**Abstract**

We use Monte Carlo simulations to analyze the effectiveness of costless information shocks in reducing one source of limits to arbitrage in the market during asset mispricings and bubbles. We build on the synchronization risk and delayed arbitrage model introduced by Abreu and Brunnermeier (2002), in which traders uniformly and sequentially become aware of mispricings but try to avoid "jumping the gun" due to holding costs. The original model assumes that each trader strategizes his trading decision based on a single piece of private information. In our model, from the time of the mispricing, traders receive new information from a common prior distribution regarding when the mispricing occurred. We show that as shocks become more frequent, the duration of the mispricing becomes shorter. Moreover, we see that as the frequency of information shocks increases, the marginal gain in market efficiency decreases. We also computationally verify the results in Abreu and Brunnermeier (2002).

# Contents

# 1 Introduction

## 1.1 Motivation

Prolonged asset mispricings are interesting phenomena in financial markets. One potential cause is noise or behavioral traders in the market who prevent immediate price correction. However, this fails to explain the situation when sufficient rational arbitrageurs become aware of the mispricing to correct it; in such occasions, the mispricing sometimes still persists.

An intuition for why these arbitrageurs are unable to coordinate an attack on the over- or under-valued asset is the lack of common knowledge. Suppose each trader knows some private information, common knowledge implies that he knows about the information of every other trader and understands other traders know this as well. The lack of common knowledge refers to the fact that information is not transparent to everyone in the market. More specifically, in the realm of asset bubbles, each trader may only know about his own information and is aware that others only know about their own information.

Although each rational trader may know his own view of the market, he may be hesitant to change his position since he does not have information on the views of other rational traders. Since any one trader alone would lack enough market power to move asset prices, each would be wary of changing their position too soon.

For instance, if some trader becomes aware of the mispricing at a specific time, he would not immediately trade unless he knows that a sufficient number of other traders already have traded the same way. Otherwise, should the price correction fail to occur at the instance after he acts, this trader would have to suffer some sort of holding cost while waiting for enough fellow arbitrageurs to attack the bubble. This kind of friction prompts even rational traders to delay their actions to maximize profit, preventing themselves from "jumping the gun" and suffering unnecessary holding costs.

Abreu and Brunnermeier (2002) describe the lack of common knowledge using the term "synchronization risk." Specificially, synchronization risk refers to the inability of arbitrageurs to perfectly time the activity of other arbitrageurs. If no single trader can significantly move the market, he cannot single handedly correct a mispricing. So even if enough rational traders become aware of a mispricing, they may not attack due to uncertainty about the information available to other rational traders.

In their original paper on synchronization risk, Abreu and Brunnermeier (2002) solve for a symmetric trigger strategy equilibrium in a market where rational traders sequentially learn about the mispricing. Here, by symmetric trigger strategy, we refer to the idea that all arbitrageurs would delay their trade by some fixed amount of time, regardless of when they become aware of the mispricing. The strategy provides an equilibrium if, given all other rational traders delay their purchase/sale by this specified period of time, any one trader would keep with the same strategy to maximize his expected payoff. The paper focuses on what happens when each trader knows nothing but the underlying distribution of the initial mispricing time, the holding cost, and the price and value of the asset.

For our work, we would like to investigate what happens when we relax the barrier on information travel amongst traders. In other words, we allow traders to know more about the market over time and learn information available to other traders. We can then assess the connection between bubbles and degree of common knowledge. Intuitively, we expect that as the market becomes closer to a complete-information world, it achieves higher efficiency. We would expect the duration of bubbles to decrease as the degree of common knowledge increases.

## 1.2 Questions Addressed

There are two main purposes for this thesis. Foremost, we would like to investigate how information dispersion and common knowledge affect the duration of mispricings and the actions of rational traders. Abreu and Brunnermeier (2002) solve the original case and find a symmetric equilibrium strategy where each arbitrageur delays his trade by the same lag time. They prove that in their basic model, "the mispricing is never common knowledge among a positive mass of arbitrageurs." One characteristic of the market that prevents the existence of common knowledge is that traders sequentially become aware of the mispricing and do not learn about the market views of other traders.

In our model, we incorporate a way for traders to gain information as time passes. To bypass the concern of credibility, we may simply let there be a common universe of possible mispricing information available to the traders. Each trader begins with one piece of information about the mispricing and learns more as he encounters random “information shocks”, each implying an estimate for the time at which the fundamental value of an asset would deviate from its market price. Since all arbitrageurs will obtain increasingly accurate estimates of the mispricing as time passes, synchronization risk decreases when the views of traders become more condensed together. In the limit case when such “information shocks” are very common (i.e. all traders immediately know about the mispricing when it occurs), the price correction occurs instantaneously as if some credible, universal signal immediately reaches all rational traders. We would like to illustrate the effectiveness of information flow in diminishing synchronization risk, as well as derive a new equilibrium strategy that depends on the frequency of these “information shocks”. Naturally, the higher the frequency of new information, the more efficient we expect the market to be in correcting mispricings.

Secondly, we want to step away from abstraction and relax the assumption of an infinite number of traders, which has allowed Abreu and Brunnermeier (2002) to derive an elegant analytical solution. We show that when there exists a large number of traders in the market the original theoretical equilibrium approximates the symmetric equilibrium strategy in the finite case. We also look at how the number of market participants (in smaller magnitudes) affects the equilibrium strategy. Additionally, since the inclusion of information flow makes the model more complicated, we will use Monte Carlo simulations to computationally solve for the new equilibrium strategy and duration of price correction.

# 2 Related Literature

Pricing bubbles have been prevalent in financial history. A bubble is associated with extraordinary deviations in asset prices different from its fundamental value, followed by a drastic price correction. Many historical examples depict a rapid increase in asset prices which precede a crash, and the concept of a “negative bubble”—opposite phenomenon to that of typical historical examples—has also been discussed in literature (Yan, Woodard and Sornette 2010). As early as the 1630s, one of the first recorded bubbles occurred when rare species of tulip bulbs became items of luxury and status during the Dutch tulip mania. Later cases include the famous British South Sea Bubble in 1720 and the Roaring Twenties Bubble (Brunnermeier 2009). During the late 1990s, the relative price of Internet stocks greatly surpassed that of the entire market (Ofek and Richardson 2003). The recent late 2000s housing bubble is another case when we see assets being mispriced over a prolonged period of time, prior to returning to lower price levels.

The current literature concerning bubbles and asset mispricings broadly fall into four categories based on the investors' information, beliefs, and the degree of common knowledge in the market (Brunnermeier 2009). In the simplest case, all investors are rational and share the same information. Blanchard and Watson (1982) introduce a model in which asset prices can deviate from their fundamental values even when investors exhibit rational behavior and expectations. During each time period, the bubble may increase by a factor of $(1 + r)/\pi$ with probability $\pi$, or crash otherwise; in other words, there is an expected bubble component built into stock prices. Investors in rational bubbles would continue to hold onto their asset as long as they expect continuing growth.

Another branch of bubble models relaxes the assumption that all traders share identical information; instead, each player in the market receives different information from a common prior distribution of possible information. In this type of model, it is not necessary that an investor believes the asset he is holding will continue to grow in value. Instead, he would hold the asset as long as he believes there would be a net profit through reselling to some "greater fool" (Kindleberger 2005). Additionally, fund managers—on the premise that their clients are relatively uninformed—would sometimes buy overpriced assets in order to signal that they have private information (Allen and Gorton 1993).

Heterogeneous beliefs and short-sale constraints offer another explanation for prolonged mispricings. A common example of this is when market participants are overconfident or overoptimistic about their own convictions. Miller (1977) argues that even in a market that has many well-informed investors, a small group of overoptimistic traders may be able to overly bid up a stock if the well-informed investors have short-sale constraints. Scheinkman and Xiong (2004) construct static and dynamic models to describe how asset pricing may be affected by short-sale constraints and heterogeneous beliefs and suggest these disparities in beliefs may be due to overconfidence.

Short-sale constraints also relate to the last category of mispricings due to limited arbitrage. Lamont and Thaler (2003) present the pricing anomaly involving 3Com and Palm. Even when investors knew that Palm was overpriced and that 3Com was underpriced, the authors note that short-sale constraints caused the mispricing to continue for months. In limited arbitrage models, market players are broken into a dichotomy of sophisticated, well-informed investors (i.e. hedge fund managers) and behavioral, and sometimes uninformed investors (i.e. individual investors). Contrary to the notion of "efficient markets", rational investors may not always be able to fully take advantage of arbitrage opportunities. For instance, noise traders may intensify the mispricing between price correction occur; fund managers, who need to show positive performance to their clients to prevent fund outflow, may be hesitant to go against the mispricing (Shleifer and Vishny 1997).

The idea of synchronization risk also falls into the category of limited arbitrage (Abreu and Brunnermeier 2002, 2003). Even when traders know about mispricings, they are unable to guarantee whether or not other traders have the same knowledge if everyone becomes aware of the bubble at different times. As a result, rational traders may sometimes choose to ride the bubble if they perceive the bubble to grow in the future. Brunnermeier and Nagel (2004) show that even sophisticated and well-informed hedge funds chose to ride the late 1990s technology bubble instead of attacking it.

# 3 The Theoretical Model

## 3.1 Overview of Abreu and Brunnermeier's Model (2002)

### 3.1.1 Initial Setup

We want to introduce the original version of Abreu and Brunnermeier's (2002) "synchronization" risk model. In this initial setup, rational traders in the market uniformly and sequentially become aware of an asset mispricing (i.e. a possible arbitrage opportunity). Based on this knowledge, they try to strategize how to change their trading position so as to maximize their overall payoff, taking into account that trading too early leads to holding costs. The mispricing becomes corrected as soon as the cumulative trading pressure exceeds the market's ability to offset order imbalance. We outline the associated parameters and functions to represent the mispriced asset as follows.

Consider the price $p_t = e^{rt}$ of a risky asset, where time $t \in [0, +\infty)$ is continuous and unbounded and $r$ is the continuously compounded growth rate. At time $t = t_0$, the price $p_t$ of the risky asset deviates from its underlying value $v_t$. The initial mispricing time $t_0$ is a random variable from an exponential distribution with some rate $\lambda$. Prior to time $t_0$, the asset's fundamental value $v_t$ matches its price $p_t$. Afterwards, the fundamental value of the asset changes to either $v_t = (1+\beta)e^{rt}$ or $v_t = (1-\beta)e^{rt}$ with equal probabilities $\frac{1}{2}$. Here, $\beta$ is the proportional amount of the mispricing (the asset becomes either undervalued or overvalued, respectively), but the growth rate of the asset remains the same at $r$.

Post-mispricing, rational arbitrageurs sequentially become aware of the event in a uniform fashion within the time interval $[t_0, t_0 + \eta]$. In particular, we denote that trader $i$ knows at $t_i$ that a mispricing has occurred, where $t_i$ is defined to be a random variable from a uniform distribution over the interval $[t_0, t_0 + \eta]$.

Additionally, $x_i \in [-1, 1]$ denotes arbitrageur $i$'s trading position (negative means short, and positive means long). Abreu and Brunnermeier (2002) indicate that in a trigger strategy, rational arbitrageurs would exhaust the positional constraints. In other words, when he decides to buy/sell trader $i$ would shift his entire portfolio to either $x_i = 1$ or $x_i = -1$ respectively.

The holding cost per period is defined as $cp_t|x_i|$, where $c > 0$ and sufficiently large so that arbitrageurs have incentive to not immediately trade upon knowing about the mispricing. During any infinitesimal time interval $\Delta$, a holding cost of $cp_t|x_i|\Delta$ would be incurred. More specifically, prior to price correction, the cost incurred by trader $i$'s trading position is of the following structure.

$$\text{holding cost} = \begin{cases} 0 & \text{if arbitraguer } i \text{ has not yet traded} \\ cp_t\Delta & \text{otherwise (i.e. } x_i = \pm 1) \end{cases} \tag{1}$$

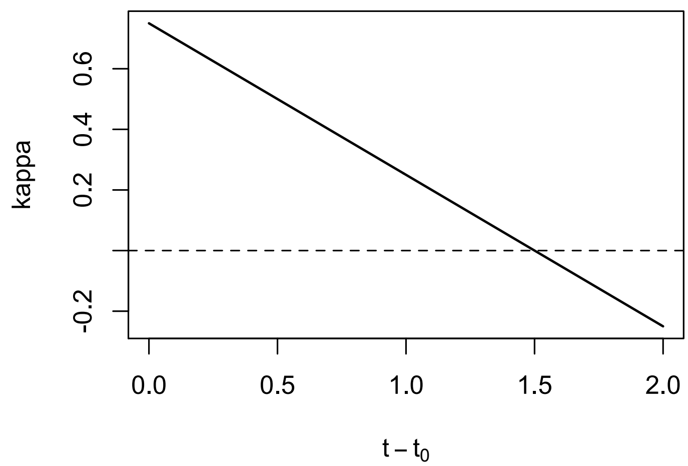


Figure 1: $\kappa$ as a Function of Time Past Initial Mispricing $t - t_0$

Lastly, we define the market's ability to offset order imbalance by a monotonically decreasing function $\kappa(t-t_0)$: $[0, +\infty] \rightarrow [0, 1]$ with $\kappa(\bar{\tau}) = 0$, i.e. the bubble will always collapse at the latest $\bar{\tau}$ past initial mispricing $t_0$. It is worth noting that this essentially means that if price correction occurs at $t$, a proportion $\kappa(t - t_0)$ of the rational traders must have already traded by then. To assist tractibility, Abreu and Brunnermeier (2002) choose the linear function

$$\kappa(t - t_0) = \kappa_0[1 - (1/\bar{\tau})(t - t_0)] \tag{2}$$

with $\bar{\tau}$ suffuciently large so that $\kappa$ declines at a relatively slow pace. By construction, at $t = t_0 + \bar{\tau}$, $\kappa$ becomes zero and the mispricing is corrrected regardless of trading pressure.

Figure 1 shows $\kappa(\cdot)$, with $\kappa_0 = 0.75$ and $\bar{\tau} = 1.5$. We see that $\kappa$ declines linearly with respect to $t - t_0$ and reaches zero at $\bar{\tau}$. This guarantees that the price correction occurs between $t_0$ and $t_0 + 1.5$.

### 3.1.2 Symmetric Trigger Strategy Nash Equilibrium

Abreu and Brunnermeier (2002) show that there exists a symmetric equilibrium defined by a trigger strategy, where each rational arbitrageur delays his/her trade for a fixed period of time $\tau^*$. In other words, trader $i$ would change his position at $t = t_i + \tau^*$. They solve for this symmetric trigger strategy equilibrium by analyzing the trading decision of each rational player, who acts based on his perceived hazard rate function of when the price correction would occur.

For any given random variable $X$ with a differentiable cumulative distribution function $F(x) = \Pr\{X \leq x\}$ and probability density function $f(x) = F'(x)$, the hazard rate function of $X$ is defined as

$$\bar{h}(x) = \frac{\frac{d}{dt}\Pr\{X \leq x\}}{\Pr\{X \geq x\}} = \frac{f(x)}{1 - F(x)} \tag{3}$$

For some hazard rate function $h(t|t_i)$, during any infinitesimal time interval $\Delta$, the expected marginal revenue from trading would then be $\beta p_t \cdot h(t|t_i)\Delta + 0 \cdot [1 - h(t|t_i)]\Delta = \beta p_t h(t|t_i)\Delta$; the expected marginal holding cost per period $\Delta$ is $0 \cdot h(t|t_i)\Delta + cp_t\Delta \cdot [1 - h(t|t_i)\Delta] = cp_t\Delta[1 - h(t|t_i)\Delta]$.

Each trader $i$ has to make a decision during each infinitesimal time step on whether or not to attack the bubble now or wait until later to do so, characteristic of trigger strategies. Once arbitrageur $i$ shifts his position from neutral (i.e. change $x_i$ from 0 to $\pm 1$), he would no longer make any changes. Therefore, one would only trade if the expected benefit exceeds the expected cost of having an undiversified portfolio.

In particular, trader $i$ will shift his position when $\beta p_t h(t|t_i)\Delta = cp_t\Delta[1 - h(t|t_i)\Delta]$, as $h(t|t_i)$ is an increasing function, $p_t$ appears on both sides of the equation, and everything else remains constant with respect to time. In the limit case where $\Delta \to 0$, we have that trade occurs when

$$\beta h(t|t_i) = c \tag{4}$$

Abreu and Brunnermeier (2002) derive the hazard rate function $h(t|t_i)$ for trader $i$'s perception of when the bubble would burst using the following set of reasoning. Foremost, since they are solving for a symmetric trigger strategy equilibrium, they consider what happens to the mispricing when each rational trader $i$ delays his trade by some $\tau'$ not necessarily equal to $\tau^*$. Since traders sequentially and uniformly become aware of the mispricing, we can derive the total time between initial mispricing and price correction to be the implicit function $\phi(\tau')$, where

$$\phi(\tau') = \tau' + \eta\kappa(\phi(\tau')) \tag{5}$$

The mispricing is corrected at time $t = t_0 + \tau' + \eta\kappa(\phi(\tau'))$, because everyone delays by $\tau'$ and $\kappa(\phi(\tau'))$ of the investors need to sell/buy to allow price correction. Since $\kappa(t - t_0)$ is a linear function (see Equation 2), we can rearrange the above equation and find a closed form for $\phi(\tau')$.

$$\phi(\tau') = \frac{\bar{\tau}(\tau' + \eta\kappa_0)}{\bar{\tau} + \eta\kappa_0} \tag{6}$$

The hazard rate function $h(t|t_i)$ is a piecewise function that depends on the value of $t$ in relation with $t_i$. Now, suppose every trader knows that everyone else is using the trigger strategy $\tau'$. Trader $t_i$ can deduce that the initial mispricing occurred within $[t_i - \eta, t_i]$. Thus, he has the belief that price correction will happen in $[t_i - \eta + \phi(\tau'), t_i + \phi(\tau')]$. If $t < t_i - \eta + \phi(\tau')$, the hazard rate is zero since $t$ lies out of trader $i$'s belief about the price correction. Otherwise if $t \geq t_i - \eta + \phi(\tau')$, trader $i$'s belief updates to having the price correction occur between $t$ and $t_i + \phi(\tau')$. Given that the initial arrival time of the mispricing $t_0$ is exponential with rate $\lambda$ and the traders are uniformly informed of the mispricing, Abreu and Brunnermeier (2002) derive the following hazard rate function.

$$h(t|[T_{i,1}, T_{i,2}]) = \begin{cases} 0 & \text{for } t < t_i - \eta + \phi(\tau') \\ \frac{\lambda}{1 - \exp\{-\lambda(t_i + \phi(\tau') - t)\}} & \text{for } t \geq t_i - \eta + \phi(\tau') \end{cases} \tag{7}$$

From the implications of Equation 4, the symmetric equilibrium trigger strategy $\tau^*$ should satisfy $\beta h(t_i + \tau^*|t_i) = c$. Combining this equation with the closed form formula for $\phi(\tau^*)$ (see Equation 6), we arrive at the equilibrium symmetric trigger strategy $\tau^* = \bar{\tau} - ((\bar{\tau} + \eta\kappa_0)/\lambda\eta\kappa_0)\ln[c/(c - \lambda\beta)] > 0$. Plugging this value back into the function $\phi(\tau^*)$ allows us to calculate the time of price correction in this equilibrium to be $t_0+\bar{\tau}(1-(1/\lambda\eta\kappa_0))\ln[c/(c-\lambda\beta)]$.

## 3.2 Information Shocks with Rate $\gamma$

We add more complexity to the original Abreu and Brunnermeier (2002) model by allowing traders to gain information beyond their initial knowledge. In particular, during each infinitesimal segment of time, any trader can receive such information shocks drawn from a pool of all possible information available to all traders. To make our problem more interesting and similar to real-world scenarios, these information shocks would be modeled as rare events. Rational traders update their plan of action over time based on the new information they receive.

For representing information shocks, we introduce an additional parameter $\gamma$ known to all rational arbitrageurs. From time $t_0$ to $t_0+\eta$, each trader may receive new information about the initial mispricing time uniformly drawn from $[t_0, t_0+\eta]$—regardless of whether he has already become aware of the mispricing (i.e. whether $t > t_i$)—at a Poisson rate $\gamma$.

Moreover, we denote the time-dependent random variable $I_{i,\min}(t)$, as the most accurate information about the initial mispricing time $t_0$ available to arbitrageur $i$ at time $t$. (To clarify, $I_{i,\min}(t)$ is a number that comes from a probability distribution dependent on time $t$.) In other words, $I_{i,\min}(t)$ is the minimum of all such received information. Each trader, at the earliest, would trade at this most accurate information $I_{i,\min}(t)$.

For clarity, we introduce an additional random variable $I_{i,\min}(t|N_{i,t})$ the value of $I_{i,\min}$ given that trader $i$ received $N_{i,t}$ pieces of information by time $t$. (Note here that trader $i$ starts receiving information as soon as the actual mispricing occurs, but his initial signal is $t_i$.) Specifically, we have following expression for $I_{i,\min}(t|N_{i,t})$.

$$I_{i,\min}(t|N_{i,t}) = \min\{t_i, t_{i,1}, t_{i,2}, \cdots, t_{i,N_{i,t}}\} \tag{8}$$

where each $t_{i,j}$ (for $j = 1, 2, \cdots, N_{i,t}$) stands for different mispricing information received by time $t$.

Note here that $N_{i,t}$ is actually the random variable associated with the Poisson process with rate $\gamma$ over the time interval $[t_0, t]$. In other words, $N_{i,t}$ comes from a Poisson distribution with mean $\gamma(t - t_0)$ and

$$\Pr\{N_{i,t} = k\} = \frac{[\gamma(t-t_0)]^k e^{-\gamma(t-t_0)}}{k!} \tag{9}$$

# 4 Results for Abreu and Brunnermeier's (2002) Model

## 4.1 Computational Replication of Results

We would like to computationally replicate the equilibrium results from Abreu and Brunnermeier (2002). (For our paper, all programming have been written and run in R, which contains packages for generating random numbers from exponential and Poisson distributions.) Specifically, we would like to show that the equilibrium strategy presented in their paper ensures that no rational players would deviate from that strategy (i.e. choose some other strategy $\tau'$), which by definition verifies the Nash equilibrium. One slight caveat here is that Monte Carlo simulation allows only an approximation of what has been done analytically, but with sufficient iterations it should be close.

In the real world, the number of rational arbitrageurs in the market is finite, different from the assumption in the previous and current papers. Intuitively, however, given a large order of magnitude of traders, the trigger strategy $\tau^*$ on average should provide close to the Nash equilibrium lag time for attacking the bubble when there are a finite but sufficiently many number of arbitrageurs in the market.

Figure 2 shows the average payoffs associated with each deviating strategy $\tau' \neq \tau^*$, from the Monte Carlo simulation with 5000 rational traders and a total of 2000 iterations (the mechanics of each iteration are described in the next section). Here, $\eta = 1, \lambda = 1, \kappa_0 = 1$, and $\beta = 0.2$. The parameters $c = 0.8$ and $\bar{\tau} = 1.25$ satisfy conditions for non-immediate price correction described in Abreu and Brunnermeier (2002). The time steps are of size $10^{-3}$. The vertical line marks the theoretical value for the optimal $\tau^* = 0.603$ (implying that the mispricing occurrs at $\phi(\tau^*) = \tau^* + \ln(c/(c - \lambda * \beta))/\lambda = 0.890$), and the horizontal line signifies the value at which the maximum payoff occurs in the simulation. We see that the intersection of these points lie—as we would expect—approximately on the global maximum of the payoff

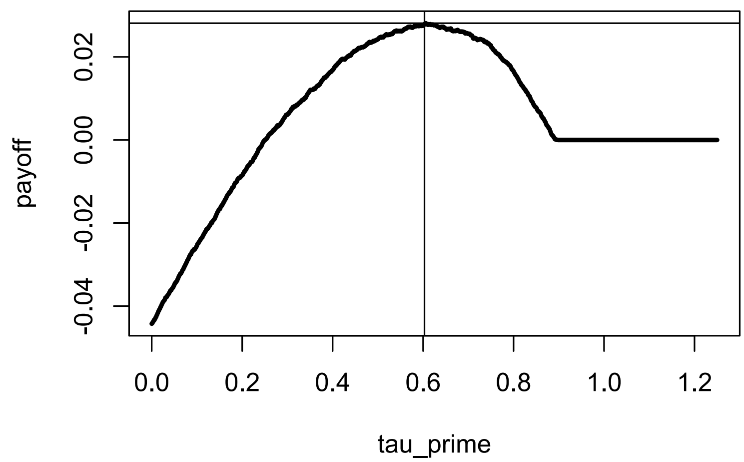


Figure 2: Payoff as a Function of Trigger Strategy $\tau'$

function.

### 4.1.1 Description of Algorithm

The definition of a Nash equilibrium strategy is for $\tau^*$ to be the optimal lag time for any trader $i$ to maximize total expected payoff, given that all other traders are also applying strategy $\tau^*$. Since each realization of random variables does not characterize what happens on average, we need to run a large number of iterations (with one iteration representing an entire process of bubble followed by price correction) and take the sample average payoffs.

Let $M$ be the number of traders that are represented in our simulation. During each iteration, we generate a new value for $t_0$ drawn from the exponential distribution with rate $\lambda$. Next, $M$ random variables are generated uniformly in the interval $[t_0, t_0 + \eta]$. Traders $i = 1, 2, \cdots, M-1$ will follow the equilibrium strategy, changing their positions when $t \geq t_i + \tau^*$. From time $t_0$ onward, the program increments time by our specified time steps, updating the $\kappa(\cdot)$ value and trading pressure at the end of each step. Once the trading pressure surpasses $\kappa(\cdot)$, the mispricing becomes corrected.

To verify that $\tau^*$ is indeed the optimal strategy, given all other traders already use it, we

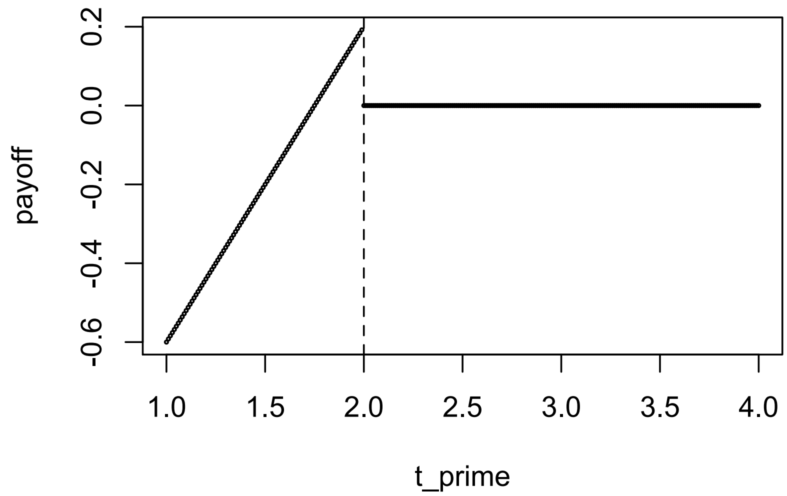


Figure 3: Payoff as a Function of Trading Time $t'$, with Price Correction $t = 2.0$

first generate a sequence of potential trigger strategy values from 0 to $\bar{\tau}$. Then we can calculate the present-value payoffs associated with each $\tau' \in [0, \bar{\tau}]$.

**Proposition 1.** *The payoff $\pi$ of trader $M$ when he uses the trigger strategy $\tau'$ is*

$$\pi(t' = t_M + \tau') = \begin{cases} \beta - c(t - t') & \text{for } t' \leq t \\ 0 & \text{for } t' > t \end{cases} \tag{10}$$

*where $t_M$ is the $M^{th}$ random variable drawn from the aforementioned uniform distribution and $t$ is the time of the price correction.*

*Proof.* We arrive at the above piecewise payoff function, if we use $r$ to discount everything to today's value. When $t' > t$, then price correction would have occurred before the $M^{th}$ trader changes his position. Therefore, in this case, the payoff would simply be zero. On the other hand, if $t' \leq t$, the trader would receive a one-shot gain of $\beta e^{rt}$ at time $t$, which discounts to $\beta$ today. The downside is that if trader $M$ acts too soon, he also incurs a trading cost of approximately $cp_{\tilde{t}}\Delta = ce^{r\tilde{t}}\Delta$ during any infinitesimal interval $[\tilde{t}, \tilde{t} + \Delta]$. Again, discounted to today's value, the trading cost associated with some small duration $\Delta$ is approximately $c\Delta$.

Hence, when $t' \leq t$, in addition to the revenue gained at $t$, there is also a cumulative loss from holding cost of $c(t - t')$. Combining these two cases gives us the piecewise function in Equation 10. $\square$

Figure 3 shows an example of trader $i$'s—with $t_i = 1.0$—payoff function in a market environment where $c = 0.8$ and $\beta = 0.2$, and the mispricing persists until time $t = 2.0$. We see that if trader $i$ has poor market timing, he sustains net losses if he trades too early (if $t_i + \tau' = t' < t - \beta/c$), and loses out on potential benefits if he waits too long (if $t_i + \tau' = t' > t$).

Now, we repeat the process many times, recording down the cumulative payoff associated with each trigger strategy $\tau'$. Finally, after all the iterations, we may calculate the sample average payoffs of each lag time and assess the optimality of $\tau^*$.

## 4.2 Computational Results for Finite Number of Traders

We examine what happens when there are a limited number of rational traders in the market. When there are only a few players in the market, the dynamics of equilibrium changes drastically from that in the infinite case. More specifically, the original solution proposed by Abreu and Brunnermeier (2002) no longer provides a close approximation when there are a small number of traders. In this section, we show that the equilibrium delay strategy $\tau^*$ increases with the number of players in the market.

One crucial assumption we make here is that every arbitrageur knows exactly how many other arbitrageurs exist in the market. In other words, the number of rational traders in the market should be common knowledge. For the extreme case of only one trader in the market, the strategy is to trade immediately upon becoming aware of the mispricing. This is true since the trader has enough market power to burst the bubble.

Figure 4 shows the equilibrium trigger strategy $\tau^*$ approximated by Monte Carlo simulations when there are $M = 1$ to 25 traders in the market. Each data point is generated from the result of 500 iterations. Here, $\eta = 1, \lambda = 1, \kappa_0 = 1$, and $\beta = 0.2$. The parameters $c = 0.816$ and $\bar{\tau} = 1.5$ satisfy conditions for non-immediate price correction described in Abreu and Brunnermeier (2002). The time steps are of size $10^{-3}$.

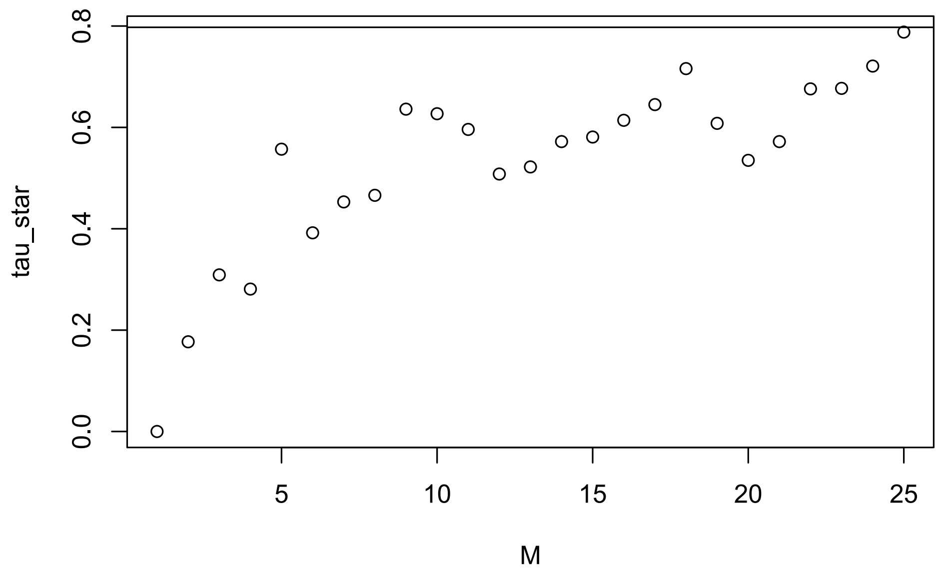


Figure 4: Equilibrium Strategy $\tau^*$ versus the Number of Rational Traders $M$

We see that in general, the equilibrium symmetric trigger strategy $\tau^*$ tends to increase with the number of players in the market. This conclusion is not too surprising if we consider the amount of market power each trader has as the number of rational players increases toward infinity. If there were only one agent in the market, any changes he make in his trading position would exert a pressure of 1—always enough to offset order imbalance $\kappa(\cdot)$. Thus, he would trade as soon as he becomes aware of the mispricing. As more agents enter the market, each trader's effect on market prices decreases as the total number of arbitrageurs increases. Therefore, we would expect the mispricing to persist longer on average, and the equilibrium strategy $\tau^*$ increases as well. When the number of players in the market becomes relatively large, we would expect $\tau^*$ tend toward the infinite-number case of $\tau^* = 0.797$ as marked by the horizontal line in Figure 4.

### 4.2.1 Description of Algorithm

We approach the problem from recalling the definition of a symmetric trigger strategy equilibrium. Suppose there are $M$ traders in the market. If $\tau^*$ is an equilibrium strategy, then if players $1, 2, \cdots, M-1$ all utilize this strategy (that is, they trade at times $t_1 + \tau^*, t_2 + \tau^*, \cdots, t_{M-1} + \tau^*$ respectively) the last player $M$ would have no incentive to deviate from strategy $\tau^*$. In other words, given all others delay their trade by $\tau^*$, the response that maximizes payoff for the $M^{th}$ player is to also delay his trade by $\tau^*$.

We first generate a sequence of potential values for $\tau^*$ equally spaced within the interval $[0, \bar{\tau}]$. We shall refer to each test case as $\tau'$. Unlike before, we do not have a benchmark value of $\tau^*$ against which we should check our solutions. We use several nested loops to find the equilibrium strategy associated with a given value of $M$. The top layer of the loops set a potential strategy $\tau'$ to be used by traders $1, 2, \cdots, M$. The middle layer is used for creating iterations, since each realization of random variables does not characterize what happens on average and we want to consider expected payoffs.

During each iteration, we generate a new value of $t_0$ from an exponential distribution with rate $\lambda$. From the interval $[t_0, t_0 + \eta]$, we generate $M$ uniformly distributed random numbers, indicating the $t_i$'s for each respective trader $i$. The first $M-1$ traders would use the proposed strategy $\tau'$ and make trades as soon as $t \geq t_i + \tau'$. The $M^{th}$ player then tests out each potential response within $[0, \bar{\tau}]$. We use a vector to keep track of the payoffs (as calculated via the function described in Equation 10) associated with each of these potential responses. We increment the time in small intervals from $t_0$ onward and stop when there exists enough trading pressure to correct the mispricing.

After all the iterations are done for each potential equilibrium strategy $\tau'$. The $M^{th}$ trader then tries to optimize his average payoff by choosing some $\tau'' \in [0, \bar{\tau}]$ (making trades as soon as $t \geq t_M + \tau''$). If this optimal $\tau''$ is sufficiently close to $\tau'$, then we have found $\tau^*$. Otherwise, we move on the next test case.

# 5 Results after Incorporation of Information Shocks

## 5.1 Intuition

We try to find a new symmetric trigger strategy for the case when traders are exposed to information shocks in addition to their initial knowledge. After the initial mispricing, each trader gains new knowledge about the mispricing through a series of information shocks, in addition to the original piece of information. Traders become aware of the mispricing as soon as the current time equates to the most accurate estimate they have of the initial mispricing time, and this is when they begin to consider trading. Throughout the bubble, the trader maintains the set of all information he received and makes inferences on when the mispricing would collapse. We provide an intuition of what deductions traders make from out of all these information shocks.

As an example, if mispricing occurred at $t_0 = 1.2$, and some trader $i$ has information that would allow him to become aware of the mispricing at $t_i = 2.0$. During the time past $t_0$ (including prior to $t_i$), new information becomes revealed to trader $i$ at the Poisson rate $\gamma$. Suppose he receives two new pieces of information between $t = t_0$ and $t = 1.8$, one which allows him to be aware at $t_{i,1} = 1.8$ and another at $t_{i,2} = 2.3$. Then, trader $i$ would become aware of the mispricing at $t = 1.8 = \min\{t_i, t_{i,1}, t_{i,2}\}$. He would also remember the information implying $t_{i,2} = 2.3$ as the time of initial mispricing and know that other rational traders in the market may be acting according to that signal.

Suppose $I_i(t) = \{t_i, t_{i,1}, t_{i,2}, \cdots, t_{i,N_{i,t}}\}$ is the set of mispricing times revealed by the information available to trader $i$ at time $t$, where $t_{i,j}$ refers to the implication from the $j^{th}$ information that $i$ has received and $N_{i,t}$ is the quantity of new information obtained up to time $t$. Additionally, denote $I_{i,\min}(t) = \min\{t_i, t_{i,1}, \cdots, t_{i,N_{i,t}}\}$ and $I_{i,\max}(t) = \max\{t_i, t_{i,1}, \cdots, t_{i,N_{i,t}}\}$. We will conjecture—and later verify—that in equilibrium the price correction happens at time $t_0 + \rho$ for some constant $\rho$. Assuming that this is the case, we can derive the arbitrageurs' optimal strategies. Foremost, we may then have the following proposition.

**Proposition 2.** *Trader $i$ knows that the price correction will occur between time $T_{i,1} = \max\{I_{i,\max}(t) - \eta + \rho, t\}$ and $T_{i,2} = I_{i,\min}(t) + \rho$.*

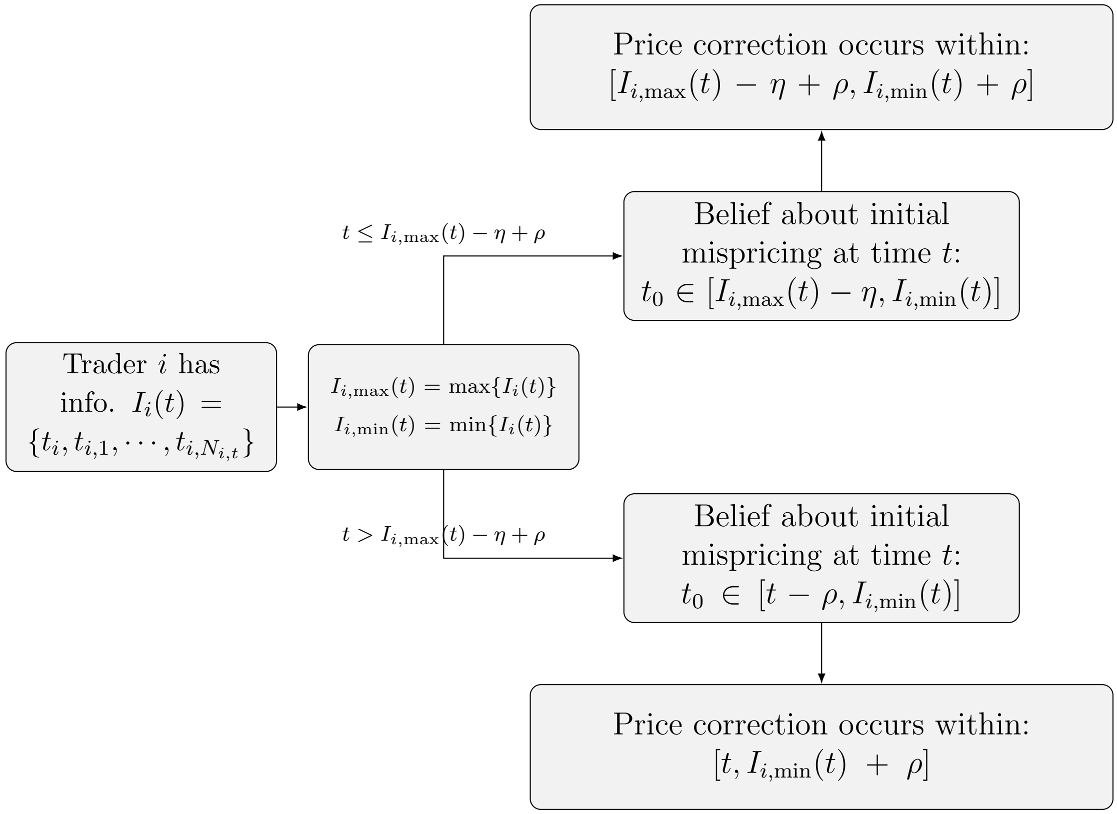


Figure 5: Trader $i$'s deduction process in narrowing his belief of price correction time.

*Proof.* (Refer to flow chart in Figure 5) Suppose that $t < I_{i,\max}(t) - \eta + \rho$. Trader $i$ knows that someone will become aware of the mispricing at time $I_{i,\max}(t)$. Trader $i$ knows that the price mispricing could not have occurred at $t_0 < I_{i,\max}(t) - \eta$. Assuming everyone uses the strategy $\tau'$, the price correction would happen after $I_{i,\max}(t) - \eta + \rho$. The more significant term here, as we will later demonstrate, is $I_{i,\min}(t)$. From trader $i$'s information at time $t$, at the latest the price mispricing began at $I_{i,\min}(t)$. In other words, $t_0 \leq I_{i,\min}(t)$ and the price correction will happen no later than $I_{i,\min}(t) + \rho$. It is also be possible that we have $t \geq I_{i,\max}(t) - \eta + \rho$. Then it is clear that—assuming the mispricing still persists up to time $t$—that price correction will fall between $t + \rho$ and $I_{i,\max}(t) - \eta + \rho$. □

We may now use the results of the traders' deductions to define their perceived hazard rate function for price correction and ultimately describe their equilibrium strategy. As a head start, from our aforementioned reasoning, it is clear that—given trader $i$ has not already changed his position prior to time $t$—he would perceive a hazard rate of zero if $t < T_{i,1}$. From

$t = T_{i,1}$ onward, we would expect trader $i$'s hazard rate function to increase, at some point, from zero toward infinity at $t = T_{i,2}$. Therefore, trader $i$ is guaranteed to have either shifted from neutral position or missed the price correction by $T_{i,2}$.

## 5.2 Hazard Rate Function

During each infinitesimal time period, traders decide whether or not to change their trading position. Once they have traded, they will hold on to their position until price correction. In order to ensure an optimal strategy, each player considers the benefits and costs of waiting until a later period to attack. When his expected benefits exceed the costs, the arbitrageur would trade. Rational players use their perceived hazard rate functions for the time of price correction to evaluate the tradeoff between trading and not trading. We will also use this method in finding the strategy rational traders employ during a mispricing.

**Proposition 3.** *At time $t$, trader $i$'s hazard rate function that the mispricing will be corrected in the immediate following instance is*

$$h(t|[T_{i,1}, T_{i,2}]) = \begin{cases} 0 & \text{for } t < I_{i,\max}(t) - \eta + \rho \\ \frac{\lambda}{1-\exp\{-\lambda(I_{i,\min}(t)+\rho-t)\}} & \text{for } t \geq I_{i,\max}(t) - \eta + \rho \end{cases} \tag{11}$$

*Proof.* According to the previous section, we have that trader $i$ infers price correction to be between the times $T_{i,1}$ and $T_{i,2}$. Now we may start our analysis by breaking the time $t$ into two different cases, for the value of $T_{i,1}$ is dependent on $t$'s relating with the statistic $I_{i,\max}(t)$.

The simple case is when $t \leq I_{i,\max}(t) - \eta + \rho$. Then, it must be that $T_{i,1} = I_{i,\max}(t) - \eta + \rho$, and trader $i$ knows price correction will only happen at $t \in [I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \rho]$. Therefore, the hazard rate must be zero when $t \leq I_{i,\max}(t) - \eta + \rho$.

Less trivially, we now consider the case for $t \geq I_{i,\max}(t) - \eta + \rho$. Recall that all traders know $t_0$ comes from an exponential distribution function with rate $\lambda$. The hazard rate is defined as the probability that price correction occurs at the next instant after $t$, given it has not yet occurred prior to $t$.

Suppose that the price correction time correspond to the random variable $T$. Then, the probability of the mispricing sustaining up to time $t \leq T_{i,2}$ is

$$
\begin{aligned}
\Pr\{T \geq t | T \in [T_{i,1}, T_{i,2}]\} &= \frac{\Pr\{T \geq t, T_{i,1} \leq T \leq T_{i,2}\}}{\Pr\{T_{i,1} \leq T \leq T_{i,2}\}} \\
&= \frac{\Pr\{t \leq T \leq T_{i,2}\}}{\Pr\{T_{i,1} \leq T \leq T_{i,2}\}} \\
&= \frac{1 - \exp\{-\lambda(T_{i,2} - \rho)\} - [1 - \exp\{-\lambda(t - \rho)\}]}{1 - \exp\{-\lambda(T_{i,2} - \rho)\} - [1 - \exp\{-\lambda(T_{i,1} - \rho)\}]} \\
&= \frac{\exp(-\lambda t) - \exp(-\lambda T_{i,2})}{\exp(-\lambda T_{i,1}) - \exp(-\lambda T_{i,2})} \qquad (12)
\end{aligned}
$$

From this, we can find the hazard rate function $h(\cdot|[T_{i,1}, T_{i,2}])$ for arbitrageur $i$.

$$
\begin{aligned}
h(t|[T_{i,1}, T_{i,2}]) &= \frac{\frac{d}{dt}\left[1 - \Pr\{T \geq t | T \in [T_{i,1}, T_{i,2}]\}\right]}{\Pr\{T \geq t | T \in [T_{i,1}, T_{i,2}]\}} \\
&= \frac{\frac{\lambda \exp(-\lambda t)}{\exp(-\lambda T_{i,1}) - \exp(-\lambda T_{i,2})}}{\frac{\exp(-\lambda t) - \exp(-\lambda T_{i,2})}{\exp(-\lambda T_{i,1}) - \exp(-\lambda T_{i,2})}} \\
&= \frac{\lambda \exp(-\lambda t)}{\exp(-\lambda t) - \exp(-\lambda T_{i,2})} \\
&= \frac{\lambda}{1 - \exp\{-\lambda(T_{i,2} - t)\}} \\
&= \frac{\lambda}{1 - \exp\{-\lambda(I_{i,\min}(t) + \rho - t)\}} \qquad (13)
\end{aligned}
$$

□

Notice that when $t \geq I_{i,\max}(t) - \eta + \rho$, the hazard rate function becomes positive. In the limit case when $t$ approaches $T_{i,2} = I_{i,\min}(t) + \rho$, the hazard rate rises toward infinity. This again asserts the fact that trader $i$ will change his position sometime between $I_{i,\max}(t) - \eta + \rho$ and $I_{i,\min}(t) + \rho$. In Figure 6, we demonstrate the shape of the hazard rate function at a snapshot in time. Here, we take everything except for $t$ as fixed ($\eta = 1$, $\lambda = 1$, $\rho = 0.75$, $I_{i,\max}(t) = 1.25$, and $I_{i,\min}(t) = 0.75$). We see that for all $t < I_{i,\max}(t) - \eta + \rho = 1$, the hazard rate remains at zero. At $t = 1$, the hazard rate jumps up and goes toward $+\infty$ as $t$ approaches $I_{i,\min}(t) + \rho = 1.5$.

Now that we have each trader's hazard rate functions, we can derive the following result regarding their equilibrium strategy.

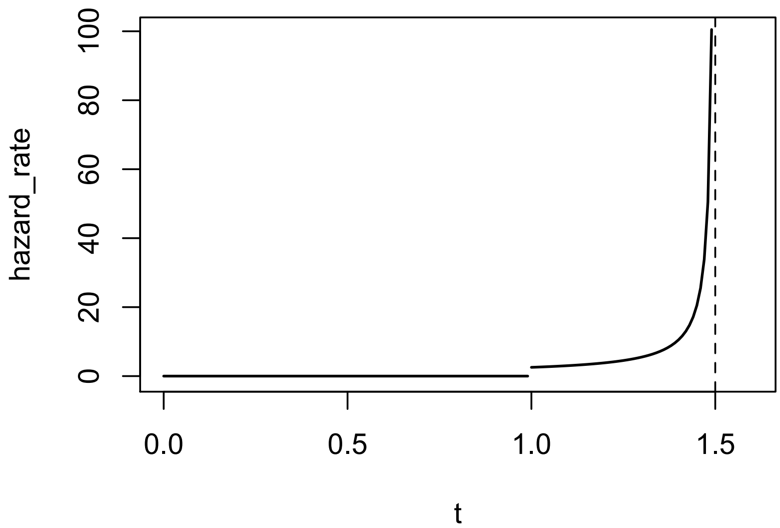


Figure 6: Hazard Rate as a Function of Time $t$

**Proposition 4.** *Arbitrageur $i$ will attack the bubble the first instance when $t \geq \max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\}$, with $\tau^* = \rho - \ln[c/(c - \lambda\beta)]/\lambda$.*

*Proof.* Similar to what has been done in Abreu and Brunnermeier (2002), we want to find the first instance in which the expected benefit of trading exceeds the holding cost. In other words, given $[T_{i,1}, T_{i,2}]$, we want to find the time $t$ when $\beta h(t|[T_{i,1}, T_{i,2}]) \geq c$ (see the argument leading to condition shown in Equation 4).

From the hazard rate function (Equation 11), we know that $\beta h(t|[T_{i,1}, T_{i,2}]) = 0 < c$ for $t < I_{i,\max}(t) - \eta + \rho$. Thus, our $t$ of interest must be in the interval $[I_{i,\max}(t) - \eta + \rho, I_{i,\min} + \rho]$. In particular $t$ must satisfy

$$
\begin{aligned}
\frac{\beta\lambda}{1 - \exp\{-\lambda(I_{i,\min}(t) + \rho - t)\}} &\geq c \\
\frac{\beta\lambda}{c} &\geq 1 - \exp\{-\lambda(I_{i,\min}(t) + \rho - t)\} \\
-\lambda(I_{i,\min}(t) + \rho - t) &\geq \ln\left(\frac{c - \beta\lambda}{c}\right) \\
t &\geq I_{i,\min}(t) + (\rho - \ln[c/(c - \lambda\beta)]/\lambda)
\end{aligned}
$$

Denote $\tau^* = \rho - \ln[c/(c - \lambda\beta)]/\lambda$. Then we see that trader $i$ would choose to change his position at $t = I_{i,\min}(t) + \tau^*$ unless $I_{i,\min}(t) + \tau^* < I_{i,\max}(t) - \eta + \rho$, in which case he would trade at $t = I_{i,\max}(t) - \eta + \rho$ instead.

$\square$

## 5.3 Analytical Approach

We have established how traders would strategize when they know the equilibrium duration of the mispricing. The only missing piece is deriving an expression for the equilibrium duration itself. We can approach this problem by looking at the overall market condition under which the bubble would collapse; this happens when the cumulative trading pressure from the arbitrageurs overcomes the market's ability to offset the trading balance.

In Abreu and Brunnermeier's (2002) model, each trader uses a symmetric trigger strategy $\tau'$, resulting in the mispricing duration to be $\phi(\tau')$ as defined in Equation 5. If we rearrange that equation to solve for $\kappa$, we have the following form.

$$\kappa(\phi(\tau')) = \frac{\phi(\tau') - \tau'}{\eta} \tag{14}$$

This provides another way to interpret the total market delay function $\phi(\cdot)$

The left hand side of Equation 14 is just the ability of the market to absorb trading imbalance when $t - t_0 = \tau'$. The right hand side, which is somewhat less obvious, is the amount of trading pressure present in the market at time $t_0$ (i.e. the proportion of traders who have already traded by time $t_0 + \phi(\tau')$). Since each trader acts after a delay of $\tau'$, only traders who became aware of the mispricing during $[t_0, t_0 + \phi(\tau') - \tau']$ would have acted prior to price correction at $t_0 + \phi(\tau')$. In other words, the right hand side of Equation 14 is the proportion of the rational arbitrageurs who have changed their position prior to price correction, i.e. the trader pressure when the bubble disappears.

Similarly, we need an expression for the equilibrium duration $\rho$ in our information shock model. The left hand side of the equation would be as presented in Equation 14. The right hand side should be, comparable to the original model, the proportion of the rational traders that have acted by the time $t = t_0 + \rho$.

Let $g(t - t_0)$ denote the proportion of people who will have traded by time $t$, following the strategy of trading at the first instance that $t \geq \max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\}$. Then we must have that (similar to the intuition behind Equation 14)

$$\kappa(\rho) = g(\rho) \tag{15}$$

The derivation of $g(\cdot)$ is somewhat tedious, and we provide some intuition as to how to approach the problem analytically in the following section. However, we will ultimately solve for $\rho$ computationally.

#### 5.3.1 Cumulative Trading Pressure

To solve our problem analytically, we need to calculate the proportion of traders that have changed their position at any instance in time. The key difficulty is that each trader bases his decision on information that—unlike in Abreu and Brunnermeier's (2002) model—dynamically change over time, and he only shift their position once. We develop an answer for what happens at some snapshot in time $t$ if we inhibit traders from trading before $t$. From this, we can use limits and Bayesian probabilities to search for an analytical solution for the cumulative trading pressure at $t$ when we allow rational players to trade as they wish (i.e. possibly before $t$).

In the original problem, each rational trader's perception of the mispricing event remains the same throughout the duration of the bubble; arbitrageur $i$ only knows about his own $t_i$ up until price correction. Since each $t_i$ is drawn uniformly and independently from $[t_0, t_0 + \eta]$, the probability density function of the random variable $t_i$ is simply just $1/\eta$. Knowing this, the derivation of the delay between the start and the end of the mispricing $\phi(\tau')$, given all traders use strategy $\tau'$, is implicitly defined as $\phi(\tau') = \tau' + \eta\kappa(\phi(\tau'))$. The intuition behind this formula is that we know $\kappa(\rho(\tau'))$ of the traders need to change their positions to end the mispricing. Thus, if everyone delays by $\tau'$, the price correction would then occur at $t = t_0 + \tau' + \eta\kappa(\rho(\tau'))$.

The problem becomes more complicated when each trader $i$ receives more information than just the initial signal $t_i$. More specifically, we would expect the price correction to occur, on average, earlier than that in the original problem. We know that when the bubble bursts at $t_0 + \rho$, there must have been $\kappa(\rho)$ proportion of the traders who changed their position. More specifically, when each trader uses some our derived equilibrium strategy and trades when $t \geq \max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\}$, this proportion $\kappa(\rho)$ should equate to the proportion of traders who already traded according to this strategy.

With this in mind, it is of great interest to find the relation between $I_{i,\min}(t)$ and $I_{i,\max}(t)$ given $t_i$. To do so, we may assume that the arrival of information events—associated with the Poisson rate $\gamma$—are independent of the information received. Consider the conditional random variable $I_{i,\min}(t|N_{i,t})$. When $N_{i,t} = k$ is fixed, $I_{i,\min}(t)$ is the minimum of $k+1$ independent, uniform random variables in $[t_0, t_0 + \eta]$ (similarly for $I_{i,\max}(t|N_{i,t})$).

The problem is easier to interpret if we think about what happens if trader $i$ does not do anything prior to some time $t$. Suppose that at time $t$, player $i$ has not yet traded. Then the probability of a trade occurring immediately after $t$ would be (fixing $N_{i,t} = k$)

$$\begin{aligned}\Pr\{\max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\} \leq t\} &= \Pr\{I_{i,\max}(t) - \eta + \rho \leq t, I_{i,\min}(t) + \tau^* \leq t\} \\ &= \Pr\{I_{i,\min}(t) \leq t - \tau^* | I_{i,\max}(t) \leq t + \eta - \rho\} \Pr\{I_{i,\max} \leq t + \eta - \rho\}\end{aligned}$$

where

$$\Pr\{I_{i,\max}(t) \leq t + \eta - \rho\} = \left(\frac{(t - t_0) + \eta - \rho}{\eta}\right)^{k+1}$$

and

$$\Pr\{I_{i,\min}(t) \leq t - \tau^* | I_{i,\max}(t) \leq t + \eta - \rho\} \quad = \quad 1 - \left(\frac{\eta - \rho + \tau^*}{(t - t_0) + \eta - \rho}\right)^{k+1}$$

In the end, we may rewrite the above as

$$\Pr\{\max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\} \leq t\} = \left(\frac{(t - t_0) + \eta - \rho}{\eta}\right)^{k+1} - \left(\frac{\eta - \rho + \tau^*}{\eta}\right)^{k+1}$$

where $N_{i,t}$ is fixed at $k$ and the trade has not yet occurred up to point $t$. Note that this value will be always positive for $t \geq t_0+\tau^*$. We know that no trader would make trades prior to $t_0+\tau^*$ (for $I_{i,\min}(t)+\tau^* \geq t_0+\tau^*$ for all $t$ and traders only trade if $t \geq \max\{I_{i,\max}(t)-\eta+\rho, I_{i,\min}(t)+\tau^*\}$), so this makes sense.

Next, together with the distribution of $N_{i,t}$ (see Equation 9), we can solve for the total probability (unconditional on the value of $N_{i,t}$) of player $t_i$ trading at time $t$, given he has not yet traded. For simplicity let $a = [(t-t_0)+\eta-\rho]/\eta$ and $b = [\eta-\rho+\tau^*]/\eta$. Then we have that $\Pr\{\max\{I_{i,\max}(t)-\eta+\rho, I_{i,\min}(t)+\tau^*\} \leq t | N_{i,t} = k\} = a^{k+1} - b^{k+1}$, and

$$\begin{aligned}
\Pr\{\max\{I_{i,\max}(t)-\eta+\rho, I_{i,\min}(t)+\tau^*\} \leq t\} &= \sum_{k=0}^{\infty}(a^{k+1}-b^{k+1}) \Pr\{N_{i,t}=k\} \\
&= \sum_{k=0}^{\infty}(a^{k+1}-b^{k+1}) \cdot \left[\frac{[\gamma(t-t_0)]^k e^{-\gamma(t-t_0)}}{k!}\right] \\
&= ae^{(a-1)\gamma(t-t_0)} - be^{(b-1)\gamma(t-t_0)} \\
&= e^{-\gamma(t-t_0)}(ae^{a\gamma(t-t_0)} - be^{b\gamma(t-t_0)}) \qquad (16)
\end{aligned}$$

However, we should take into account that, at $t$, we need to condition the above probabilities by the fact that for all $s < t$ (otherwise, a trade would have occurred before time t). The analytical expression for this—which would equate to $g(\cdot)$ in the previous section—is beyond the scope of this paper. Instead, we will use Monte Carlo simulations to solve for $\rho$.

## 5.4 Computational Approach

We want to find the equilibrium total delay $\rho$ between the initial mispricing time and the final price correction. Assuming this number to be a known constant to the traders, we derived the equilibrium trading strategy the traders would use during a mispricing. Our computational approach hinges on the fact that all traders would use this strategy, which depends on the trader's perception of $\rho$ itself. The Monte Carlo simulations we run involve guesses for $\rho$ corresponding to some given information shock rate parameter $\gamma$. If some guess approximately equates to the average simulated time lag from the start to the collapse of the bubble, then we know that this guess is the actual equilibrium mispricing duration.

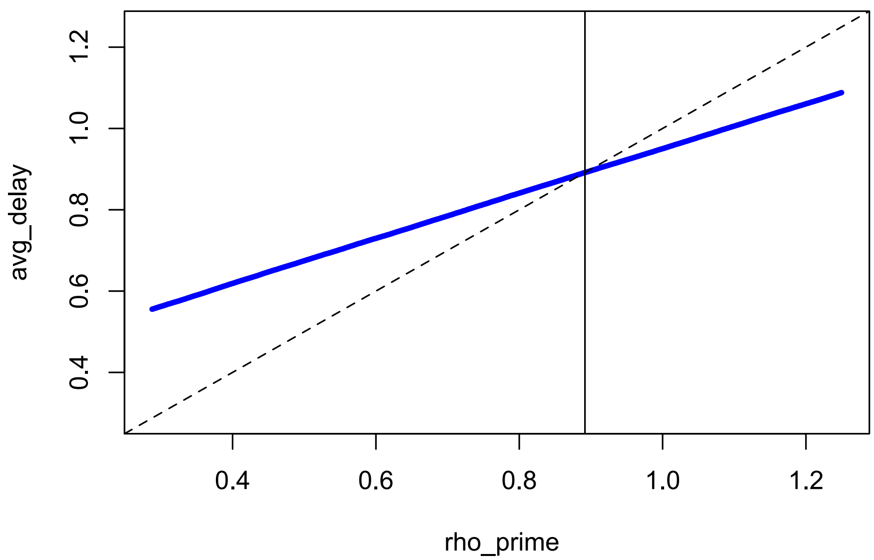


Figure 7: Mispricing Duration (avg-delay) as a Function of Potential $\rho'$, with $\gamma = 0$

Suppose we guess some $\rho'$ to be the rational traders' perception for the bubble duration. By Proposition 4, each arbitrageur $i$ would change his position the first instant when $t \geq \max\{I_{i,\max}(t) - \eta + \rho', I_{i,\min}(t) + \tau'$, where $\tau' = \rho' - \ln[c/(c - \lambda\beta)]/\lambda$. With this in mind, we move on to our computational results.

### 5.4.1 Finding Equilibrium Duration

We use Monte Carlo simulations to find an estimate for the new value of bubble duration when we incorporate information shocks into the synchronization risk model. Given an initial guess $\rho'$ for the total duration, each trader would act accordingly to the strategy described in Proposition 4. We know that a guess is correct if it is within a small margin of error of the simulated actual duration of the bubble when the traders employ the associated strategy. Additionally, we may check the results of our simulation by seeing if it makes sense in the limit case where there are no information shocks at all.

Figure 7 ($\gamma = 0$) and Figure 8 ($\gamma = 1.6$) show the results of the simulation with 2000 rational traders and 50 iterations per potential $\rho'$. Here, $\eta = 1, \lambda = 1, \kappa_0 = 1$, and $\beta = 0.2$.

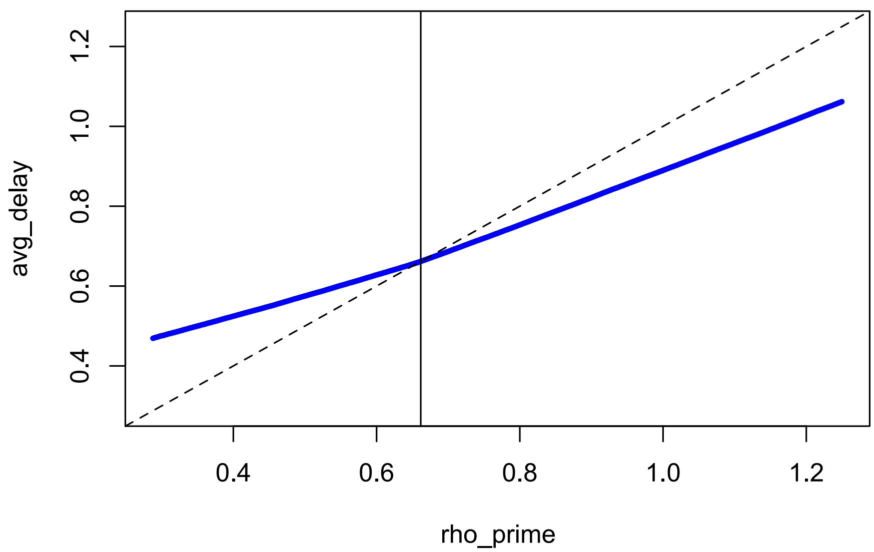


Figure 8: Mispricing Duration (avg-delay) as a Function of Potential $\rho'$, with $\gamma = 1.6$

The parameters $c = 0.8$ and $\bar{\tau} = 1.25$ satisfy conditions for non-immediate price correction described in Abreu and Brunnermeier (2002). The solid, sloped curve in both figures shows the average simulated duration of the mispricing as a function of the guess $\rho'$. The dashed line is simply the 45-degree line, which intersects the curve previously described when the simulated duration matches the guess $\rho'$. The vertical line allows easy comparison of the two figures, marking the right guess for $\rho$.

We see that when $\gamma = 0$, we have that $\rho = \phi(\tau^*) = 0.890$, which agrees with what is shown from Figure 2. With the allowance of information learning, however, the duration of the mispricing decreases drastically. For instance, in Figure 8, when $\gamma = 1.6$, the mispricing persists for a much shorter time; specifically, $\rho = 0.662$.

*Description of Algorithm.* For any simulation, to ensure accuracy we should have a large number of iterations (for we are using a less than infinite number of traders $M$) to minimize noise errors. During each iteration, we generate a new value for $t_0$ from an exponential distribution with parameter $\lambda$. Then draw $M$ uniformly distributed random numbers $t_1, t_2, \cdots, t_M$ from the interval $[t_0, t_0 + \eta]$. We also have a sequence of potential values for $\rho$ evenly distributed inside the interval $[\ln[c/(c - \lambda\beta)]/\lambda, \bar{\tau}]$. For each guess $\rho'$, we calculate the corresponding values for $\tau^*$. Each trader $i$ keeps track of two values—$I_{i,\max}(t)$ and $I_{i,\min}(t)$—and updates them according to time. Initially, we would of course have those two values be equal to $t_i$.

We step forward in time from $t_0$ with time steps of $\Delta$. Since information shocks can occur during this interval, we must generate $M$ Poisson numbers corresponding to each trader (the Poisson distribution would have the mean $\gamma\Delta$; this method is valid since we are considering a stationary Poisson process with constant rate).

Suppose trader $i$ receives $m$ number of messages during some time step, then we draw $m$ uniform variables from the distribution $[t_0, t_0 + \eta]$. Trader $i$ then updates his values of $I_{i,\max}(t)$ and $I_{i,\min}(t)$ as needed. For instance, suppose trader $i$ received information $t_{i,1}, t_{i,2}, \cdots, t_{i,m}$ in the time interval from time $t - \Delta$ to $t$, then we would have $I_{i,\max}(t) = \max\{I_{i,\max}(t - \Delta), t_{i,1}, \cdots, t_{i,m}\}$ and $I_{i,\min}(t) = \min\{I_{i,\min}(t - \Delta), t_{i,1}, \cdots, t_{i,m}\}$.

After the updating is done, we see if any player—or, rather, the ones who have not traded—has decided to trade during this time step. More specifically, arbitrageur $i$ would trade if $t \geq \max\{I_{i,\min}(t) + \tau', I_{i,\max}(t) - \eta + \rho'\}$, where $\tau' = \rho' - \ln[c/(c - \lambda\beta)]/\lambda$. We keep track of how many people have traded (to reassess trading pressure at the start of each time step) and make sure that each person will only trade once. Additionally, we must update the value of $\kappa(t - t_0)$ at the beginning of each time step as well.

When the trading pressure exceeds the current value of $\kappa$, price correction occurs. If our guess of $\rho$ is correct, then it should approximately equal to (within some margin of error) the delay between the simulated initial mispricing and the final price correction.

We repeat this process for many iterations, keeping track of the average correction time as well as the sum of the absolute deviations of the correction time from the guess for $\rho$ during each iteration. The right guess for $\rho$ is then the one that is closest in value to the corresponding time taken to price correction.

### 5.4.2 Relationship Between Bubble Duration and Symmetric Strategy

The relationship between bubble duration and rational players' symmetric strategy is fairly endogenous. Up until this point of the analysis, we have only looked at how players would react if they agree on the total duration of the bubble. Yet, it is important to realize that the bubble duration is inherently dependent on the strategy employed by the players in the market as well. In Abreu and Brunnermeier (2002), given that all traders use the same trading strategy, the authors derive a closed form expression $\phi(\cdot)$ for the time delay between mispricing and price correction. We explore a similar idea in this section and show the effect of the symmetric (but not necessarily equilibrium) strategy affects the outcome of the bubble when we incorporate information shocks.

We allow each arbitrageur to trade when $t \geq \max\{I_{i,\max}(t) - \eta + \rho', I_{i,\min}(t) + \tau'\}$ for different values of $\tau'$. We take $\tau'$ to be given and computationally find the perceived duration $\rho'$ that would induce each player to choose this value of $\tau'$. More specifically, for each $\tau'$ within the interval $[0, \bar{\tau}]$, we repeat what has been done in the previous section to find the corresponding value for $\rho'$. Each trader $i$ from 1 to $M$ will trade according to this guess, and we record when the mispricing actually corrects itself in the simulation. After some iterations, for each $\tau'$ we find the value of $\rho'$ closest in absolute distance from the actual observed delay time between initial mispricing and final price correction (in our simulation, these values are less than $10^{-3}$ apart). Accordingly, we can computationally derive $\rho'$.

Figure 9 shows the Monte Carlo simulation results using parameters M = 2000 rational traders and 50 iterations for each $\tau'$ to find the approximated value of $\rho'$. Here, $\eta = 1, \lambda = 1, \kappa_0 = 1$, and $\beta = 0.2$. The parameters $c = 0.8$ and $\bar{\tau} = 1.25$. The dashed line represents the original total delay function $\phi(\tau') = \bar{\tau}(\tau' + \eta\kappa_0)/(\bar{\tau} + \eta\kappa_0)$ as derived in Abreu and Brunnermeier's (2002) initial model (i.e. $\gamma = 0$). The discrete points are the estimates for $\phi(\tau')$ for a sequence of sample points for $\tau'$ within $[0, \bar{\tau}]$, with a nonzero $\gamma = 5.0$. The solid diagonal line depicts the line $\rho = \tau^* + \ln(c/(c - \lambda\beta))/\lambda$. We would expect this line to intersect at the trigger strategies $\tau^*$ corresponding to $\gamma = 0$ and $\gamma = 5.0$ As we would expect, the equilibrium trigger strategy and mispricing duration is significantly smaller when $\gamma = 5.0$ compared to $\gamma = 0$.

We see that the mispricing persists for a shorter time when $\gamma$ is nonzero, given each trader

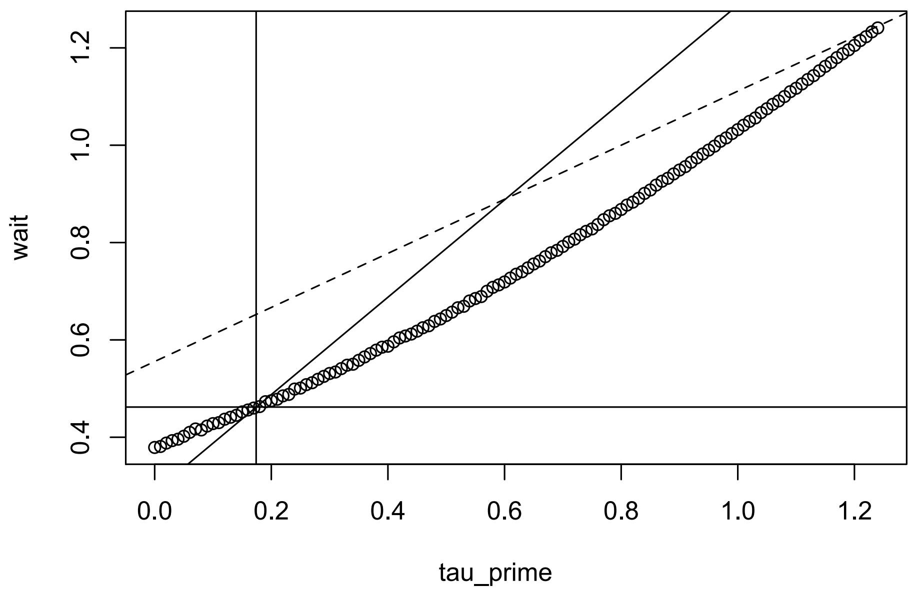


Figure 9: Bubble Duration (wait) when Traders Use Symmetric Strategy Associated with $\tau'$

uses the same strategy associated with $\tau'$. This is due to the fact that higher $\gamma$'s allows all traders to have more accurate gauge of when initial mispricing $t_0$ occurred. Therefore, more arbitrageurs would have traded by some time $t$ in the case where $\gamma$ is nonzero compared to the original model.

Additionally, the effect of $\gamma$ on the duration of asset mispricings also changes in magnitude in terms of $\tau'$. When $\tau'$ is relatively small, we expect that $\rho'$ would also be relatively small. From earlier, we derived that rational player $i$ would change his position at the first time $t = \max\{I_{i,\max}(t) - \eta + \rho', I_{i,\min}(t) + \tau'\}$. Since $I_{i,\max}(t) - \eta \geq I_{i,\min}(t)$ for all $t$, we see that $I_{i,\min}(t)$ weighs in more on trader $i$'s decision to trade for relatively small values of $\tau'$. For nonzero values of $\gamma$, trader $i$ receives more and more accurate information of $t_0$ over time (i.e. $I_{i,\min}(t)$ becomes smaller and approaches $t_0$ over time). Consequently, compared to the situation where $\gamma = 0$, more people would have traded by $t$ in the case when $\gamma > 0$, and the difference is more pronounced for smaller values of $\tau'$.

On the other hand, when $\tau'$ is large—especially pronounced in the case where $\tau' = \bar{\tau}$—the effect of $\gamma$ on the duration of the mispricing diminishes. This is illustrated by the decrease in the distance between the dashed line ($\gamma = 0$) and the estimated discrete-point curve ($\gamma = 5.0$), as $\tau'$ increases. This is due to the fact that as $\tau'$ increases, more players would wait longer at each point $t$. However, as time $t$ passes, $I_{i,\max}(t)$ starts to weigh more on trader $i$'s decision to change his position. Arbitrageur $i$'s incentive to trade at $I_{i,\min}(t) + \tau'$ may be offset by the disincentive to trade knowing mispricing must have occurred after $I_{i,\max}(t) - \eta$ (note here that $I_{i,\min}(t)$ becomes smaller over time and $I_{i,\max}(t)$ becomes larger). In the extreme case where $\tau' = \bar{\tau}$, we see that the mispricing becomes corrected at $\bar{\tau}$ regardless of $\gamma$.

### 5.4.3 Payoff Analysis

Similar to how we computationally replicate the theoretical results from Abreu and Brunnermeir (2002), we shall check the validity of the equilibrium strategy that we have solved for the case when there exist information shocks. In other words, we must check to see if any player would—for the sake of obtaining a higher expected payoff—deviate from this strategy given all other players already utilize the proposed equilibrium strategy. We will computationally show that when everyone one else trades according to the strategy described in Proposition 4, the last player would also do the same in order to maximum his expected payoff.

In the original model, trader $i$ would in equilibrium change his position at time $t = t_i + \tau^*$. When $\gamma > 0$, however, an arbitrageur trades at time $t = \max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\}$. The latter term is akin to the original model, and the term that involves $I_{i,\max}(t)$ sometimes prevents a rational player from trading too early if he knows for sure that the initial mispricing occurred after $I_{i,\max}(t) - \eta$ (i.e. the correction would not occur until after $I_{i,\max}(t) - \eta + \rho$).

Since there are many players in the market, no single player would have a significant effect on price movements. More specifically, even if the last arbitrageur $M$ (assuming the market consists of rational players $1, 2, \cdots, M$) changes his strategy from $\tau^*$ to some other value in $[0, \bar{\tau}]$, we would still expect the price correction to occur at $\rho$ especially when $M$ becomes infinitely large. Rational player $M$ knows this fact, and thus the overall strategy he employs will be trading at $t = \max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau'\}$. (Note here that player $M$ knows his change in strategy would not affect the overall duration of the mispricing.)

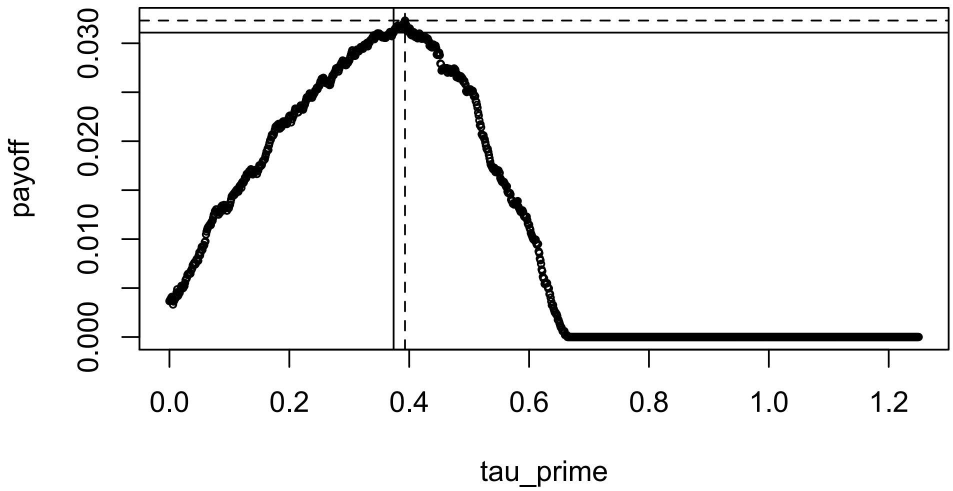


Figure 10: Payoff as a Function of Strategy Associated with $\tau'$, with $\gamma = 1.6$

Figure 10, comparable to Figure 2, shows the results from the Monte Carlo simulation with M = 2000 rational traders and a total of 1000 iterations. Due to computational intensity, we were not able to run the Monte Carlo simulations with higher magnitudes of traders without significantly slowing down the running time. Here, $\eta = 1, \lambda = 1, \kappa_0 = 1$, and $\beta = 0.2$. The parameters $c = 0.8$ and $\bar{\tau} = 1.25$. The time steps $\Delta$ are of size $10^{-3}$. All parameters are the same as that in Figure 2, except now we incorporate the learning of new information over time, i.e. $\gamma = 1.6 > 0$. From Section 5.4.1, we have estimated that $\rho = 0.662$ and $\tau^* = \rho - \ln(c/(c - \lambda\beta))/\lambda = 0.374$ when $\gamma = 1.6$.

The solid lines in Figure 10 depict the theoretical equilibrium strategy: the vertical line corresponds to $\tau^* = 0.374$ and the horizontal line corresponds to the average payoff associated with this strategy $\pi^* = 0.0310$. Similarly, the dashed vertical and horizontal lines in the figure shows the trader $M$'s actual best response strategy $\tau' = 0.393$ (and respectively the payoff $\pi' = 0.0323$) as estimated by the simulation. The margin of error may be explained by the error associated with the estimation for $\rho$ as well as the limitations from not being able to simulate with larger magnitudes $M$ of traders and/or smaller time steps $\Delta$.

Both Figure 2 and Figure 10 correspond to simulations with the same underlying parameters for the market variables. We see that the equilibrium expected payoff for each player is higher when $\gamma = 1.6$ compared to when $\gamma = 0$. Additionally, unlike when $\gamma = 0$, player $M$ would never receive less than 0 as his payoff regardless of his choice of $\tau'$.

*Description of Algorithm.* For the Monte Carlo simulation, we can approximate the infinite traders case by simulating with a large $M$ number of traders in the market. Without loss of generality, we just have to show that the $M^{th}$ trader would not change his strategy $\tau^*$ to any other in $[0, \bar{\tau}]$ in order to maximize payoff. In other words, we would like to show that $\tau^*$ is player $M$'s best response to when players $1, 2, \cdots, M-1$ also play $\tau^*$.

As usual, each iteration begins with the generation of initial mispricing $t_0$ drawn from the exponential distribution with parameter $\lambda$. We have the initial information available to the $M$ players $t_1, t_2, \cdots, t_M$ uniformly generated from the interval $[t_0, t_0 + \eta]$. Initially, we set $I_{i,\min}(t_0) = I_{i,\max}(t_0) = t_i$. From time $t_0$ onward, we step ahead in time in intervals of $\Delta$, updating $\kappa(\cdot)$ and the amount of cumulative trading activity at each time $t$.

During each interval $t-\Delta$ to $t$, each trader $i$ receives some number of shocks, generated from a Poisson distribution with mean $\Delta\gamma$. Each trader updates his $I_{i,\min}(t)$ and $I_{i,\max}(t)$ according to the new information he receives. The behavior of arbitrageurs $i = 1, 2, \cdots, M-1$ is what we have derived as the equilibrium trigger strategy. They would trade the first instance when time $t = \max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\}$ and not change their positions from there after. For player $M$, however, we keep a vector of times for him to change his position depending on the choice of $\tau' \in [0, \bar{\tau}]$. We generate a sequence of potential trigger strategy values from 0 to $\bar{\tau}$ and store each respective time when trader $M$ makes a trade.

At the end of the iteration—corresponding to when the total proportion of rational players who have traded exceeds $\kappa(t - t_0)$—we calculate the payoff associated with each $\tau'$ according to Equation 10. That is, suppose strategy $\tau'$ leads trader $M$ to change his position at time $t'$ and the price correction occurs at $t$, then trader $M$'s payoff would be $\pi(t') = \beta - c(t - t')$ if $t' < t$ and zero otherwise.

We repeat this entire process for many iterations, recording the cumulative sum of payoffs associated with strategy $\tau' \in [0, \bar{\tau}]$ for player $M$. After all iterations, we calculate overall averages and assess the optimality of $\tau^*$ compared to alternative strategies. By definition, if $\tau^*$ is player $M$'s best, payoff-maximizing response to the other players employing the same trigger strategy, then $\tau^*$ is an symmetric equilibrium trigger strategy.

# 6 Conclusion

For our paper, we analyze the effect of introducing an information shock rate $\gamma$ to the synchronization risk model presented in Abreu and Brunnermeier (2002). The variable $\gamma$ serves as a measure of the degree of common knowledge in the market: the higher the expected information shock frequency the more rational trader coordination and the lesser the amount of synchronization risk. In particular, we solve for a comparable equilibrium symmetric trigger strategy solution as that presented for the original model in the cases where we allow some degree of common knowledge in the assessment of initial mispricing of an asset.

We observe that each trader $i$ would ultimately base his decisions on two time-varying random variables $I_{i,\max}(t)$ and $I_{i,\min}(t)$. Comparable to the original model in which trade occurs at $t = t_i + \tau^*$, arbitrageur $i$ in equilibrium changes his position at time $t = I_{i,\min}(t) + \tau^*$, where $\tau^*$ is the symmetric equilibrium trigger strategy. However, the rational player may sometimes delay his decision based on $I_{i,\max}(t)$. In general, player $i$ trades the first instance when $t = \max\{I_{i,\max}(t) - \eta + \rho, I_{i,\min}(t) + \tau^*\}$.

From Proposition 4, we must have the $\tau^*$ and $\rho$ satisfy the relationship $\rho = \tau^* + \ln(c/(c - \lambda\beta))/\lambda$. Additionally, from the argument in Section 5.3 the general function that depicts the delay between initial mispricing and final price correction $\rho$ should satisfy $\kappa(\rho) = g(\rho)$. Here, $\kappa(t - t_0)$ denotes the market's ability to offset order imbalance (see Equation 2), and $g(t - t_0)$ is the amount of cumulative trading pressure at time $t$. As suggested by the equivalence condition, the mispricing would happen as soon as $\kappa(t - t_0) \leq g(t - t_0)$.

If we find a closed form expression for $g(\cdot)$, we may proceed to an analytical solution for $\rho$ in terms of all the market parameters and the information rate $\gamma$. Our paper, instead, uses computational methods—specifically, Monte Carlo simulations—to solve for the new mis-

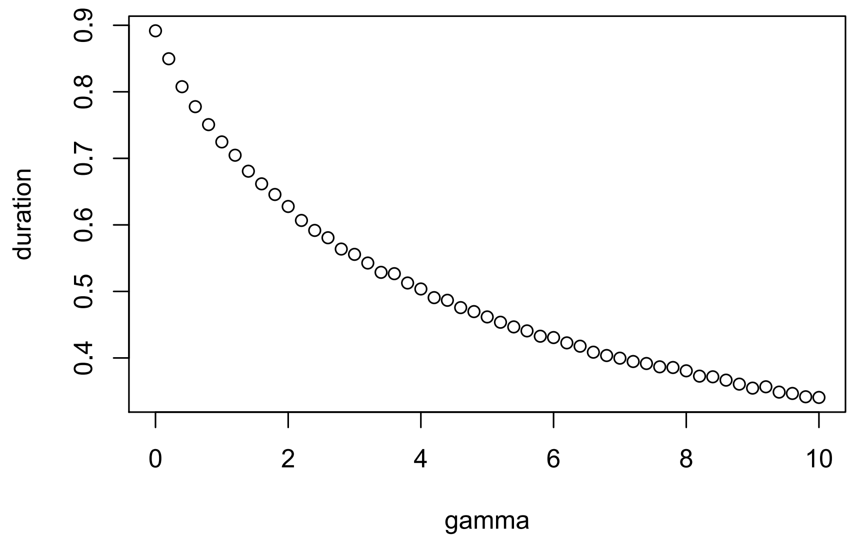


Figure 11: Bubble Duration as a Function of Information Shock Rate $\gamma$

pricing duration $\rho$ and its associated equilibrium strategy. We have discussed computational methods for finding $\rho$ for a specified value of $\gamma$ and deriving the relationship between traders' symmetric strategy and total bubble duration. Both exercises provide evidence that a nonzero $\gamma$ in equilibrium leads to a shorter delay $\tau^*$ in terms of trader's individual decisions to trade, compared to the original model where $\gamma = 0$. This would also lead to a shorter total duration between the initial mispricing of the asset to its final price correction $\rho$. Additionally, we have shown that, in non-equilibrium but symmetric trigger strategy cases, the effect of $\gamma$ on total mispricing duration decreases as the symmetric individual delay time increases.

Moreover, we can utilize the methods described in our paper to examine the relationship between total duration time of asset mispricings and information shock rate $\gamma$. Figure 11 illustrates this relationship using a Monte Carlo simulation with $M = 2000$ traders, $\eta = 1, \lambda = 1, \kappa_0 = 1, \beta = 0.2, c = 0.8$ and $\bar{\tau} = 1.25$. We sample the mispricing duration $\rho$ for 51 values of $\gamma$ equally spaced between 0 and 10 (i.e. steps of size 0.2), and plotted the average mispricing duration over 50 iterations per sample point.

We see that as the degree of common knowledge (i.e. $\gamma$) increases, the duration of the mispricing decreases. The relationship between mispricing duration and $\gamma$ is clearly nonlinear, diminishing rapidly when $\gamma$ is relatively small. This shows that the introduction of information shocks assist in trader coordination, but the marginal effects of the shocks diminishes with the rate $\gamma$.

Lastly, we have some evidence that a higher equilibrium expected payoff can be achieved when $\gamma > 0$ compared to the original model. In other words, the introduction of (free) information shocks enables each rational trader to gain more profit on average. However, in real world markets, the attainment of new information is often costly and would probably offset these additional profits.

# 7 Future Research and Extensions

## 7.1 Complexity of Information Dispersion

Our model can certainly be improved if we increase the complexity of information dispersion. In the current setup, all traders learn information based on a common prior distribution of information regarding the timing of the initial mispricing. However, we did not consider allowing traders to (perhaps strategically) communicate with each other in order to coordinate an attack on the bubble.

Another possible extension in this vein is to take into account the heterogeneity among rational investors in terms of the amount of effort they devote to gathering new information. It might also be of interest to take into account the cost of obtaining information and see if traders would decide on an optimal level of effort to devote to discovering mispricings.

## 7.2 Measuring the Benefits of Information

As shown in Abreu and Brunnermeier (2002), arbitrage opportunities do exist and can be profitable in this model, albeit with some delay. With the incorporation of information shocks, we see that equilibrium profits may increase when the degree of common knowledge is greater among rational traders. In the future, we may analyze the relationship between the rate of information shocks and equilibrium arbitrage profits.

If we want to be consistent with efficient markets, we may try to see what happens whether the incorporation of information costs offset the benefits of the increase in availability of information.

### 7.3 Analytical Solution

For our analysis, we have not solved for an expression for the fraction of rational traders who have changed their position by time $t$. This formula would be what was referred to as $g(\cdot)$ in Section 5.3. Ideally, if possible, obtaining this expression would allow us to find an analytical expression for the equilibrium duration of mispricings. Furthermore, we may then do comparative statistics on how the degree of common knowledge affects the equilibrium solutions, further confirming our computational results.

## 8 Bibliography


[1] Abreu, D., and Brunnermeier, M. K. Synchronization risk and delayed arbitrage. *Journal of Financial Economics 66*, 2-3 (2002), 341 – 360.

[2] Abreu, D., and Brunnermeier, M. K. Bubbles and crashes. *Econometrica 71*, 1 (2003), 173–204.

[3] Allen, F., and Gorton, G. Churning bubbles. *Review of Economic Studies 60*, 4 (1993), 813–36.

[4] Blanchard, O. J., and Watson, M. W. Bubbles, rational expectations and financial markets. In *Crisis in the Economic and Financial Structure*. Lexington, MA: Lexington, 1982.

[5] Brunnermeier, M. K. *Bubbles: Entry in New Palgrave Dictionary of Economics*. 2009.

[6] Brunnermeier, M. K., and Nagel, S. Hedge funds and the technology bubble. *The Journal of Finance 59*, 5 (2004), 2013–2040.

[7] Kindleberger, C. P., and Aliber, R. Z. *Manias, Panics, and Crashes: A History of Financial Crises*, 5 ed. John Wiley & Sons, 2005.

[8] Lamont, O. A., and Thaler, R. H. Can the market add and subtract? mispricing in tech stock carve-outs. *Journal of Political Economy 111*, 2 (2003), pp. 227–268.

[9] Miller, E. M. Risk, uncertainty, and divergence of opinion. *The Journal of Finance 32*, 4 (1977), pp. 1151–1168.

[10] Ofek, E., and Richardson, M. Dotcom mania: The rise and fall of internet stock prices. *The Journal of Finance 58*, 3 (2003), pp. 1113–1137.

[11] Scheinkman, J., and Xiong, W. Heterogeneous beliefs, speculation and trading in financial markets. In *Paris-Princeton Lectures on Mathematical Finance 2003*, R. A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, J. A. Scheinkman, and N. Touzi, Eds., vol. 1847 of *Lecture Notes in Mathematics*. Springer Berlin / Heidelberg, 2004, pp. 223–233. 10.1007/978-3-540-44468-8_3.

[12] Shleifer, A., and Vishny, R. W. The limits of arbitrage. *The Journal of Finance 52*, 1 (1997), pp. 35–55.

[13] Yan, W., Woodard, R., and Sornette, D. Diagnosis and prediction of market rebounds in financial markets. Tech. rep., 2010.