = 0;$$

i.e., if the output is constant or if

$$\text{(10)} \qquad \frac{\partial \log (x_1/x_2)}{\partial \log z} = 0;$$

i.e., if, with the ratio p_1/p_2 constant, x_1/x_2 is constant at any output. Restriction (10) means that the production function is such that the

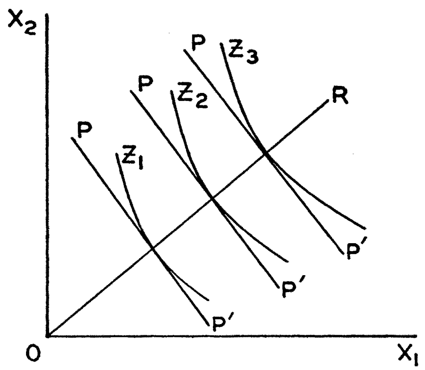

FIGURE 2

isoquants differ only in scale, so that any straight line through the origin intersects each isoquant at an angle which is constant for the given line. Such a production function is illustrated by Figure 2, in which the price lines, PP', have the slope p_1/p_2; and the line OR, representing a constant ratio between the quantities of X_1 and X_2, intersects the isoquants z_1, z_2, and z_3 at the points of tangency with the price lines. If this condition holds for any straight line OR through the origin, the measure of E_s is the same, whether or not movement is confined to an isoquant.

The class of functions which satisfies (10) includes all functions homogeneous in the factors, which can be shown as follows. Let

$$\text{(11)} \qquad f(x_1, x_2) = z,$$

where z is the quantity of the output, Z, and let f be a homogeneous function of degree n. Then $\partial x_2/\partial x_1 = -f_1/f_2$ (where f_1 and f_2 are the partial derivatives of f with respect to x_1 and x_2, respectively, and f_2 is not zero), which is homogeneous of degree zero, since f_1 and f_2 are both homogeneous of degree $n - 1$. Therefore the value of $\partial x_2/\partial x_1$ $(= p_1/p_2)$ remains unchanged when x_1 and x_2 change proportionately, i.e., when the ratio x_1/x_2 is constant.

More generally, let F be any monotonic function, and let

$$F[f(x_1, x_2)] = z, \tag{12}$$

where z is the quantity of output as before and f is a homogeneous function of degree n, as before. If G is the inverse function of F,

$$G(z) = f(x_1, x_2) \tag{13}$$

and $\partial x_2/\partial x_1$ is homogeneous of degree zero as before. Equation (12) includes production functions which give increasing and decreasing returns to scale, and the increasing or decreasing returns may be either regular or erratic. Thus (12) represents a more general form than (11), satisfying (10). It has sometimes been said that (10) is true only for linear homogeneous production functions; but (10) holds not only for *all* homogeneous functions but also for all functions of the type (12).

Condition (8) may also be put in terms appropriate for demand problems, as follows: Let Y be real income, and substitute y for z in (7). Then, for the case of two goods, (8) will hold if

$$\frac{d \log y}{d \log (p_1/p_2)} = 0, \tag{14}$$

i.e., if real income is constant (the individual or community stays on an indifference curve) or, if

$$\frac{\partial \log (x_1/x_2)}{\partial \log y} = 0, \qquad \frac{\partial \log x_1}{\partial \log y} = \frac{\partial \log x_2}{\partial \log y}, \tag{15}$$

i.e., if the income elasticities of the two goods are equal. Otherwise, an individual's or a community's response to a change in the price ratio p_1/p_2 will not give a measure of E_s as defined by (2), the "basic" definition.

IV. THE USE OF THE CONCEPT IN PRODUCTION THEORY

In the earliest theoretical uses of E_s, the production functions were assumed to be linear and homogeneous. J. R. Hicks, in *The Theory of Wages* [**7**, pp. 117–18], defined E_s with respect to changes in one factor while other factors remain constant. This definition is equivalent to the restriction (9) which confines movement to a "community" iso-product curve, by virtue of Hicks' assumption of "constant returns" for the economy as a whole. (He did not assume constant returns to individual firms, since this would make the output of firms indeterminate in his purely competitive model.)

Joan Robinson, in *The Economics of Imperfect Competition* [**20**, pp. 258–62], used E_s in a similar manner, to investigate the effects of price changes on output and on factor quantities employed by an *industry*, assuming constant returns to the industry. Mrs. Robinson stressed the point that "elasticity of substitution is determined by the *technical* conditions of production" (p. 256; italics added), and that a definition in terms of the ratio of marginal physical productivities, rather than of the ratio of factor prices, is "more fundamental" (p. 330n). She did not explain what was meant by "more fundamental"; I have interpreted this as a realization on her part that market prices are helpful in measuring E_s only under certain restrictive conditions. When Mrs. Robinson relaxes the assumption of constant returns to scale (pp. 330ff.), it is not clear what effect this has on her definition of E_s. She made a great deal of use of the concept of "efficiency units," a device which transfers variations in output as scale is changed into variations in factor prices as factor quantities change, and which thus makes every production function linear and homogeneous (p. 345). But E_s defined in terms of such a production function becomes an elusive concept. Kahn [11; see especially footnote 7 on p. 72], in discussing E_s, also measured factors in "efficiency units," thus converting his production functions into a linear homogeneous form.

If constant returns do not prevail and the production function is not of some other form which satisfies (10), then (6), which measures E_s in terms of prices, does not give the same value as (1) or (2). The value of (6) will, in fact, depend on the particular changes which occur in the prices and on how these price changes affect the scale of operation and the proportions in which the various factors are employed. The price changes, in turn, will depend on conditions of supply of the factors. E_s, then, is no longer "determined by technical conditions of production," but depends on the production function *and* on market conditions.

One may still wish to define E_s by (6), i.e., as $d \log (x_1/x_2)/d \log (p_1/p_2)$ without restricting the definition to movement along an isoquant or to a production function for which (10) is satisfied. But such a definition would seem to require that some assumptions be made about supply conditions; and if the model on which such a definition is based requires some specifications concerning both the production function and factor supply conditions, it should be recognized that E_s so defined depends only in part on "technical conditions of production."

V. THE USE OF THE CONCEPT IN INDIVIDUAL DEMAND THEORY

The discussion thus far has been primarily in terms of E_s as applied to substitution between two factors of production. Let us turn now to a consideration of how the concept has been used in the study of substitution between two commodities.

If E_s is defined with respect to movement along an individual's indifference curve, the meaning is clear (if we accept the rationale of indifference curves). E_s is then defined by (1) or (2), where X_1 and X_2 are two commodities and dx_2/dx_1 is the ratio of marginal utilities of X_1 and X_2, or the marginal rate of substitution between X_1 and X_2. If we wish to measure E_s by reference to observed behavior, we must use actual prices and quantities as they appear in the individual's purchases. But there is usually little justification for assuming that the individual remains on a given indifference curve when prices change. Can we (following the analogy with the production problem) help the situation by assuming that the indifference map satisfies (15)? Can we say that the indifference map is "homogeneous" in the sense that income elasticities are equal? It would be difficult to establish a priori grounds for such an assumption, even if we have only two commodities with which to deal. If more than two goods are considered, the conditions under which (8) (now considered for an indifference map rather than for a production function) is true are even more restrictive. Income elasticities of X_1 and X_2 must be equal, *and* one of the following must be true for each $X_i (1 \neq i \neq 2)$, namely

$$\frac{d \log x_i}{d \log (p_1/p_2)} = 0, \tag{16}$$

i.e., there must be no changes in the quantity of X_i as a result of changes in p_1/p_2; or

$$\frac{\partial \log (x_1/x_2)}{\partial \log x_i} = 0, \qquad \frac{\partial \log x_1}{\partial \log x_i} = \frac{\partial \log x_2}{\partial \log x_i}, \tag{17}$$

i.e., there must be a certain symmetry in the relation between X_1 and X_i as compared with the relation between X_2 and X_i. Condition (16) means that X_i must be independent of X_1 and X_2, in the sense that the quantity of X_i purchased is not affected by changes in p_1 or in p_2; while (17) means that, if X_i is not independent of X_1 and X_2 in the sense just described, substitution between X_1 and X_i and between X_2 and X_i must take place in such a way that the indifference curves in the (X_1, X_2) plane remain "homogeneous." A type of indifference map which would satisfy (17) is

$$x_1 x_2 x_3 \cdots x_n = U[u(y)], \tag{18}$$

where y is real income, u is utility, and U is a utility index. In (18), income elasticities of all goods are equal to +1,[4] the relations between all goods are symmetrical, and the indifference curve relating any pair of goods is a rectangular hyperbole. A more general form is

$$(19)\quad x_{11}x_{12}\cdots x_{1n_1} + x_{21}x_{22}\cdots x_{2n_2} + \cdots + x_{m1}x_{m2}\cdots x_{mn_m} = U[u(y)],$$

where substitution is nil between groups of goods and completely symmetrical within groups. It does not seem likely that indifference maps approximating (18) or (19), in which each pair of goods is either independent or have equal income elasticities, would exist. In studying the relationship between two particular commodities, at any rate, one would scarcely be justified in beginning with the assumption that the two goods are either independent or have equal income elasticities.

VI. THE USE OF THE CONCEPT IN AGGREGATE DEMAND THEORY

The foregoing remarks apply to measurement of E_s for an individual, using individual quantity and price data. If we attempt to measure E_s between two goods for the economy, using market quantities and prices, we run into all the difficulties encountered in measuring E_s for an individual—that is, the necessity of assuming that the community stays on a given indifference curve, or that the indifference map is of a restricted form (19)—plus the conceptual difficulties of "community indifference maps." In view of these great difficulties, it may be desirable to define E_s anew (as was suggested above with respect to substitution of factors) as $d \log (x_1/x_2)/d \log (p_1/p_2)$ without reference to indifference maps. E_s would then depend partly on tastes and partly on supply conditions. Tinbergen [**23,** p. 110] adopted this definition; attributing it, erroneously I think, to Allen and Hicks. Allen and Hicks were clear in defining E_s with respect to movement along an indifference curve, rather than with respect to market prices (see [**9,** especially pp. 59, 61, and 199]). Tse Chu Chang [**3,** p. 110] also attributed this definition to Hicks and Allen, referring specifically to their 1934 *Economica* article, and J. J. Polak [**19,** p. 16] used the same definition, following Tinbergen, in his study of E_s.

Let us examine the implications of such a definition, first for the simplified case of two goods, using the following model:

$$(20)\quad \begin{aligned} &(a)\ x_1 = f_1(p_1, p_2, y),\\ &(b)\ x_2 = f_2(p_1, p_2, y),\\ &(c)\ y\ = p_1x_1 + p_2x_2, \end{aligned}$$

[4] For y constant, $\partial x_i/\partial x_j = -x_i/x_j$. But $\partial x_i/\partial x_j = -p_j/p_i$ in equilibrium; therefore $x_i/x_j = p_j/p_i$ and $p_i x_i = p_j x_j$. That is, an equal amount of money, y/n, is spent on each good. Therefore $x_i = y/p_i n$, and the income elasticity for good i,

$$E_{x_iy} = \frac{\partial x_i}{\partial y}\frac{y}{x_i} = \frac{1}{p_i n}\frac{y}{x_i} = \frac{y}{(y/n)n} = 1.$$

where x_1 and x_2 are the quantities of the two goods, the p's are prices, and y is money income. In terms of (20),

$$E_s = \frac{d \log (x_1/x_2)}{d \log (p_1/p_2)} = \frac{d \log x_1 - d \log x_2}{d \log p_1 - d \log p_2}$$

$$= \frac{\dfrac{\partial \log x_1}{\partial \log p_1} d \log p_1 + \dfrac{\partial \log x_1}{\partial \log p_2} d \log p_2 + \dfrac{\partial \log x_1}{\partial \log y} d \log y}{\dfrac{- \dfrac{\partial \log x_2}{\partial \log p_1} d \log p_1 \dfrac{\partial \log x_2}{\partial \log p_2} d \log p_2 - \dfrac{\partial \log x_2}{\partial \log y} d \log y}{d \log p_1 - d \log p_2}}.$$

Therefore

$$E_s = \frac{E_{11} - E_{21}}{1 - \dfrac{d \log p_2}{d \log p_1}} + \frac{E_{22} - E_{12}}{1 - \dfrac{d \log p_1}{d \log p_2}} + \frac{E_{1Y} - E_{2Y}}{\dfrac{d \log p_1}{d \log y} - \dfrac{d \log p_2}{d \log y}}, \tag{21}$$

where E_{11} stands for price elasticity of demand for X_1, E_{21} for cross-elasticity of demand for X_2 with respect to the price of X_1, E_{1Y} for income elasticity of X_1, etc. By dividing and multiplying the denominator of the last term of (21) by $d \log p_1/d \log y$, we get

$$E_s = \frac{E_{11} - E_{21}}{1 - \dfrac{d \log p_2}{d \log p_1}} + \frac{E_{22} - E_{12}}{1 - \dfrac{d \log p_1}{d \log p_2}} + \frac{E_{1Y} - E_{2Y}}{\dfrac{d \log p_1}{d \log y}\left(1 - \dfrac{d \log p_2}{d \log p_1}\right)}. \tag{22}$$

Equation (22) expresses E_s between X_1 and X_2 in terms of elasticities, cross-elasticities, and income elasticities, but it also contains the terms $d \log p_2/d \log p_1$ and $d \log p_1/d \log y$. The equations (20) make it possible to determine the elasticities E_{11}, E_{1Y}, etc., but do not tell us the values of $d \log p_2/d \log p_1$ and $d \log p_1/d \log y$; thus, E_s defined by the empirical definition is indeterminate with respect to the model (20). The problem might also be viewed in another way: the model (20) contains three equations and five variables, hence it is likely that two more relationships will be required to completely determine the relationship among the variables. Or, to state the problem still another way, E_s on our present definition is not uniquely defined until we know something about how p_1 and p_2 are related to each other and how at least one of the prices is related to income.

If we make the assumption

$$y = \text{constant}, \tag{20d}$$

the last term of (22) becomes zero and we are still lacking one relationship, which will make it possible to evaluate $d \log p_2/d \log p_1$, in order to define E_s uniquely. It should be noted here that the argument would not be changed if the assumption were made that real income rather than money income is constant. In the latter case, a price index, $P = f_P(x_1, x_2, p_1, p_2)$ would be introduced, (20d) would become y/P = constant, and there would then be six variables with five equations.

There are several possible assumptions that could be added to (20a–20d) to make the definition of E_s determinate:

$$f_3(x_1, x_2) = \text{constant.} \tag{23a}$$

This can be interpreted as the assumption, borrowed from the "indifference map model," that utility remains constant; i.e., the community stays on a given indifference curve. However, we have rejected such an assumption in the present analysis.

$$E_s = \text{constant.} \tag{23b}$$

I know of no a priori basis for assuming that E_s is constant, unless it might be that one kind of linear relationship is as good as any other in the difficult business of estimating relationships between prices and quantities. *This assumption is the one commonly used in the estimation of* E_s ; in fact, the making of this assumption is the *only* means which I have seen used in numerous efforts to measure E_s empirically and to circumvent the difficulty which I have just outlined. (Most of the empirical analyses have assumed constant income; and they have also, as a rule, made assumptions which reduced the analysis to the two-good case.) The assumption that E_s is constant has important implications, which will be discussed in Section VIII.

Assumptions (23a) and (23b) are related to demand. It is also possible to introduce additional restrictions on the supply side to make E_s determinate:

$$x_1 = \text{constant} \quad \text{or} \quad x_2 = \text{constant.} \tag{23c}$$

These are the simplest supply conditions—that supplies are fixed.

$$x_1 = f_4(p_1) \quad \text{or} \quad x_2 = f_5(p_2). \tag{23d}$$

These are the assumptions that quantities supplied are functions of prices. A variation of (23d) was used, for example, by J. E. Meade [**15**] in a theoretical discussion in which E_s, defined by the "empirical" definition (6), was used as a tool in studying the incidence of taxation on two "goods," land and buildings. Meade assumed supply elasticities to be known. J. J. Polak [**19**, pp. 17, 20] also introduced known supply elasticities in his discussion of the measurement of E_s. More complicated supply conditions could, of course, be introduced.

We can now turn to the general case, using the following model:

$$
\begin{aligned}
x_1 &= f_1(p_1, p_2, p_3, \cdots, p_n, y) \\
x_2 &= f_2(p_1, p_2, p_3, \cdots, p_n, y) \\
x_3 &= f_3(p_1, p_2, p_3, \cdots, p_n, y) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_n &= f_n(p_1, p_2, p_3, \cdots, p_n, y) \\
y &= \sum p_i x_i .
\end{aligned} \tag{24}
$$

In terms of (24), E_s may be expressed as

$$
\begin{aligned}
\frac{d \log (x_1/x_2)}{d \log (p_1/p_2)} &= \frac{E_{11} - E_{21}}{1 - \dfrac{d \log p_2}{d \log p_1}} + \frac{E_{22} - E_{21}}{1 - \dfrac{d \log p_1}{d \log p_2}} \\
&+ \frac{E_{13} - E_{23}}{\dfrac{d \log p_1}{d \log p_3}\left(1 - \dfrac{d \log p_2}{d \log p_1}\right)} + \cdots + \frac{E_{1n} - E_{2n}}{\dfrac{d \log p_1}{d \log p_n}\left(1 - \dfrac{d \log p_2}{d \log p_1}\right)} \\
&+ \frac{E_{1Y} - E_{2Y}}{\dfrac{d \log p_1}{d \log y}\left(1 - \dfrac{d \log p_2}{d \log p_1}\right)} .
\end{aligned} \tag{25}
$$

In general, there are $n + 1$ equations and $2n + 1$ variables; therefore it might be necessary to add as many as n additional relationships of the kind suggested by (23a–23d), in order to define E_s uniquely.

VII. THE SIGNIFICANCE OF THE CONCEPT FOR DEMAND THEORY

Let us suppose for the moment that additional assumptions are introduced so that the empirical definition of E_s is determinate. What significance can be attached to E_s thus defined?

Schultz [**21,** pp. 571–72] proposed that a comparison of variations in the quantity ratio with variations in the price ratio be used as a "rough test" of the relationship between the demands for two goods. He argued that in the case of completely complementary goods the ratio of x_1 to x_2 cannot change, and therefore the numerator of (6) and E_s itself must always be zero. Also, when X_1 and X_2 are perfect substitutes the price ratio cannot change (as long as some quantity of each good is sold), and therefore the denominator of (6) must be zero and E_s must be infinite. It is tempting to conclude from these considerations (as Schultz did) that values of E_s which lie between zero and infinity measure demand relationships which are intermediate between perfect complementarity and perfect substitution. But this is not a satisfactory procedure since E_s is undefined, in the sense explained on pages 50 and 51 above, for all values between zero and infinity. In the absence of simplifying assumptions such as (23a–23d) it is not even possible to say that E_s is a monotonic function of the degree of relationship between two demands, since the value of E_s changes with changes in supply conditions. It would be possible to specify conditions under which E_s would turn out to be, say, -0.5 for two goods with demands which are closely related (in the sense of having high cross-elasticities) and -4.0 for two goods which are quite independent. Further, it should be noted that the cases in which E_s does measure the degree of relationship between goods, i.e., in the cases of perfect complements and perfect substitutes, the measure is of little interest since each such pair of commodities is for all practical purposes a single commodity. There is little point, for example, in studying the relationship between the demands for right and left shoes or between Kansas and Nebraska wheat.[5]

Tinbergen [**23,** p. 112], in explaining the meaning of an estimate of -2.0 for E_s between all exports of a country and world exports, stated "this means that a ten per cent reduction in the export price level only induces a twenty per cent increase in the volume of exports." But such an interpretation assumes that p_2 remains constant, in which case E_s reduces (if y also is constant) to $E_{11} - E_{21}$ [see equation (21)]; also, if the trade of the country under consideration is small compared to total world trade, E_{21} is small and E_s under these assumptions is very nearly the same thing as E_{11}, the price elasticity of demand. Although it is clear that Tinbergen did not intend that E_s always should be interpreted under restrictions of this kind, the phrase quoted is the only verbal explanation which he gave of the significance of E_s.

Chang [**3**] gave a number of estimates of E_s, but he did not discuss the significance of the concept and in his summary discussion of the effects of exchange depreciation he relied almost entirely on his estimates of price elasticities.

In some of the studies which have related quantity ratios to price ratios, the rationale has been presented principally in terms of the "reasonableness" of such a procedure. Clawson [**4,** p. 265], for example, in an extensive empirical study of price and quantity ratios, said "if the quantity of beef consumed rose in comparison with the quantity of pork consumed, then it would not be unreasonable to expect the price of beef to fall in comparison with the price of pork," adding that "the relation of this type of inquiry to such theoretical concepts as indifference curves, elasticity of substitution, and other concepts of recent popularity cannot be attempted in a single article." Brems [**2,** p. 37] also

[5] Arguments for rejecting Schultz's "rough test" are also given by Adolf Kozlik [**12,** pp. 654–55] and by Sidney Hoos [**10,** p. 528].

appealed to the reasonableness of the assumption that the ratio of Ford to Chevrolet sales is determined by the ratio of Ford to Chevrolet prices. Both Clawson [**4**, pp. 300–301] and Brems [**2**, pp. 36–37] looked upon this procedure as a means of handling the deflation problem, and Clawson pointed to the additional feature that it reduces the number of variables. One or both of these features have also commended the method to others engaged in the study of interrelated demands.

Perhaps the most obvious interpretation of E_s as defined in this section is in terms of market shares, which indeed is similar to the purpose for which Hicks originally invented the concept.[6] If b is the measure of E_s, the relation between x_1, x_2, p_1, and p_2 in the immediate neighborhood of a set of given values for these four variables is approximated by $x_1/x_2 = a(p_1/p_2)^b$, which can be written as $p_1x_1/p_2x_2 = a(p_1/p_2)^{b+1}$. If $b = -1$, market shares (in value terms) will remain constant even though the ratio of prices changes. If $b = -10$, the market shares will change very rapidly in favor of X_2 if the ratio p_1/p_2 increases. If $b = -0.5$, the market shares will change slowly in favor of X_1 if the ratio p_1/p_2 increases. A value of zero for b would indicate that market shares are directly proportional to the price ratio. Positive values of b would indicate that changes in the ratio of p_1 to p_2 would result in more than proportionate changes in the market share of X_1.

One might imagine circumstances under which a measure of E_s based on the empirical definition might be useful: (i) A monopoly or collusive oligopoly might wish to know what would happen to market shares if relative prices of two of its products were changed. (ii) A nation contemplating devaluation, and confident that there would be no retaliation, might wish to know the effect of such devaluation on market shares for certain of its products or for all its exports. However, a measure of E_s would not, in itself, be useful in these ways if one could change one price but were uncertain about what would happen to the other price, as in noncollusive oligopoly or as when the repercussions of devaluation could not be predicted. Also, this measure would give no indication of what would happen to absolute demand for one or both products.

VIII. METHODS OF ESTIMATION

We come now to the methods which have been used to estimate E_s. In every instance that I have seen, b has been estimated by fitting a line of least squares of the form

$$\log (x_1/x_2) = a + b(\log p_1/p_2). \tag{26}$$

[6] Hicks first used E_s to aid his description of the conditions under which the shares of capital and labor in the national dividend would remain constant (see [**7**, p. 117 and Appendix]).

Differentiating, $b = d \log (x_1/x_2)/d \log (p_1/p_2) = E_s$ as defined by (6), Chang [**3**, p. 110] introduced a variation, which explicitly accounted for income by using the form

$$\log (x_1/x_2) = a + b(\log p_1/p_2) + c \log y. \tag{27}$$

The latter method brings income specifically into the analysis, which is theoretically better than assuming it to be constant.

The method of estimation indicated by (26) [and also by (27)] introduces the assumption that E_s is constant. The implications of this assumption are extremely interesting and important for our analysis. Taking the total differential of (26), we get

$$d \log x_1 - d \log x_2 = b(d \log p_1) - b(d \log p_2). \tag{28}$$

From equations (20a) and (20b), for the case of two goods, we get

$$d \log x_1 = E_{11}\, d \log p_1 + E_{12}\, d \log p_2 + E_{1Y}\, d \log y,$$

$$d \log x_2 = E_{21}\, d \log p_1 + E_{22}\, d \log p_2 + E_{2Y}\, d \log y.$$

Subtracting,

$$\begin{aligned} d \log x_1 - d \log x_2 = (E_{11} - E_{21})\, d \log p_1 - (E_{22} - E_{12})\, d \log p_2 \\ + (E_{1Y} - E_{2Y})\, d \log y. \end{aligned} \tag{29}$$

Equating coefficients of (28) and (29), we find that

$$\begin{aligned} &\text{(a)} \quad b = E_{11} - E_{21}\,, \\ &\text{(b)} \quad b = E_{22} - E_{12}\,, \\ &\text{(c)} \quad E_{1Y} = E_{2Y}\,. \end{aligned} \tag{30}$$

[If (27) is used instead of (26), the only difference is that (30c) becomes $E_{1Y} = E_{2Y} + c$.] By a similar procedure for the case of n goods, we find that, in addition to the relations shown by (30),

$$E_{1i} = E_{2i} \qquad (i = 3, 4, \cdots, n). \tag{31}$$

It should be stressed that all of the relations of (30) and (31) are implied by the single assumption that E_s is constant.[7] It is worth while

[7] (30) and (31) include n independent assumptions; (30a) and (30b) can be combined into $E_{11} - E_{21} = E_{22} - E_{12}$, giving 2 equations from (30) and $n - 2$ equations from (31). Compare the last sentence of Section VI, in which it is stated that it might be necessary to add n assumptions to the general equilibrium model for n goods, (24), in order to define E_s uniquely. The assumption that E_s is constant implies the n independent assumptions of (30) and (31).

dwelling on these relations for a moment. (i) If the cross-elasticities are not zero, the assumption of this model for the estimation of E_s is that the sum of the price elasticity of X_1 and the cross-elasticity of X_1 to P_2 is equal to the sum of the price elasticity of X_2 and the elasticity of X_2 to P_1, and that each of these sums is equal to E_s. (ii) If the cross-elasticities are zero, the assumption of this model is that the price elasticities are equal to each other and to E_s. Thus it is not surprising that values of E_s estimated by this method are, as a rule, negative and not close to zero; whether one is estimating E_s for closely related goods or for remotely related goods (that is, whether cross-elasticities are large or small), the estimate is in every case importantly influenced by the price elasticities. From another point of view, we may look upon b as a simultaneous estimator of the elasticities of demand for two unrelated products (for which both cross-elasticities may be assumed to be zero). Anyone familiar with Schultz's monumental efforts to measure one elasticity at a time will agree that this b is a remarkable estimator. The method appears to be even more remarkable when one views it also as a means of eliminating trend and simplifying the analysis by reducing the number of variables.[8] (iii) Equations (31) imply that X_1 and X_2 have a symmetrical relationship to all other goods, the cross-elasticities of demand for X_1 and X_2 to the price of any other good, P_i, being equal.

It is interesting to note that some of these implications—namely, (30c) and (31)—which follow from the assumption that E_s is constant, are the same as the assumptions (15) and (17) which were found necessary to make E_s empirically measurable using the indifference map model.

As would be expected on the basis of comment (ii) just above, most of the values estimated for E_s from empirical data have, in fact, been negative and not close to zero. This fact has given encouragement to some investigators, by seeming to indicate a certain consistency in the data and in the method. J. N. Morgan [**17**, pp. 37–38], for example, took comfort from the fact that his measures of E_s between butter and margarine were all in the neighborhood of -2.0. But a crude a priori argument may be given for the belief that E_s for any two commodities selected at random is likely to be numerically greater than -1. If the relation between all goods were symmetrical, income elasticities of all goods would be $+1$, and there would be a weak substitute relation (i.e., small positive cross-elasticities) between every pair of goods. Slutzky ([**22**, p. 15]; cited by Schultz [**21**, p. 621]) has shown that

$$\sum_{j=1}^{n} E_{ij} = -E_{iY}, \tag{32}$$

where n is the number of commodities; i.e., that the sum of the elasticities of a good to its own and all other prices equals the negative of the

[8] See p. 54 above.

good's income elasticity. If all n goods are symmetrically related as posited, with income elasticities of $+1$ and all cross-elasticities equal, (32) implies that[9]

$$(33) \qquad -E_{ii} = (n - 1)E_{ij} + 1, \qquad (i \neq j).$$

If E_{ij} is positive, as is also implied by the symmetry assumption, $-E_{ii} > +1$, i.e., the minimum numerical value of any price elasticity is one. If E_{ij} is, for example, equal to $1/(n - 1)$, then E_{ii} equals -2. Unless the cross-elasticities are extremely small, price elasticities must be sensibly greater than unity. It would not be surprising, therefore, if two unrelated commodities had an E_s of -2 when E_s has been calculated by the method of (26).

Tinbergen [**23**, p. 112] also commented on the fact that his measurements of E_s, where X_1 and X_2 referred to total exports of a particular country and of the world, respectively, centered around the figure -2.0. He inferred that the low numerical value of this figure indicates that competition in world trade is quite imperfect. I am sure this conclusion is correct, and that Tinbergen's data are compatible with such a conclusion; but I wonder to what extent this method of analysis actually lends support to such a conclusion.

Tinbergen also noted, after reviewing a number of studies in which attempts were made to measure E_s, that most of the estimates of other investigators also centered around -2.0. Tinbergen's review, unfortunately, does not deal with the significance of the results which he summarized.

The estimates of E_s calculated by D. J. Morgan and Corlett [**16**, pp. 309–10] were widely scattered in value, about three-fourths of the estimates being negative. In their theoretical discussion, they indicated the conditions under which the method of (26) gives positive estimates.

IX. SOME RECENT CONTRIBUTIONS

Most of the theoretical difficulties of measuring E_s which have been considered here have been recognized and discussed from somewhat different viewpoints in two recent and excellent articles by Polak [**19**] and by D. J. Morgan and Corlett [**16**]. Polak, although limiting his analysis to the simpler case of two goods, indicated clearly some of the difficulties involved in the efforts of Tinbergen and Chang to measure E_s. Since Polak defined E_s as "the change in the ratio of the volume of exports caused by a change in the price ratio," one is not sure what he means by the "true" as opposed to the "quasi" E_s; but it turns out that the true E_s ($\sigma_1 + \sigma_2$ in his symbols) is that which would result if the

[9] From (32), $\sum_{j,\, j\neq i} E_{ij} + E_{ii} = -E_{iY}$. The symmetry assumption implies that $E_{iY} = +1$ and $\sum_j E_{ij} = (n - 1)\, E_{ij}$ for any $i, j \neq i$. Therefore $(n - 1)$ $E_{ij} + E_{ii} = -1$, and $-E_{ii} = (n - 1)\, E_{ij} + 1$.

income elasticities of the two goods were equal. Thus his "true" and "quasi" E_s correspond to my "basic" and "empirical" definitions, respectively. If the income elasticities of the two goods are not equal, it is necessary to introduce some additional conditions, and Polak assumed the supply elasticities to be known. He then showed that the estimates of E_s depend on the income and supply elasticities as well as on the true E_s ; and that the estimates of E_s may vary from $-\infty$ to $+\infty$ even if the true E_s is zero.

The contribution of D. J. Morgan and Corlett is, I believe, the most useful one that has been made on this subject. Making use of Polak's theoretical developments, they examined in great detail the difficulties involved in estimating E_s by the method of correlating price and quantity ratios. Whereas my criticism of the method has dealt primarily with the form of the estimate and the assumptions implicit in this form, Morgan and Corlett discussed briefly [**16,** pp. 312–13] a much broader range of difficulties, including the auto-correlation of error terms; the inadequacy of average prices due to the nonhomogeneity of the "commodities" considered; and correlation of the error term with the price ratio (such correlation must exist if one considers supply conditions to be included in the error term). Following their theoretical discussion, Morgan and Corlett presented the results of extensive calculations of estimates of E_s for goods entering international trade from alternative sources. For simple correlations between price ratios and quantity ratios, the estimates of E_s varied from $+4.87$ to -12.15, with 36 out of 47 estimates negative. There was considerable variation within groups of estimates which, on a priori reasoning concerning values of E_s, one would expect to be of comparable value. The apparent randomness of the estimates was decreased somewhat by the introduction of income and time as specific variables, and by the use of first differences.

X. SUMMARY

1. The concept of elasticity of substitution originally appeared in Hicks' *Theory of Wages* [**7,** pp. 117ff.], in connection with a discussion of the shares of capital and labor in the national income. The definition and discussion given there were cumbersome, and both soon became "obsolete."

2. Within a short time after the publication of *The Theory of Wages*, there was general agreement (among, e.g., Hicks, Lerner, Allen, and Pigou) that (i) E_s is a measure of the rate of change of the ratio of two inputs or two outputs, relative to the marginal rate of substitution between the two inputs or outputs, and (ii) E_s must be measured along an isoquant. Although most of the early discussion was in terms of production functions, it was clear that the concept could be extended, as

appropriate, to transformation and indifference curves, and to individuals, firms, industries, and the whole economy. If the system of isoquants is "homogeneous," the value of E_s is the same whether or not movement is confined to an isoquant. Because of the assumption of linear homogeneous production functions in most of the early discussions, the necessity of confining movement to an isoquant in the general case was sometimes glossed over. The practice of limiting discussion to the case of two factors introduced another ambiguity, as to whether E_s subsumed substitution of the two factors in question with other factors or confined substitution to shifts between the two factors under study.

3. A notable exception to the generalization that theoretical discussions of E_s were in terms of production functions was the 1934 *Economica* article of Hicks and Allen [**9**]. Throughout this lengthy article, Hicks and Allen were very specific in defining E_s in terms of movement along an *individual*'s indifference curve, a fact that has been overlooked by some later writers who have used the *Economica* article as the principal theoretical support for their empirical investigations. Hicks and Allen developed the implications of their definition of E_s at great length, and predicted that the concept would play a major role in demand theory. The conspicuous absence of E_s from Hicks' later *Value and Capital* [**8**] appears as a tacit refutation of that prediction.

4. In a number of articles in the last ten years (e.g., in Chang [**3**], Clawson [**4**], J. N. Morgan [**17**], and Tinbergen [**23**]), the concept of E_s has been used to measure demand interrelations empirically. In all of these studies, E_s has been defined as

$$\frac{d \log (x_1/x_2)}{d \log (p_1/p_2)}. \tag{6}$$

This definition is equivalent to the definition used by Allen, Lerner, and others,—namely,

$$\frac{d \log (x_1/x_2)}{d \log (dx_2/dx_1)} \tag{2}$$

—only under certain conditions. The conditions under which the two definitions are equivalent frequently have not been examined; some writers have almost completely ignored the distinction between the two definitions and have erroneously attributed the first definition, (6), to Hicks and Allen.

5. The conditions under which the two definitions are equivalent are unlikely to exist, as may be seen by examining (6) with the aid of an indifference map model and a general equilibrium model.

6. If E_s is defined *de nouveau* as (6), without regard for whether (6) is equivalent to (2), the significance of the concept is not clear. There has been virtually no theoretical discussion of this definition, its justification having rested implicitly on the theoretical discussions concerning definition (2).

7. The method commonly used to estimate E_s uses an assumption of linearity which appears to be fairly innocent and reasonable on the surface, but which implies a number of unlikely assumptions concerning the indifference maps or demand functions. The use of this method tends to create a certain consistency in the results which has given reassurances of dubious value to some investigators.

XI. CONCLUSIONS

My own conclusions from this historical and theoretical investigation can be stated as follows:

1. Elasticity of substitution, as originally defined by Allen and others for the purpose of measuring interrelated demands of individual consumers, is unambiguous when referred to individual indifference curves (the case of two goods). With care in extending the definition, it may be kept unambiguous for the case of more than two goods. Extension of the concept to community indifference curves, requiring definition of E_s in terms of "movement along a community indifference curve," seems to me a dubious procedure. I have not argued the rationale of community indifference maps in this article, although I have alluded to the fact that most theoretical discussions of E_s have referred to individual indifference curves or maps, whereas all of the empirical studies have referred to market responses.

2. Estimates of E_s derived from market data by the methods reviewed here probably give a very poor approximation to E_s defined by (2). The reason is that the conditions which are necessary for a good approximation are not likely to exist; at any rate, the empirical studies reviewed here have not adequately discussed the question of whether the necessary conditions do exist or have been accounted for.

3. E_s defined *de nouveau* by (6) is of limited value as a tool of analysis. It is a combination, with little intuitive appeal, of price elasticities, cross-elasticities, income elasticities, and supply elasticities.

In most of the studies which have been mentioned here, it would be possible to estimate the relationship of each of two rival goods to the prices; i.e., to estimate the functions (20a) and (20b). From these functions, one could read off demand elasticities, cross-elasticities, and income elasticities; and, if one wished to make additional assumptions or measurements of the kind suggested by (23a)–(23d), one could calculate E_s (which would not necessarily be constant in this model). In view of the great difficulties inherent in efforts to measure demands and cross-demands empirically, it is possible that the results of the latter method would not be considered satisfactory. But whatever defects exist in this method (e.g., the necessity of assuming (i) that quantities bought are equilibrium quantities corresponding to the price data, and (ii) that the relationships being measured do not change during the period covered by the data) are present—along with additional defects—in the method of estimation which has been criticized here.

My conclusion (3) above parallels conclusions reached by Pigou in a discussion in 1934 in *The Economic Journal* [**18**, pp. 240–41] of the usefulness of the concept of E_s in problems of production and distribution. In the simple case of two factors and a homogeneous production function, in which no information about supply functions is necessary to make E_s determinate, Pigou "found it more convenient to deal with the elasticities of partial productivity individually rather than with combinations of them." The parallel statement for demand problems would be: there is no gain in working with E_s, which is a combination of price elasticities and cross-elasticities, rather than working with the latter elasticities directly. In the more complex case of three or more factors, Pigou concluded that "for problems, in respect of which the relevant elasticity of substitution is a complex describing characteristics of one or more supply functions interwoven with characteristics of the *productivity function*, the concept, as so far defined, is of no service. If and when we have expressed the conditions of some event in a formula embodying the elasticity of substitution we have merely posed our problem in a new form; we have done nothing towards solving it." For a parallel statement relevant to the problems discussed in this paper, one need only substitute the phrase "preference pattern" for the phrase (which I have italicized) "productivity function."

University of California, Berkeley

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