GRAVITATIONAL WAVES

Evan Craft

DARTMOUTH

1. Derive Gravitational Waves

2. Detection
    1. Geodesic Deviation
    2. Example

3. Solutions to the equations with source
    1. Quadrupole Approximation
    2. Example

4. Energy transported by the waves
    1. Depends on the amplitude
    2. Example

DERIVATION

$$g_{ab} = \eta_{ab} + h_{ab} \qquad ||h_{ab}|| \ll 1$$

Perturb the Minkowski metric

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$$G_{ab} = R_{ab} - \frac{1}{2}\eta_{ab}R$$

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$$\Box = \partial_c \partial^c = \nabla^2 - \partial_t^2$$

Hope this gives some sort of wave equation

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$$R = R^a{}_a = \left(\partial_c \partial^a h^c{}_a - \Box h\right)$$

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$$\left. - \eta_{ab}\partial_c \partial^d h^c{}_d + \eta_{ab}\Box h\right)$$

Plugging in

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Plugging in

Really only want **this** part

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$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

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Simplifies a bit more

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$$\partial^a\bar{h}_{ab} = 0$$

Is there any way to do this?

DERIVATION

Introduce a new **coordinate system** or "Gauge"

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For this to be satisfied

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$$\partial^a \bar{h}_{ab} = 0 \quad \Rightarrow \quad \Box \xi_b = \partial^a \bar{h}_{ab}$$

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We must have:

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This is just a homogenous wave equation which can be satisfied by integrating over the sources

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For this to be satisfied

$$\Box \xi_b = \partial^a \bar{h}_{ab}$$

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Using this result,

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When propagating in a certain direction z

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Two Polarization states

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{yy} & h_{yz} \\ 0 & 0 & h_{yz} & -h_{yy} \end{pmatrix}$$

Q: HOW DO GRAVITATIONAL WAVES AFFECT MATTER?

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IDEA: COULD THEY AFFECT THE PATHS PARTICLES TRAVEL ON?

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Equations of Motion

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No effect, a single particle is **not sufficient** to measure the effects of gravitational waves!

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Relative acceleration depends on the curvature tensor which is nonzero (in our case)!

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$$\Delta l = \int ds = \int_{y_A}^{y_B} |g_{yy}|^{\frac{1}{2}} dy = \int_{y_A}^{y_B} |1 + h^{TT}{}_{yy}(t - x/c)|^{\frac{1}{2}} dy \neq \text{constant}$$

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We will have a nonzero change in separation which can be measured

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$\frac{dx^{\mu'}}{d\tau}_{|A} = (1, 0, 0, 0)$

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Choose locally inertial frame centered on the geodesic of one of the particles, say A

Locally: $$ds^2 = \eta_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} + O(|\delta x|^2)$$

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$g_{\mu'\nu',\alpha'\,|A} = 0 \ \ (\text{i.e. } \Gamma^{\alpha'}_{\mu'\nu'\,|A} = 0)$

GEODESIC DEVIATION (EXAMPLE)

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Just need to calculate the Riemann tensor

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$$+ \quad g_{\nu\sigma}\left(\Gamma^\nu{}_{\kappa\lambda}\Gamma^\sigma{}_{\alpha\mu} - \Gamma^\nu{}_{\kappa\mu}\Gamma^\sigma{}_{\alpha\lambda}\right)$$

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Neglect terms second order in the perturbation

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$$\rightarrow \quad R_{i00m} = \frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{00}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{i0}}{\partial x^0 \partial x^m} - \frac{\partial^2 h_{0m}}{\partial x^i \partial x^0}\right) = \frac{1}{2} h^{TT}_{im,00}$$

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Working in the transverse traceless Gauge,

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Hence we may now rewrite our equation

$$\frac{d^2}{dt^2}\delta x^{\lambda} = \frac{1}{2}\,\eta^{\lambda i}\,\frac{\partial^2 h^{TT}{}_{im}}{\partial t^2}\delta x^m$$

GEODESIC DEVIATION (EXAMPLE)

Initial Conditions: $t \leq 0$

$$\delta x_0^j = \quad \text{const}$$

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$$\frac{d^2}{dt^2}\delta x_1^j = \frac{1}{2}\ \eta^{ji}\frac{\partial^2 h^{TT}{}_{ik}}{\partial t^2}\delta x_0^k \quad \Rightarrow \quad \delta x^j = \delta x_0^j + \frac{1}{2}\ \eta^{ji}\ h^{TT}{}_{ik}\ \delta x_0^k$$

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$$\delta x^1 = \delta x_0^1 + \frac{1}{2}\,\eta^{11} h^{TT}{}_{1k}\,\delta x_0^k = \delta x_0^1$$

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Only accelerated in plane orthogonal to the direction of propagation

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By our choice of Gauge:

$$h_{yy} = -h_{zz} = 2\Re\left\{A_+ e^{i\omega(t-\frac{x}{c})}\right\}$$

$$h_{yz} = h_{zy} = 2\Re\left\{A_\times e^{i\omega(t-\frac{x}{c})}\right\}$$

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Choose coordinates

$(0, y_0, 0)$ $(0, 0, z_0)$

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Make an assumption:

$A_+ \neq 0$ $A_\times = 0$

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Choose coordinates

$(0, y_0, 0)$ $(0, 0, z_0)$

Make an assumption:

$A_+ \neq 0$ $A_\times = 0$

$$h_{yy} = -h_{zz} = 2A_+ \cos\omega(t - \frac{x}{c})$$

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Initial Conditions: $t = 0 \quad \omega(t - \frac{x}{c}) = \frac{\pi}{2}$

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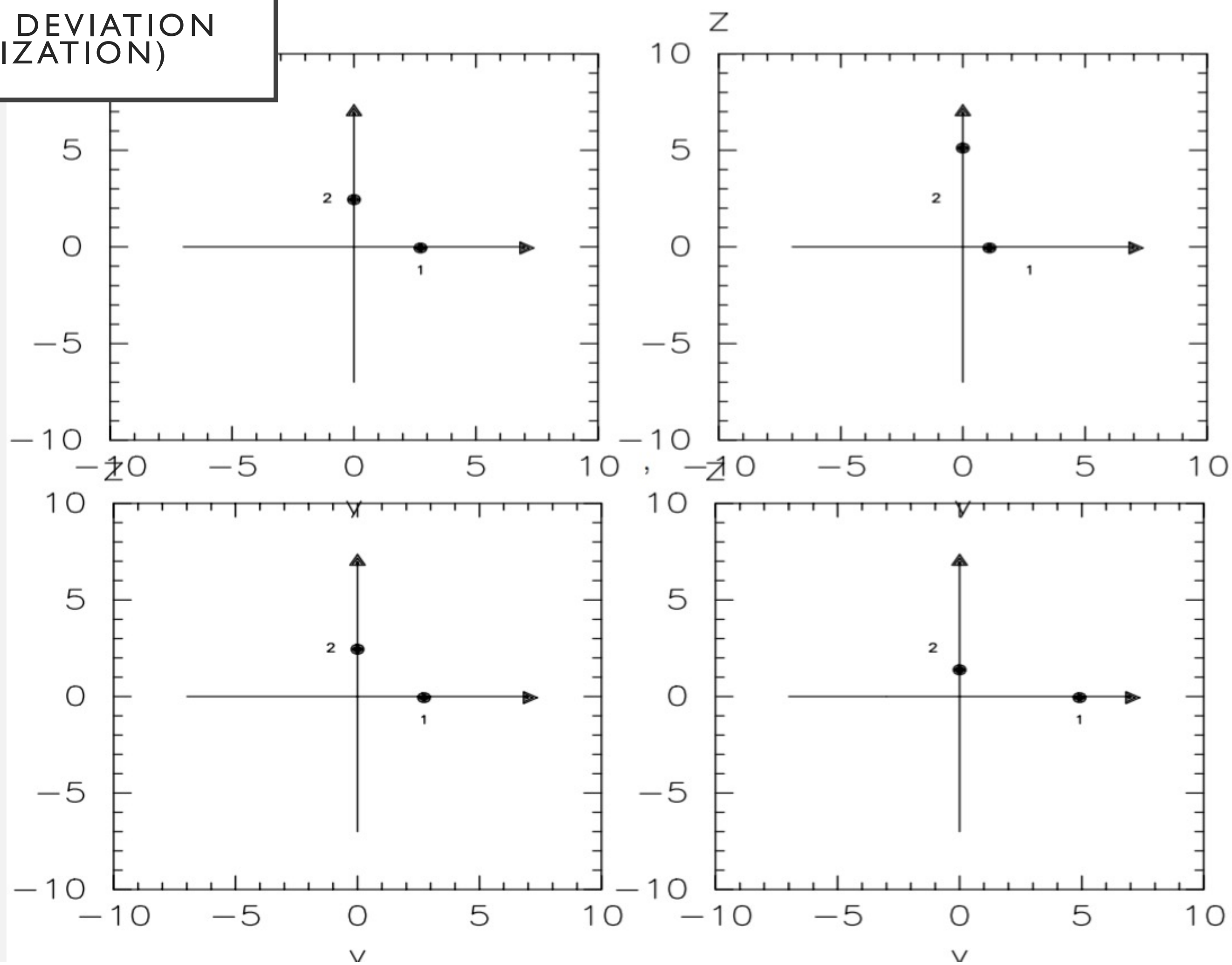
$t > 0$

$$
\begin{aligned}
&1) \quad z = 0, \quad y = y_0 + \frac{1}{2}h_{yy}\, y_0 = y_0\,[1 + A_+ \cos\omega(t - \frac{x}{c})], \\
&2) \quad y = 0, \quad z = z_0 + \frac{1}{2}h_{zz}\, z_0 = z_0[1 - A_+ \cos\omega(t - \frac{x}{c})].
\end{aligned}
$$

GEODESIC DEVIATION (POLARIZATION)



Q: HOW DO SOURCES AFFECT THINGS?

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IDEA: GREEN'S FUNCTIONS

SOLUTIONS WITH SOURCES

SOURCES

Want to solve: $\Box \bar{h}_{ab} = -16\pi T_{ab}$

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General Wave Equation:

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And you obtain a solution:

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And you obtain a solution: $f(t, \mathbf{x}) = \int dt' d^3x' \, G(t, \mathbf{x}; t', \mathbf{x}') \, s(t', \mathbf{x}')$

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Green's Function:

SOURCES

Want to solve: $\Box \bar{h}_{ab} = -16\pi T_{ab}$

Green's Function:

$$G(t, \mathbf{x}; t', \mathbf{x}') = -\frac{\delta(t' - [t - |\mathbf{x} - \mathbf{x}'|/c])}{4\pi|\mathbf{x} - \mathbf{x}'|}$$

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Solution:

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Solution:

$$\bar{h}_{ab}(t, \mathbf{x}) = 4 \int d^3x' \frac{T_{ab}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

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Kind of a tough integral!

QUADRUPOLE APPROXIMATION

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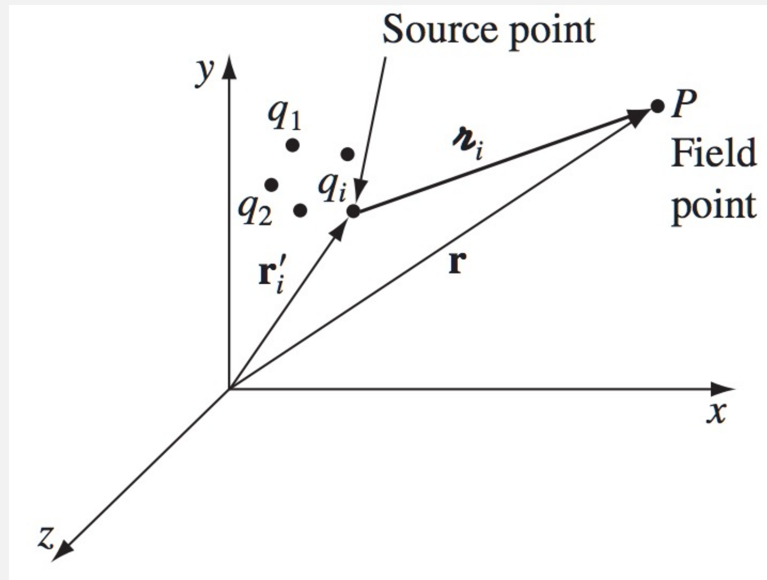

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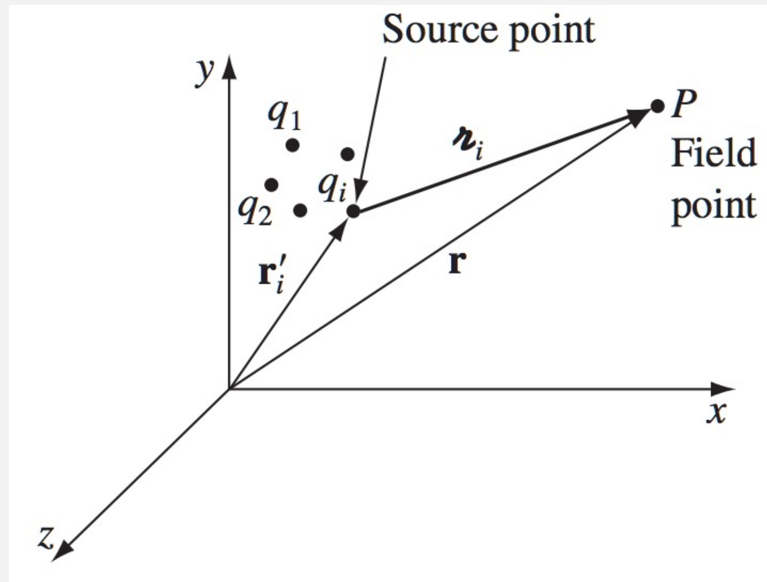
Work Far from the source:



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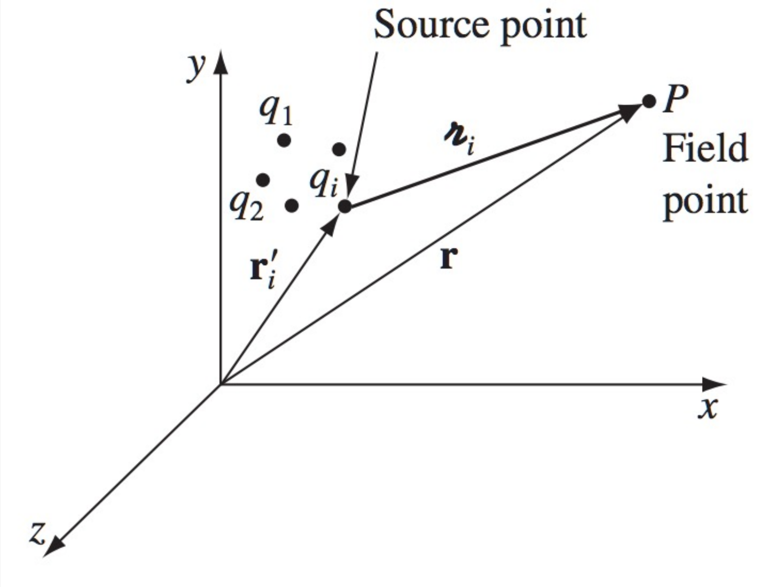
$$r = |\mathbf{x}|$$



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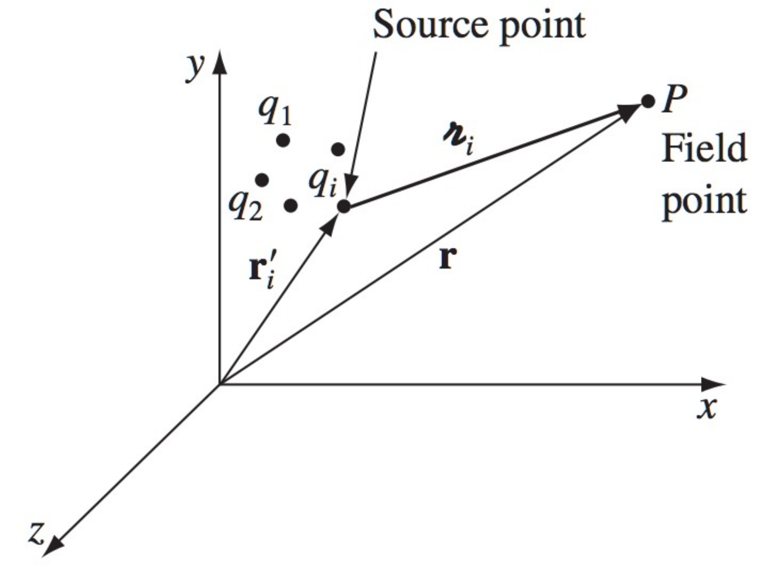
$$T_{ij}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') \approx T_{ij}(t-r,\mathbf{x}')$$



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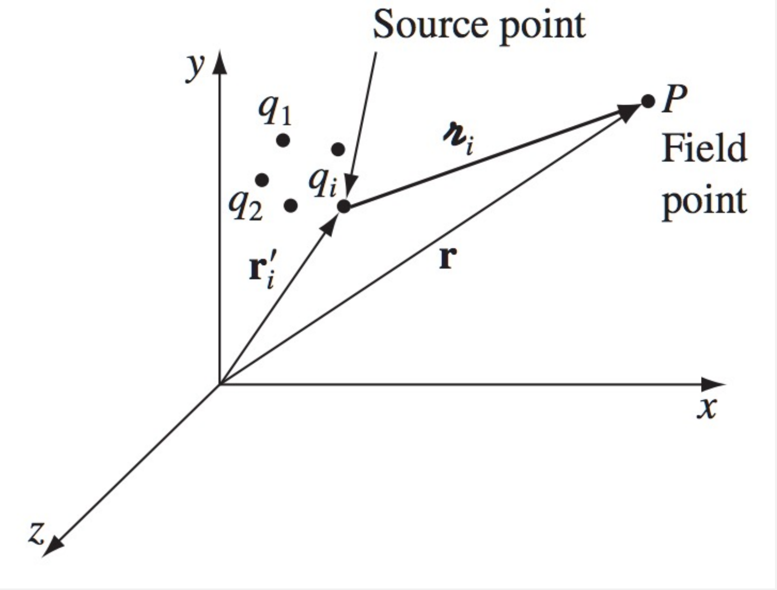

Simplifies a bit

$$\bar{h}_{ij}(t,\mathbf{x}) = \frac{4}{r}\int d^3x'\, T^{ij}(t-r,\mathbf{x}')$$

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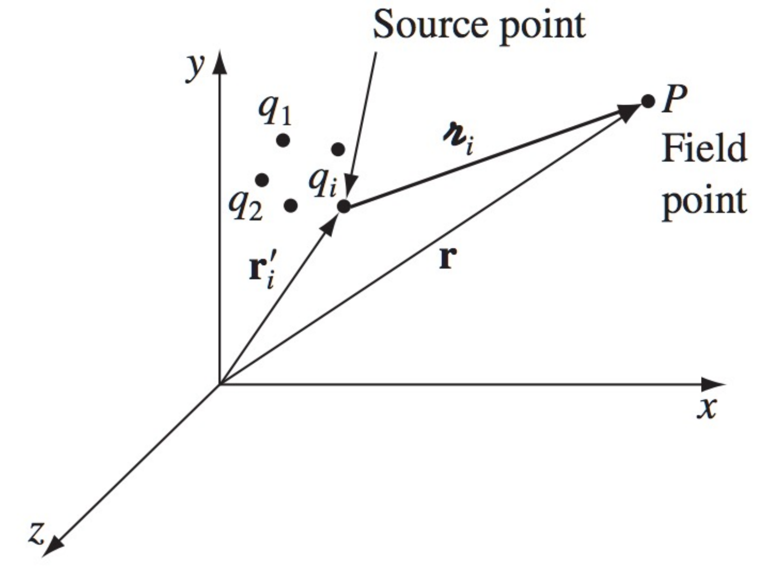

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Is this useful? $\partial_a T^{ab} = 0$

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$$\partial_a T^{ab} = 0$$

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$$\partial_a T^{ab} = 0$$

Separate into spacial and temporal parts

$$\partial_t T^{tt} + \partial_i T^{ti} = 0$$
$$\partial_t T^{ti} + \partial_j T^{ij} = 0$$

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Combine: $\partial_t^2 T^{tt} = \partial_k \partial_l T^{kl}$

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$$\partial_t T^{ti} + \partial_j T^{ij} = 0$$

Combine: $\partial_t^2 T^{tt} = \partial_k \partial_l T^{kl}$

Multiply by a product: $\partial_t^2 T^{tt} x^i x^j = \partial_t^2 \left(T^{tt} x^i x^j \right)$

QUADRUPOLE APPROXIMATION

$$\partial_a T^{ab} = 0$$

Separate into spacial and temporal parts

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Combine: $\partial_t^2 T^{tt} = \partial_k \partial_l T^{kl}$

Multiply by a product: $\partial_t^2 T^{tt} x^i x^j = \partial_t^2 \left(T^{tt} x^i x^j\right)$

Know: $\partial_k \partial_l T^{kl} x^i x^j = \partial_k \partial_l \left(T^{kl} x^i x^j\right) - 2\partial_k \left(T^{ik} x^j + T^{kj} x^i\right) + 2T^{ij}$

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QUADRUPOLE APPROXIMATION (EXAMPLE)

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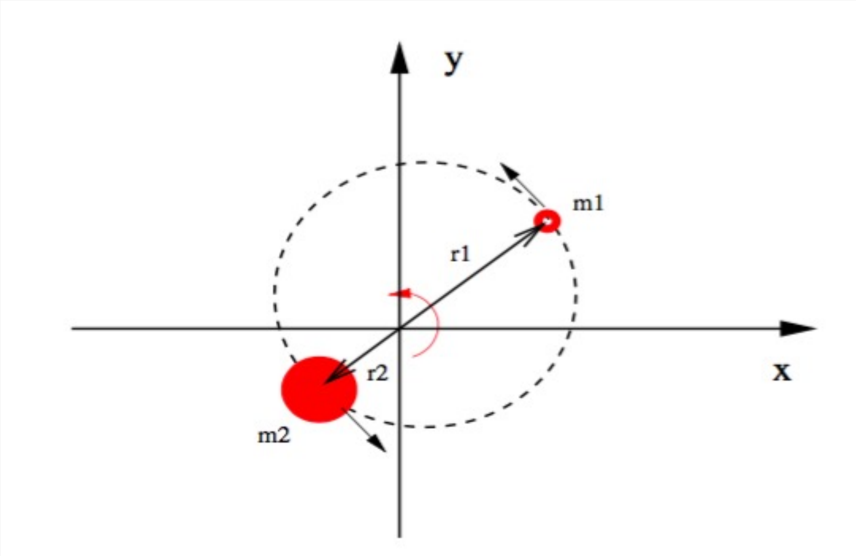
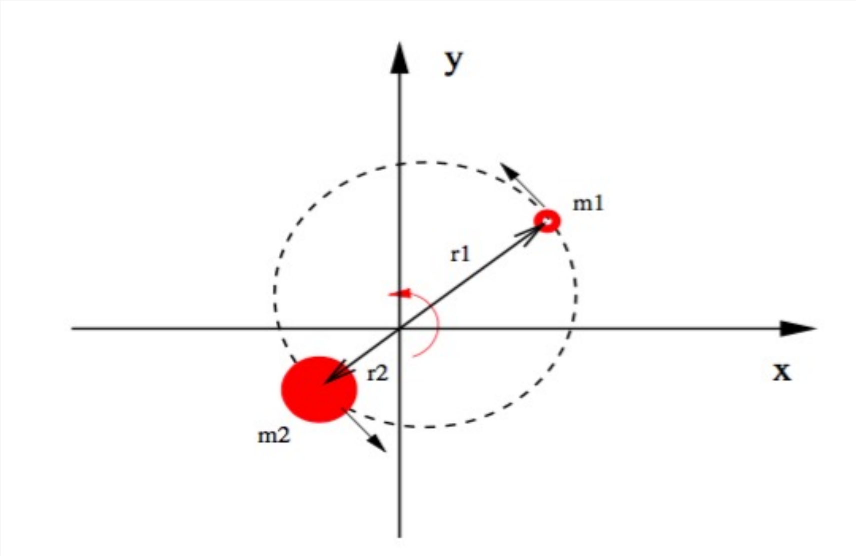


QUADRUPOLE (EXAMPLE)

Classical Mechanics Variables:



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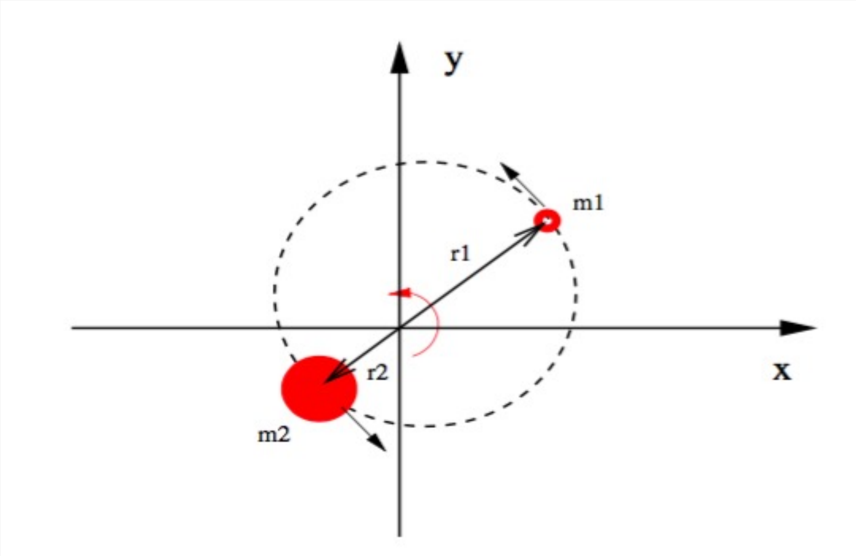

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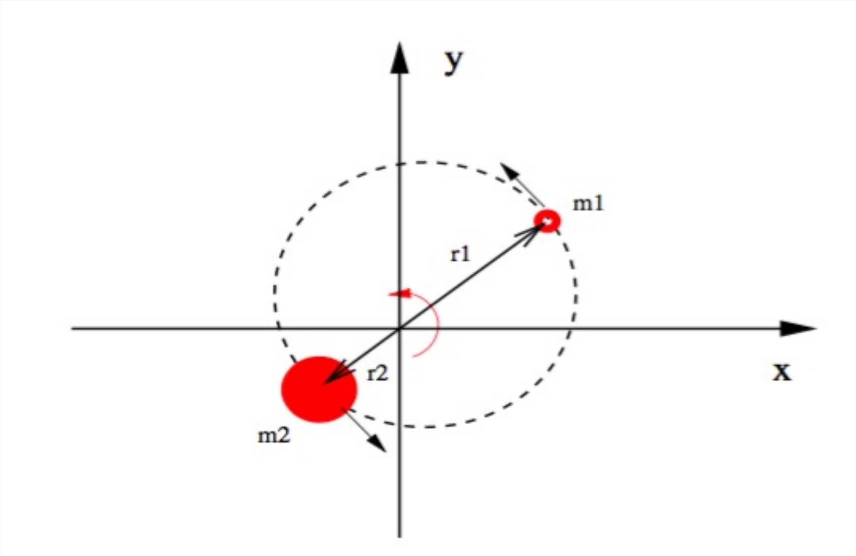

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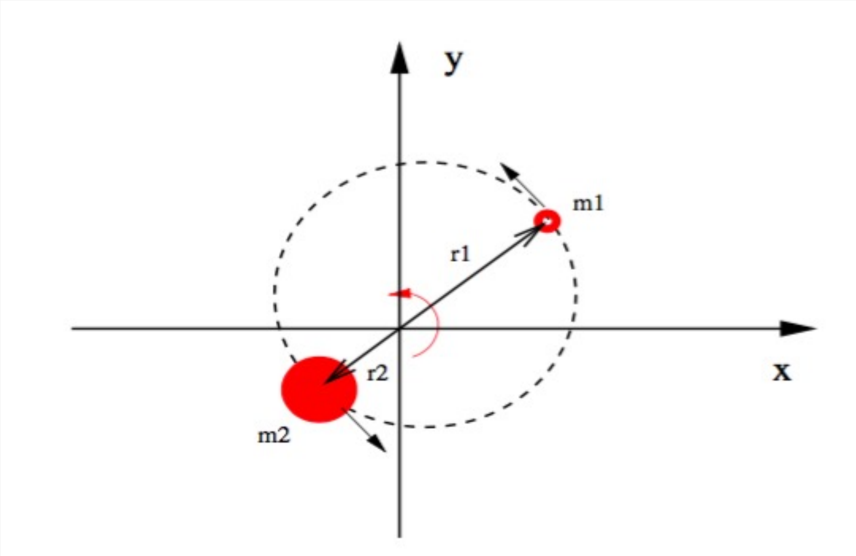

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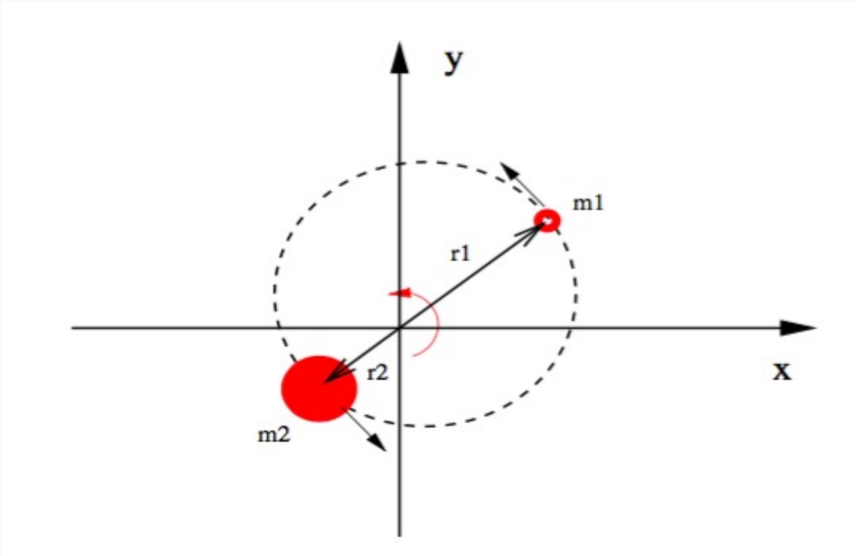

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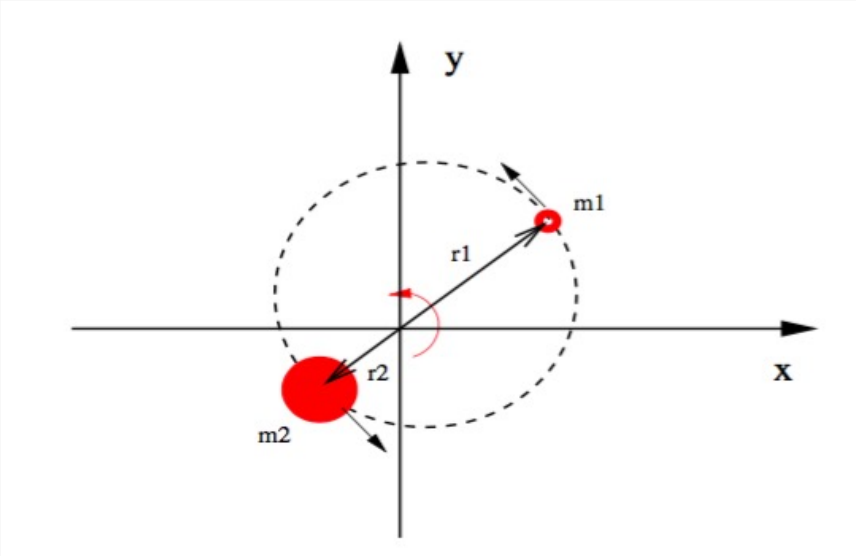

Use Kepler's Law to get the orbital frequency

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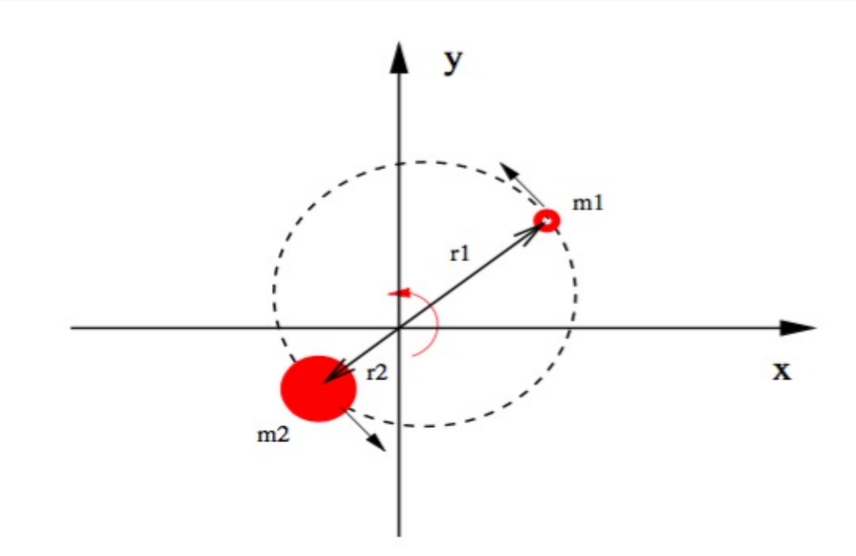

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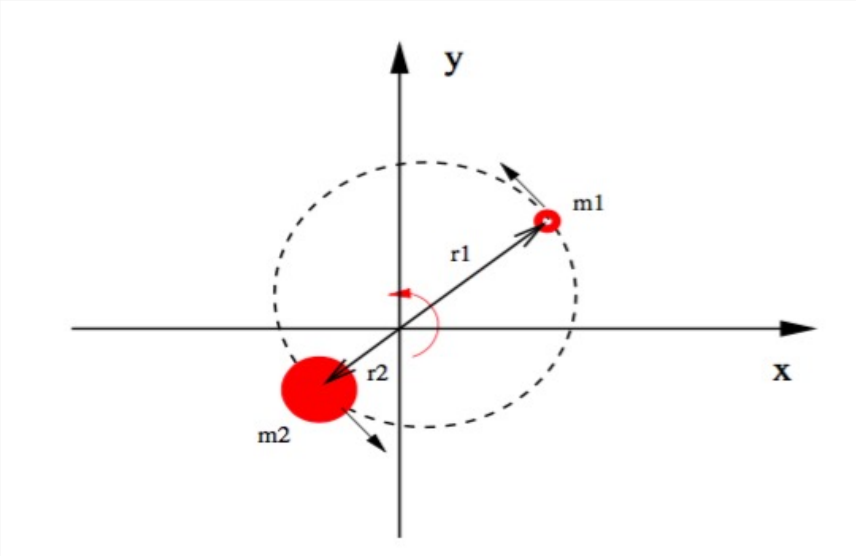

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Where:

$$A_{ij}(t) = \begin{pmatrix} \cos \, 2\omega_K t & \sin \, 2\omega_K t & 0 \\ \sin \, 2\omega_K t & -\cos \, 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Remember:

$$h_{ij}^{\mathbf{TT}}(t,r) = \frac{2G}{rc^4}\frac{d^2}{dt^2}\left[Q_{ij}^{\mathbf{TT}}(t-\frac{r}{c})\right]$$

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A: IT’S COMPLICATED

STRESS ENERGY PSEUDO-TENSOR OF THE GRAVITATIONAL FIELD

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Let's try this where the R.H.S quantity acted on is antisymmetric in it's final two indices

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Riemann Tensor:

$$R_{\gamma\alpha\delta\beta} = \frac{1}{2}\left[\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right]$$

$$+g_{\sigma\rho}\left(\Gamma^\sigma_{\alpha\delta}\Gamma^\rho_{\gamma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\gamma\delta}\right)$$

GRAVITATIONAL PSEUDO-TENSOR

Einstein's Equations:

$$T^{\mu\nu} = \frac{c^4}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)$$

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GRAVITATIONAL PSEUDO-TENSOR

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$$R^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} = g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}R_{\gamma\alpha\delta\beta}$$

GRAVITATIONAL PSEUDO-TENSOR

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$$R^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} = g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}R_{\gamma\alpha\delta\beta}$$

$$= \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}\left(\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right)$$

GRAVITATIONAL PSEUDO-TENSOR

Little bit of work…

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\left\{\frac{c^4}{16\pi G}\frac{1}{(-g)}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right]\right\}$$

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But we found what we were looking for

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\eta^{\mu\nu\alpha}$$

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Define:

$$\zeta^{\mu\nu\alpha} = (-g)\eta^{\mu\nu\alpha} = \frac{c^4}{16\pi G}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right]$$

GRAVITATIONAL PSEUDO-TENSOR

In this locally inertial frame: $$\frac{\partial}{\partial x^\beta}\frac{1}{(-g)} = 0$$

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GRAVITATIONAL PSEUDO-TENSOR

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Using this, $\Rightarrow$

$$\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} = (-g)T^{\mu\nu}$$

Define:

$$(-g)t^{\mu\nu} = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha} - (-g)T^{\mu\nu}$$

GRAVITATIONAL PSEUDO-TENSOR

$$\begin{aligned}
t^{\mu\nu} &= \frac{c^4}{16\pi G}\Big\{\Big(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\Big)\Big(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\Big) \\
&+ g^{\mu\alpha}g^{\beta\delta}\left(\Gamma^{\nu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\nu}{}_{\beta\delta}\Gamma^{\sigma}{}_{\alpha\sigma} - \Gamma^{\nu}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma}\right) \\
&+ g^{\nu\alpha}g^{\beta\delta}\left(\Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\sigma}{}_{\alpha\sigma} - \Gamma^{\mu}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma}\right) \\
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In any frame (not just a locally inertial one) we have:

$$(-g)\left(t^{\mu\nu} + T^{\mu\nu}\right) = \frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}$$

GRAVITATIONAL PSEUDO-TENSOR

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\begin{aligned}
t^{\mu\nu} \;=\;& \frac{c^4}{16\pi G}\Big\{\Big(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\Big)\Big(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\Big) \\
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And since:

$$\frac{\partial}{\partial x^{\mu}}\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^{\alpha}} = 0,$$

GRAVITATIONAL PSEUDO-TENSOR

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We then have:

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GRAVITATIONAL PSEUDO-TENSOR

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This object vanishes identically in a locally inertial frame, interpreted as containing information about the SE of the Gravitational field

GRAVITATIONAL PSEUDO-TENSOR

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t^{\mu\nu} \;=\; & \frac{c^4}{16\pi G}\Big\{\Big(2\Gamma^{\delta}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\delta\sigma} - \Gamma^{\delta}{}_{\alpha\sigma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\delta}{}_{\alpha\delta}\Gamma^{\sigma}{}_{\beta\sigma}\Big)\Big(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\Big) \\
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But it's not a tensor….

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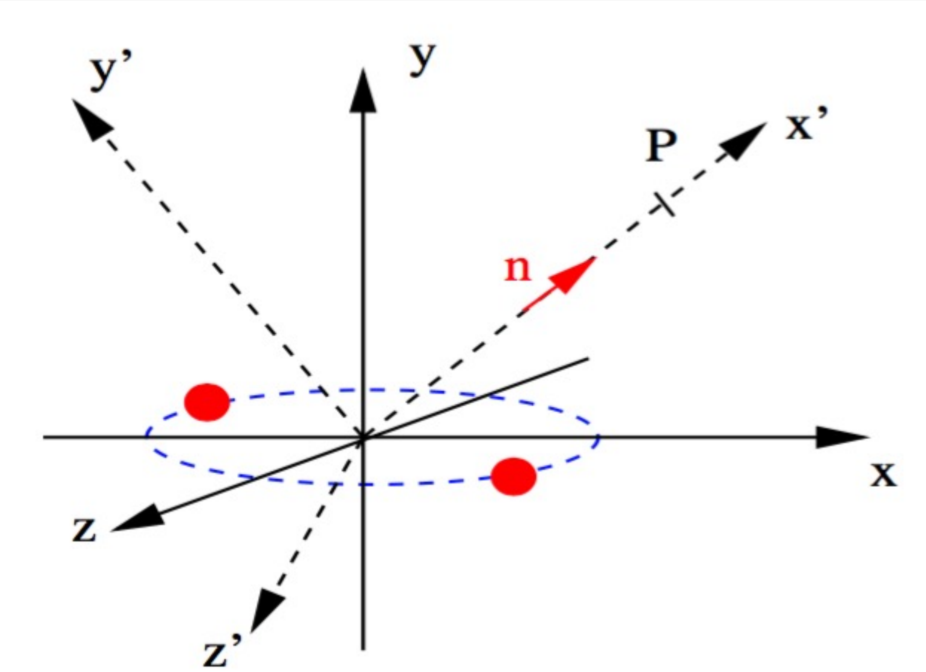
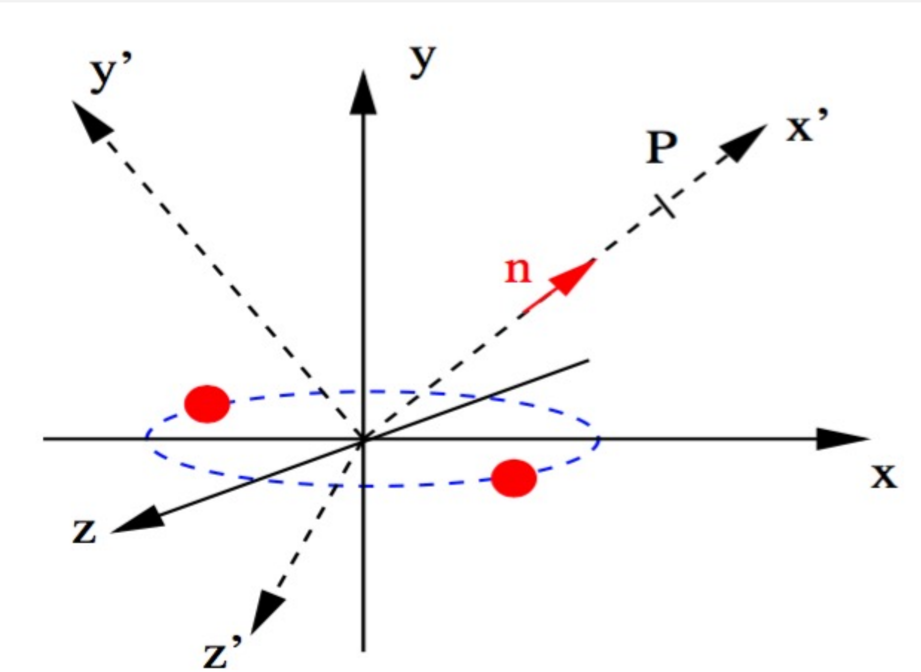
STRESS ENERGY PSEUDO-TENSOR (EXAMPLE)

PSEUDO-TENSOR
(EXAMPLE)



PSEUDO-TENSOR
(EXAMPLE)

$$
g_{\mu'\nu'} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{pmatrix} (ct) & (x') & (y') & (z') \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & [1+h_{+}^{\mathbf{TT}}(t,x')] & h_{\times}^{\mathbf{TT}}(t,x') \\ 0 & 0 & h_{\times}^{\mathbf{TT}}(t,x') & [1-h_{+}^{\mathbf{TT}}(t,x')] \end{pmatrix}
$$



PSEUDO-TENSOR
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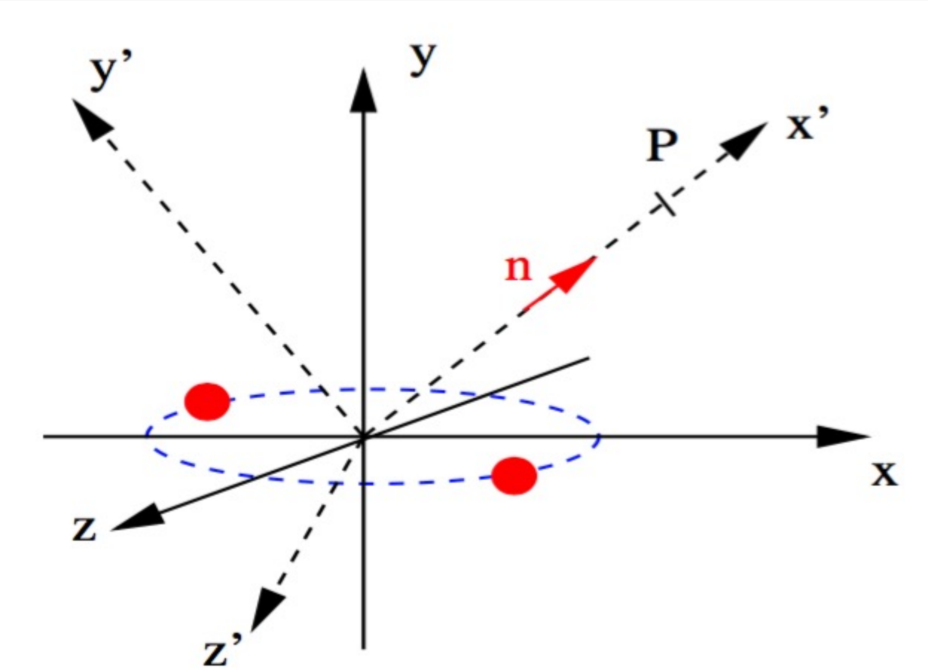

Remember: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

PSEUDO-TENSOR (EXAMPLE)

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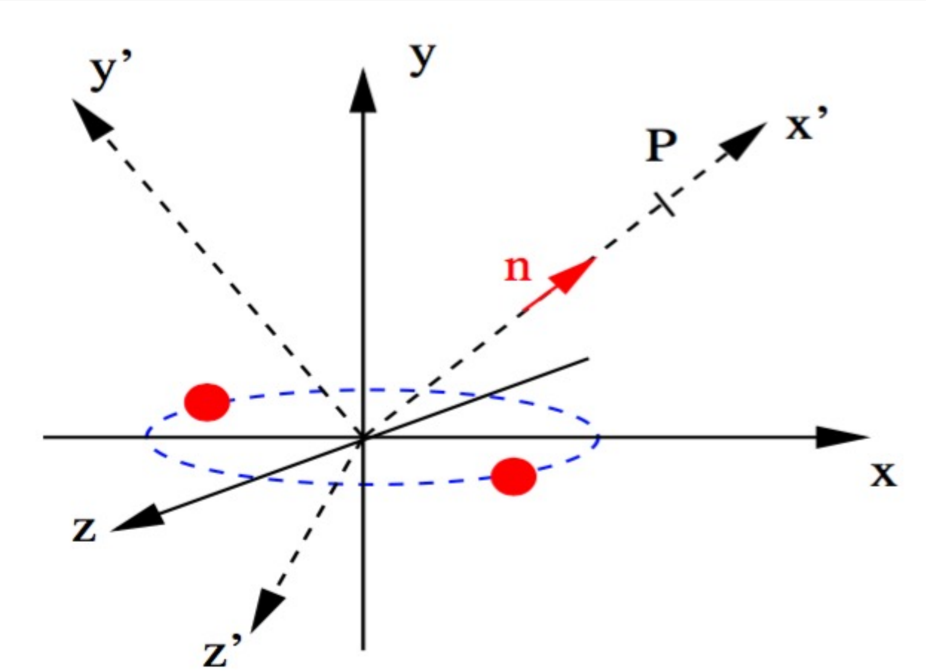

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Energy flux in the direction of propagation?:

PSEUDO-TENSOR (EXAMPLE)

Calculate Christoffel symbols from metric, pseudo-tensor follows

PSEUDO-TENSOR (EXAMPLE)

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$$ct^{0x'} = \frac{dE_{GW}}{dtdS} = \frac{c^3}{16\pi G}\left[\left(\frac{dh_{+}^{\mathbf{TT}}(t,x')}{dt}\right)^2 + \left(\frac{dh_{\times}^{\mathbf{TT}}(t,x')}{dt}\right)^2\right]$$

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$$t^{0r} = \frac{c^2}{32\pi G}\left[\sum_{jk}\left(\frac{dh_{jk}^{\mathbf{TT}}(t,r)}{dt}\right)^2\right]$$

Axis was arbitrary, new frame just rewrites perturbation in TT gauge associated with direction

PSEUDO-TENSOR (EXAMPLE)

Energy of Gravitational field cannot be defined locally, average over wavelengths

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PSEUDO-TENSOR (EXAMPLE)

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Using the gauge along with our results,

$$\begin{cases} h^{\mathbf{TT}}{}_{\mu 0} = 0, & \mu = 0,3 \\ h_{ik}^{\mathbf{TT}}(t,r) = \dfrac{2G}{c^4 r} \cdot \left[\dfrac{d^2}{dt^2} Q_{ik}^{\mathbf{TT}} \left(t - \dfrac{r}{c} \right) \right] \end{cases}$$

PSEUDO-TENSOR (EXAMPLE)

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$$\frac{dE_{GW}}{dtdS} = \frac{G}{8\pi c^5\, r^2} \left\langle \sum_{jk} \left(\dddot{Q}^{\mathbf{TT}}_{jk} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle$$

PSEUDO-TENSOR (EXAMPLE)

How about a simpler quantity?

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$$\Rightarrow \quad L_{GW} = \int \frac{dE_{GW}}{dtdS}\, dS = \int \frac{dE_{GW}}{dtdS}\, r^2 d\Omega$$

PSEUDO-TENSOR (EXAMPLE)

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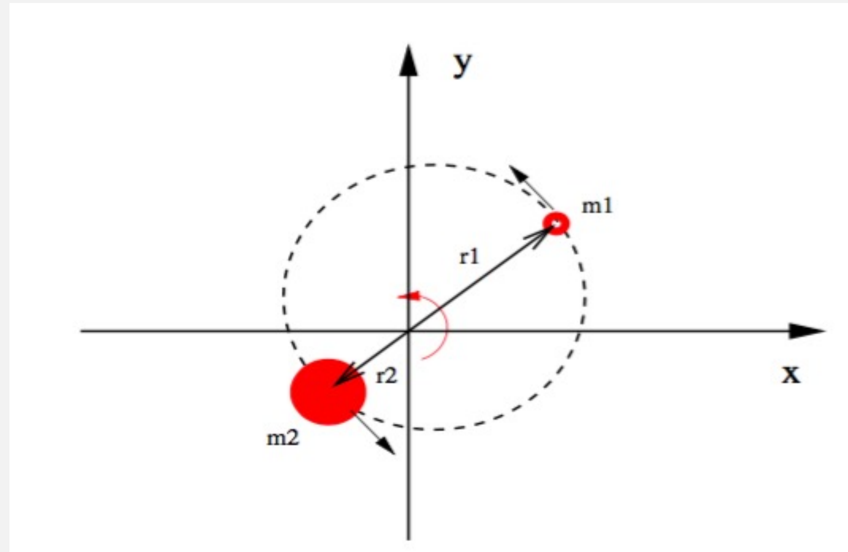
$$\frac{dE_{GW}}{dt dS} = \frac{G}{8\pi c^5\, r^2} \left\langle \sum_{jk} \left(\dddot{Q}^{\mathrm{TT}}_{jk} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle$$

$$L_{GW} = \int \frac{dE_{GW}}{dt dS}\, dS = \int \frac{dE_{GW}}{dt dS}\, r^2 d\Omega$$

$$L_{GW} = \frac{G}{5c^5} \left\langle \sum_{k,n=1}^{3} \dddot{Q}_{kn} \left(t - \frac{r}{c} \right) \dddot{Q}_{kn} \left(t - \frac{r}{c} \right) \right\rangle$$

PSEUDO-TENSOR (EXAMPLE)

Remember our binary system?



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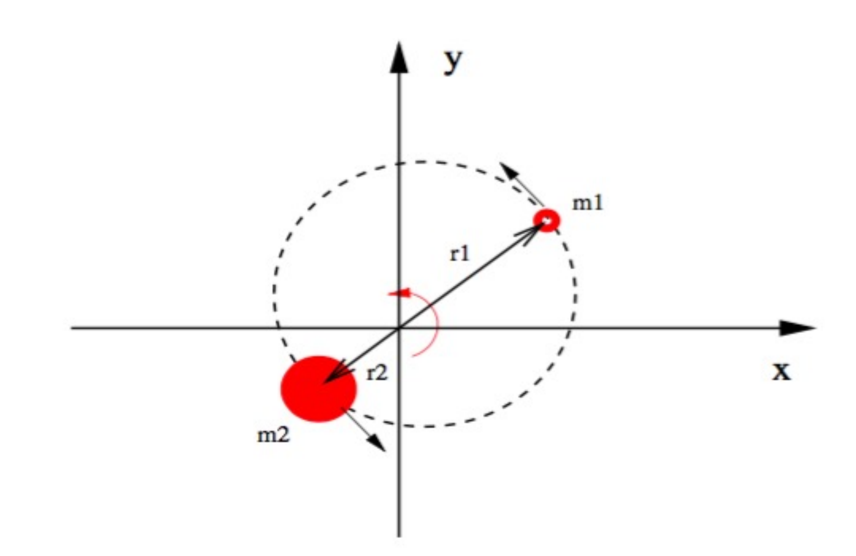

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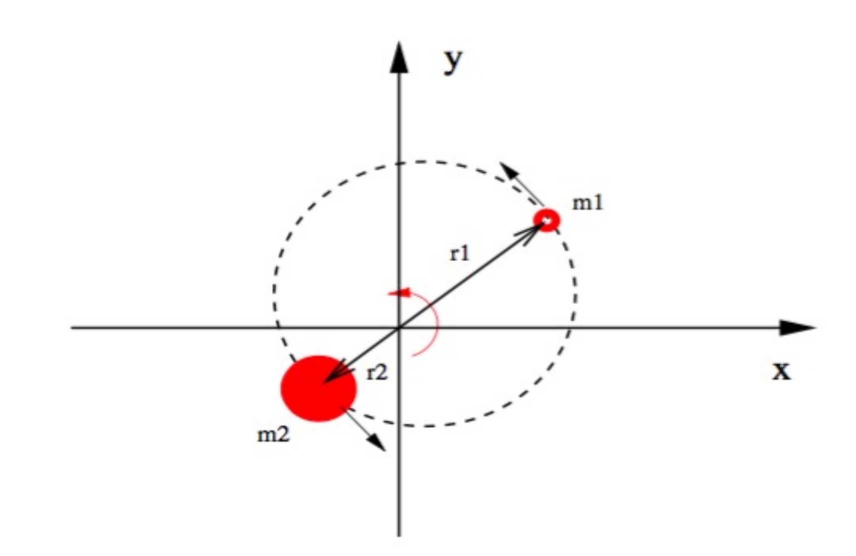
$$\sum_{k,n=1}^{3} \dddot{Q}_{kn} \dddot{Q}_{kn} = 32\ \mu^2\ l_0^4\ \omega_K^6 = 32\ \mu^2\ G^3\ \frac{M^3}{l_0^5}$$



PSEUDO-TENSOR
(EXAMPLE)

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$$L_{GW} = \frac{G}{5c^5}\left\langle \sum_{k,n=1}^{3} \dddot{Q}_{kn}\left(t-\frac{r}{c}\right)\dddot{Q}_{kn}\left(t-\frac{r}{c}\right)\right\rangle$$

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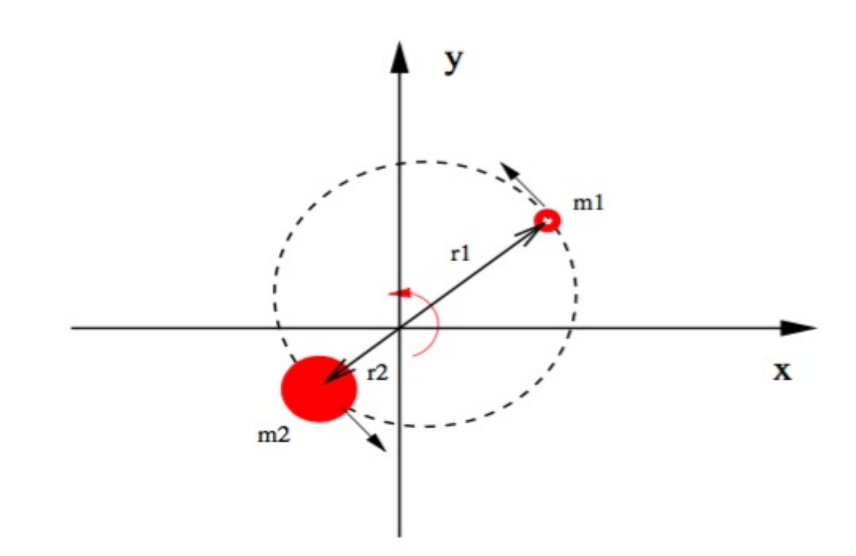
$$\Rightarrow\quad L_{GW} \equiv \frac{dE_{GW}}{dt} = \frac{32}{5}\ \frac{G^4}{c^5}\ \frac{\mu^2 M^3}{l_0^5}$$



PSEUDO-TENSOR (EXAMPLE)

Change in the period?

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Adiabatic Approximation: Orbital parameters do not change

$$\frac{dE_{orb}}{dt} + L_{GW} = 0$$

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Know:

$$E_{orb} = E_K + U$$

PSEUDO-TENSOR (EXAMPLE)

Change in the period?

$$E_K = \frac{1}{2}m_1\omega_K^2\, r_1^2 + \frac{1}{2}m_2\omega_K^2\, r_2^2$$

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PSEUDO-TENSOR (EXAMPLE)

Change in the period?

$$
\begin{aligned}
E_K &= \frac{1}{2}m_1\omega_K^2\, r_1^2 + \frac{1}{2}m_2\omega_K^2\, r_2^2 \\
&= \frac{1}{2}\omega_K^2\left[\frac{m_1 m_2^2 l_0^2}{M^2} + \frac{m_2 m_1^2 l_0^2}{M^2}\right] \\
&= \frac{1}{2}\omega_K^2 \mu l_0^2 = \frac{1}{2}\frac{G\mu M}{l_0}
\end{aligned}
$$

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(EXAMPLE)

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$$= \frac{1}{2} \omega_K^2 \mu l_0^2 = \frac{1}{2} \frac{G \mu M}{l_0}$$

$$U = -\frac{G m_1 m_2}{l_0} = -\frac{G \mu M}{l_0}$$

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$$\Rightarrow \quad E_{orb} = -\frac{1}{2}\frac{G\mu M}{l_0}$$

PSEUDO-TENSOR (EXAMPLE)

Change in the period?

Taking the derivative, only variable is the separation:

$$\frac{dE_{orb}}{dt} = \frac{1}{2}\frac{G\mu M}{l_0}\left(\frac{1}{l_0}\frac{dl_0}{dt}\right) = -E_{orb}\left(\frac{1}{l_0}\frac{dl_0}{dt}\right)$$

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$$\Rightarrow \quad \frac{dE_{orb}}{dt} = \frac{2}{3}\frac{E_{orb}}{\omega_K}\frac{d\omega_K}{dt}$$

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$$\frac{dP}{dt} = -\frac{3}{2}\frac{P}{E_{orb}}\frac{dE_{orb}}{dt}$$

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PSEUDO-TENSOR (EXAMPLE)

PSR 1913+16

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Adjust for eccentricity
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Hulse and Taylor win
1993 Nobel Prize:

$$\frac{dP}{dt} = -(2.4184 \pm 0.0009) \cdot 10^{-12}$$

PSEUDO-TENSOR (EXAMPLE)

Change in the separation?

PSEUDO-TENSOR
(EXAMPLE)

Change in the separation?

From our previous results: $$\frac{1}{l_0}\frac{dl_0}{dt} = \frac{L_{GW}}{E_{orb}} = -\left[\frac{64}{5}\frac{G^3}{c^5}\,\mu\, M^2\right]\cdot\frac{1}{l_0^4}$$

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Define: $$t_{coal} = \frac{5}{256}\frac{c^5}{G^3}\frac{(l_0^{in})^4}{\mu M^2}$$

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(EXAMPLE)

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Define: $$t_{coal} = \frac{5}{256}\frac{c^5}{G^3}\frac{(l_0^{in})^4}{\mu M^2}$$

Position's are equal at t-coalesce as objects have been treated as point-like

$$l_0(t) = l_0^{in}\left[1 - \frac{t}{t_{coal}}\right]^{1/4}$$

1. Derive Gravitational Waves

2. Detection
    1. Geodesic Deviation
    2. Example

3. Solutions to the equations with source
    1. Quadrupole Approximation
    2. Example

4. Energy transported by the waves
    1. Depends on the amplitude
    2. Example

STRESS ENERGY TENSOR (ALTERNATE)

$$S[\phi_a, \partial_\mu \phi_a] \equiv \int d^4x \, \mathcal{L}\big(\phi_a(x), \partial_\mu \phi_a(x)\big)$$

$$\delta S = \int \Big(\delta\big(d^4x\big)\mathcal{L} + d^4x \, \delta\mathcal{L} \Big)$$

$$\delta S = \int d^4x \, \partial_\mu \mathcal{J}^\mu = 0$$

$$\mathcal{J}^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta\phi_a - T^{\mu\nu} \, \delta x_\nu$$

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \partial^\nu \phi_a - \mathcal{L} \, \eta^{\mu\nu}$$

$$\mathcal{J}^{\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_a)}\,\delta\phi_a - T^{\mu\nu}\,\delta x_{\nu}$$

$$\delta\phi_a = 0$$

$$\partial_{\mu}T^{\mu\nu} = 0$$

$$\delta S = \int d^4x\,\partial_{\mu}\mathcal{J}^{\mu} = 0$$

$$\mathcal{L} = \frac{1}{64\pi G}\left[\partial_\alpha h\,\partial^\alpha h + 2\partial_\alpha h_{\beta\gamma}\,\partial^\beta h^{\alpha\gamma}\right.$$

$$\left.-2\partial^\alpha h\,\partial_\beta h^\beta{}_\alpha - \partial_\gamma h_{\alpha\beta}\,\partial^\gamma h^{\alpha\beta}\right]$$

$$\frac{\partial\mathcal{L}}{\partial h_{\alpha\beta}} - \partial_\gamma\frac{\partial\mathcal{L}}{\partial(\partial_\gamma h_{\alpha\beta})} = \frac{1}{2}\partial_\gamma\partial^\gamma h^{\mathrm{TT}}_{\alpha\beta} = 0$$

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a - \mathcal{L}\,\eta^{\mu\nu}$$

$$\Theta^\alpha{}_\beta = \frac{1}{32\pi G}\left\langle\partial^\alpha h^{TT}_{\gamma\delta}\,\partial_\beta h^{\gamma\delta}_{TT}\right\rangle$$

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - \mathcal{L}\, \eta^{\mu\nu}$$

$$T_B^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda (S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu})$$

$$\Theta^\alpha{}_\beta = \frac{1}{32\pi G} \left\langle \partial^\alpha h^{TT}_{\gamma\delta}\, \partial_\beta h^{\gamma\delta}_{TT} \right\rangle$$

$$\Theta^{\mu\nu} + \Delta^{\mu\nu}$$

$$S = \int \left[\frac{1}{2\kappa} R + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g} \, \mathrm{d}^4 x$$

$$0 = \delta S$$

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -2\kappa \frac{1}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_{\mathrm{M}})}{\delta g^{\mu\nu}},$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_{\mathrm{M}})}{\delta g^{\mu\nu}}$$

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \, \partial^\nu \phi_a - \mathcal{L} \, \eta^{\mu\nu}$$

$$T_B^{\mu\nu} = \quad \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\mathrm{M})}{\delta g^{\mu\nu}}$$

$$\Theta^{\alpha}{}_{\beta} = \frac{1}{32\pi G}\left\langle \partial^\alpha h^{TT}_{\gamma\delta}\, \partial_\beta h^{\gamma\delta}_{TT}\right\rangle \qquad \Theta^{\mu\nu} + \Delta^{\mu\nu}$$

$$g_{ab} = g^{\mathrm{B}}_{ab} + \varepsilon h_{ab} + \varepsilon^2 j_{ab} + O(\varepsilon^3)$$

$$0 = G_{ab}$$

$$= G_{ab}[g^{\mathrm{B}}_{cd}] + \varepsilon G^{(1)}_{ab}[h_{cd}; g^{\mathrm{B}}_{ef}] + \varepsilon^2 G^{(1)}_{ab}[j_{cd}; g^{\mathrm{B}}_{ef}] + \varepsilon^2 G^{(2)}_{ab}[h_{cd}; g^{\mathrm{B}}_{ef}]$$

$$+ O(\varepsilon^3)$$

Einstein Tensor of the background metric

$$G_{ab}[g^{\mathrm{B}}_{cd}]$$

First order pertubation about the background metric

$$G^{(1)}_{ab}[\ldots; g^{\mathrm{B}}_{ef}]$$

Quadratic terms in the pertubation

$$G^{(2)}_{ab}[h_{cd}; g^{\mathrm{B}}_{ef}]$$