GRAVITATIONAL WAVES

Evan Craft

1. Derive Gravitational Waves

2. Detection
   1. Geodesic Deviation
   2. Example

3. Solutions to the equations with source
   1. Quadrupole Approximation
   2. Example

4. Energy transported by the waves
   1. Depends on the amplitude
   2. Example

DERIVATION

$$g_{ab} = \eta_{ab} + h_{ab} \qquad ||h_{ab}|| \ll 1$$

Perturb the Minkowski metric

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$$G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab}R$$

Want to calculate the Einstein Tensor

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$$\Box = \partial_c\partial^c = \nabla^2 - \partial_t^2$$

Hope this gives some sort of wave equation

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$$R = R^a{}_a = \left(\partial_c \partial^a h^c{}_a - \Box h\right)$$

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Plugging in

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Plugging in

Really only want **this** part

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Simplifies a bit more

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$$\partial^a \bar{h}_{ab} = 0$$

Is there any way
to do this?

DERIVATION

Introduce a new **coordinate system** or "Gauge"

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Is this enough?

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For this to be satisfied

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$$\partial^a \bar{h}_{ab} = 0 \quad \Rightarrow \quad \Box \xi_b = \partial^a \bar{h}_{ab}$$

For this to be satisfied

We must have:

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This is just a homogenous wave equation which can be satisfied by integrating over the sources

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For this to be satisfied

We must have:

$$\Box \xi_b = \partial^a \bar{h}_{ab}$$

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Using this result,

$$G_{ab} = \frac{1}{2}\left(\partial_c\partial_b\bar{h}^c{}_a + \partial^c\partial_a\bar{h}_{bc} - \Box\bar{h}_{ab} - \eta_{ab}\partial_c\partial^d\bar{h}^c{}_d\right)$$

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$$\Box \bar{h}_{ab} = -16\pi T_{ab}$$

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When propagating in a certain direction z

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Two Polarization states

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{yy} & h_{yz} \\ 0 & 0 & h_{yz} & -h_{yy} \end{pmatrix}$$

Q: HOW DO GRAVITATIONAL WAVES AFFECT MATTER?

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IDEA: COULD THEY AFFECT THE PATHS PARTICLES TRAVEL ON?

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Equations of Motion

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No effect, a single particle is **not sufficient** to measure the effects of gravitational waves!

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Relative acceleration depends on the curvature tensor which is nonzero (in our case)!

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$$\Delta l = \int ds = \int_{y_A}^{y_B} |g_{yy}|^{\frac{1}{2}} dy = \int_{y_A}^{y_B} |1 + h^{TT}{}_{yy}(t - x/c)|^{\frac{1}{2}} dy \neq \text{constant}$$

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We will have a nonzero change in separation which can be measured

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$\left.\frac{dx^{\mu'}}{d\tau}\right|_A = (1, 0, 0, 0)$

GEODESIC DEVIATION (EXAMPLE)

Choose locally inertial frame centered on the geodesic of one of the particles, say A

Locally: $ds^2 = \eta_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} + O(|\delta x|^2)$

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$g_{\mu'\nu',\alpha'\,|A} = 0 \ \ (\text{i.e. } \Gamma^{\alpha'}_{\mu'\nu'\,|A} = 0)$

GEODESIC DEVIATION (EXAMPLE)

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The separation vector has the coordinates of the second particle, B

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Just need to calculate the Riemann tensor

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$$+ \quad g_{\nu\sigma}\left(\Gamma^\nu{}_{\kappa\lambda}\Gamma^\sigma{}_{\alpha\mu} - \Gamma^\nu{}_{\kappa\mu}\Gamma^\sigma{}_{\alpha\lambda}\right)$$

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Neglect terms second order in the perturbation

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$$R_{i00m} = \frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{00}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{i0}}{\partial x^0 \partial x^m} - \frac{\partial^2 h_{0m}}{\partial x^i \partial x^0}\right) = \frac{1}{2}h^{TT}_{im,00}$$

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Working in the transverse traceless Gauge,

$$h_{i0} = h_{00} = 0$$

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Hence we may now rewrite our equation

$$\frac{d^2}{dt^2}\delta x^{\lambda} = \frac{1}{2}\,\eta^{\lambda i}\,\frac{\partial^2 h^{TT}{}_{im}}{\partial t^2}\delta x^m$$

GEODESIC DEVIATION (EXAMPLE)

Initial Conditions: $t \leq 0$

$$\delta x_0^j = \quad \text{const}$$

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Using our eq: $$\frac{d^2}{dt^2}\delta x^\lambda = \frac{1}{2}\,\eta^{\lambda i}\,\frac{\partial^2 {h^{TT}}_{im}}{\partial t^2}\delta x^m$$

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$$\delta x^1 = \delta x_0^1 + \frac{1}{2}\,\eta^{11} {h^{TT}}_{1k}\,\delta x_0^k = \delta x_0^1$$
$$\delta x^2 = \delta x_0^2 + \frac{1}{2}\,\eta^{22} {h^{TT}}_{2k}\,\delta x_0^k = \delta x_0^2 + \frac{1}{2}\left({h^{TT}}_{22}\,\delta x_0^2 + {h^{TT}}_{23}\,\delta x_0^3\right)$$
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Only accelerated in plane orthogonal to the direction of propagation

GEODESIC DEVIATION (POLARIZATION)

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By our choice of Gauge:

$$h_{yy} = -h_{zz} = 2\Re\left\{A_+ e^{i\omega(t-\frac{x}{c})}\right\}$$

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Choose coordinates

$(0, y_0, 0)$ $(0, 0, z_0)$

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Make an assumption:

$A_+ \neq 0$ $A_\times = 0$

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Choose coordinates

$(0, y_0, 0)$ $(0, 0, z_0)$

Make an assumption:

$A_+ \neq 0$ $A_\times = 0$

$$h_{yy} = -h_{zz} = 2A_+ \cos\omega(t - \frac{x}{c})$$

$$h_{yz} = h_{zy} = 0$$

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Initial Conditions: $t = 0 \quad \omega(t - \frac{x}{c}) = \frac{\pi}{2}$

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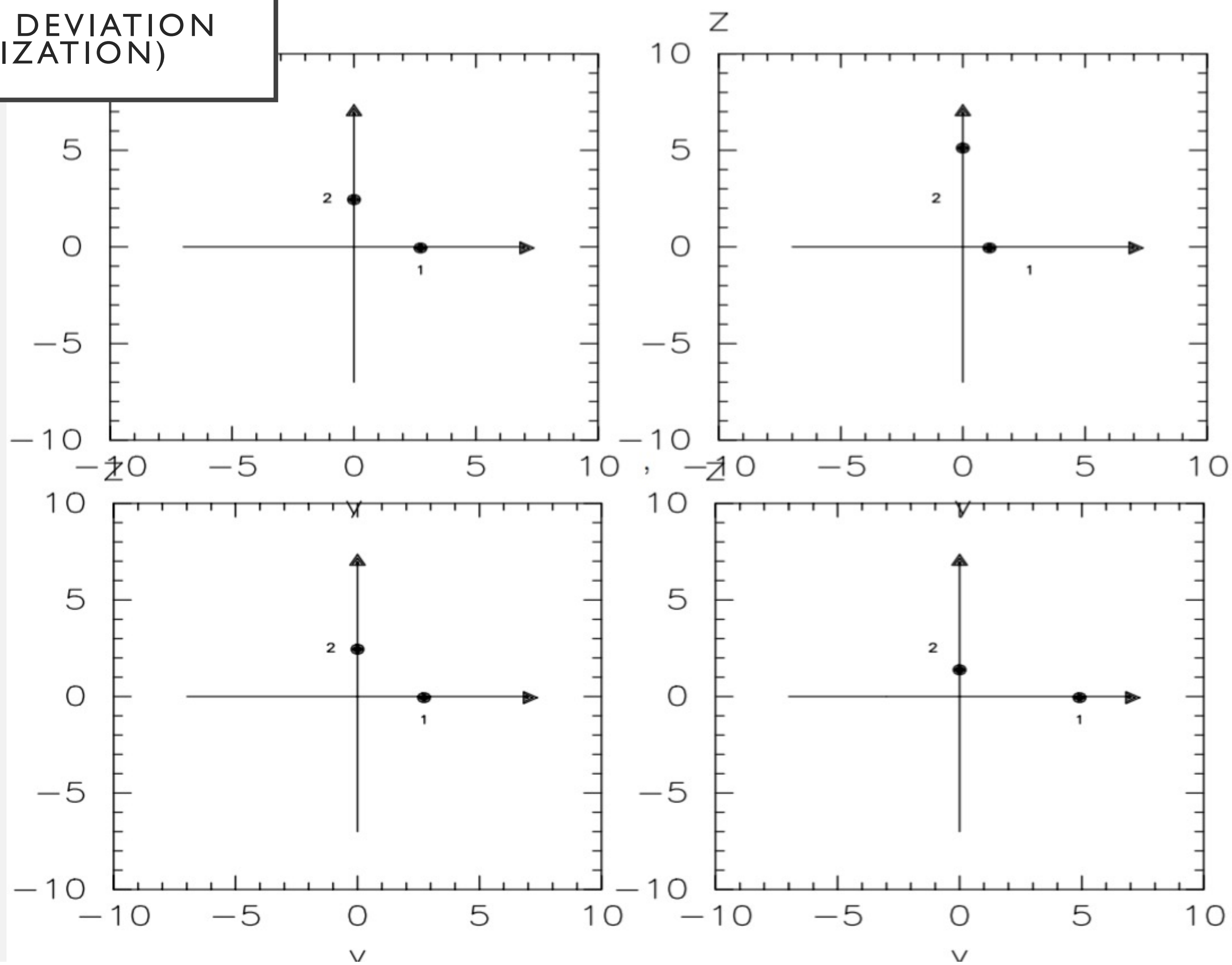
$t > 0$

$$1) \qquad z = 0, \quad y = y_0 + \frac{1}{2} h_{yy}\, y_0 = y_0\, [1 + A_+ \cos \omega (t - \frac{x}{c})],$$

$$2) \qquad y = 0, \quad z = z_0 + \frac{1}{2} h_{zz}\, z_0 = z_0 [1 - A_+ \cos \omega (t - \frac{x}{c})].$$

GEODESIC DEVIATION (POLARIZATION)



Q: HOW DO SOURCES AFFECT THINGS?

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IDEA: GREEN’S FUNCTIONS

SOLUTIONS WITH SOURCES

SOURCES

Want to solve: $\Box \bar{h}_{ab} = -16\pi T_{ab}$

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General Wave Equation:

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And you obtain a solution:

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And you obtain a solution: $f(t, \mathbf{x}) = \int dt' d^3x' \, G(t, \mathbf{x}; t', \mathbf{x}') \, s(t', \mathbf{x}')$

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Green's Function:

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Green's Function:

$$G(t, \mathbf{x}; t', \mathbf{x}') = -\frac{\delta(t' - [t - |\mathbf{x} - \mathbf{x}'|/c])}{4\pi|\mathbf{x} - \mathbf{x}'|}$$

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Solution:

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Solution:

$$\bar{h}_{ab}(t, \mathbf{x}) = 4 \int d^3x' \frac{T_{ab}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

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Kind of a tough integral!

QUADRUPOLE APPROXIMATION

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General Solution:

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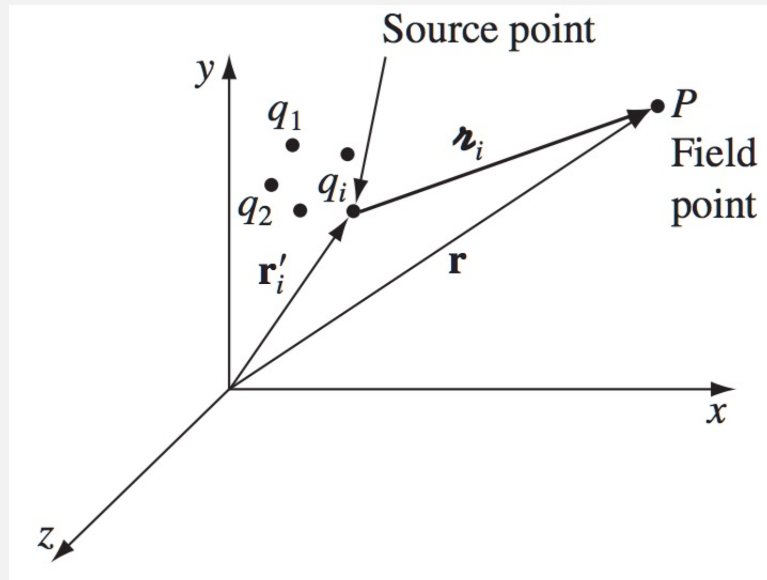

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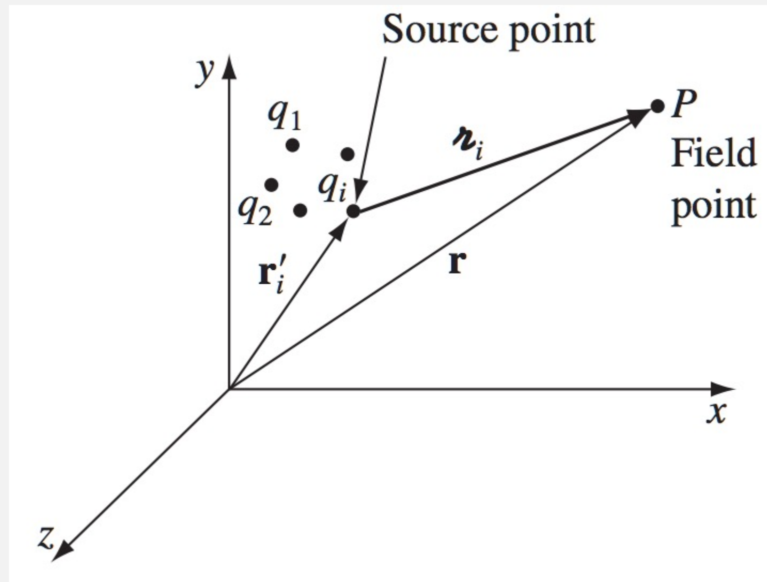
Work Far from the source:



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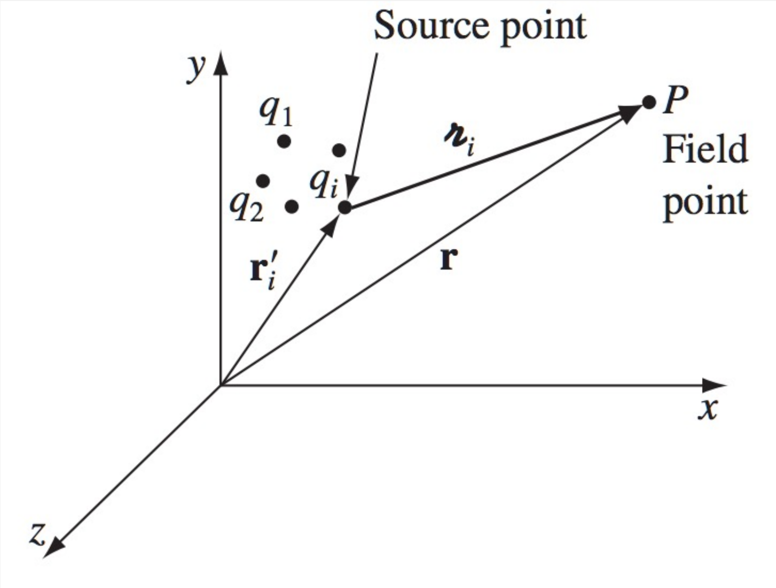
$\Rightarrow$ $r = |\mathbf{x}|$



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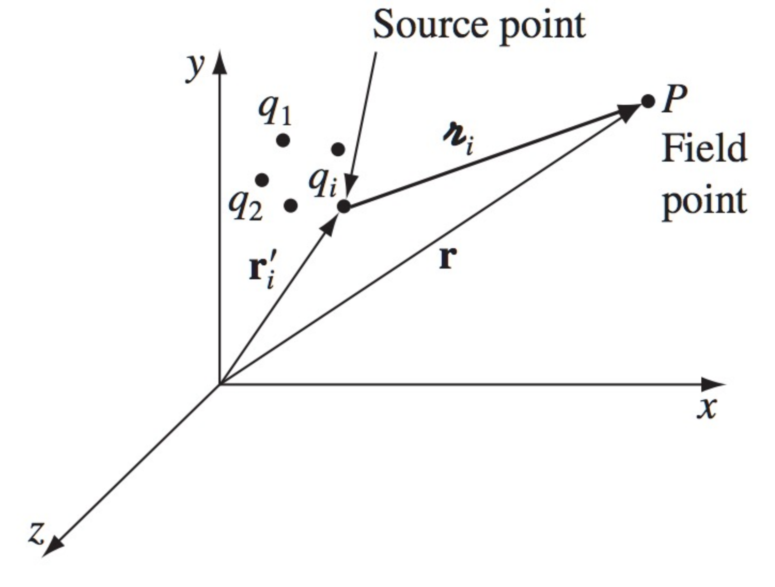
Work Far from the source:

$$r = |\mathbf{x}|$$

$$T_{ij}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') \approx T_{ij}(t-r,\mathbf{x}')$$



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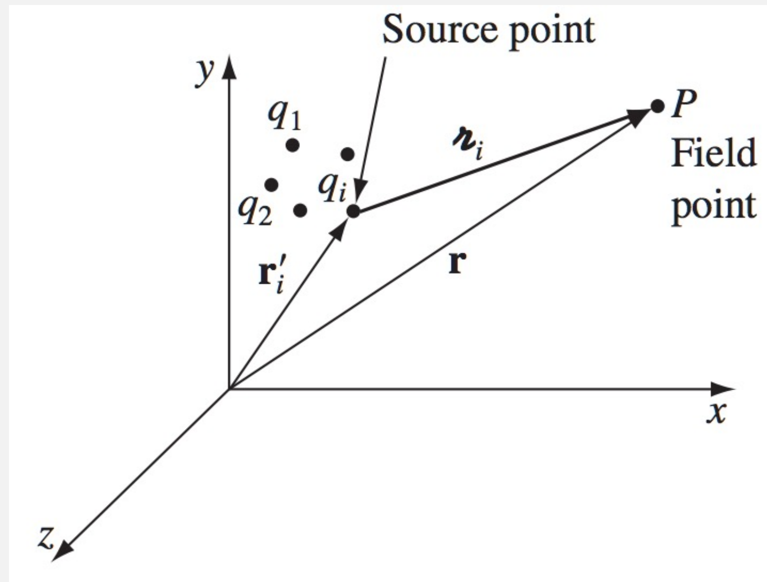

Simplifies a bit

$$\bar{h}_{ij}(t,\mathbf{x}) = \frac{4}{r}\int d^3x' \, T^{ij}(t-r,\mathbf{x}')$$

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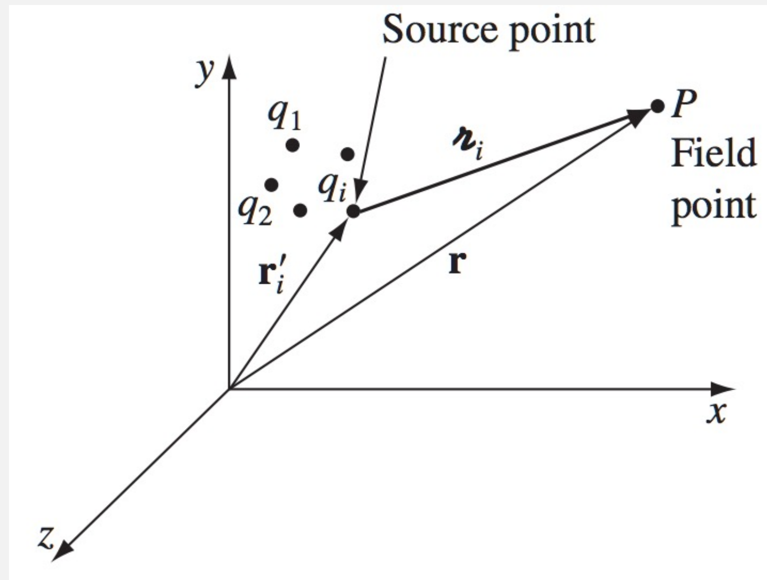

Simplifies a bit

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Is this useful? $\partial_a T^{ab} = 0$

QUADRUPOLE APPROXIMATION

$$\partial_a T^{ab} = 0$$

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$$\partial_a T^{ab} = 0$$

Separate into spacial and temporal parts

$$\partial_t T^{tt} + \partial_i T^{ti} = 0$$
$$\partial_t T^{ti} + \partial_j T^{ij} = 0$$

QUADRUPOLE APPROXIMATION

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Combine: $\partial_t^2 T^{tt} = \partial_k \partial_l T^{kl}$

QUADRUPOLE APPROXIMATION

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$$\partial_t T^{ti} + \partial_j T^{ij} = 0$$

Combine: $\partial_t^2 T^{tt} = \partial_k \partial_l T^{kl}$

Multiply by a product: $\partial_t^2 T^{tt} x^i x^j = \partial_t^2 \left(T^{tt} x^i x^j\right)$

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QUADRUPOLE APPROXIMATION (EXAMPLE)

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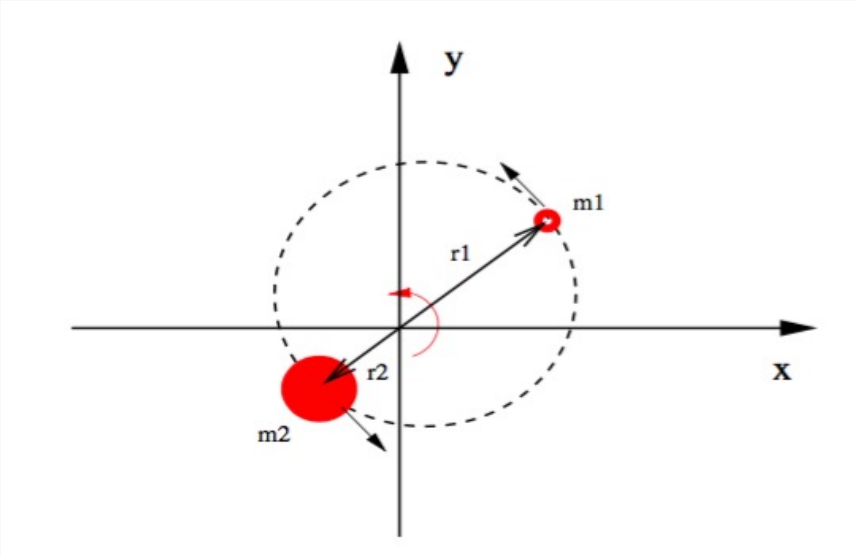

QUADRUPOLE (EXAMPLE)

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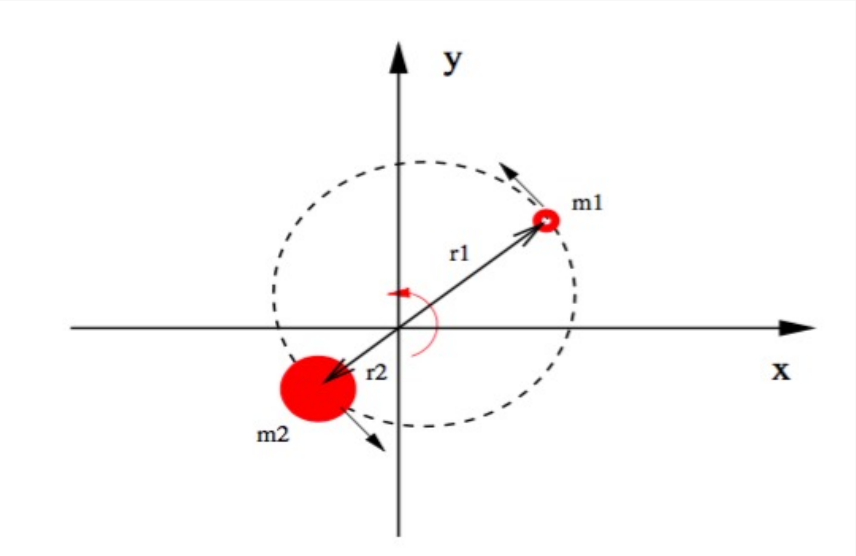

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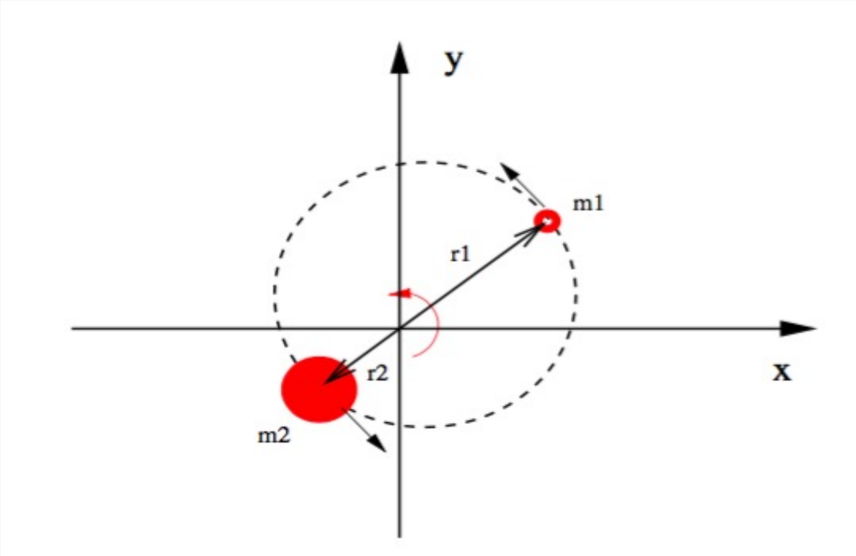

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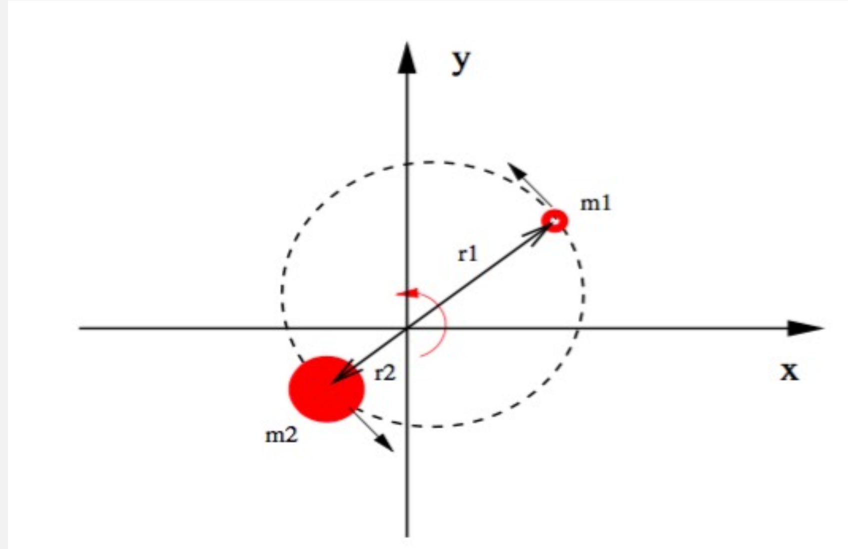

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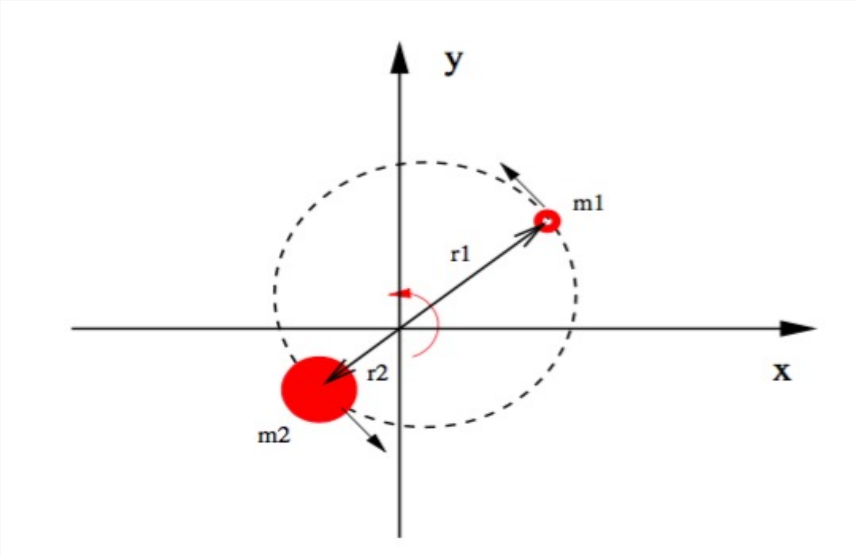

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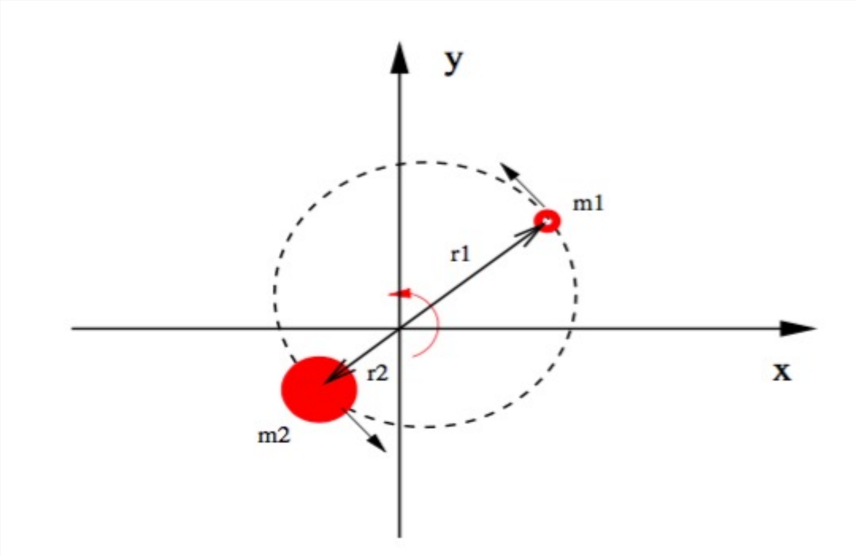

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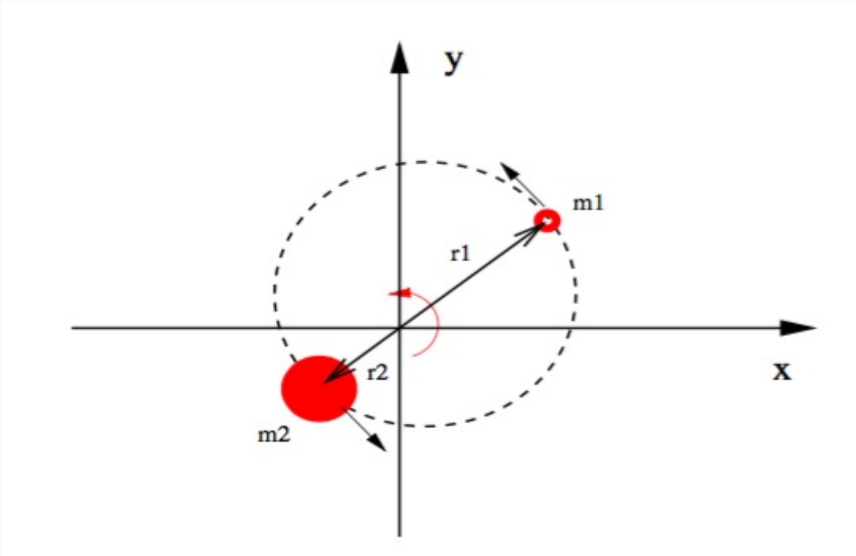

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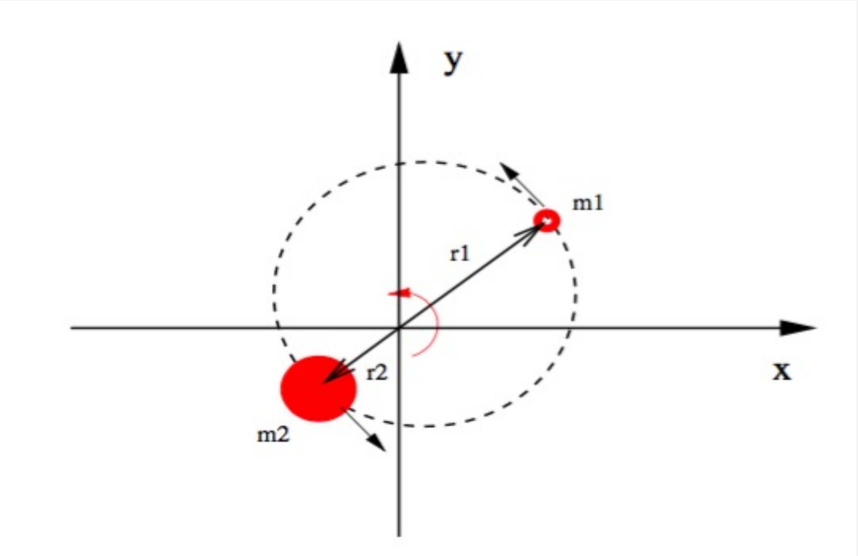

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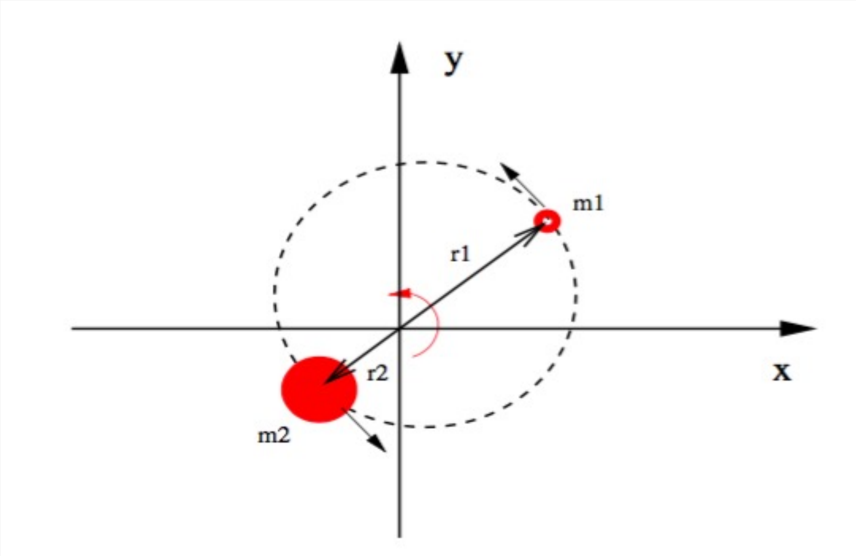

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Where:

$$A_{ij}(t) = \begin{pmatrix} \cos \, 2\omega_K t & \sin \, 2\omega_K t & 0 \\ \sin \, 2\omega_K t & -\cos \, 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Remember: $$h_{ij}^{\mathrm{TT}}(t,r) = \frac{2G}{rc^4}\frac{d^2}{dt^2}\left[Q_{ij}^{\mathrm{TT}}(t-\frac{r}{c})\right]$$

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A: IT’S COMPLICATED

STRESS ENERGY PSEUDO-TENSOR OF THE GRAVITATIONAL FIELD

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Let's try this where the R.H.S quantity acted on is antisymmetric in it's final two indices

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Einstein's Equations: $$T^{\mu\nu} = \frac{c^4}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)$$

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Riemann Tensor: $$R_{\gamma\alpha\delta\beta} = \frac{1}{2}\left[\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right]$$

$$+g_{\sigma\rho}\left(\Gamma^\sigma_{\alpha\delta}\Gamma^\rho_{\gamma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\gamma\delta}\right)$$

GRAVITATIONAL PSEUDO-TENSOR

Einstein's Equations:

$$T^{\mu\nu} = \frac{c^4}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)$$

Riemann Tensor:

$$R_{\gamma\alpha\delta\beta} = \frac{1}{2}\left[\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right]$$

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In a Locally Inertial Frame:

GRAVITATIONAL PSEUDO-TENSOR

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$$R^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} = g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}R_{\gamma\alpha\delta\beta}$$

GRAVITATIONAL PSEUDO-TENSOR

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$$R^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} = g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}R_{\gamma\alpha\delta\beta}$$

$$= \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}\left(\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right)$$

GRAVITATIONAL PSEUDO-TENSOR

Little bit of work…

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\left\{\frac{c^4}{16\pi G}\frac{1}{(-g)}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right]\right\}$$

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But we found what we were looking for

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\eta^{\mu\nu\alpha}$$

GRAVITATIONAL PSEUDO-TENSOR

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But we found what we were looking for

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\eta^{\mu\nu\alpha}$$

$$\Rightarrow \quad \eta^{\mu\nu\alpha} = \frac{c^4}{16\pi G}\frac{1}{(-g)}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right]$$

GRAVITATIONAL PSEUDO-TENSOR

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Define:

$$\zeta^{\mu\nu\alpha} = (-g)\eta^{\mu\nu\alpha} = \frac{c^4}{16\pi G}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right]$$

GRAVITATIONAL PSEUDO-TENSOR

In this locally inertial frame:

$$\frac{\partial}{\partial x^\beta}\frac{1}{(-g)} = 0$$

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Using this, $\Rightarrow$

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GRAVITATIONAL PSEUDO-TENSOR

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Using this, $\Rightarrow$

$$\frac{\partial\zeta^{\mu\nu\alpha}}{\partial x^\alpha} = (-g)T^{\mu\nu}$$

Define:

$$(-g)t^{\mu\nu} = \frac{\partial\zeta^{\mu\nu\alpha}}{\partial x^\alpha} - (-g)T^{\mu\nu}$$

GRAVITATIONAL PSEUDO-TENSOR

$$\begin{aligned}
t^{\mu\nu} &= \frac{c^4}{16\pi G}\Big\{\left(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\right)\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\right) \\
&+ g^{\mu\alpha}g^{\beta\delta}\left(\Gamma^{\nu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\nu}{}_{\beta\delta}\Gamma^{\sigma}{}_{\alpha\sigma} - \Gamma^{\nu}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma}\right) \\
&+ g^{\nu\alpha}g^{\beta\delta}\left(\Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\sigma}{}_{\alpha\sigma} - \Gamma^{\mu}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma}\right) \\
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In any frame (not just a locally inertial one) we have:

$$(-g)\left(t^{\mu\nu} + T^{\mu\nu}\right) = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^{\alpha}}$$

GRAVITATIONAL PSEUDO-TENSOR

$$
\begin{aligned}
t^{\mu\nu} \;=\; & \frac{c^4}{16\pi G}\Big\{\left(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\right)\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\right) \\
+ \; & g^{\mu\alpha}g^{\beta\delta}\left(\Gamma^{\nu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\nu}{}_{\beta\delta}\Gamma^{\sigma}{}_{\alpha\sigma} - \Gamma^{\nu}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma}\right) \\
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And since:

$$\frac{\partial}{\partial x^{\mu}}\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^{\alpha}} = 0,$$

GRAVITATIONAL PSEUDO-TENSOR

$$\begin{aligned} t^{\mu\nu} &= \frac{c^4}{16\pi G}\Big\{\left(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\right)\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\right) \\ &+ g^{\mu\alpha}g^{\beta\delta}\left(\Gamma^{\nu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\nu}{}_{\beta\delta}\Gamma^{\sigma}{}_{\alpha\sigma} - \Gamma^{\nu}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma}\right) \\ &+ g^{\nu\alpha}g^{\beta\delta}\left(\Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\sigma}{}_{\alpha\sigma} - \Gamma^{\mu}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma}\right) \\ &+ g^{\alpha\beta}g^{\delta\sigma}\left(\Gamma^{\mu}{}_{\alpha\delta}\Gamma^{\nu}{}_{\beta\sigma} - \Gamma^{\mu}{}_{\alpha\beta}\Gamma^{\nu}{}_{\delta\sigma}\right)\Big\} \end{aligned}$$

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And since:

$$\frac{\partial}{\partial x^\mu}\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} = 0,$$

We then have:

$$\frac{\partial}{\partial x^\mu}\left[(-g)\left(t^{\mu\nu} + T^{\mu\nu}\right)\right] = 0.$$

GRAVITATIONAL PSEUDO-TENSOR

$$
\begin{aligned}
t^{\mu\nu} &= \frac{c^4}{16\pi G}\Big\{\left(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\right)\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\right) \\
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This object vanishes identically in a locally inertial frame, interpreted as containing information about the SE of the Gravitational field

GRAVITATIONAL PSEUDO-TENSOR

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t^{\mu\nu} \;=\; & \frac{c^4}{16\pi G}\Big\{\Big(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\Big)\Big(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\Big) \\
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But it's not a tensor….

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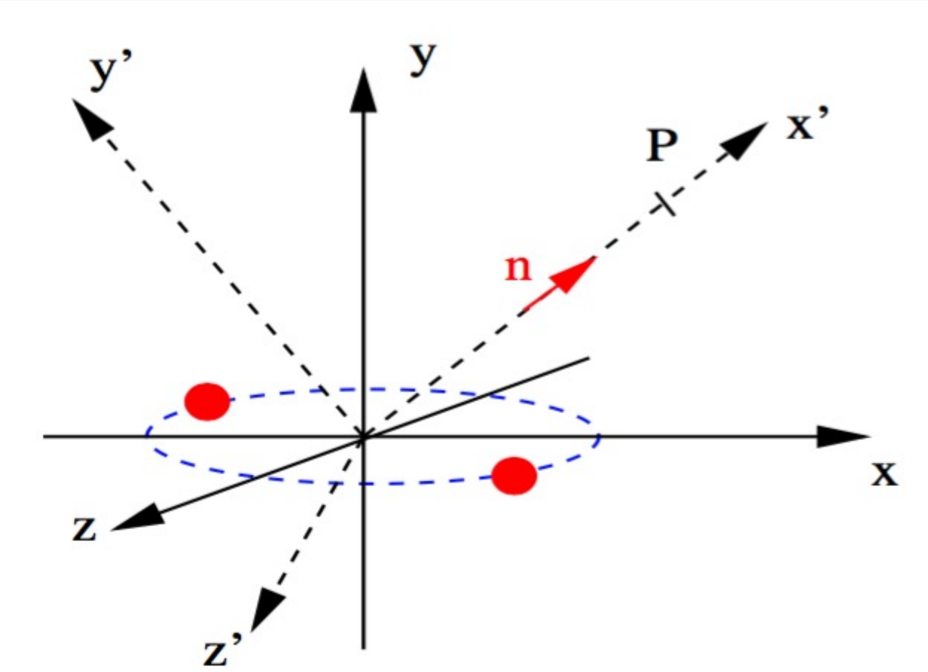
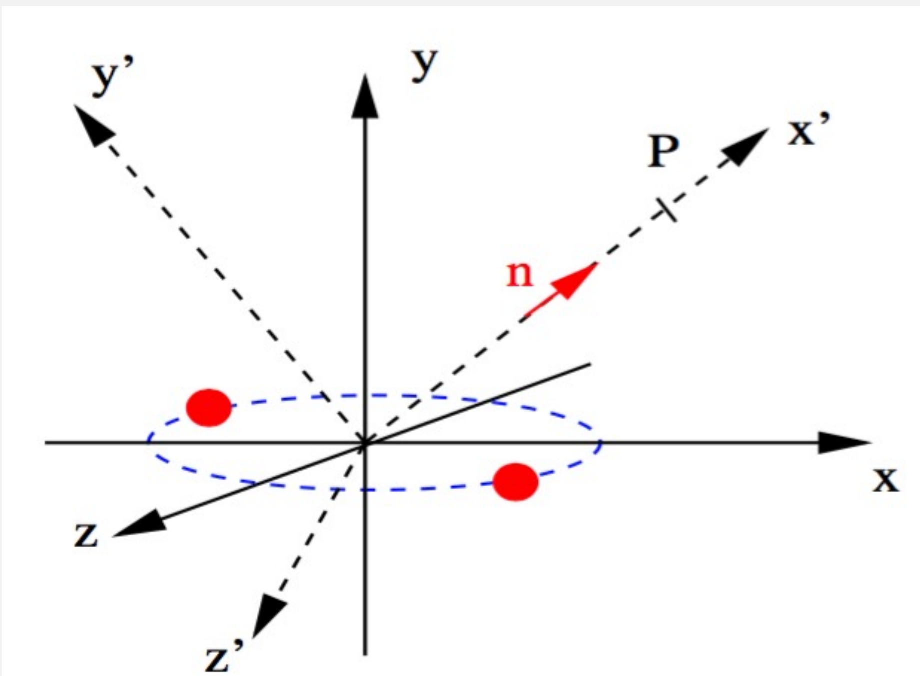
STRESS ENERGY PSEUDO-TENSOR (EXAMPLE)

PSEUDO-TENSOR
(EXAMPLE)



PSEUDO-TENSOR (EXAMPLE)

$$
g_{\mu'\nu'} = \begin{array}{c}
\begin{array}{cccc} (ct) & (x') & (y') & (z') \end{array} \\
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & [1 + h_{+}^{\mathbf{TT}}(t, x')] & h_{\times}^{\mathbf{TT}}(t, x') \\
0 & 0 & h_{\times}^{\mathbf{TT}}(t, x') & [1 - h_{+}^{\mathbf{TT}}(t, x')]
\end{pmatrix}
\end{array}
$$



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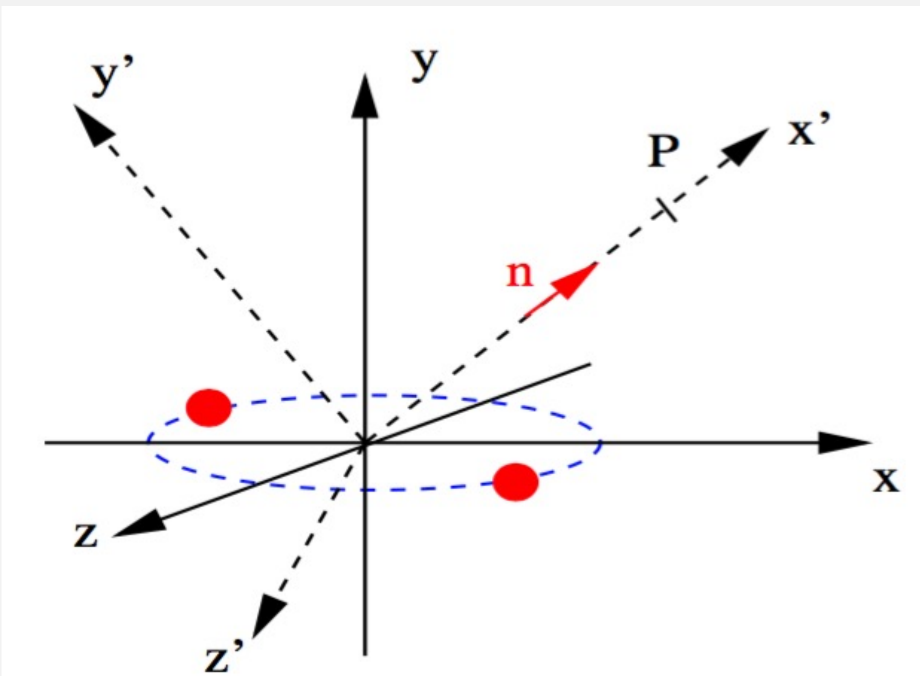

Remember: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

PSEUDO-TENSOR (EXAMPLE)

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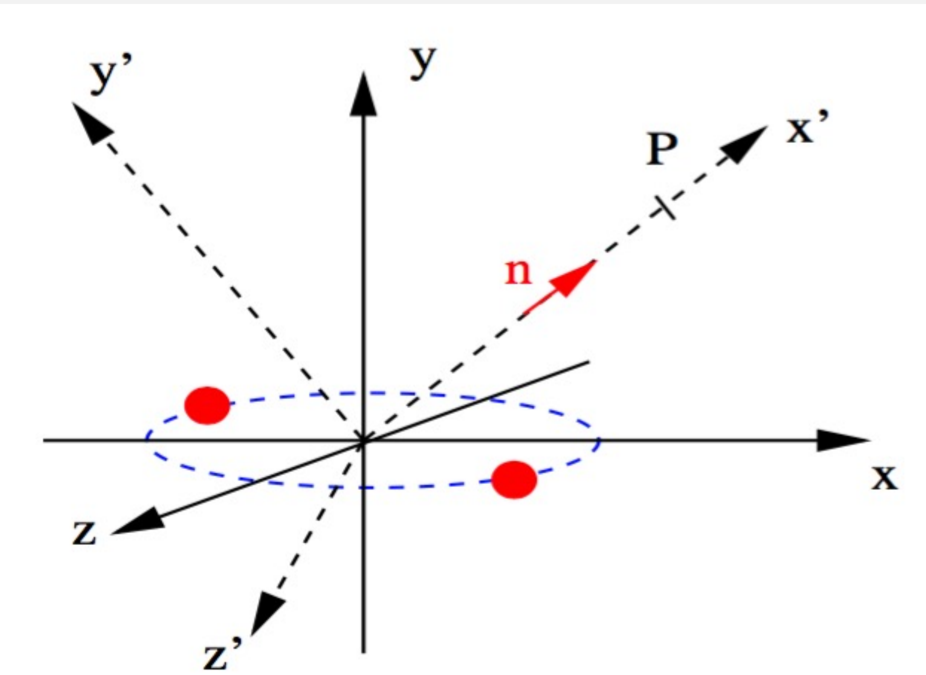

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Energy flux in the direction of propagation?:

PSEUDO-TENSOR (EXAMPLE)

Calculate Christoffel symbols from metric, pseudo-tensor follows

PSEUDO-TENSOR (EXAMPLE)

Calculate Christoffel symbols from metric, pseudo-tensor follows

$$ct^{0x'} = \frac{dE_{GW}}{dtdS} = \frac{c^3}{16\pi G}\left[\left(\frac{dh_+^{\mathbf{TT}}(t,x')}{dt}\right)^2 + \left(\frac{dh_\times^{\mathbf{TT}}(t,x')}{dt}\right)^2\right]$$

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$$= \frac{c^3}{32\pi G}\left[\sum_{jk}\left(\frac{dh_{jk}^{\mathbf{TT}}(t,x')}{dt}\right)^2\right]$$

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Calculate Christoffel symbols from metric, pseudo-tensor follows

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$$= \frac{c^3}{32\pi G}\left[\sum_{jk}\left(\frac{dh_{jk}^{\mathbf{TT}}(t,x')}{dt}\right)^2\right]$$

$$t^{0r} = \frac{c^2}{32\pi G}\left[\sum_{jk}\left(\frac{dh_{jk}^{\mathbf{TT}}(t,r)}{dt}\right)^2\right]$$

Axis was arbitrary, new frame just rewrites perturbation in TT gauge associated with direction

PSEUDO-TENSOR (EXAMPLE)

Energy of Gravitational field cannot be defined locally, average over wavelengths

PSEUDO-TENSOR (EXAMPLE)

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PSEUDO-TENSOR (EXAMPLE)

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Using the gauge along with our results,

$$\begin{cases} h^{\mathbf{TT}}{}_{\mu 0} = 0, \qquad \mu = 0,3 \\ h^{\mathbf{TT}}_{ik}(t,r) = \dfrac{2G}{c^4 r} \cdot \left[\dfrac{d^2}{dt^2} \, Q^{\mathbf{TT}}_{ik} \left(t - \dfrac{r}{c} \right) \right] \end{cases}$$

PSEUDO-TENSOR (EXAMPLE)

Energy of Gravitational field cannot be defined locally, average over wavelengths

$$\frac{dE_{GW}}{dtdS} = \left\langle ct^{0r} \right\rangle = \frac{c^3}{32\pi G} \left\langle \sum_{jk} \left(\frac{dh_{jk}^{\mathbf{TT}}(t,r)}{dt} \right)^2 \right\rangle$$

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$$\Rightarrow \quad \frac{dE_{GW}}{dtdS} = \frac{G}{8\pi c^5\, r^2} \left\langle \sum_{jk} \left(\dddot{Q}_{jk}^{\mathbf{TT}} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle$$

PSEUDO-TENSOR (EXAMPLE)

How about a simpler quantity?

$$L_{GW} = \frac{dE_{GW}}{dt}$$

PSEUDO-TENSOR (EXAMPLE)

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$$\Rightarrow \quad L_{GW} = \int \frac{dE_{GW}}{dtdS}\, dS = \int \frac{dE_{GW}}{dtdS}\, r^2 d\Omega$$

PSEUDO-TENSOR (EXAMPLE)

How about a simpler quantity?

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Know:

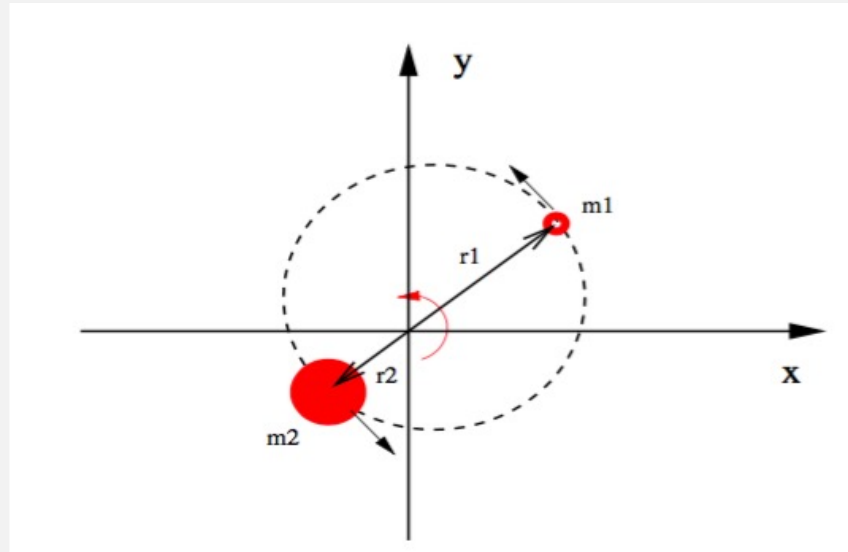
$$\frac{dE_{GW}}{dt\,dS} = \frac{G}{8\pi c^5\, r^2} \left\langle \sum_{jk} \left(\dddot{Q}^{\mathrm{TT}}_{jk} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle$$

$$L_{GW} = \int \frac{dE_{GW}}{dt\,dS}\, dS = \int \frac{dE_{GW}}{dt\,dS}\, r^2 d\Omega$$

$$L_{GW} = \frac{G}{5c^5} \left\langle \sum_{k,n=1}^{3} \dddot{Q}_{kn} \left(t - \frac{r}{c} \right) \dddot{Q}_{kn} \left(t - \frac{r}{c} \right) \right\rangle$$

PSEUDO-TENSOR (EXAMPLE)

Remember our binary system?



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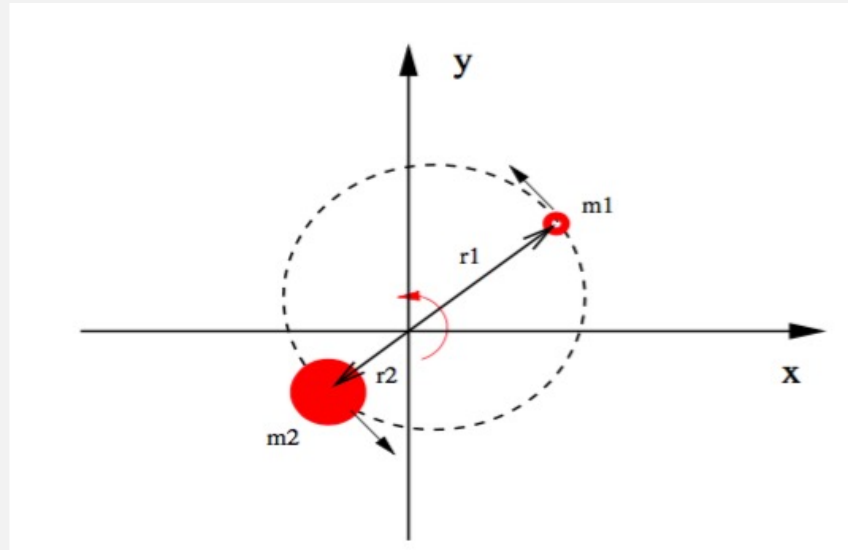

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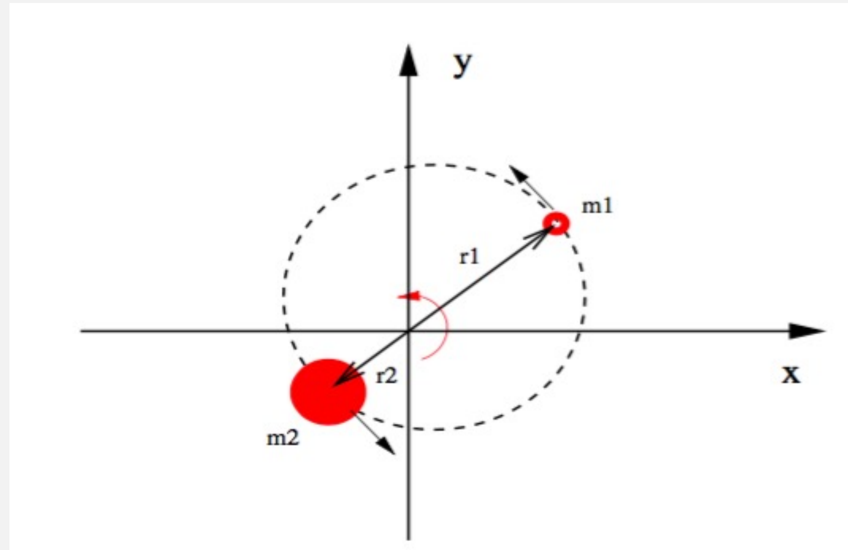

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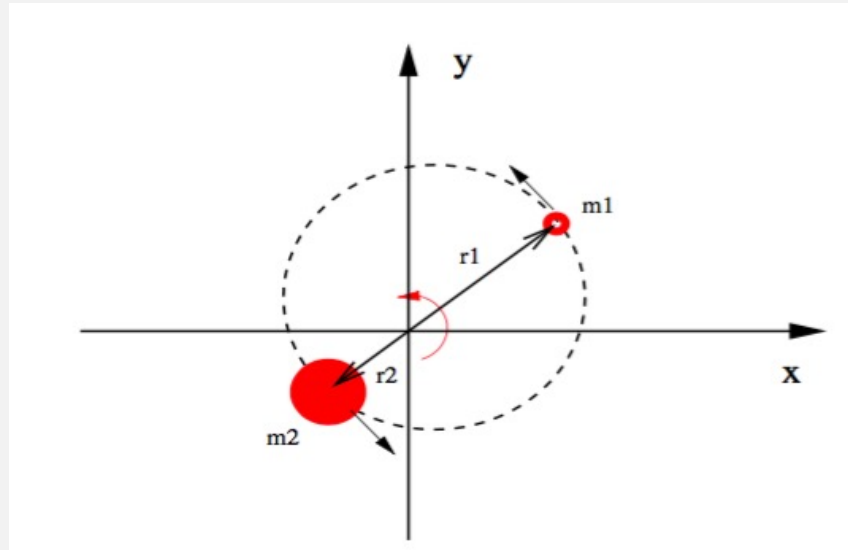

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$$\Rightarrow \quad L_{GW} \equiv \frac{dE_{GW}}{dt} = \frac{32}{5}\ \frac{G^4}{c^5}\ \frac{\mu^2 M^3}{l_0^5}$$

PSEUDO-TENSOR (EXAMPLE)

Change in the period?

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Adiabatic Approximation: Orbital parameters do not change

$$\frac{dE_{orb}}{dt} + L_{GW} = 0$$

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Know: $E_{orb} = E_K + U$

PSEUDO-TENSOR (EXAMPLE)

Change in the period?

$$E_K = \frac{1}{2}m_1\omega_K^2\, r_1^2 + \frac{1}{2}m_2\omega_K^2\, r_2^2$$

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$$= \frac{1}{2}\omega_K^2\left[\frac{m_1 m_2^2 l_0^2}{M^2} + \frac{m_2 m_1^2 l_0^2}{M^2}\right]$$

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PSEUDO-TENSOR (EXAMPLE)

Change in the period?

$$
\begin{aligned}
E_K &= \frac{1}{2} m_1 \omega_K^2\, r_1^2 + \frac{1}{2} m_2 \omega_K^2\, r_2^2 \\
&= \frac{1}{2} \omega_K^2 \left[\frac{m_1 m_2^2 l_0^2}{M^2} + \frac{m_2 m_1^2 l_0^2}{M^2} \right] \\
&= \frac{1}{2} \omega_K^2 \mu l_0^2 = \frac{1}{2} \frac{G\mu M}{l_0}
\end{aligned}
$$

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$$= \frac{1}{2}\omega_K^2 \mu l_0^2 = \frac{1}{2}\frac{G\mu M}{l_0}$$

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$$\Rightarrow \quad E_{orb} = -\frac{1}{2} \frac{G\mu M}{l_0}$$

PSEUDO-TENSOR (EXAMPLE)

Change in the period?

Taking the derivative, only variable is the separation:

$$\frac{dE_{orb}}{dt} = \frac{1}{2}\frac{G\mu M}{l_0}\left(\frac{1}{l_0}\frac{dl_0}{dt}\right) = -E_{orb}\left(\frac{1}{l_0}\frac{dl_0}{dt}\right)$$

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Using Kepler's Law for the orbital frequency

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$$\omega_K^2 = GMl_0^{-3}$$

$$\Rightarrow \quad \frac{dE_{orb}}{dt} = \frac{2}{3}\frac{E_{orb}}{\omega_K}\frac{d\omega_K}{dt}$$

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Change in the period?

By definition: $\omega_K = 2\pi P^{-1}$

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From the previous result we then have,

$$\frac{dP}{dt} = -\frac{3}{2}\frac{P}{E_{orb}}\frac{dE_{orb}}{dt}$$

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PSEUDO-TENSOR
(EXAMPLE)

PSR 1913+16

PSEUDO-TENSOR (EXAMPLE)

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$$P = 27907\ s, \qquad E_{orb} \sim -1.4 \cdot 10^{48}\ erg, \qquad L_{GW} \sim 0.7 \cdot 10^{31}\ erg/s$$

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$$\frac{dP}{dt} \sim -2.2 \cdot 10^{-13}$$

Adjust for eccentricity
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$$\epsilon \simeq 0.617$$

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Hulse and Taylor win
1993 Nobel Prize:

$$\frac{dP}{dt} = -(2.4184 \pm 0.0009) \cdot 10^{-12}$$

PSEUDO-TENSOR (EXAMPLE)

Change in the separation?

PSEUDO-TENSOR (EXAMPLE)

Change in the separation?

From our previous results: $$\frac{1}{l_0}\frac{dl_0}{dt} = \frac{L_{GW}}{E_{orb}} = -\left[\frac{64}{5}\frac{G^3}{c^5}\,\mu\,M^2\right]\cdot\frac{1}{l_0^4}$$

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Integrating: → $$l_0^4(t) = (l_0^{in})^4 - \frac{256}{5}\frac{G^3}{c^5}\,\mu\, M^2\, t.$$

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Integrating:
$$\Longrightarrow \quad l_0^4(t) = (l_0^{in})^4 - \frac{256}{5}\frac{G^3}{c^5}\,\mu\,M^2\,t.$$

Define:
$$t_{coal} = \frac{5}{256}\frac{c^5}{G^3}\frac{\left(l_0^{in}\right)^4}{\mu M^2}$$

PSEUDO-TENSOR
(EXAMPLE)

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Define: $$t_{coal} = \frac{5}{256}\frac{c^5}{G^3}\frac{(l_0^{in})^4}{\mu M^2}$$

Position's are equal at t-coalesce as objects have been treated as point-like

→ $$l_0(t) = l_0^{in}\left[1 - \frac{t}{t_{coal}}\right]^{1/4}$$

1. Derive Gravitational Waves

2. Detection
   1. Geodesic Deviation
   2. Example

3. Solutions to the equations with source
   1. Quadrupole Approximation
   2. Example

4. Energy transported by the waves
   1. Depends on the amplitude
   2. Example

STRESS ENERGY TENSOR (ALTERNATE)

$$S[\phi_a, \partial_\mu \phi_a] \equiv \int d^4x \, \mathcal{L}\big(\phi_a(x), \partial_\mu \phi_a(x)\big)$$

$$\delta S = \int \Big(\delta\big(d^4x\big)\mathcal{L} + d^4x \, \delta \mathcal{L}\Big)$$

$$\delta S = \int d^4x \, \partial_\mu \mathcal{J}^\mu = 0$$

$$\mathcal{J}^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta\phi_a - T^{\mu\nu} \, \delta x_\nu$$

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \partial^\nu \phi_a - \mathcal{L} \, \eta^{\mu\nu}$$

$$\mathcal{J}^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta\phi_a - T^{\mu\nu} \, \delta x_\nu$$

$$\delta\phi_a = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\delta S = \int d^4x \, \partial_\mu \mathcal{J}^\mu = 0$$

$$\mathcal{L} = \frac{1}{64\pi G} \left[\partial_\alpha h \, \partial^\alpha h + 2\partial_\alpha h_{\beta\gamma} \, \partial^\beta h^{\alpha\gamma}\right.$$

$$\left. -2\partial^\alpha h \, \partial_\beta h^\beta{}_\alpha - \partial_\gamma h_{\alpha\beta} \, \partial^\gamma h^{\alpha\beta}\right]$$

$$\frac{\partial \mathcal{L}}{\partial h_{\alpha\beta}} - \partial_\gamma \frac{\partial \mathcal{L}}{\partial(\partial_\gamma h_{\alpha\beta})} = \frac{1}{2}\partial_\gamma \partial^\gamma h^{\mathrm{TT}}_{\alpha\beta} = 0$$

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \partial^\nu \phi_a - \mathcal{L} \, \eta^{\mu\nu}$$

$$\Theta^\alpha{}_\beta = \frac{1}{32\pi G} \left\langle \partial^\alpha h^{TT}_{\gamma\delta} \, \partial_\beta h^{\gamma\delta}_{TT} \right\rangle$$

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \partial^\nu \phi_a - \mathcal{L} \, \eta^{\mu\nu}$$

$$T_B^{\mu\nu} = T^{\mu\nu} + \frac{1}{2} \partial_\lambda (S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu})$$

$$\Theta^\alpha{}_\beta = \frac{1}{32\pi G} \left\langle \partial^\alpha h_{\gamma\delta}^{TT} \, \partial_\beta h_{TT}^{\gamma\delta} \right\rangle$$

$$\Theta^{\mu\nu} + \Delta^{\mu\nu}$$

$$S=\int\left[\frac{1}{2\kappa}R+\mathcal{L}_{\mathrm{M}}\right]\sqrt{-g}\,\mathrm{d}^4x$$

$$0=\delta S$$

$$\frac{\delta R}{\delta g^{\mu\nu}}+\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}=-2\kappa\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{M}})}{\delta g^{\mu\nu}},$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi G}{c^4}T_{\mu\nu}$$

$$T_{\mu\nu}:=\frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{M}})}{\delta g^{\mu\nu}}$$

$$T^{\mu\nu}\equiv\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a-\mathcal{L}\,\eta^{\mu\nu}$$

$$T_B^{\mu\nu} = \qquad \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{M}})}{\delta g^{\mu\nu}}$$

$$\Theta^{\alpha}{}_{\beta} = \frac{1}{32\pi G} \left\langle \partial^{\alpha} h^{TT}_{\gamma\delta}\, \partial_{\beta} h^{\gamma\delta}_{TT} \right\rangle \qquad \Theta^{\mu\nu} + \Delta^{\mu\nu}$$

$$g_{ab} = g_{ab}^{\mathrm{B}} + \varepsilon h_{ab} + \varepsilon^2 j_{ab} + O(\varepsilon^3)$$

$$0 = G_{ab}$$

$$= G_{ab}[g_{cd}^{\mathrm{B}}] + \varepsilon G_{ab}^{(1)}[h_{cd}; g_{ef}^{\mathrm{B}}] + \varepsilon^2 G_{ab}^{(1)}[j_{cd}; g_{ef}^{\mathrm{B}}] + \varepsilon^2 G_{ab}^{(2)}[h_{cd}; g_{ef}^{\mathrm{B}}]$$

$$+ O(\varepsilon^3)$$

Einstein Tensor of the background metric

$$G_{ab}[g_{cd}^{\mathrm{B}}]$$

First order pertubation about the background metric

$$G_{ab}^{(1)}[\ldots; g_{ef}^{\mathrm{B}}]$$

Quadratic terms in the pertubation

$$G_{ab}^{(2)}[h_{cd}; g_{ef}^{\mathrm{B}}]$$