# Homework 1

*Evan David (ead955)*

## Problem 1

We want to prove $(A^{-1})^T = (A^T)^{-1}$

Let us define $B = (A^{-1})$

Then $B^T = (A^{-1})^T$

By definition, $AB = A(A^{-1}) = I$, the identity matrix

It follows that $(AB)^T = I^T = I$, or $B^T A^T = I$

Then multiplying both sides by $(A^T)^{-1}$, we have $B^T A^T (A^T)^{-1} = I(A^T)^{-1}$

That gives $B^T I = (A^T)^{-1}$

Then we have $B^T = (A^T)^{-1}$

And so $B^T = (A^{-1})^T = (A^T)^{-1}$

## Problem 2

We will first set up the problem. We'll define $x_1$ as first mortgage loans, $x_2$ as second mortgage loans, $x_3$ as home improvement loans, and $x_4$ as personal overdraft loans.

From the problem we know the following:

1. In total \$250 million is lent out. This is represented by the equation:

$$x_1 + x_2 + x_3 + x_4 = 250$$

2. First mortgages are 55% of all mortgages (i.e., first and second mortgage) issued.

$$x_1 = 0.55(x_1 + x_2)$$

   Or,

$$0.45x_1 - 0.55x_2 = 0$$

3. Second mortgages are 25% of all loans issued. This gives:

$$x_2 = 0.25(x_1 + x_2 + x_3 + x_4)$$

   Or,

$$-0.25x_1 + 0.75x_2 - 0.25x_3 - 0.25x_4 = 0$$

4. The average interest rate on all loans is 15%. From this, and the information we know about the interest rates of each loan type, we can say:

$$0.14x_1 + 0.2x_2 + 0.2x_3 + 0.1x_4 = 0.15(x_1 + x_2 + x_3 + x_4)$$

   Or,

$$-0.01x_1 + 0.05x_2 + 0.05x_3 - 0.05x_4 = 0$$

   From these equations we can set up a matrix equation

$$Ax = y$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.45 & -0.55 & 0 & 0 \\ -0.25 & 0.75 & -0.25 & -0.25 \\ -0.01 & 0.05 & 0.05 & -0.05 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 250 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we solve for x using matrix inversion:

$$x = A^{-1}y$$

The solution then gives:

```
##          [,1]
## [1,] 76.38889
## [2,] 62.50000
## [3,] 31.94444
## [4,] 79.16667
```

And these are the values of $x_1$, $x_2$, $x_3$, and $x_4$, in millions, that give the best lending strategy.

## Problem 3

In this problem, we want to maximize profit by determining the best number of units manufactured for each variant.

### Decision Variables

$x_1$, $x_2$, $x_3$, and $x_4$ will represent the number of units manufactured for each type of variant.

### Constraints

First, the non-negative constraints:

$$x_1, x_2, x_3, x_4 \geqslant 0$$

Additionally, we have constraints for assembly, polishing, and packing, which are as follows:

$$2x_1 + 4x_2 + 3x_3 + 7x_4 \leqslant 100000$$

$$3x_1 + 2x_2 + 3x_3 + 4x_4 \leqslant 50000$$

$$2x_1 + 3x_2 + 2x_3 + 5x_4 \leqslant 60000$$

### Objective

We want to maximize profit, so we will maximize:

$$1.5x_1 + 2.5x_2 + 3x_3 + 4.5x_4$$

## Problem 4

### Part a

The following R code will generate a 20 by 20 Lehmer matrix A:

```r
A = matrix(0,nrow = 20,ncol = 20)

for (i in 1:nrow(A)) {
  for (j in 1:ncol(A)){
    A[i,j] = min(i,j)/max(i,j)

  }
}
```

### Part b

Test whether A is symmetric, by checking if $A = A^T$

```r
# checks whether the transpose of A is equal to A
all.equal(t(A),A)
```

```
## [1] TRUE
```

Therefore A is symmetric.

### Part c

```r
C = solve(A)
all.equal(C %*% A, diag(20))
```

```
## [1] TRUE
```

Therefore C is the correct inverse of A.

### Part d

```r
d = c(1:10,10:1)
d
```

```
##  [1]  1  2  3  4  5  6  7  8  9 10 10  9  8  7  6  5  4  3  2  1
```

### Part e

$$x = A^{-1}Cd$$

```r
x = solve(A) %*% C %*% d
x
```

```
##                [,1]
##  [1,] -5.304478e-15
##  [2,]  1.248443e-14
##  [3,]  1.737561e-14
##  [4,] -7.210962e-14
##  [5,]  9.849542e-15
##  [6,] -1.489992e-14
##  [7,]  1.127494e-13
##  [8,] -3.373619e-15
```

```
##  [9,] -2.481203e+01
## [10,]  2.006424e+01
## [11,]  3.581375e+01
## [12,] -3.006263e+01
## [13,] -3.736996e-04
## [14,] -2.772044e-04
## [15,] -2.099688e-04
## [16,] -1.619541e-04
## [17,] -1.269228e-04
## [18,] -1.008779e-04
## [19,]  9.505933e+01
## [20,] -1.000629e+02
```