THE DERIVED DEMAND FOR URBAN RESIDENTIAL LAND

RICHARD F. MUTH

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Introduction

An understanding of the demand for urban residential land is necessary for many problems of city structure and growth and for appraising the effects of certain governmental housing policies. In an earlier work I found it both analytically convenient and empirically fruitful to treat land and built structures as inputs into the production of a commodity called housing (1969). So viewed, the demand function for land is derived from the demand for housing and the supply of built structures. Its parameters depend upon the elasticity of housing demand, the elasticity of supply of structures, the relative importance of land, and the elasticity of substitution of land for structures in producing housing. While plausible inferences could be drawn about the values of the first three of these, I had only very sketchy information upon which to base an estimate of the elasticity of substitution. Furthermore, while the view of land described above lent itself well to interpreting many of my empirical findings, I was unable directly to subject it to an empirical test.

Many other writings on urban land problems, notably Alonso (1964), and Wingo (1961), have treated land as an argument of household utility functions. This treatment, though more general, yields weaker empirical implications. Indeed, in the absence of subsidiary hypotheses, the demand for urban residential land in this view need only obey very general restrictions on consumer demand functions. One widespread subsidiary hypothesis about the demand for urban residential land is that it has a higher income elasticity than the demand for structures (Hoover and Vernon, 1959, pp. 169 and 222). This last hypothesis is frequently held to account for the fact that higher income persons live at the lowest population densities and at greater distances from the central business districts of U.S. cities.

In this paper I make use of data that has recently become available to estimate the demand function for urban residential land. I will also test whether this estimated demand function appears to agree with the notion of land as an input into the production of housing. The data used, which refer to new single-family houses financed by Federal Housing Administration (FHA)-insured mortgages, were first published for 1966. They are the only set of data known to me for which the unit price of land is available along with necessary other data for estimating a demand function for urban residential land. The comparisons were limited to data for new houses for the obvious reason that long-run production relationships will be more closely realised for this group than for existing homes. So far as consumer demand for housing is concerned, though, there is no reason to omit existing homes. The second part of the article presents the derivation of demand relationships for urban

Professor Muth is a Professor at Stanford University.

residential land and related magnitudes which follow from the treatment of land as an input along with structures into the production of housing. After describing the data used in more detail, my empirical findings are presented in the next section. My major findings are:

(1) the coefficients of land price per square foot in the demand functions for land and for structures agree quite closely with the hypothesis that land and structures are inputs into the production of a commodity called housing,
(2) the elasticity of substitution of land for structures in producing housing appears to be about 0·5, and
(3) if anything, the income elasticity of the demand for residential land appears to be smaller than that for structures rather than the reverse.

Finally, in the last section, I will discuss some of the implications of my findings for problems of city structure and for evaluating certain effects of U.S. governmental housing programmes. First I consider the effect of differences in the elasticity of housing supply per unit of land in different parts of a city given the tendency for urban populations to become more decentralised as they grow. Using my previous estimate of the elasticity of substitution of land for structures (1969), I concluded that differences in the elasticity of housing supply could account for only about half of the tendency for population densities to decline at relatively slower rates in larger cities. However, using the value calculated here for the substitution elasticity, I find that differences in housing supply elasticities would produce greater decentralisation as city size increases than I actually observed. Next I show how the analysis of this paper can be used to determine the effects of a housing subsidy. I conclude that, even if the supply of residential land were perfectly inelastic, housing subsidies to lower-income families would have little effect upon housing prices. Finally, using a constant elasticity of substitution production function for housing and the parameter values found in this paper, I evaluate the effects on the cost per dwelling of building public housing on cleared slum sites as compared with others. My calculations indicate that a public housing unit built upon slum land costs about 60% more and that the resource cost of the U.S. public housing programme was about 20% greater during the 'fifties than it would have been if all units had been built on non-slum sites.

The Derivation of Demand

Assume that land, L, and built structures, N, are used as inputs into the production of the commodity called housing, Q. Q is measured in terms of capacity to provide a flow of housing services, which may vary from dwelling to dwelling, rather than as a number of dwelling units. The production function $Q = f(L, N)$ describes the maximum quantity of housing that can be produced using given inputs of land and structures with a given technology. Let r and n be the unit prices of land and structures, respectively, and p be the unit price of housing. Assuming that producers of housing are competitive in both product and factor markets, the following conditions must hold if these producers are to maximise their incomes:

$$\frac{\partial Q}{\partial L} = \frac{r}{p} \quad \text{and} \quad \frac{\partial Q}{\partial N} = \frac{n}{p}.$$

To determine the impact of changes in underlying conditions in the market for new houses, the production function and the two marginal conditions for income maximisation above can be differentiated. Letting the superscript * designate the logarithmic differential of the variable so designated (Muth, 1964),

$$Q^* - k_L L^* + k_N N^* = 0 \qquad (1)$$

$$-k_N L^* + k_N N^* + \sigma p^* = \sigma r^* \qquad (2)$$

$$k_L L^* - k_L N^* + \sigma p^* = \sigma n^*, \qquad (3)$$

where σ is the elasticity of substitution of land for structures in producing housing, k_L is the fraction of payments to land in the total value of housing (or site value relative to property value), and k_N is the share of payments for structures in the total value of the house. Here it is assumed that $f(L, N)$ is homogeneous of degree one in L and N so that $k_L + k_N = 1$.

A final condition for determining Q, L, N and p for given values of r and n is the demand function for housing. Where η_p is the price elasticity of housing demand, η_y the income elasticity, and y is income,

$$Q^* - \eta_p p^* = \eta_y y^*. \tag{4}$$

It should be stressed that the data used here refer to housing per family. Hence, there is no need to include a variable describing the existing housing stock, as would be needed if I were examining data on aggregate new building in some place such as a city in a given interval of time. Similarly, the estimated values of η_p and η_y from the per family data refer to stock demand rather than flow demand elasticities.

I solve equations (1)-(4) for rL, nN, and pQ along with p because the basic data used in the next section relate to expenditures for housing and land. The solutions are:

$$(rL)^* = \{1-(k_N\sigma - k_L\eta_p)\}r^* + k_N(\sigma+\eta_p)n^* + \eta_y y^* \tag{5}$$

$$(nN)^* = k_L(\sigma+\eta_p)r^* + \{1-(k_L\sigma - k_N\eta_p)\}n^* + \eta_y y^* \tag{6}$$

$$(pQ)^* = k_L(1+\eta_p)r^* + k_N(1+\eta_p)n^* + \eta_y y^* \tag{7}$$

$$p^* = k_L r^* + k_N n^*. \tag{8}$$

In (8) the unit price of housing is implicitly defined as an appropriately weighted average of the unit prices of land and structures. Unless the shares of land and structures are fixed, which occurs if and only if $\sigma = 1$, (8) holds only approximately for finite changes in input prices. Making use of (8), (7) reduces essentially to (4) and can be used to estimate the elasticities of housing demand.

From the definition of the elasticity of substitution,

$$\left(\frac{N}{L}\right)^* = \sigma\left(\frac{r}{n}\right)^*,$$

so

$$\left(\frac{rL}{nN}\right)^* = (1-\sigma)\left(\frac{r}{n}\right)^*. \tag{9}$$

Thus, σ may be estimated from a regression analysis of the ratio of site value to structure value on the ratio of land to structure prices. The resulting value of σ together with housing demand elasticities obtained as described in the preceding paragraph may be used along with sample averages of k_L and k_N to infer the coefficients of r^* and n^* in (5) and (6). Comparison of these inferred or expected values with actual regression estimates of them, as presented in Table 3 below, provides evidence for the consistency of the model presented here with real-world behaviour.

If $f(L, N)$ has a constant elasticity of substitution, then it may be written as

$$Q = [aL^{-c} + bN^{-c}]^{-1/c}, \tag{10}$$

where a, b, and c are constants and $\sigma = 1/(1+c)$. The marginal product of land is thus

$$Q_L = aQ^{c+1}L^{-(c+1)},$$

and similarly for structures. Equating the ratio of marginal products to the factor price ratio yields

$$\frac{b}{a}\left(\frac{L}{N}\right)^{c+1} = \frac{n}{r},$$

so

$$\ln\left(\frac{rL}{nN}\right) = -\sigma \ln\left(\frac{b}{a}\right) + (1-\sigma)\ln\left(\frac{r}{n}\right). \tag{11}$$

For a constant elasticity of substitution production function, then, (9) is not merely a first-order approximation but holds exactly. Equation (11) then enables one to estimate the ratio of the distribution parameters, (b/a), from known values of relative factor shares, relative factor prices and the elasticity of substitution.

Empirical Findings

With the exception of the construction cost index, all of the data used for the analysis in this section were obtained from tabulations of characteristics of new single-family houses financed by FHA-insured mortgages in 1966. The data are averages for all such houses for a given FHA housing area, essentially Standard Metropolitan Statistical Areas, which shall be somewhat loosely termed cities, in what follows. Unless otherwise noted, all variables are measured in dollars, and natural logs of all variables were used in the regression analysis.

Since the sample data refer to FHA-insured homes only, they may be subjected to some limitations. It seems quite likely that, since the FHA imposes a fixed insurance charge regardless of the terms of the loan, some owners who prefer to make higher down-payments and borrow for shorter periods than average, would find conventional mortgages less costly than FHA-insured mortgages. Such borrowers are likely to have higher incomes and greater accumulated non-human wealth than FHA borrowers. This could mean that among buyers of a given average income, those financing home purchases by FHA mortgages would tend to have higher costs of capital and thus purchase less housing. The higher the average income of a group of buyers, the smaller presumably would be the fraction choosing FHA financing and the higher the costs of capital of FHA borrowers relative to the average for the whole group. Closely related to the above is the effect of upper limits on the amount of an FHA-insured mortgage. As such limits become effective, with increases in average income the average expenditures on housing of FHA borrowers would increase less than the average for all borrowers. Thus, it is possible that the income elasticity of housing demand obtained from the FHA data would tend to understate the true value for all new homes. There is no particular reason, however, why this understatement should influence the estimated income elasticity of the demand for land more strongly than the demand for structures or vice-versa. Nor is there any reason to anticipate distorted responses to factor price differentials.

The value of new homes, pQ, is measured by the sales price of single-family owner-occupant homes. This is the amount stated in the sales agreement adjusted to exclude closing costs and non-real estate items which the sales agreement indicated were to be assumed by the seller. Expenditures on land, rL, are the FHA estimated market price of the site including street improvements, utilities, and landscaping. The data thus considerably overestimate, probably by a factor of at least two, the raw land value. In addition, there are probably fewer possibilities for substituting improved land for structures than raw land for non-land capital. Hence, the data probably yield an underestimate of the raw land for non-land capital substitution elasticity. Expenditures on structures, nN, were obtained by subtracting expenditures on land from the sales price of new homes.

The variable hereafter designated simply as net family income or income, y, is that obtained by deducting federal income taxes from what the FHA terms 'total effective income'.[1] The latter is defined as the 'FHA-estimated amount . . . that is likely to prevail during approximately the first third of the mortgage term' (FHA, 1967). No details of how the estimates are made were given. Based on my experience with FHA loan applications, however, I suspect that some earnings of the household head from other than his or her principal employment and some income of secondary earners are probably deducted from total family income. As such, some transitory income elements are no doubt eliminated from the income variable, but it is by no means certain that this variable closely approximates to normal or permanent income, upon which housing expenditures are likely to be based. Hence, the estimated income elasticities shown below are likely to be downward biased. In addition, because housing prices tend to be positively correlated with income, the estimated price elasticity is likely to be biased toward zero as well. There is no reason to suspect, though, that the demand for land will be differentially

[1] The monthly figure reported was multiplied by twelve to convert it to the more usual annual basis.

affected as compared with the demand for structures on this account. Neither would one expect distorted responses to land price changes relative to changes in structure prices.

The last variable obtained from the FHA tabulations is the average price of site in dollars per square foot, r. As was indicated earlier, the FHA data are unique in presenting such a measure. This measure showed considerable variation over the sample, ranging from as low as \$0·20 in some cities to around \$1 or more in the California cities included in the sample. The price of structures, n, was measured by the Boeckh index of brick residential structures, 1926-29 U.S. average = 100. This measure varied considerably less than land prices over the sample, ranging from only about 260 or a little more in some southern cities to about 375 in certain north-eastern ones. The limited variability of the Boeckh index relative to the variation of observed land costs among different cities may be partly responsible for its rather disappointing showing in some of the regression results that follow. Availability of the Boeckh index limited the sample to forty-seven U.S. cities.[2] Since the index is reported for large urban areas, in three cases the same Boeckh index value was used for two FHA housing areas in the same urban area—Chicago and Gary in the Chicago area, for example.

My estimates of the elasticity of substitution of land for structures are given in Table 1. (Throughout the tables, standard errors are shown below the coefficients.) The coefficient of the ratio of land price to construction cost in equation (A) implies an elasticity of substitution of almost exactly 0·5. This value is significantly smaller, both in the statistical sense as indicated by its standard error and in a practical sense as will be indicated later, than my previous estimate (1964, p. 229) based upon very scant data. Like the earlier estimate, the new one implies that with a rise in land prices relative to structure costs, less land in physical units is used relative to structures but more is spent on land relative to expenditures on structures. The factor price ratio alone explains about seven-tenths of the variation in the ratio of expenditure on factors.

In equation (B) the income of home buyers was also used as a determinant of relative expenditures on land and structures. Rather surprisingly the coefficient of income is strongly negative, though its inclusion has little effect upon the goodness-of-fit of the regression. Since I have no explanation for the strongly negative coefficient of income, I prefer simply to accept the null hypothesis that the income elasticities of land and structure demand are the same. Clearly, though, the income coefficient in equation (B) is inconsistent with the notion that the income elasticity of demand for land is higher than that for structures.

Equation (C) shows the results obtained when the number of occupant purchase cases in the FHA-sample in each area for 1966 is used as a weighting factor in the regression. The number of observations varied substantially as among the different cities—from about 70 to 1,000. In principle, the weighted regression results would be preferred, but in this instance I feel the standard regression gives better ones for reasons to be noted soon. Here, though, the weighted regression yields a factor price ratio coefficient which is almost identical to that in equation (A). As is to be expected, the goodness-of-fit of the weighted regression is noticeably better. Finally, in equation (D) land price was assumed to be endogenous, and the coefficients were estimated using the method of instrumental variables. This last comparison was made to examine the possibility of least-square bias due to simultaneous determination of land prices and expenditures on land. The results of equation (D) indicate such bias as truly negligible, for an almost identical estimate to that of equation (A) is obtained. Thus, regardless of the method of estimation, a land for structures substitution elasticity of very close to 0·5 is obtained. The results of adding income to either (C) or (D),

2 I would be happy to supply a list of cities included to any interested reader.

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which are not shown, were also virtually identical to those obtained by adding income to the standard regression.

Table 2 explores the results of relating per family expenditures on housing to measures of housing cost and to net family income. In equation (E) land price and the construction cost index are included as separate variables. In equation (F), however, the two separate factor prices were combined into a single housing price variable using equation (8) and the sample geometric mean of the ratio of site expenditures to sales price, which was about 0·18. Comparing equations (E) and (F), the latter's standard error of estimate is actually a little smaller, the reduction in the explained sum of squares being somewhat more than compensated for by the gain of one residual degree of freedom. Quite similar results were obtained using weighted regression and instrumental variable estimates, not all of which are shown in Table 2. These results are quite consistent, then, with the hypothesis that land and structures are inputs into the production of something called housing, and they provide no support for the notion that an independent demand for land exists.

For all three methods of estimation, the estimated price and income elasticities of demand for housing are substantially smaller than one numerically. (The coefficient of housing price is one plus the price elasticity of housing demand, since the dependent variable is expenditure on housing.) On the basis of other evidence, much of which is noted in my recent book *Cities and Housing* (1969), I would conclude that both elasticities are at least as great as unity in absolute value. The weighted regression estimates, equation (G), yield the smallest elasticities, each about 0·7 numerically, which is why I prefer the standard regression estimates to the weighted ones. The differences, though, are of little practical importance. The small values found here may result from the two factors noted previously in this section, namely the nature of the FHA sample and the failure completely to control for transitory income components in the income variable. The standard regression calculations for the earlier version of this paper (1968), however, yielded price and income elasticities which were insignificantly smaller than unity. These earlier calculations were based upon data for twenty-two cities only, mainly larger ones. Thus, the apparent downward bias in the price and income elasticity estimates here may result from the omission of city size or some factor correlated with it. All the other results of the earlier calculations, though, were essentially identical with the ones presented for the larger number of cities in Tables 1-3.

Table 1

RELATION OF RELATIVE INPUT SHARES[a] TO INPUT PRICE RATIO

Regression Equation	*Constant*	*Explanatory Variables* Land Price/Construction Cost Ratio	*Explanatory Variables* Net Family Income	*Standard Error of Estimate*	R^2
Standard Regression:					
Equation (A)	1·92	0·507	—	0·135	0·712
	(0·32)	(0·048)			
Equation (B)	5·65	0·552	−0·381	0·131	0·736
	(1·86)	(0·052)	(0·187)		
Weighted Regression[b]:					
Equation (C)	1·80	0·492	—	1·24	0·797
	(0·24)	(0·037)			
Instrumental Variables[c]:					
Equation (D)	1·91	0·506	—	0·135	—
	(0·47)	(0·070)			

[a] Dependent variable is the land/structure expenditure ratio.
[b] Using number of occupant purchase cases in sample to weight the observations.
[c] Using 1960 SMSA population, 1960 relative to 1950 SMSA population, and 1960 urbanised area median family income as instrumental variables and treating land price as endogenous.

Table 2

RELATION OF AVERAGE SALES PRICE[a] TO NET FAMILY INCOME AND HOUSING COSTS

Regression Equation	*Constant*	*Land Price*	*Construction Cost Index*	*Housing Price*	*Net Family Income*	*Standard Error of Estimate*	R^2
			Explanatory Variables				
Standard Regressions							
Equation (E)	2·77	0·0692	0·100	—	0·722	0·0758	0·689
	(1·21)	(0·0303)	(0·118)		(0·109)		
Equation (F)	1·96	—	—	0·238	0·748	0·0757	0·682
	(0·88)			(0·084)	(0·106)		
Weighted Regression[b]:							
Equation (G)	2·00	—	—	0·313	0·706	0·0726	0·764
	(0·78)			(0·088)	(0·106)		
Instrumental Variables[c]:							
Equation (H)	1·97	—	—	0·197	0·768	0·0760	—
	(0·88)			(0·092)	(0·108)		

[a] Dependent variable is average sales price of one-family owner occupant new homes.
[b, c] Same as Table 1.

Table 3

RELATION OF LAND AND STRUCTURE OUTLAYS TO INPUT PRICES AND INCOME

Regression Equation	*Constant*	*Land Price*	*Construction Cost Index*	*Net Family Income*	*Standard Error of Estimate*	R^2
			Explanatory Variables			
Land Outlay[a]:						
Standard Regression:						
Equation (I)	4·79	0·488	0·137	0·328	0·141	0·764
	(2·25)	(0·057)	(0·221)	(0·204)		
Expected Coefficient[c]	—	0·458	−0·220	—	—	—
Structure Outlay[b]:						
Standard Regression:						
Equation (J)	1·90	−0·0388	0·110	0·778	0·0695	0·614
	(1·11)	(0·0278)	(0·108)	(0·100)		
Expected Coefficient[c]	—	−0·0492	0·288	—	—	—

[a] Dependent variable is the average market price of site of one-family new homes.
[b] Dependent variable is sales price minus average market price of site.
[c] Assuming $\eta_p = -0{\cdot}76$, $\sigma = 0{\cdot}49$, $\bar{k}_L = 0{\cdot}18$.

The FHA data also permit one to estimate separate demand functions for land and for structures, which are shown in Table 3. Only standard regressions equations are presented there, but quite similar results were found using weighted or instrumental variable regressions. In addition to the standard regression equations in full, expected land price and construction cost index coefficients are shown. These were derived from equations (5) and (6) together with the price and substitution elasticity estimates derived from regression equations (F) and (A), respectively, and the mean share of land for the sample as a whole. These expected values, of course, were derived from the same body of data as the regression estimates in Table 3. However, they provide a useful check on the consistency of the unrestricted factor demand estimates with the model presented earlier. Examining Table 3 one sees that the coefficients of land price in both the land and structure demand regressions agree quite closely with those expected if land is an input into the production of housing. Neither estimated coefficient differs by as much as one standard error from its expected value. The coefficients of the construction cost index differ somewhat more, though both by less than two standard errors. Because of the limited variability of the construction cost index, however, the standard errors of the construction cost coefficients are relatively large, and little can be concluded from these coefficients. Finally, note that the estimated income elasticity of land demand is less than half that for structures. As was noted in connection with equation (B), the FHA data certainly provided no evidence that the demand for land rises relatively to that for structures as income increases.

Some Implications

Results such as were obtained earlier can be employed for a variety of purposes. Here only a few are sketched out. Of greatest interest are the implications of the elasticity of substitution of land for structures, here estimated at 0·5, for the tendency for cities to spread out as they grow and become richer. In an earlier work on the spatial pattern of residential land use in cities (1969), it was argued that, if the substitution elasticity of land for structures in producing housing is less than unity, the elasticity of housing supply per unit of land would be greater in the outer parts of cities than in their inner zones. Consequently, as the demand for housing in a particular city grows, the output of housing per unit of land and thus population density grows more rapidly in the outer parts of cities. In interpreting my empirical findings, I used a value of the substitution elasticity of 0·75, derived from very limited data. I concluded that differential responsiveness of housing output in different locations was not sufficient to account fully for the tendency for population densities to decline at relatively slower rates with distance from the CBD in larger cities (1969, pp. 317-318). The value found here for the substitution elasticity of land for structures, however, implies a still greater responsiveness of housing output to price changes in the outer parts of cities. This, in itself, would produce a numerically larger elasticity of the population density gradient with respect to city size than I previously observed.

In *Cities and Housing* I argued that the decline in population density with distance from the CBD results largely from the decline in the value of housing produced per unit of land. The relative change in the latter was shown to be equal to what I called the elasticity of value of housing produced per unit of land multiplied by the relative change in housing prices, or the price gradient (Ch. 4, pp. 54-55). If $\sigma = 0{\cdot}75$, the elasticity of value of housing output per square mile would be about $+15$, but for $\sigma = 0{\cdot}5$ it is only about $+10$.[3] The median density gradient observed for U.S. cities in 1950 was about 0·3; hence the price gradient would be about 3% per mile rather than 2%, and the residential land rental gradient about 0·6 (Ch. 4, p. 52). Likewise, substituting into my expression for the change in the elasticity of housing value produced (Ch. 4, p. 57) for $\sigma = 0{\cdot}5$ and $(r_k/r) = -0{\cdot}6$, one finds the elasticity to increase by about 2·8 per mile. The last means in particular that, for a city whose radius is six miles, the elasticity of housing value produced per square mile would be only about 1·5 at the centre but 18·5 at its edge.

With an elasticity of housing value per square mile of about $+10$, a 10% increase in population and hence housing demand would imply a rise in housing prices everywhere of about 8·3% (Ch. 4, p. 317). Consequently, the value of housing output would grow by about 1·25% at the centre and by about 15·4% at the edge of the hypothetical city described above. These values, in turn, imply the density gradient would decline by about 0·022 or about 7% with a 10% population increase. The differential output elasticities alone, then, imply a density gradient elasticity of about $-0{\cdot}7$, as compared with

[3] In the notation used here this elasticity is equal to $[1+(k_N/k_L)\sigma]$.

$-0{\cdot}27$ in my earlier calculations using $\sigma = 0{\cdot}75$ and an observed elasticity of $-0{\cdot}47$.

Thus, rather than having to search for additional decentralising forces associated with population growth, the problem now appears to be one of explaining why cities, as they grow, apparently don't decentralise as much as anticipated. A possible reason is one which originally led me to anticipate a positive relation between the density gradient and city size, namely that greater traffic congestion in larger cities means marginal transport costs are higher there. It may also be the case, as suggested in the third section, that, because the measure of land used here includes some non-land capital investment in streets, utilities, and landscaping, the substitution elasticity estimate of 0·5 underestimates the one between raw land and non-land capital. If so, the calculated value of the density gradient elasticity with respect to city size which was just presented would be too large numerically. I also note that this revised substitution elasticity estimate implies that the differential output response to increased housing demand and an income elasticity of housing demand of only +1 are sufficient to account for my observed elasticity of the density gradient with respect to income, which was $-0{\cdot}63$.

The system of equations (5) through (8) can be used in evaluating the effects of a variety of influences upon housing markets. Of course, to complete the model one would need supply schedules of land and structures to the housing industry. In two important special cases, however, the factor supply schedules may be neglected. First, for studying the immediate impact of changes, the elasticities of supply of land and of structures are zero. In this case, $L^* = N^* = 0$ (hence by (1), $Q^* = 0$), so (5), (6) and (8) yield r^*, n^* and p^* as functions of the exogenous changes. Second, and of much greater interest, in the long run, it appears that the supply of structures is highly elastic or n is approximately fixed (Muth, 1969, pp. 57-58). The supply of land to the housing industry undoubtedly has a positive elasticity, both because of conversion of urban land from non-residential uses and because of growth of the city. Bounds on other variables may be obtained, though, by assuming $L^* = 0$, so that the model determines r^*, N^*, Q^* and p^*. Indeed, because land's share is relatively small, the results of calculations such as are shown two paragraphs below are not very sensitive to the elasticity of residential land supply.

For working through some of the implications of the model presented in the second section, it is useful to substitute approximate numerical estimates of the parameters into equations (5)-(8). I will use the value of the substitution elasticity of land for structures obtained here, namely 0·5. The mean value of the share of land in the value of houses for the cities used here was 0·18, as noted earlier. Since the measure of land includes non-land capital in the form of streets, utilities, and landscaping, which could easily be as much as raw land value itself, 0·1 seems a more reasonable value for K_L. I mentioned in the third section that, in my judgment, the price and income elasticities obtained here are too low, and I will assume instead that both are equal to unity. Making these substitutions into (5)-(8) one finds:

$$L^* = -0{\cdot}55\, r^* - 0{\cdot}45\, n^* + y^* \qquad (12)$$

$$N^* = -0{\cdot}05\, r^* - 0{\cdot}95\, n^* + y^* \qquad (13)$$

$$p^* = 0{\cdot}1\, r^* + 0{\cdot}9\, n^* \qquad (14)$$

$$Q^* = -p^* + y^*. \qquad (15)$$

As an example, consider the impact of a 10% housing price or income subsidy to, say, the lowest 10% of families in the income distribution. Many persons have argued that the impact of such subsidies would tend to be dissipated on higher site value or 'profits' to landlords (Grigsby, 1963, pp. 303-304). This is the case, of course, when rentals have adjusted to the higher demand with a fixed quantity of housing services produced. In the long run, however, it is unlikely to be the case even with a fixed quantity of residential land provided the supply of structures is highly elastic, because land's share in the production of housing is small. With

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either subsidy, the quantity of housing demanded, by (15), increases by 10% for the lowest income 10% of the families or by less than 1% in the aggregate. If the production function for housing is homogenous of degree one, the quantity of land demanded at given factor prices likewise rises by less than 1%. If the quantity of land available to the housing industry were fixed, then by (12) $0{\cdot}01 - 0{\cdot}55\, r^* = 0$ or $r^* \cong 0{\cdot}018$. The increase in land prices of less than 2% would, in turn, by (14) result in a housing price increase of less than 0·2%. (With a price subsidy, of course, housing prices would have fallen to the recipients by more than 9·8%.) In either case, however, little of the subsidy would be dissipated through higher site costs.

As a final example of the uses of which the analysis of this paper might be put, consider the U.S. federal public housing programme. In his excellent evaluation of the public housing programme, Edgar Olsen noted what is an apparent substantial waste in its production of dwellings, amounting to around 40% or more of the average expenditure per dwelling during the 'fifties (Olsen, 1968, pp. 70-71).[4] He pointed to the fact that some public housing units are built on cleared slum land, whose price per square foot is much higher than non-slum land, as one source of waste. His estimate of the waste due to building projects on slum land, however, is simply the additional expenditure for land of those units built on slum land and takes no account of additional expenditures on construction as structures are substituted for land.[5]

Properly to evaluate the resource cost of building on slum land, consider the constant elasticity of substitution production function for housing, equation (10). This may be rewritten as

$$L = a^{1/c} Q \left[1 + \frac{b}{a} \left(\frac{N}{L} \right)^{-c} \right]^{1/c}. \tag{16}$$

By equating the ratio of marginal physical products to the ratio of factor prices,

$$\frac{N}{L} = \left(\frac{b}{a} \times \frac{r}{n} \right)^{1/(c+1)}, \tag{17}$$

so substituting (17) into (16) yields

$$\frac{L}{Q} = a^{(\sigma/1-\sigma)} \left[1 + \left(\frac{b}{a} \right)^{\sigma} \left(\frac{r}{n} \right)^{\sigma - 1} \right]^{(\sigma/1-\sigma)}. \tag{18}$$

Similarly,

$$\frac{N}{Q} = b^{(\sigma/1-\sigma)} \left[1 + \left(\frac{b}{a} \right)^{-\sigma} \left(\frac{r}{n} \right)^{-(\sigma - 1)} \right]^{(\sigma/1-\sigma)}. \tag{19}$$

By making these substitutions, of course, I am assuming that public housing units are built at the minimum resource cost for the prevailing ratio of factor prices. Knowing the amount of land and structure per unit of housing as a function of the factor price ratio, the resource cost per unit of housing may readily be calculated.

Making such calculations, though, requires numerical values of the parameters of the production function. One may normalise or define units by taking $a = 1$ and $n = \$1$. From the FHA data used for the calculations in the third section, the geometric mean for the forty-seven cities used of (rL/nN) was about 0·22; that of r was \$0·39 per square foot. Substituting into (11) one finds for $\sigma = 0{\cdot}5$, $(b/a) \cong 7{\cdot}7$ and $(b/a)^{\sigma} \cong 2{\cdot}8$. Hence, (18) and (19) become

$$\frac{L}{Q} = \left[1 + 2{\cdot}8 \times \left(\frac{r}{n} \right)^{-0{\cdot}5} \right] \tag{20}$$

$$\frac{N}{Q} = 7{\cdot}7 \left[1 + 2{\cdot}8^{-1} \times \left(\frac{r}{n} \right)^{0{\cdot}5} \right]. \tag{21}$$

As shown in Table 4, the average square foot prices paid for slum and non-slum land were \$1·12 and 0·091, respectively (Table 4). Substituting these values in turn into (20) and (21) and

[4] Proper measurement of the extent of this waste turns out to be a rather complicated problem. As of now I am inclined to value it higher than Olsen does, but I cannot go into the details of my evaluation here.

[5] Indeed, if the elasticity of substitution of land for structures were large enough, expenditures for land per unit of housing might decline with a rise in land price, as seen by setting $\eta_p = 0$ in equation (5). Building on slum land, though, would still involve a higher resource cost per dwelling unit.

Table 4

SELECTED CHARACTERISTICS OF PUBLIC HOUSING UNITS BUILT IN THE U.S., 1950-59, UPON SLUM *vs.* NON-SLUM LAND

Built upon	*Land Price per sq. ft.* ($s)	*Expenditures on Land per Dwelling Unit* ($s)	*Dwelling Units per Acre*
Slum land	1·116	1,945	25·2
Non-slum land	0·091	350	11·3

Source: U.S. Housing and Home Finance Agency Fourteenth Annual Report 1960. Washington D.C.: U.S. Government Printing Office. 1961.

then evaluating the resource cost per unit of housing, one finds it to be $14·7 per unit of housing built upon slum land as compared with only about $9·5 for that built upon non-slum land. The resource cost per unit of public housing would thus be about 55% greater for units built upon cleared slum land. Additional expenditure on structures is about 40% of the excess cost of building on slum land, according to these calculations. Since about 36·5% of all units built during the 'fifties were built upon slum land, the resource cost of the public housing programme was about 20% greater than it would have been had all units been built upon non-slum land. Since demolition of slum housing to provide sites for public housing not only leads to displacement of, but also makes housing more costly to lower-income persons, at least in the short run, there is little empirically justifiable rationale for such demolitions.[6]

A comparison of certain other actual characteristics of public housing units built on slum *vs.* non-slum land are shown in Table 4 along with ratios calculated using equations (20) and (21). The agreement between the actual and calculated ratios is close enough for me to put some confidence in my calculations in the paragraph above but divergent enough to make me suspect some real source of discrepancy exists. In making comparisons it is in effect assumed that public housing units built on slum and non-slum land contain the same number of units of housing per dwelling. While such by no means need be the case, it is not at all clear what kind of differences exist and what effects they might have on the calculations. Second, the production function parameter estimates used in deriving (20) and (21) were based upon data for single-family houses only. Many, if not most, public housing units are in multi-family structures, however. I know of no data which could be used to assess the differences in production possibilities for single- *vs.* multi-family housing directly. In an earlier study (Muth, 1969, pp. 314-316), though, I examined some of the implications of different production functions for different types of structures. The evidence, admittedly indirect, seems clearly to favour the hypothesis that different types of structures reflect different ratios of structure to land combined in accordance with a single production function.

Finally, Table 5 indicates that the intensity of land use on slum sites is lower relative to non-slum sites than anticipated. This observation suggests a third possibility. In building public housing on slum sites, its producers may not have substituted structures for land, or built housing as densely, as private producers of housing would have given the higher relative price of land. Indeed, public producers may have felt that to do

[6] Studies by former students of mine suggest to me that the external effects of public housing and similar redevelopments, while somewhat uncertain, are probably small. See in particular Nourse (1963).

Table 5

ACTUAL AND CALCULATED[a] RATIOS OF SELECTED CHARACTERISTICS OF PUBLIC HOUSING UNITS BUILT UPON SLUM *vs.* NON-SLUM LAND

Characteristic	*Actual*[b]	*Calculated*[a]
Cost per dwelling	—	1·6
Dwellings per unit of land	2·2	2·8
Expenditures on land per dwelling	5·6	4·5

[a] Using equations (20) and (21) and assuming comparability in public housing units built upon slum and non-slum land.
[b] From Table 4.

so would have produced adverse external effects, either for project residents themselves or upon users of surrounding properties. The calculations made above, while taking into account any lesser desirability of denser housing from its inhabitants' viewpoint, fail to make any allowance for possible external effects. Calculations similar to those made two paragraphs above, however, suggest that the failure of public housing to be built as densely on slum sites as anticipated would have relatively little effect upon my estimate of the relative cost of building public housing on cleared slum sites.[7]

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[7] Using (16) one can calculate (N/L) on slum sites as a function of that on non-slum sites, assuming the ratio of housing per unit of land on the two sites is 2·2. Assuming further that housing is produced at minimum cost on non-slum sites determines (N/L) on the former and hence on slum sites. Dividing the ratio of cost per unit of land by that of housing per unit of land on slum relative to non-slum sites then yields a relative cost on slum sites of 1·58, as compared with 1·55 if production on slum sites is also at minimum cost.