A New Formulation of the Gravitational Action in terms of Spinor Variables

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Abstract

We rewrite the gravitational action solely in terms of the gamma matrices and spinor connection (as defined in the curved space Dirac equation). The requirement that the variation vanishes gives the condition that the affine connection be Levi-Civita along Einstein's equations in vacuum. We then propose an extension of this theory to the case of the gravitational field interacting with another, external field.

1. Preliminary Remarks

Hilbert used the action principle to formalize the gravitational field equations and extend them further. His original approach, however, was not unique. It was soon realized that different pairs of variables, such as the vierbein and spin connection or the metric and affine connection, could be used to rewrite Hilbert's action. From these, one could also reproduce Einstein's theory. In the current discourse, we describe another formulation. Dirac's variables seemingly coincide with the vierbein formalism. Both the tetrad and gamma matrices can be used to construct the metric. And, the spin and spinor connections are intimately related by formulae. From this, we are led to believe that a symmetry of must exist between the theories. This paper is the result of that conviction and serves as a proof of the aforementioned correspondence. In Appendix A, we derive the closed form action in terms of Dirac's variables. Taking the variation, one arrives at the desired result. This was not my original direction, however. Presented is my initial proof.

2. Gravitation

We will first show that the Palatini Lagrangian may be rewritten so that

$$S = \int g_{\mu\nu} R^{\mu\nu}(\Gamma) \rightarrow \int \mathscr{L}(\gamma, \Pi) \tag{1}$$

where all integrations are over a certain four-dimensional volume. Here, we are referring the spinor connection and gamma matrices as used in the covariant Dirac equation

$$\gamma^\mu D_\mu \psi = m\psi \tag{2}$$

where the covariant derivative of spinors is defined as

$$D_\mu \equiv \partial_\mu - \Pi_\mu \tag{3}$$

A. Re-expressing the metric

The defining property of the gamma matrices are the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}\, \mathrm{I} \tag{4}$$

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Inverting this, we find

$$g^{\mu\nu} = \frac{1}{8}\,\mathrm{tr}\{\gamma^{\mu}, \gamma^{\nu}\} \tag{5}$$

B. Re-expressing the affine connection

We have that gamma matrices are covariantly constant,

$$\nabla_{\nu}\gamma^{\mu} = \partial_{\nu}\gamma^{\mu} + \Gamma^{\mu\nu}_{\ \ \ell}\gamma^{\ell} + \left[\gamma^{\mu}, \Pi_{\nu}\right] = 0 \tag{6}$$

This assertion and choice of covariant derivative is due to equation (8) of [1]. We can solve this for $\Gamma^{\rho}_{\mu\nu}$. Rearranging,

$$-\Gamma^{\mu}_{\nu\ell}\gamma^{\ell} = \partial_{\nu}\gamma^{\mu} + \left[\gamma^{\mu}, \Pi_{\nu}\right] \tag{7}$$

Take the anticommutator on the L.H.S.

$$\{\Gamma^{\mu}_{\nu\ell}\gamma^{\ell}, \gamma^{\rho}\} = \Gamma^{\mu}_{\nu\ell}\{\gamma^{\ell}, \gamma^{\rho}\} \tag{8a}$$

$$= 2\,\Gamma^{\mu}_{\nu\ell}\; g^{\ell\rho}\ \mathrm{I} \tag{8b}$$

Hence raising the index, we get

$$\{\Gamma^{\mu}_{\nu\ell}\gamma^{\ell}, \gamma^{\rho}\} = 2\,\Gamma^{\mu\rho}_{\nu}\ \mathrm{I} \tag{9}$$

We can then take the anticommutator on the R.H.S. of (7) so that

$$\Gamma^{\mu\rho}_{\nu} = -\frac{1}{8}\,\mathrm{tr}\{\partial_{\nu}\gamma^{\mu} + \left[\gamma^{\mu}, \Pi_{\nu}\right], \gamma^{\rho}\} \tag{10}$$

where we have taken the trace over the spinor indices. Hence both metric and affine connection can be written as functions of the spinor connection and gamma matrices alone.

3. Equations of Motion

In order for $\delta S = 0$, we must have that

$$\int d^4x\,[\frac{\delta S}{\delta\gamma^{\mu}}\delta\gamma^{\mu} + \frac{\delta S}{\delta\Pi_{\mu}}\delta\Pi_{\mu}] = 0 \tag{11}$$

Each term must vanish separately so that

1. $\dfrac{\delta S}{\delta\gamma^{\mu}}\delta\gamma^{\mu} = 0$ for all variations $\delta\gamma^{\mu}$ (12a)

2. $\dfrac{\delta S}{\delta\Pi_{\mu}}\delta\Pi_{\mu} = 0$ for all variations $\delta\Pi_{\mu}$ (12b)

A. Variation with respect to the gamma matrices

From above, the first condition must be true for any arbitrary variation of the gamma matrices. We must then require

$$\frac{\delta S}{\delta \gamma^{\mu}} = 0 \tag{13}$$

This can be expanded in terms of the metric and affine connection

$$\frac{\delta S}{\delta g_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\delta \gamma^{\rho}} + \frac{\delta S}{\delta \Gamma^{\ell}_{\mu\nu}} \frac{\delta \Gamma^{\ell}_{\mu\nu}}{\delta \gamma^{\rho}} = 0 \tag{14}$$

after a change of indices $\mu \to \rho$ in the gamma matrices. We can do this chain rule expansion as the original Palatini Action could be written solely as a function of both the metric and affine connection. The metric can then be rewritten in terms of the gamma matrices by formula (5). Furthermore, since the spinor connection is fixed, the affine connection is completely determined by the these same matrices using (10).

For the second term in (14), we find

$$\frac{\delta S}{\delta \Gamma^{\ell}_{\mu\nu}} = \delta^{\nu}_{\ell} \nabla_{\lambda} \mathfrak{g}^{\mu\lambda} - \nabla_{\ell}\, \mathfrak{g}^{\mu\nu} \tag{15}$$

where we have abbreviated

$$\mathfrak{g}_{\mu\nu} \equiv g_{\mu\nu}\sqrt{g} \tag{16}$$

Since Dirac's matrices are covariantly constant (6), we may conclude

$$\nabla g_{\mu\nu} = 0 \tag{17}$$

And therefore

$$\frac{\delta S}{\delta \Gamma^{\ell}_{\mu\nu}} = 0 \tag{18}$$

Hence the second term in (14) is vanishing. For the first term, we can compute the variation of the metric using (5). This is done in text Appendix B. The result is

$$\frac{\delta g^{\mu\nu}}{\delta \gamma^{\rho}} = \frac{1}{4}\, [\gamma^{\mu}\delta^{\nu}_{\rho} + \gamma^{\nu}\delta^{\mu}_{\rho}]^{\mathrm{T}} \tag{19}$$

We may then say

$$\frac{\delta S}{\delta g_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\delta \gamma^{\rho}} = [R_{\mu\nu} - g_{\mu\nu}R\ ][\gamma^{\mu}\delta^{\nu}_{\rho} + \gamma^{\nu}\delta^{\mu}_{\rho}]^{\mathrm{T}}\, \frac{1}{4} \tag{20}$$

where we have used Palatini's Variation

$$\frac{\delta S}{\delta g_{\mu\nu}} = R_{\mu\nu} - g_{\mu\nu}R \tag{21}$$

to evaluate the first element. Now taking the transpose over the spinor indices:

$$\left(\frac{\delta S}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta\gamma^{\rho}}\right)\mathrm{T} = G_{\mu\nu}[\gamma^{\mu}\delta^{\nu\rho} + \gamma^{\nu}\delta^{\mu\rho}]\frac{1}{4} \tag{22}$$

where we have abbreviated Einstein's tensor as

$$G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R \tag{23}$$

We may take the anticommutator with γ^{λ}

$$\{\gamma^{\mu}\delta^{\nu}_{\rho} + \gamma^{\nu}\delta^{\mu}_{\rho}\,, \gamma^{\lambda}\} \tag{24a}$$

$$= \{\gamma^{\mu}\delta^{\nu}_{\rho}, \gamma^{\lambda}\} + \{\gamma^{\nu}\delta^{\mu}_{\rho}, \gamma^{\lambda}\} \tag{24b}$$

$$= \delta^{\nu}_{\rho}\{\gamma^{\mu}, \gamma^{\lambda}\} + \delta^{\mu}_{\rho}\{\gamma^{\nu}, \gamma^{\lambda}\} \tag{24c}$$

$$= 2\,[\delta^{\nu}_{\rho}g^{\mu\lambda}\mathrm{I} + \delta^{\mu}_{\rho}g^{\nu\lambda}\mathrm{I}] \tag{24d}$$

Using this result in (22),

$$\{G_{\mu\nu}\,[\gamma^{\mu}\delta^{\nu}_{\rho} + \gamma^{\nu}\delta^{\mu}_{\rho}]\,, \gamma^{\lambda}\} \tag{25a}$$

$$= G_{\mu\nu}\{[\gamma^{\mu}\delta^{\nu}_{\rho} + \gamma^{\nu}\delta^{\mu}_{\rho}]\,, \gamma^{\lambda}\} \tag{25b}$$

$$= 2\,G_{\mu\nu}[\delta^{\nu}_{\rho}g^{\mu\lambda}\mathrm{I} + \delta^{\mu}_{\rho}g^{\nu\lambda}\mathrm{I}] \tag{25c}$$

$$= 2\,[G_{\alpha\rho} + G_{\rho\alpha}]\,\mathrm{I} \tag{25d}$$

In the next section, we will show that the connection is symmetric and hence the Einstein tensor is symmetric so that this reduces to

$$\{G_{\mu\nu}[\gamma^{\mu}\delta^{\nu}_{\rho} + \gamma^{\nu}\delta^{\mu}_{\rho}]\,, \gamma^{\lambda}\} = 4\,G_{\alpha\rho}\;\mathrm{I} \tag{26}$$

Taking the trace over the spinor indices we get

$$\frac{1}{16}\,\mathrm{tr}\{G_{\mu\nu}[\gamma^{\mu}\delta^{\nu}_{\rho} + \gamma^{\nu}\delta^{\mu}_{\rho}]\,, \gamma^{\lambda}\} = G_{\alpha\rho} \tag{27}$$

From (14), (18), and (22) we conclude

$$G_{\alpha\rho} = 0 \tag{28}$$

B. Variation with respect to the spinor connection

We may now focus on the spinor connection. From before,

$$\frac{\delta S}{\delta \Pi_\mu}\delta \Pi_\mu = 0 \tag{29}$$

$\forall$ variations $\delta \Pi_\mu$

We can obtain a formula for $\delta \Pi_\mu$ in terms of the other variables. We will need equation (6), and for convenience, we restate it here:

$$\nabla_\nu \gamma^\mu = \partial_\nu \gamma^\mu + \Gamma^\mu_{\nu\ell}\gamma^\ell + \left[\gamma^\mu, \Pi_\nu\right] = 0 \tag{30}$$

Contracting with γ_μ

$$-\gamma_\mu \gamma^\mu \Pi_\nu = \gamma_\mu[\partial_\nu \gamma^\mu + \gamma_\mu \Gamma^\mu_{\nu\ell}\gamma^\ell] - \gamma_\mu \Pi_\nu \gamma^\mu \tag{31}$$

The contraction on the L.H.S. can be evaluated with the help of the tetrad. The tetrad (or vierbein) projects the curved geometry down to flat space and vice versa. In the new coordinate system, the curved space vector A_μ becomes

$$A^a = e^a_\mu A^\mu \tag{32}$$

And its scalar product gives

$$\langle A, B\rangle = A^a B^b \eta_{ab} \tag{33}$$

This results in the condition

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{34}$$

which is equation (3) of [2]. We may use (32) and (34) to re-express the anticommutation relation (4)

$$\{\gamma^\mu, \gamma^\nu\} = \{e^\mu_a \gamma^a, e^\nu_b \gamma^b\} \tag{35a}$$

$$= e^\mu_a e^\nu_b \{\gamma^a, \gamma^b\} \tag{35b}$$

We can multiply by the inverse vierbein and use (4) and (34) on the L.H.S to get

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathrm{I} \tag{36}$$

Contracting over the latin indices, we find $\gamma^a\gamma_a = 4\mathrm{I}$. Since the inner product is a scalar, it will not depend on coordinates so that $\gamma^\mu\gamma_\mu = 4\mathrm{I}$. Using this fact in (31),

$$-4\Pi_\nu = \gamma_\mu[\partial_\nu \gamma^\mu + \gamma_\mu \Gamma^\mu_{\nu\ell}\gamma^\ell] - \gamma_\mu \Pi_\nu \gamma^\mu \tag{37}$$

In Appendix C, we show that the solution to the above equation is given by

$$\Pi_\nu = -\frac{1}{4}\gamma_\mu[\partial_\nu \gamma^\mu + \Gamma^\mu_{\nu\ell}\gamma^\ell] - ieA_\nu \mathrm{I} \tag{38}$$

where A_ν is a four vector. Hence the spinor connection can be written solely as a function of the gamma matrices, affine connection, and an arbitrary four vector. We can now rewrite the $\delta \Pi_\mu$ in (29) as

$$\delta\Pi_\rho = \frac{\delta\Pi_\rho}{\delta\gamma^\mu}\,\delta\gamma^\mu + \frac{\delta\Pi_\rho}{\delta\Gamma^\lambda_{\mu\nu}}\,\delta\Gamma^\lambda_{\mu\nu} + \frac{\delta\Pi_\rho}{\delta A_\nu}\,\delta A_\nu \tag{39}$$

after a change of indices $\mu \rightarrow \rho$ in the spinor connection. Since the affine connection transforms as a connection, it can be written solely as a function of the transformation matrix (the vierbein) and its corresponding connection in flat space (the spin connection). We may therefore express its variation as

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{\delta\Gamma^\lambda_{\mu\nu}}{\delta e^\sigma_b}\,\delta e^\sigma_b + \frac{\delta\Gamma^\lambda_{\mu\nu}}{\delta\omega^\sigma_{ab}}\,\delta\omega^\sigma_{ab} \tag{40}$$

where we have used ω^σ_{ab} to denote the spin connection. We can now rewrite the variation of the spinor connection. Using this in (39) we get

$$\delta\Pi_\rho = \frac{\delta\Pi_\rho}{\delta\gamma^\mu}\,\delta\gamma^\mu + \frac{\delta\Pi_\rho}{\delta A_\nu}\,\delta A_\nu \tag{41}$$

$$+\frac{\delta\Pi_\rho}{\delta\Gamma^\lambda_{\mu\nu}}[\frac{\delta\Gamma^\lambda_{\mu\nu}}{\delta e^\sigma_b}\,\delta e^\sigma_b + \frac{\delta\Gamma^\lambda_{\mu\nu}}{\delta\omega^\sigma_{ab}}\,\delta\omega^\sigma_{ab}]$$

The curved space gamma matrices are being held fixed so that the first term vanishes. At this point we may also eliminate the variation in the vector potential. The reason for this is explained in Appendix D. We are left with

$$\delta\Pi_\rho = \frac{\delta\Pi_\rho}{\delta\Gamma^\lambda_{\mu\nu}}[\frac{\delta\Gamma^\lambda_{\mu\nu}}{\delta e^\sigma_b}\,\delta e^\sigma_b + \frac{\delta\Gamma^\lambda_{\mu\nu}}{\delta\omega^\sigma_{ab}}\,\delta\omega^\sigma_{ab}] \tag{42}$$

In Appendix D, we also show that whenever the curved space gamma matrices are held fixed, the vierbein is also fixed. The above equation then implies

$$\delta\Pi_\rho = \frac{\delta\Pi_\rho}{\delta\Gamma^\lambda_{\mu\nu}}[\frac{\delta\Gamma^\lambda_{\mu\nu}}{\delta\omega^\sigma_{ab}}\,\delta\omega^\sigma_{ab}] \tag{43}$$

The vector potential is fixed and the gamma matrices are fixed so the spinor connection is solely a function of the affine connection. We can therefore eliminate the chain rule in this variable leaving

$$\delta\Pi_\rho = \frac{\delta\Pi_\rho}{\delta\omega^\sigma_{a_b}}\,\delta\omega^\sigma_{ab} \tag{44}$$

Now that we have simplified the variation of the spinor connection, we can insert it into (29):

$$\frac{\delta S}{\delta\Pi_\rho}\frac{\delta\Pi_\rho}{\delta\omega^\sigma_{ab}}\,\delta\omega^\sigma_{ab} = 0 \tag{45}$$

Since this variation holds the curved space gamma matrices fixed, the action is solely a function of the spinor connection. Removing the chain rule,

$$\frac{\delta S}{\delta \omega^{\sigma}_{ab}} \delta \omega^{\sigma}_{ab} = 0 \tag{46}$$

$\forall$ variations $\delta \omega^{\sigma}_{ab}$ hence,

$$\frac{\delta S}{\delta \omega^{\sigma}_{ab}} = 0 \tag{47}$$

The tetradic version of Palatini's Theorem allows us to write the action solely as a function of the vierbein and spin connection. The requirement that the variation vanish with respect to the spin connection gives the condition that the connection be symmetric. Applying this to (47), since the vierbein is fixed (Appendix D), we would then have that the connection is symmetric. This along with (17) gives the Levi-Civita connection.

4. A Comprehensive Action Principle

For the total action, one could propose

$$S = \int \mathscr{L}_g(\gamma^\mu, \Pi) + S' \tag{48}$$

where the primed functional corresponds to the external field. In the case of a Dirac particle we get

$$S = \int \mathscr{L}_g(\gamma^\mu, \Pi) + \int \bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi \tag{49}$$

where we treat the gamma matrices, spinor connection, spinor, and adjoint spinor as independent variables. The variation then gives

$$\delta S = \int \delta \mathscr{L}_g(\gamma^\mu, \Pi) \tag{50}$$

$$+ \int (\delta\bar{\psi})\gamma^\mu D_\mu \psi + \bar{\psi}\delta(\gamma^\mu) D_\mu \psi + \bar{\psi}\gamma^\mu \delta(D_\mu \psi) - m(\delta\bar{\psi})\psi - m\bar{\psi}(\delta\psi)$$

We can compute the variation of the covariant derivative as

$$\delta(D_\mu \psi) = \bar{\psi}\gamma^\mu \delta(\partial_\mu \psi) + \bar{\psi}\gamma^\mu \delta(\Pi_\mu)\psi + \bar{\psi}\gamma^\mu \Pi_\mu \delta(\psi) \tag{51}$$

Forcing the total variation to vanish, we obtain the equations of motion for the gravitational field

$$\frac{\delta \mathscr{L}_g}{\delta \gamma} = -\bar{\psi} D_\mu \psi \tag{52a}$$

$$\frac{\delta \mathscr{L}_g}{\delta \Pi} = -\bar{\psi}\gamma^\mu \psi \tag{53b}$$

And the equations of motion for the spinor field

$$\bar{\psi}\gamma^\mu D_\mu - m\bar{\psi} = 0 \tag{53a}$$

$$\gamma^{\mu} D_{\mu}\psi - m\psi = 0 \tag{53b}$$

We may add extra terms to the external action by using the relations (5) and (10). Metric and affine functionals are simply rewritten in terms of Dirac's variables.

Appendix A. The Closed Form Action

Here, we derive the closed form of the Einstein-Hilbert Action in terms of the gamma matrices and spinor connection alone. The scalar density $\sqrt{g}$ can be written solely as a function of these variables using (4). All that is left is re-expressing the Ricci Scalar.

1. Spinor curvature and torsion

The covariant derivative of spinors is of the form

$$\nabla_{\mu}\psi = \partial_{\mu}\psi - \Pi_{\mu}\psi \tag{A1}$$

We want to calculate the commutator of covariant derivatives $[\nabla_{\mu}, \nabla_{\nu}]\psi$. This is analogous to how the Riemann curvature tensor is obtained. For the first term, we have

$$\nabla_{\mu}\nabla_{\nu}\psi = \nabla_{\mu}[\partial_{\nu}\psi - \Pi_{\nu}\psi] \tag{A2}$$

Expanding the covariant derivative,

$$\nabla_{\mu}\nabla_{\nu}\psi = \partial_{\mu}[\partial_{\nu}\psi - \Pi_{\nu}\psi] - \Gamma^{\rho}_{\nu\mu}[\partial_{\rho}\psi - \Pi_{\rho}\psi] \tag{A3}$$

$$-\Pi_{\mu}[\partial_{\nu}\psi - \Pi_{\nu}\psi]$$

Similarly,

$$\nabla_{\nu}\nabla_{\mu}\psi = \partial_{\nu}[\partial_{\mu}\psi - \Pi_{\mu}\psi] - \Gamma^{\rho}_{\mu\nu}[\partial_{\rho}\psi - \Pi_{\rho}\psi] \tag{A4}$$

$$-\Pi_{\nu}[\partial_{\mu}\psi - \Pi_{\mu}\psi]$$

Subtracting, we obtain the commutator

$$[\nabla_{\mu}, \nabla_{\nu}]\psi = \Phi_{\mu\nu}\psi + T^{\rho}_{\mu\nu}\nabla_{\rho}\psi \tag{A5}$$

where

$$\Phi_{\mu\nu} \equiv \partial_{\nu}\Pi_{\mu} - \partial_{\mu}\Pi_{\nu} + [\Pi_{\mu}, \Pi_{\nu}] \tag{A6}$$

$$T^{\rho}_{\mu\nu} \equiv \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} \tag{A7}$$

2. Two-Spinors

We can create a two spinor by contracting onto the gamma matrices. Choose some arbitrary four vector A_{ρ} and let

$$\sigma \equiv A_{\rho}\gamma^{\rho} \tag{A8}$$

This can be inverted to

$$A^{\rho} = \frac{1}{8}\,\mathrm{tr}\{\sigma, \gamma^{\rho}\} \tag{A9}$$

We can follow suit with the previous section and take the commutator on this object $[\nabla_\mu, \nabla_\nu]\,\sigma$. Using (6), we have for the first term:

$$\nabla_\mu \nabla_\nu \sigma = \nabla_\mu(\partial_\nu \sigma + \left[\sigma, \Pi_\nu\right]) \tag{A10}$$

Expanding the covariant derivative,

$$\nabla_\mu \nabla_\nu \sigma = \partial_\mu \left(\partial_\nu \sigma + \left[\sigma, \Pi_\nu\right]\right) - \Gamma^{\lambda}_{\nu\mu}\left(\partial_\lambda \sigma + \left[\sigma, \Pi_\lambda\right]\right)$$

$$+[\left(\partial_\nu \sigma + \left[\sigma, \Pi_\nu\right]\right), \Pi_\mu] \tag{A11}$$

Similarly,

$$\nabla_\nu \nabla_\mu \sigma = \partial_\nu \left(\partial_\mu \sigma + \left[\sigma, \Pi_\mu\right]\right) - \Gamma^{\lambda}_{\mu\nu}\left(\partial_\lambda \sigma + \left[\sigma, \Pi_\lambda\right]\right) \tag{A12}$$

$$+[\left(\partial_\mu \sigma + \left[\sigma, \Pi_\mu\right]\right), \Pi_\nu]$$

Subtracting these, we arrive at the commutator

$$[\nabla_\mu, \nabla_\nu]\,\sigma = [\Phi_{\mu\nu}, \sigma] + T^{\lambda}_{\mu\nu}\nabla_\lambda \sigma \tag{A13}$$

3. The Riemann tensor in terms of the gamma matrices and spinor connection

We can use the result for two-spinors to help calculate the commutator on the gamma matrices. From (6),

$$\nabla_\nu \gamma^{\rho} = \partial_\nu \gamma^{\rho} + \Gamma^{\rho}_{\nu\ell}\gamma^{\ell} + \left[\gamma^{\rho}, \Pi_\nu\right] = 0 \tag{A14}$$

So that the first term in the commutator gives

$$\nabla_\mu \nabla_\nu \gamma^{\rho} = \nabla_\mu\left(\partial_\nu \gamma^{\rho} + \Gamma^{\rho}_{\nu\ell}\gamma^{\ell} + \left[\gamma^{\rho}, \Pi_\nu\right]\right) \tag{A15}$$

We can expand out the covariant derivative:

$$\nabla_\mu \nabla_\nu \gamma^{\rho} = \partial_\mu\left(\partial_\nu \gamma^{\rho} + \Gamma^{\rho}_{\nu\ell}\gamma^{\ell} + \left[\gamma^{\rho}, \Pi_\nu\right]\right) \tag{A16}$$

$$+\Gamma^{\rho}_{\mu\lambda}\left(\partial_\nu \gamma^{\lambda} + \Gamma^{\lambda}_{\nu\ell}\gamma^{\ell} + \left[\gamma^{\lambda}, \Pi_\nu\right]\right)$$

$$-\Gamma^{\lambda}_{\nu\mu}\left(\partial_\lambda \gamma^{\rho} + \Gamma^{\rho}_{\lambda\ell}\gamma^{\ell} + \left[\gamma^{\rho}, \Pi_\lambda\right]\right)$$

$$+\Big[(\partial_\nu \gamma^\rho + \Gamma^\rho_{\nu\ell}\gamma^\ell + \big[\gamma^\rho, \Pi_\nu\big]), \Pi_\mu\Big]$$

Taking the commutator and using (A5) we find

$$[\,\nabla_\mu, \nabla_\nu\,]\,\gamma^\rho = [\Phi_{\mu\nu}, \gamma^\rho] + T^\lambda_{\nu\mu}\nabla_\lambda \gamma^\rho + R^\rho_{\ell\mu\nu}\gamma^\ell \tag{A17}$$

The gamma matrices are covariantly constant so that the entire L.H.S and the middle term on the R.H.S vanish. We are left with

$$0 = [\Phi_{\mu\nu}, \gamma^\rho] + R^\rho_{\ell\mu\nu}\gamma^\ell \tag{A18}$$

Rearranging,

$$R^\rho_{\ell\mu\nu}\gamma^\ell = -\,[\Phi_{\mu\nu}, \gamma^\rho] \tag{A19}$$

On the L.H.S of the above equation, we may take the anticommutator with γ^λ

$$\{R^\rho_{\ell\mu\nu}\gamma^\ell, \gamma^\lambda\} \tag{A20a}$$

$$= R^\rho_{\ell\mu\nu}\{\gamma^\ell, \gamma^\lambda\} \tag{A20b}$$

$$= 2\,R^\rho_{\ell\mu\nu}\, g^{\ell\lambda}\,\mathrm{I} \tag{A20c}$$

$$= 2\,R^{\rho\lambda}_{\mu\nu}\,\mathrm{I} \tag{A20d}$$

The anticommutator on the R.H.S of (A19) gives

$$2\,R^{\rho\lambda}_{\mu\nu}\,\mathrm{I} = -\,\{[\Phi_{\mu\nu}, \gamma^\rho], \gamma^\lambda\} \tag{A21}$$

so that the Riemann Tensor,

$$R^{\rho\lambda}_{\mu\nu} = -\,\frac{1}{8}\,\mathrm{tr}\{[\Phi_{\mu\nu}, \gamma^\rho], \gamma^\lambda\} \tag{A22}$$

the Ricci Tensor,

$$R^\lambda_\nu \equiv R^{\rho\lambda}_{\rho\nu} = -\,\frac{1}{8}\,\{[\Phi_{\rho\nu}, \gamma^\rho], \gamma^\lambda\} \tag{A23}$$

and the Ricci Scalar

$$R\,\equiv R^\lambda_\lambda = -\,\frac{1}{8}\,\mathrm{tr}\{[\Phi_{\rho\lambda}, \gamma^\rho], \gamma^\lambda\} \tag{A24}$$

can all be written in terms of the gamma matrices and spinor connection alone.

Appendix B. Variation of the Metric

The metric is related to the gamma matrices by the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}\,\mathrm{I} \tag{B1}$$

This can be inverted to

$$g^{\mu\nu} = \frac{1}{8}\,\mathrm{tr}\{\gamma^\mu, \gamma^\nu\} \tag{B2}$$

We can use this to take the variation of the metric:

$$\delta g^{\mu\nu} = \frac{1}{8}\,\delta\mathrm{tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] \tag{B3}$$

Use the fact that tr(AB) = tr(BA) so that

$$\delta g^{\mu\nu} = \frac{1}{4}\,\delta\mathrm{tr}[\gamma^\mu\gamma^\nu] \tag{B4}$$

The trace also commutes with the variation:

$$\delta g^{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,\delta[\gamma^\mu\gamma^\nu] \tag{B5}$$

Working out the parenthesis,

$$\mathrm{tr}\,\delta[\gamma^\mu\gamma^\nu] = \mathrm{tr}\,[\gamma^\mu\left(\delta\gamma^\nu\right) + \left(\delta\gamma^\mu\right)\gamma^\nu] \tag{B6a}$$

$$= \mathrm{tr}\,[\gamma^\mu\left(\delta\gamma^\nu\right) + \gamma^\nu\left(\delta\gamma^\mu\right)] \tag{B6b}$$

$$= \mathrm{tr}\,[\gamma^\mu\delta^\nu_\rho\left(\delta\gamma^\rho\right) + \gamma^\nu\delta^\mu_\rho\left(\delta\gamma^\rho\right)] \tag{B6c}$$

$$= \mathrm{tr}\,[\{\gamma^\mu\delta^\nu_\rho + \gamma^\nu\delta^\mu_\rho\}\left(\delta\gamma^\rho\right)] \tag{B6d}$$

We can use this in (B5) so that

$$\delta g^{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\{\gamma^\mu\delta^\nu_\rho + \gamma^\nu\delta^\mu_\rho\}\left(\delta\gamma^\rho\right)] \tag{B7a}$$

$$= \frac{1}{4}\,\mathrm{tr}\,[\left(\delta\gamma^\rho\right)\{\gamma^\mu\delta^\nu_\rho + \gamma^\nu\delta^\mu_\rho\}] \tag{B7b}$$

By the definition of the trace,

$$\delta g^{\mu\nu} = \frac{1}{4}\sum_{A,B}\,[\left(\delta\gamma^\rho\right)_{AB}\{\gamma^\mu\delta^\nu_\rho + \gamma^\nu\delta^\mu_\rho\}_{BA}] \tag{B8}$$

Giving the coefficient

$$\frac{\delta g^{\mu\nu}}{\delta\gamma^\rho} = \frac{1}{4}\,[\gamma^\mu\delta^\nu_\rho + \gamma^\nu\delta^\mu_\rho]^{\mathrm{T}} \tag{B9}$$

Appendix C. Solving for the Spinor Connection

Begin with differential equation (37)

$$-4\Pi_\nu = \gamma_\mu[\partial_\nu\gamma^\mu + \gamma_\mu\Gamma^\mu_{\nu\ell}\gamma^\ell] - \gamma_\mu\Pi_\nu\gamma^\mu \tag{C1}$$

Assume there exists a solution

$$\gamma_\mu W_\nu \gamma^\mu = 0 \tag{C2}$$

We would then have

$$W_\nu \equiv -\frac{1}{4}\gamma_\mu[\partial_\nu\gamma^\mu + \Gamma^\mu_{\nu\ell}\gamma^\ell] \tag{C3}$$

From this, we use the ansatz

$$\Pi_\nu = W_\nu + Q_\nu \tag{C4}$$

where Q_ν is to be determined. Inserting equation (C4) in (C1) we are left with the condition

$$4Q_\nu = \gamma_\mu[W_\nu + Q_\nu]\gamma^\mu \tag{C5}$$

So that

$$4Q_\nu = \gamma_\mu Q_\nu \gamma^\mu \tag{C6}$$

The solution to which is given by

$$Q_\nu = -ieA_\nu \mathrm{I} \tag{C7}$$

for an arbitrary four vector A_ν. So in total:

$$\Pi_\nu = -\frac{1}{4}\gamma_\mu[\partial_\nu\gamma^\mu + \Gamma^\mu_{\nu\ell}\gamma^\ell] - ieA_\nu \mathrm{I} \tag{C8}$$

Using the anticommutation properties of the gamma matrices, one can show that an object of the form (C3) exists and satisfies (C2) so that the above expression is valid.

Appendix D. Some comments on the variation with respect to the spinor connection

1. Fundamental variables

Given the vierbein, the choice of metric for the curved space is completely unambiguous. One simply takes the Minkowski metric and constructs it by projecting up,

$$g_{\mu\nu} \equiv e_\mu{}^a e_\nu^b \eta_{ab} \tag{D1}$$

Now, given the vierbein, it would seem ambiguous what to choose for the curved space gamma matrices. However, this ambiguity does not exist. In flat space, in order to solve the Dirac equation, one makes a choice of gamma matrices to use. From these, we can then construct the curved space gamma matrices by projecting up:

$$\gamma^{\mu} \equiv e_a^{\mu} \gamma^a \tag{D2}$$

The tetrad is the only variable.

2. A relation between the gamma matrices and vierbein

We can now use the argument of the preceding section. The claim is that whenever the curved space gamma matrices are fixed, the vierbein must also be fixed. Begin by taking the variation of (D2),

$$\delta\gamma^{\mu} = e_a^{\mu} \delta\gamma^a + \gamma^a \delta e_a^{\mu} \tag{D3}$$

If the curved space gamma matrices are held fixed then the L.H.S vanishes leaving

$$0 = e_a^{\mu} \delta\gamma^a + \gamma^a \delta e_a^{\mu} \tag{D4}$$

Now, since the flat space gamma matrices are not variables, we get

$$0 = \gamma^a \delta e_a^{\mu} \tag{D5}$$

Take the anticommutator with γ^b:

$$0 = \{\gamma^a \delta e_a^{\mu}, \gamma^b\} \tag{D6a}$$

$$= (\delta e_a^{\mu})\{\gamma^a, \gamma^b\} \tag{D6b}$$

$$= (\delta e_a^{\mu})\, \eta^{ab}\, \mathrm{I} \tag{D6b}$$

We may take the trace over the spinor indices to eliminate the identity matrix. Finally, we can contract with the Minkowski metric to obtain

$$0 = (\delta e_a^{\mu})\, \eta^{ab} \eta_{bc} \tag{D7a}$$

$$= (\delta e_a^{\mu})\, \delta_c^a \tag{D7b}$$

$$= \delta e_c^{\mu} \tag{D7c}$$

which is the desired result. From (D3) we also see that whenever the vierbein is fixed, the curved space gamma matrices are also fixed. Finally, we note that the vierbein is completely determined by the gamma matrices as

$$\{\gamma^{\mu}, \gamma_b\} = e_a^{\mu}\{\gamma^a, \gamma_b\} \tag{D8a}$$

$$= e_a^{\mu} \eta_b^a \tag{D8b}$$

$$= e_a^{\mu} \delta_b^a \tag{D8c}$$

$$= e_b^{\mu} \tag{D8d}$$

A similar process gives the inverse.

3. The vector potential

We may also say things about vector potential. Recall that the connection for spinors is given by (C8),

$$\Pi_\mu = \pi_\mu - ieA_\mu \mathrm{I} \tag{D9}$$

where

$$\pi_\mu \equiv -\frac{1}{4}\gamma_\nu[\partial_\mu \gamma^\nu + \Gamma^\nu_{\mu\ell}\gamma^\ell] \tag{D10}$$

Any choice of four vector could be used in (D9) to satisfy the equation (C1). It would therefore seem ambiguous how to determine the nature of A_μ.

This ambiguity, however, does not exist. Given a particular choice of Dirac spinor, there is a fixed, predetermined four potential with which it is associated. It is precisely the four potential contained in the interaction term of its Lagrangian [3]. From this, we can then construct the covariant derivative. The variation of this object is therefore vanishing; it is not a variable.

References

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