A Proof Extending Noether's Theorem to an Isomorphism

Evan Craft*

Abstract

Noether's theorem relating symmetries of the action and conserved quantities is extended to an isomorphism. We then give a number of complementary theorems before applying these ideas to theories which are in correspondence. We conclude with an extension to the quantum theory.

1. Preliminary Remarks

Noether found that for every continuous transformation G which leaves the action invariant there is a corresponding conserved quantity [1]. By invariance, we mean

$$\delta S \equiv G \vdash S - S = 0 \tag{1}$$

where S is the action functional

$$S = \int \sqrt{g} d^4 x \mathscr{L} \tag{2}$$

and $G \vdash S$ is the action of G on S. Such transformations are called symmetries of the action. Let J be the current operator which maps a symmetry G of S to its corresponding current $J \vdash G$. We would then have

$$\nabla (J \vdash G) = 0 \tag{3}$$

In the following sections, we extend this theorem to a number of different isomorphisms.

2. Internal Symmetries

We can decompose any transformation G into a field and Lorenz part so that

$$G = \begin{pmatrix} \sigma \\ \Lambda \end{pmatrix} \tag{4}$$

The field component, of course, act on the field while the Lorenz component acts on the coordinates. We may can now apply G to the action

$$G \vdash S \equiv G \vdash S[\phi, x] \tag{5a}$$

$$= \begin{pmatrix} \sigma \\ \Lambda \end{pmatrix} \vdash S[\phi, x] \tag{5b}$$

$$= S[\sigma \vdash \phi, \Lambda \vdash x] \tag{5c}$$

$$= S'[\phi', x'] \tag{5d}$$

Where we have taken

* evan.d.craft.20@dartmouth.edu

$$\phi' \equiv \sigma \vdash \phi \tag{6a}$$

$$x' \equiv \Lambda \vdash x \tag{6c}$$

The composition of two field transformations is itself a field transformation. We can assert a similar result for Lorenz transformations. Therefore, each components is itself a part of a larger group. We can promote the set of all symmetries to a group by defining

$$G \star H = \begin{pmatrix} \sigma \\ \Lambda \end{pmatrix} \star \begin{pmatrix} \sigma' \\ \lambda' \end{pmatrix} \tag{7a}$$

$$= \begin{pmatrix} \sigma \circ \sigma' \\ \lambda \circ \lambda' \end{pmatrix} \tag{7b}$$

where we are taking the composition of each transformation. Our goal now is to find a group isomorphic to this one. Begin with Noether's theorem. It is well known that the conserved current associate with a symmetry G is given by

$$J \vdash G = \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a - T^{\mu\nu} \delta x_\nu \tag{8}$$

We can simplify this current by simplifying G. To first order,

$$\phi' = (\mathrm{I} + \epsilon \omega) \vdash \phi \tag{9a}$$

$$x' = (\mathrm{I} + \epsilon \lambda) \vdash x \tag{9b}$$

so that

$$\delta \phi = \omega \vdash \phi \tag{10a}$$

$$\delta x = \lambda \vdash x \tag{10b}$$

We can now decompose the current as

$$\overrightarrow{J}_G \equiv \begin{pmatrix} \omega \\ \lambda \end{pmatrix} \tag{11}$$

where we have introduced a shorthand notation to differentiate the results from the coming section. The basis vectors are given by

$$e_1 = \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \phi_a \tag{12a}$$

$$e_2 = T^{\mu\nu} x_\nu \tag{12b}$$

Using this, we may promote the set of all currents into a group by forcing

$$\overrightarrow{J_G} \star \overrightarrow{J_H} = \begin{pmatrix} a \\ b \end{pmatrix} \star \begin{pmatrix} c \\ d \end{pmatrix} \tag{13a}$$

$$= \begin{pmatrix} a \cdot c \\ b \cdot d \end{pmatrix} \tag{13b}$$

For some choice of product. If we choose composition, we have

$$\overrightarrow{J_G} \star \overrightarrow{J_H} = \begin{pmatrix} \omega \circ \omega' \\ \lambda \circ \lambda' \end{pmatrix} \tag{14}$$

So that

$$\overrightarrow{J_G} \star \overrightarrow{J_H} = \overrightarrow{J}_{G\star H} \tag{15}$$

Hence the set of all symmetries is isomorphic to the set of all currents as they are representations of the same vector field. In particular, they are both vector fields over the symmetry group.

3. A Complimentary Theorem

Again consider

$$J \vdash G = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta\phi_a - T^{\mu\nu}\delta x_\nu \tag{16}$$

We can choose a different decomposition. We can represent this current as

$$\overrightarrow{J_{\mathscr{L}}} \equiv \begin{pmatrix} \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a\right)} \phi_a \\ T^{\mu\nu} x_\nu \end{pmatrix} \tag{17}$$

where the basis vectors are given by

$$e_1 = \omega \tag{18a}$$

$$e_2 = \lambda \tag{18b}$$

Again, we may turn the currents into a group by forcing

$$\overrightarrow{J_{\mathscr{L}}} \star \overrightarrow{J_{\mathscr{F}}} = \begin{pmatrix} a \\ b \end{pmatrix} \star \begin{pmatrix} c \\ d \end{pmatrix} \tag{19a}$$

$$= \begin{pmatrix} a \cdot c \\ b \cdot d \end{pmatrix} \tag{19b}$$

for some choice of product. If we choose addition, we have

$$\overrightarrow{J_{\mathscr{L}}} \star \overrightarrow{J_{\mathscr{F}}} = \frac{1}{2} \begin{pmatrix} \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_a\right)} \phi_a + \frac{\partial \mathscr{F}}{\partial\left(\partial_\mu \phi_a\right)} \phi_a \\ T^{\mu\nu} x_\nu + Q^{\mu\nu} x_\nu \end{pmatrix} \tag{20}$$

Where the factor of 1/2 will be justified shortly. Here, $Q^{\mu\nu}$ is the stress tensor for the second Lagrangian. We would now like to construct a group isomorphic to the above. To do this, consider to Lagrangians $\mathscr{L}$ and $\mathscr{F}$. We will say that $\mathscr{L}$ and $\mathscr{F}$ are equivalent if they give the same equations of motion so that

$$\delta S_{\mathscr{L}}[\phi, x] = \delta S_{\mathscr{F}}[\phi, x] \tag{21}$$

We can then define the product of two equivalent Lagrangians to be

$$\mathscr{L} \star \mathscr{F} = \frac{1}{2}[\mathscr{L} + \mathscr{F}] \tag{22}$$

so that the product again gives the same equations of motions: elevating our set of equivalent Lagrangians to a group. Choose a symmetry of the first action

$$\delta_G S_{\mathscr{L}}[\phi, x] = G \vdash S_{\mathscr{L}}[\phi, x] - S_{\mathscr{L}}[\phi, x] = 0 \tag{23}$$

The second Lagrangian would yield

$$\delta_G S_{\mathscr{F}}[\phi, x] = G \vdash S_{\mathscr{F}}[\phi, x] - S_{\mathscr{F}}[\phi, x] \tag{24}$$

But from (21) we would then have

$$\delta_G S_{\mathscr{F}}[x, \phi] = 0 \tag{25}$$

since the variations are equal. Therefore, for a set of equivalent Lagrangians, the symmetry group is fixed so that the current map becomes a homomorphism

$$\overrightarrow{J_{\mathscr{L}}} \star \overrightarrow{J_{\mathscr{F}}} = \overrightarrow{J}_{\mathscr{L}\star\mathscr{F}} \tag{26}$$

where the same symmetry is chosen for both currents. This can easily be extended to the action functionals so that

$$\overrightarrow{J_S} \star \overrightarrow{J}_I = \overrightarrow{J}_{S\star I} \tag{27}$$

as the set of all Lagrangians can be mapped isomorphically to an action functional by taking

$$\mathscr{L} \to S_{\mathscr{L}}[\phi, x] \tag{28}$$

4. External Symmetries

Many times, a theory is independent of representative. That is, given two actions, they imply the same physical results. In the previous examples, this would require $S[\varphi, x]$ and $I[\psi, y]$ so that

$$\delta S[\varphi, x] = 0 \iff \delta I[\psi, y] = 0 \tag{29}$$

for some choice of φ and ψ. We say that these theories are in correspondence. We focus on the case where the correspondence can be extended to a bijection between action functionals

$$q \vdash S = I \tag{30}$$

We will call q the correspondence function. In the proofs that follow, S and I need not be action functionals. Indeed, any such theories related by (31) will have the isomorphism structure we will describe.

A. Action Invariants

We will now show that the requirement (31) implies that the symmetry groups of S and I are isomorphic. Consider a symmetry

$$G \vdash S = S \tag{31}$$

Since the theories are in correspondence, we can construct the map

$$\alpha(G) = qGq^{-1} \tag{32}$$

This is a homomorphism as

$$\alpha(FG) = qFGq^{-1} \tag{33a}$$

$$= qFq^{-1}qGq^{-1} \tag{33b}$$

$$= \alpha(F) \cdot \alpha(G) \tag{33c}$$

It is an isomorphism as the composition of bijective functions is bijective. This is indeed a symmetry as of I since

$$\alpha(G) \vdash I = qGq^{-1} \vdash I \tag{34a}$$

$$= qG \vdash S \tag{34b}$$

$$= q \vdash S \tag{34c}$$

$$= I \tag{34d}$$

In theories following Noether's result, we have that for every symmetry of the system, we can construct a corresponding conserved current. Again, let J be the current operator which maps a symmetry of S to its corresponding conserved current

$$J \vdash G = \overrightarrow{J}_G \tag{35}$$

Where we have used the shorthand notation from the previous sections. We would have a corresponding current operator for I and one of its symmetries H

$$K \vdash H = \overrightarrow{K}_H \tag{36}$$

In the case that J and K are bijections, we could form the map

$$\beta \vdash \overrightarrow{J}_G = K\,\alpha\,J^{-1} \vdash \overrightarrow{J}_G \tag{37}$$

Since K maps symmetries to conserved currents then $\beta \vdash \overrightarrow{J}_G$ is also conserved. Furthermore, if J and K can be extended to isomorphisms (as is possible for theories in Section 2) then β would the composition of isomorphisms and therefore an isomorphism itself. Finally, this would imply that the symmetry group of S is isomorphic to the current group of I.

5. Isomorphic Variables

There are a number of ways to construct such a correspondence function. We can define the coordinate operator such that

$$a \vdash S = \begin{pmatrix} \varphi \\ x \end{pmatrix} \tag{38a}$$

$$b \vdash I = \begin{pmatrix} \psi \\ y \end{pmatrix} \tag{38b}$$

Furthermore, if we insist on a relation between the variables

$$p \vdash \begin{pmatrix} \varphi \\ x \end{pmatrix} = \begin{pmatrix} \psi \\ y \end{pmatrix} \tag{39}$$

We can construct a correspondence function function

$$q = b^{-1} p a \tag{40}$$

so that

$$q \vdash S = I \tag{41}$$

so long as the coordination functions and variable relations are invertible. Many times, the coordinate relation may be a homomorphism. We call such variables homomorphic and the definition of isomorphic variables then follows.

A. Path Integrals

We may also use this symmetry of the action to produce a symmetry of the path integral. Given [2]

$$I(\phi) \equiv \int D\varphi \; e^{iS(\varphi)} \tag{42}$$

This is the infinite limit of

$$I = \int \prod_{n=1}^{N} d\varphi_n \; e^{i\sum_{n=1}^{\infty} \mathscr{L}\Delta t/\hbar} \tag{43}$$

We know that

$$D\varphi = \prod_{i=0}^{N} d\varphi_n = \prod_{i=0}^{N} \frac{d\varphi_n}{d\psi_a} d\psi_a \equiv \frac{D\varphi}{D\psi} D\psi \tag{44}$$

So that

$$I(\phi) = \int \frac{D\varphi}{D\psi} D\psi \; e^{iS(\psi)} \equiv I(\psi) \tag{45}$$

So that there is an extra factor in the definition of the path integral. We may also consider

$$\star\, I(\psi) \equiv \int D\psi \; e^{iS(\psi)} \tag{46}$$

We may call the two theories faithful if

$$\star I = I \tag{47}$$

We can again say that the symmetry groups are isomorphic as we have have a correspondence between the path integrals. And hence the conserved current groups of the path integral also correspond. These ideas can be extended to higher order theories

$$I(\phi, \Gamma) \equiv \int D\varphi D\Gamma \, e^{iS(\varphi,\Gamma)} \tag{48a}$$

$$K(\psi, \Pi) \equiv \int D\psi D\Pi \, e^{iS(\psi,\Pi)} \tag{48b}$$

6. Quantum Mechanics

All of the above results can be seen in quantum theory. Consider a unitary operator T in Hilbert space.

$$T\,|\psi\rangle = |\varphi\rangle \tag{49}$$

Infinitesimally

$$T = 1 + ieG \tag{50}$$

If T preserves the Hamiltonian,

$$T^{\dagger}HT = 0 \tag{51}$$

we would then have

$$[G, H] = 0 \tag{52}$$

or rather

$$\frac{d(Q \vdash T)}{dt} = 0 \tag{53}$$

where we have defined the charge operator

$$Q \vdash T \equiv G \tag{54}$$

A. Internal Symmetries

We can consider this time two infinitesimal symmetries of the action

$$T = 1 + i\epsilon G \tag{55a}$$

$$R = 1 + i\epsilon P \tag{55b}$$

We then have

$$R^{\dagger}T^{\dagger}HTR = 0 \tag{56}$$

Calculating this out, we have

$$TR = 1 + i\epsilon(G + P) \tag{57}$$

to first order. Hence we may associate the product with the infinitesimal translation

$$A = 1 + ie(G + P) \tag{58}$$

up to first order so that

$$A^{\dagger} H A = 0 \tag{59}$$

and hence

$$[G + P, H] = 0 \tag{60}$$

Hence

$$Q \vdash GF = Q \vdash G + Q \vdash F \tag{61}$$

And hence we have a homomorphism between the charge symmetry symmetry groups.

B. External Symmetries

Suppose that two separate Hamiltonians H and K give the same equations of motion. Furthermore, assume that there exists some correspondence between the two theories such that

$$q \vdash H = K \tag{62}$$

By the previous sections, we will have a four part isomorphism between the current groups and symmetry groups.

References

[1] E. Noether, Kgl. Ges. d. Wiss. Nachrichten, Math.-phys. Klasse (1918).
[2] R. P. Feynman, Rev. of Mod. Phys., **20**, 367 (1948)