A Proof of a Galois Correspondence in General Relativity

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Abstract

We show that in Minkowski space, the gamma matrices are invariant under a certain group of permutations. This symmetry is then projected up to the curved geometry using the vierbein. We use this permutation group to construct a Galois correspondence which can also be projected up to the curved geometry using the vierbein.

1. Preliminary Remarks

In flat space, the defining property of gamma matrices are the the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab}\mathrm{I} \tag{1}$$

where η^{ab} is the Minkowski metric. This relation is invariant under spatial permutations. If we are given a solution set to the above

$$\{\gamma\} = \{\gamma_t, \gamma_x, \gamma_y, \gamma_z\} \tag{2}$$

Then for instance

$$\{\beta\} = \{\gamma_t, \gamma_z\, \gamma_y, \gamma_x\} \tag{3}$$

would also be a solution to (1). This is a tetrahedral symmetry. We can project this symmetry of flat space up to the curved geometry using the vierbein. For the gamma matrices, we can define [1]

$$\gamma^\mu \equiv e^\mu_a \gamma^a \tag{4}$$

where the latin indices are curved space indices. We would then have

$$\{\gamma^\mu, \gamma^\nu\} = \{e^\mu_a\gamma^a, e^\nu_b\gamma^b\} \tag{5a}$$

$$= e^\mu_a e^\nu_b \{\gamma^a, \gamma^b\} \tag{5b}$$

$$= 2\, g^{\mu\nu}\,\mathrm{I} \tag{5c}$$

We will call this the curved space anti commutation relation. We would expect the symmetry of flat space to be preserved. To verify this, let γ^a be a solution to the anticommutation relations (1) and β^a a spatial permutation of these matrices. We can project up using the vierbein

$$\gamma^\mu = e^\mu_a \gamma^a \tag{6a}$$

$$\beta^\mu = e^\mu_a \beta^a \tag{6b}$$

Both γ^a and β^a are solutions to the flat space anti-commutation relations, and therefore both will be solutions to the curved space relation also. This symmetry group will then be used to

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construct a Galois Correspondence in Section 2. In the following parts, we analyze further our tetrahedral symmetry.

A. Permutation Groups

In what follows, we will use $\{x, y, z\}$ to refer to arbitrary spatial coordinates in curved or flat space. For a given set of gamma matrices, we associate the symbol

$$\gamma_x \gamma_y \gamma_z \sim \{\gamma_t, \gamma_x, \gamma_y, \gamma_z\} \tag{7}$$

This is a bijection by design. We can form another symbol

$$\gamma_y \gamma_z \gamma_x \sim \{\gamma_t, \gamma_y, \gamma_z, \gamma_x\} \tag{8}$$

which can be rewritten as

$$\gamma_y \gamma_z \gamma_x \sim (y, x, z)\{\gamma_t, \gamma_x, \gamma_y, \gamma_z\} \tag{9}$$

for $(y, x, z) \in S_3$ so that each symbol can be represented uniquely by a permutation.

$$\sigma \vdash \gamma_a \gamma_b \gamma_c \equiv \sigma_{abc}$$

$\sigma_{abc} \in S_3$. Particularly, σ_{abc} is the permutation needed to produce the set $\{a, b, c\}$ from the base set $\{x, y, z\}$. Since the matrices can be permuted in any way, we can produce all of S_3. If two symbols give the same permutation, then we have two cases. The first is if the produced permutation is the identity. The only symbol that maps to the identity is the base set as all other symbols are non-identity cyclic permutations of the base set. For the second case, if the produced permutation is not the identity, then the two symbols must again be identical as the the cycle representation of the permutation must match the indices on the symbols. Hence we have a bijection.

Since the map is invertible, then to each permutation, we can also associate a unique symbol. We can therefore define the product of two symbols to be the unique symbol associated with the product of their permutations

$$\gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \equiv \gamma \vdash (\sigma_{abc} \sigma_{pqs}) \tag{10}$$

Hence the set of all symbols $\{\gamma_a \gamma_b \gamma_c\}$ is isomorphic to the symmetric group on 3 letters as

$$(\sigma \vdash \gamma_a \gamma_b \gamma_c) \circ (\sigma \vdash \gamma_p \gamma_q \gamma_s) = \sigma_{abc} \sigma_{pqs} \tag{11a}$$

$$= \gamma^{-1} \vdash \gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \tag{11b}$$

$$= \sigma \vdash \gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \tag{11c}$$

The subsets of S_3 are known and hence we can replace each subgroup lattice of S_3 with its corresponding symbol group.

B. Operator Groups

Each symbol is associated with a set of gamma matrices and each of these sets can be associated with an operator. We have

$$\gamma_x \gamma_y \gamma_z \sim \{\gamma_t, \gamma_x, \gamma_y, \gamma_z\} \tag{12a}$$

$$\sim \gamma^{\mu} \nabla_{\mu} - \Gamma_{\mu} \tag{12b}$$

$$\equiv D_{xyz} \tag{12c}$$

Here, ∇_{μ} is the covariant derivative operator and Γ_{μ} is the spinor connection which is only a function of the choice of gamma matrices [2]. We can therefore define the derivative operator

$$D \vdash \gamma_a \gamma_b \gamma_c \equiv D_{abc} \tag{13}$$

The map is a surjection by construction. The only symbol which maps to the identity is the base symbol. Otherwise the indices of the symbol must match the indices of the operator so that we have a bijection, just as before. The derivative operator can also act on the corresponding permutation by composition

$$D_{abc} \equiv D\sigma^{-1} \vdash \sigma_{abc} \tag{14}$$

the product of two operators is defined as the operator associated with the product of their corresponding permutations

$$D_{abc} \cdot D_{pqs} = D\sigma^{-1} \vdash (\sigma_{abc}\, \sigma_{pqs}) \tag{15}$$

so that the operator group is also isomorphic to the symbol group

$$(D \vdash \gamma_a \gamma_b \gamma_c) \circ (D \vdash \gamma_p \gamma_q \gamma_s) = D_{abc} D_{pqs} \tag{16a}$$

$$= D\sigma^{-1} \vdash (\sigma_{abc}\, \sigma_{pqs}) \tag{16b}$$

$$= D \vdash \gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \tag{16c}$$

And therefore isomorphic to S_3 by composition.

2. Proof of the Main Theorem

We've already shown that every permutation of S_3 corresponds to a symbol. Therefore, if we have a subgroup of permutations this corresponds to a subgroup of symbols. We can promote this to a Galois correspondence by instead associating a subgroup of permutations to a field extension of the corresponding symbols.

When Dirac was solving the to figure out his namesake equation, what he really did was to extend the field of operators to include those of Dirac type. The original field of operators was given by

$$\hat{\mathbb{C}} \equiv \mathbb{C} \cdot \mathrm{I} \tag{17}$$

Everything in this set is a constant diagonal operator so that for some $\hat{z} \in \hat{\mathbb{C}}$

$$\hat{z} = z \cdot \mathrm{I} \tag{18}$$

for $z \in \mathbb{C}$. We can extend this field with the Dirac operator solutions. Let $\{\gamma_a \gamma_b \gamma_c\}$ denote a set of symbols. We define the field extension so that

$$\hat{q} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) = a\hat{z} + \sum_{\sigma\{a,b,c\}} c_{abc}\gamma_a \gamma_b \gamma_c + \text{prod.} \tag{19}$$

Where c_{abc} are coefficients and we are summing over all permutations in our symbol set. Here, prod. stands for higher order products of the base field and symbols. We can think of $\hat{q}$ as being a linear superposition of vectors. In what follows, we work to first order in the base field and symbols and omit the prod. for simplicity. We address higher order products in the final section. A field must be closed under multiplication and addition. Choose $\hat{q}, \hat{p} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\})$

$$\hat{q} + \hat{p} = a\hat{z} + \sum_{\sigma\{a,b,c\}} c_{abc}\gamma_a \gamma_b \gamma_c + a'\hat{z}' + \sum_{\sigma\{p,q,s\}} c'_{pqs}\gamma_p \gamma_q \gamma_s \tag{20a}$$

$$= (aq + a'q') \cdot \mathrm{I} + \sum_{\sigma\{a,b,c\}} (c_{abc} + c'_{abc}) \gamma_a \gamma_b \gamma_c \tag{20b}$$

so that the field is closed under addition. Now

$$\hat{q} \cdot \hat{p} = a\hat{z} \cdot a'\hat{z}' + a\hat{z} \cdot \sum_{\sigma\{a,b,c\}} c'_{abc}\gamma_a \gamma_b \gamma_c + a'\hat{z}' \cdot \sum_{\sigma\{a,b,c\}} c'_{abc}\gamma_a \gamma_b \gamma_c + \sum_{\sigma\{a,b,c\}} c_{abc}\gamma_a \gamma_b \gamma_c \cdot \sum_{\sigma\{a,b,c\}} c'_{abc}\gamma_a \gamma_b \gamma_c \tag{21}$$

The products over the base field are naturally defined as the product over the matrices. To define the product of a diagonal operator and symbol, we choose a basis. In the coming sections, we define possibilities so that the product is symmetric and includes inverses. We would therefore have our field extension corresponding to the symmetry group.

A. Field Extensions

We can work in the set basis. Each symbol corresponds to some set so that

$$\gamma_x \gamma_y \gamma_z \sim \{\gamma_t, \gamma_x, \gamma_y, \gamma_z\} \tag{22}$$

We can define addition and multiplication using vector notation. For instance, we let

$$\hat{q} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} \alpha \mathrm{I} \\ \beta \gamma_t \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} c_{abc} \begin{pmatrix} \gamma_a \\ \gamma_b \\ \gamma_c \end{pmatrix} \tag{23}$$

Where the $\wedge$ denotes the Cartesian product. We can then define addition to be vector addition

$$\hat{q} + \hat{p} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} (\alpha + \alpha')\mathrm{I} \\ (\beta + \beta')\gamma_t \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} (c_{abc} + c'_{abc}) \begin{pmatrix} \gamma_a \\ \gamma_b \\ \gamma_c \end{pmatrix} \tag{24}$$

We define multiplication to be component wise so that

$$\hat{q} \cdot \hat{p} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} (\alpha \cdot \alpha')\mathrm{I}^2 \\ (\beta \cdot \beta')\gamma_t^2 \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\},\sigma\{a',b',c'\}} (c_{abc} \cdot c'_{a'b'c'}) \begin{pmatrix} (-1)^i \gamma_a \cdot \gamma_{a'} \\ (-1)^j \gamma_b \cdot \gamma_{b'} \\ (-1)^k \gamma_c \cdot \gamma_{c'} \end{pmatrix} \tag{25}$$

Where we have $i = 0$ if and only if (a, a') is a positive permutation (order preserving) on $\{x, y, z\}$. Else $i = 1$. A similar relationship holds for the other indices. This comes from the relation

$$\{\gamma^a, \gamma^{\neg a}\} = 0 \tag{26a}$$

$$\implies \gamma^a \cdot \gamma^{\neg a} = -\gamma^{\neg a} \cdot \gamma^a \tag{26b}$$

Where the $\neg a$ stands for not a. So that we always get the positively ordered product no matter the indices. We can therefore define

$$\gamma_a \gamma_b \gamma_c \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) = \begin{pmatrix} 0 \\ \beta\gamma_t \end{pmatrix} \wedge c_{abc} \begin{pmatrix} \gamma_a \\ \gamma_b \\ \gamma_c \end{pmatrix} \tag{27}$$

and

$$\hat{z} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) = \begin{pmatrix} \hat{z}\mathrm{I} \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{28}$$

In flat space, the gamma matrices are invertible so that inverses are well defined.

B. Higher Order Products

We can naturally define

$$\hat{q} \cdot \hat{p} \cdot \hat{g} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} (\alpha \cdot \alpha' \cdot \alpha'')\mathrm{I}^3 \\ (\beta \cdot \beta' \cdot \beta'')\gamma_t^3 \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\},\sigma\{a',b',c'\},\sigma\{a'',b'',c''\}} (c_{abc} \cdot c'_{a'b'c'} \cdot c''_{a''b''c''}) \begin{pmatrix} (-1)^i \gamma_a \cdot \gamma_{a'} \cdot \gamma_{a''} \\ (-1)^j \gamma_b \cdot \gamma_{b'} \cdot \gamma_{b''} \\ (-1)^k \gamma_c \cdot \gamma_{c'} \cdot \gamma_{c''} \end{pmatrix} \tag{29}$$

Since the square of any gamma matrix is either unity or negative unity, all products of order >3 can be reduced to products of order three. Negative permutations will be associated with a negative sign and positive permutations with a positive one so that the i, j, k follow the definition from before.

C. Curved Geometry

Though $\{a, b, c\}$ have been arbitrary spatial coordinates, in what follows, we will use these to exclusively refer to flat space coordinates and reserve $\{p, q, s\}$ for curved space. In flat space, to first order, we have

$$\hat{q} \cdot \hat{p} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) = \begin{pmatrix} (\alpha \cdot \alpha')\mathrm{I}^2 \\ e_t^p e_t^p (\beta \cdot \beta')\gamma_p^2 \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\},\sigma\{a',b',c'\}} (c_{abc} \cdot c'_{a'b'c'}) \begin{pmatrix} e_a^q e_{a'}^{q'}(-1)^i \gamma_q \cdot \gamma_{q'} \\ e_b^r e_{b'}^{r'}(-1)^j \gamma_r \cdot \gamma_{r'} \\ e_c^s e_{c'}^{s'}(-1)^k \gamma_s \cdot \gamma_{s'} \end{pmatrix} \tag{30a}$$

$$\hat{q} + \hat{p} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} (\alpha + \alpha')\mathrm{I}^2 \\ (e_t^p \beta + e_t^p \beta')\gamma_p^2 \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} (c_{abc} + c'_{abc}) \begin{pmatrix} e_a^q \gamma_q \\ e_b^r \gamma_r \\ e_c^s \gamma_s \end{pmatrix} \tag{30b}$$

We therefore define

$$e \vdash \hat{q} \equiv \begin{pmatrix} \alpha \mathrm{I} \\ \beta e_\mu^t \gamma_t \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} c_{abc} \begin{pmatrix} e_\mu^a \gamma_a \\ e_\mu^b \gamma_b \\ e_\mu^c \gamma_c \end{pmatrix} \tag{31}$$

As for the inverses, we have previously let $\{t, a', b', c'\}$ stand for a permutation of the coordinates. To avoid confusion, we now used different symbols, $\{\bar{a}, \bar{b}, \bar{c}\}$, to be the result of a usual coordinate transformation so that

$$e^{-1}e \vdash \hat{q} \equiv \begin{pmatrix} \alpha \mathrm{I} \\ \beta e_{\bar{t}}^\mu e_\mu^t \gamma_t \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} c_{abc} \begin{pmatrix} e_{\bar{a}}^\mu e_\mu^a \gamma_a \\ e_{\bar{b}}^\mu e_\mu^b \gamma_b \\ e_{\bar{c}}^\mu e_\mu^c \gamma_c \end{pmatrix} \tag{32a}$$

$$= \begin{pmatrix} \alpha \mathrm{I} \\ \beta \gamma_{\bar{t}} \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} c_{abc} \begin{pmatrix} \gamma_{\bar{a}} \\ \gamma_{\bar{b}} \\ \gamma_{\bar{c}} \end{pmatrix} \tag{32b}$$

$$= \mathrm{I} \vdash \hat{q} \tag{32c}$$

For higher order products, the vierbein operator e simply projects each gamma matrix to the corresponding coordinate system. This leaves

$$(e \vdash \hat{q}) \cdot (e \vdash \hat{q}) = e \vdash (\hat{q} \cdot \hat{p}) \tag{33}$$

and

$$(e \vdash \hat{q}) + (e \vdash \hat{q}) = e \vdash (\hat{q} + \hat{p}) \tag{34}$$

The image of a ring homomorphism is a ring. In this case, however, we have an isomorphism between the base ring and it's image. We can therefore define the curved space group by the set $\{e \vdash \hat{q}\}$. Since the base ring is a field, the image is therefore a field.

D. Correspondence Theorems

We could think of the anticommutation relation itself being a polynomial

$$\{\hat{O}^a, \hat{O}^b\} = 2\,\eta^{ab}\mathrm{I} \tag{35}$$

We then have that $\hat{\mathbb{C}}(\{\gamma_a\,\gamma_b\,\gamma_c\})$ is the splitting field of the polynomial and $\{\gamma_x\,\gamma_y\,\gamma_z\} \subset S_3$ (or S_5 as we will see) is the Galois Group. The Fundamental Theorem of Galois Theory can therefore be applied [3]. We could also associate the anticommutation relation with the polynomial

$$D^2 = -\,\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{36}$$

To apply Galois Theory, we must consider another operator field. If we define

$$\partial = \{\partial_t, \partial_x, \partial_y, \partial_z\} \tag{37}$$

and

$$\hat{\partial} \equiv \mathbb{C}(\partial)\cdot \mathrm{I} \tag{38}$$

We can then extend to the field

$$\hat{\partial}(\{\gamma_a\,\gamma_b\,\gamma_c\}) \tag{39}$$

The flat space gamma matrices commute with the all derivative operators (covariantly constant). Therefore, all of the previous results apply as the partials behave as any member of $\mathbb{C}$. We then have that $\hat{\partial}(\{\gamma_a\,\gamma_b\,\gamma_c\})$ is the splitting field of the polynomial and $\{\gamma_x\,\gamma_y\,\gamma_z\} \subset S_3$ (or S_5) is the Galois Group. The Fundamental Theorem of Galois Theory can therefore be applied. In curved space, our anti commutation polynomial becomes

$$\{\hat{O}^\mu, \hat{O}^\nu\} = 2\,g^{\mu\nu}\mathrm{I} \tag{40}$$

with the associated operator polynomial

$$D^2 = (\gamma^\mu \nabla_\mu - \Gamma_\mu)(\gamma^\nu \nabla_\nu - \Gamma_\nu) \tag{41}$$

3. The Full Symmetry Group

The anticommutation relation is invariant under uniform multiplication by -1 so that if γ^μ satisfies the relation then so does $-\gamma^\mu$. We will call the tetrahedral symmetries from before rotations. Multiplying by -1 will be called a reflection. We can use this transformation along with rotations to generate the entire symmetry group. Introduce the symbol

$$-\gamma_x\,\gamma_y\,\gamma_z \sim \{-\gamma_t, -\gamma_x, -\gamma_y, -\gamma_z\} \tag{42}$$

to be the result of a reflection and rotation. If we now associate

$$\{\gamma_t, \gamma_x, \gamma_y, \gamma_z\} \sim \{\gamma_t, \gamma_x, \gamma_y, \gamma_z\,, +\,, -\,\} \tag{43}$$

and

$$\{-\gamma_t, -\gamma_x, -\gamma_y, -\gamma_z\} \sim \{\gamma_t, \gamma_x, \gamma_y, \gamma_z\,, -\,, +\,\} \tag{44}$$

then we can generate a permutation

$$-\gamma_y \gamma_z \gamma_x \sim (+,-)(x,z,y)\{\gamma_t, \gamma_x, \gamma_y, \gamma_z, +, -\} \tag{45}$$

where $(+,-)(x,z,y) \in S_5$. This set of all such permutations is isomorphic to a subset of S_5; we are only considering transformations which don't mix the rotation and reflection coordinates. The subgroups of S_5 are known and therefore the subgroups of our symmetry group are known. In the same manner as before, we may associate each symbol with a transformation as

$$\Sigma \vdash (+\gamma_a\gamma_b\gamma_c) \equiv \sigma_{abc} \tag{46a}$$

and

$$\Sigma \vdash (-\gamma_a\gamma_b\gamma_c) \equiv \sigma_{abc}(+,-) \tag{46b}$$

so that the symbol and permutation groups are isomorphic. The map (46a) is a bijection over the positive symbols while (46b) is a bijection over the negative symbols. Since their images are disjoint, the entire map is a bijection. Therefore, to each permutation, we can associate a unique symbol, and vice versa. From henceforth, the neutral symbol $\gamma_a\gamma_b\gamma_c$ will have ambiguous charge so that

$$\Sigma \vdash \gamma_a\gamma_b\gamma_c \equiv \Sigma_{abc} \tag{47}$$

where $\Sigma_{abc} \in S_5$ and

$$\Sigma \vdash \overline{\gamma_a\gamma_b\gamma_c} \equiv \Sigma_{abc}(+,-) \tag{48}$$

We can now define the product of two symbols to be the symbol associated with the product of their corresponding permutations

$$\gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \equiv \gamma \vdash (\Sigma_{abc} \Sigma_{pqs}) \tag{49}$$

We would then have

$$(\Sigma \vdash \gamma_a\gamma_b\gamma_c) \circ (\Sigma \vdash \gamma_p\gamma_q\gamma_s) = \Sigma_{abc}\Sigma_{pqs} \tag{50a}$$

$$= \gamma^{-1} \vdash \gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \tag{50b}$$

$$= \Sigma \vdash \gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \tag{50c}$$

as before. Notice that σ also acts a homomorphism on the group $\{+\gamma_a \gamma_b \gamma_c\} \cup \{+,-\}$ as

$$\sigma \vdash (+\gamma_a\gamma_b\gamma_c) = (\sigma \vdash +)(\sigma \vdash (\gamma_a\gamma_b\gamma_c)) = (\sigma \vdash (\gamma_a\gamma_b\gamma_c)) = \sigma_{abc} \tag{51a}$$

$$\sigma \vdash (-\gamma_a\gamma_b\gamma_c) = (\sigma \vdash -)(\sigma \vdash (\gamma_a\gamma_b\gamma_c)) = (\sigma \vdash (\gamma_a\gamma_b\gamma_c)) = (+,-)\sigma_{abc} \tag{51b}$$

so that

$$\sigma \vdash (\pm\gamma_a\gamma_b\gamma_c) = (\sigma \vdash \pm)(\sigma \vdash (\gamma_a\gamma_b\gamma_c)) = (\sigma \vdash \pm)\sigma_{abc} \tag{52}$$

The subsets of S_5 are known and hence we can replace each subgroup lattice of S_5 with its corresponding symbol group. We can elevate this to a Galois Correspondence by the same methods. In particular, we will simply have

$$\hat{q} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} \alpha \mathrm{I} \\ \beta \gamma_t \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} c_{abc} \begin{pmatrix} \gamma_a \\ \gamma_b \\ \gamma_c \end{pmatrix} \tag{53}$$

As we can absorb the negative sign of $-\gamma_a \gamma_b \gamma_c$ into the coefficients. Notice that the associated derivative operator simply uses negative gamma matrices.

4. Coordinate Representations

Again consider the field

$$\hat{\mathbb{C}} \equiv \mathbb{C} \cdot \mathrm{I} \tag{54}$$

We can adjoin our symbols to this group to form $\hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\})$. This is a ring of operators; in particular, say we choose to operate on a spinor. We can choose to work in the derivative basis so that

$$\gamma_a \gamma_b \gamma_c \vdash \psi \equiv D_{abc}\psi \tag{55}$$

Notice that

$$z \cdot D_{abc} \vdash \psi = D_{abc}(z \vdash \psi) \tag{56}$$

So that this product commutes with the base ring. In this ring, we could use the symmetry operators

$$\gamma_a \gamma_b \gamma_c \cdot z \equiv (\sigma \vdash \gamma_a \gamma_b \gamma_c) \cdot z \tag{57}$$

However, we would not be acting on the spinors, we could instead act on the gamma matrices

$$\gamma_a \gamma_b \gamma_c \vdash \{\gamma\} \equiv \sigma_{abc}\{\gamma\} \tag{58}$$

A number $z \in \mathbb{C}$ would uniformly multiply the set by a constant so that

$$z \vdash \{\gamma\} \equiv \{z\gamma\} \tag{60}$$

Notice again,

$$z \cdot \sigma_{abc} \vdash \{\gamma\} = \sigma_{abc}(z \vdash \{\gamma\}) \tag{61}$$

commutes as the permutations are independent of a uniform multiplicative factor.

A. Symmetry Breaking

We could break the field symmetry of the Galois Correspondence. We do this by projecting back to the permutation group operation. In the first case, this would give

$$\gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \equiv D_{abc} \cdot D_{pqs} \tag{62a}$$

$$= D\sigma^{-1} \vdash (a,b,c)(p,q,s) \tag{62b}$$

Which is noncommuting. The permutation representation is similarly noncommuting as we have

$$\gamma_a \gamma_b \gamma_c \cdot \gamma_p \gamma_q \gamma_s \equiv \sigma_{abc} \cdot \sigma_{pqs} \tag{63}$$

This ideas can be extended to curved space as the permutation group remains the same.

B. Geometric Constructions

When considering the Dirac Operator Polynomial, we formed the set

$$\partial = \{\partial_t, \partial_x, \partial_y, \partial_z\} \tag{64}$$

and created the field extension.

$$\hat{\partial} \equiv \mathbb{C}(\partial) \cdot \mathrm{I} \tag{65}$$

Again we would have

$$\gamma_a \gamma_b \gamma_c \cdot d \equiv = d \cdot \gamma_a \gamma_b \gamma_c = \gamma_a \gamma_b \gamma_c \cdot d \tag{67}$$

in both coordinate systems. This is because the permutation is independent of a uniform derivative operator. The gamma matrices also commute with the derivative operators as they are covariantly constant so that d acts as any other member of $\mathbb{C}$. Hence we have

$$d \cdot \sigma_{abc} \vdash \{\gamma\} = \sigma_{abc}\,(d \vdash \{\gamma\}) \tag{68a}$$

$$d \cdot D_{abc} \vdash \psi = D_{abc}\,(d \vdash \psi) \tag{68b}$$

As again, both the partials and gamma matrices commute with the all derivative operators. This can be projected up by substituting the derivative operators for the covariant derivative operators. All of the preceding logic follows as the gamma are covariantly constant and the symmetry group is preserved.

C. Abelian Products

To preserve the symmetry, we could instead inherent the field operation. Recall that, to first order

$$\hat{q} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} \alpha \mathrm{I} \\ \beta \gamma_t \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\}} c_{abc} \begin{pmatrix} \gamma_a \\ \gamma_b \\ \gamma_c \end{pmatrix} \tag{69}$$

We can associate the operator

$$D \vdash \hat{q} = \alpha \mathrm{I} + \beta \gamma_t + \sum_{\sigma\{a,b,c\}} c_{abc}(\gamma_a \partial_x + \gamma_b \partial_y + \gamma_c \partial_z) \tag{70}$$

This is a bijection. For higher order products, we have

$$\hat{q} \cdot \hat{p} \in \hat{\mathbb{C}}(\{\gamma_a \gamma_b \gamma_c\}) \equiv \begin{pmatrix} (\alpha \cdot \alpha')\mathrm{I}^2 \\ (\beta \cdot \beta')\gamma_t^2 \end{pmatrix} \wedge \sum_{\sigma\{a,b,c\},\sigma\{a',b',c'\}} (c_{abc} \cdot c'_{a'b'c'}) \begin{pmatrix} (-1)^i \gamma_a \cdot \gamma_{a'} \\ (-1)^j \gamma_b \cdot \gamma_{b'} \\ (-1)^k \gamma_c \cdot \gamma_{c'} \end{pmatrix} \tag{80}$$

$$D \vdash (\hat{q} \cdot \hat{p}) \equiv (\alpha \cdot \alpha')\mathrm{I}^2 + (\beta \cdot \beta')\gamma_t^2 \partial_t^2 + \sum_{\sigma\{a,b,c\},\sigma\{a',b',c'\}} (c_{abc} \cdot c'_{a'b'c'})[(-1)^i \gamma_a \cdot \gamma_{a'} \partial_x^2 + (-1)^j \gamma_b \cdot \gamma_{b'} \partial_y^2 + (-1)^k \gamma_c \cdot \gamma_{c'} \partial_z^2] \tag{81}$$

Higher order products simply use third order derivatives. We can therefore define the product of two derivative operators to be the derivative operator associated to the product of their field representations so that

$$(D \vdash \hat{q}) \cdot (D \vdash \hat{p}) \equiv D \vdash (\hat{q} \cdot \hat{p}) \tag{82}$$

and

$$(D \vdash \hat{q}) + (D \vdash \hat{p}) \equiv D \vdash (\hat{q} + \hat{p}) \tag{83}$$

Where the sum is defined by the sum of operators. In the permutation basis, we can correlate

$$\sigma \vdash \hat{q} = \sum_{\sigma\{a,b,c\}} [\alpha, \beta, c_{abc}(a), c_{abc}(b), c_{abc}(c)] \tag{84}$$

so that

$$\sigma \vdash \hat{q} \cdot \hat{p} = \sum_{\sigma\{a,b,c\}} [\alpha, \beta, c_{abc}c_{a'b'c'}(a, a'), c_{abc}c_{a'b'c'}(b, b'), c_{abc}c_{a'b'c'}(c, c')] \tag{85}$$

and

$$\sigma \vdash \hat{q} \cdot \hat{p} = \sum_{\sigma\{a,b,c\}\sigma\{a',b',c'\}\sigma\{a'',b'',c''\}} [\alpha, \beta, c_{abc}c_{a'b'c'}(a, a', a''), c_{abc}c_{a'b'c'}(b, b', b''), c_{abc}c_{a'b'c'}(c, c', c'')] \tag{86}$$

In S_3, any permutation in 4 letters can be reduced to a permutation in 3 letters therefore we are done. The operation is symmetric as appending to the front is the same as appending to the end. The sum is naturally by the cartesian product.

References

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