A Spinor Formulation of the Einstein-Hilbert Action

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Palatini treated the metric and affine connection as independent variables. He rewrote the gravitational action, took the variation, and arrived at the Levi-Civita connection and Einstein's equations in vacuum. Subsequently, the tetrad was introduced to General Relativity. It was then realized that the action could also be written solely in terms of the vierbien and spin connection. Treating these quantities as independent, one arrives at the same result. Suprisingly, the action can be written solely in terms of the gamma matrices and spinor connection (as defined in the curved space Dirac equation). Treating these as independent, I show that one also arrives at the Levi-Civita connection and Einstein's equations in vacuum.

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I. INTRODUCTION

By simply transforming variables or re-expressing equations, the underlying physics is preserved; however, the general theory, or rather the interpretation of the equations, may change. The ever ubiquitous product rule along with the insertion of the identity in quantum mechanics are examples of such. Both are purely changes of perspective.

A reader familiar with the vierbien formalism and Dirac equation could simply skip to Appendix B. Once there, they could use the last equation presented to rewrite the usual Einstein-Hilbert Action. Treating the variables as independent, one would arrive at the desired result. I chose, however, to present a different proof. It's a much more intuitive one; and it doesn't require you to write out the explicit action. It is a bit longer, but each step is simple.

II. MOTIVATION

Define the Riemann Tensor:

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} + \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + [\,\Gamma_{\mu}\,,\,\Gamma_{\nu}\,]^{\rho}{}_{\sigma} \tag{1}$$

the Ricci Tensor:

$$R_{\sigma\mu} \equiv R^{\rho}{}_{\sigma\rho\nu} \tag{2}$$

and the Ricci Scalar:

$$R \equiv R^{\sigma}{}_{\sigma} \tag{3}$$

These objects are functions only of the affine connection and metric. Therefore, given the metric, it is ambiguous which connection to choose. Christoffel solved this problem. He used algebra to show that if:

- ∇ is torsion free ($\Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu}$)
- ∇ is metric compatible ($\nabla g_{\mu\nu} = 0$)

then $\exists$ a unique connection given by

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\, g^{\rho\lambda}\left(\partial_{\nu} g_{\lambda\mu} + \partial_{\mu} g_{\lambda\nu} - \partial_{\lambda} g_{\mu\nu}\right) \tag{4}$$

This result was further developed by Levi-Civita and the connection is formally called the Levi-Civita connection. Since this object is a function only of the metric, one can then use it to rewrite the Riemann Tensor (and therfore its contractions) in terms of the metric alone: eliminating the ambiguity.

A. The Action Principle

Hilbert then defined the action

$$S = \int_M R \equiv \int R\sqrt{g}\, d^4x \tag{5}$$

From this, he took the variation

$$\delta S(\varphi_1, ..., \varphi_n) \equiv \int d^4x\, [\frac{\delta S}{\delta \varphi_1}\delta\varphi_1 + ... + \frac{\delta S}{\delta\varphi_n}\delta\varphi_n] \tag{6}$$

And by setting $\delta S = 0$ obtained the result

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \Longrightarrow R_{\mu\nu} - g_{\mu\nu} R = 0 \tag{7}$$

ignoring the boundary term. This result relies on the action being a function only of the metric. Particularly, we've assumed that

1. ∇ is torsion free
2. ∇ is metric compatible
3. For $S = \int_M R\,(g_{\mu\nu})$ that $\delta S = 0$

Palatini then realized something fundamental. Since we may write the Ricci Scalar in a different way

$$R = g_{\mu\nu} R^{\mu\nu} \tag{8}$$

What happens if we instead assume that

1. ∇ is torsion free
2. For $S = \int_M g_{\mu\nu} R^{\mu\nu}(\Gamma)$ that $\delta S = 0$

where we treat the metric and connection as *independent*. It turns out that

- $\frac{\delta S}{\delta g_{\mu\nu}} = 0 \Longrightarrow R_{\mu\nu} - g_{\mu\nu} R = 0$
- $\frac{\delta S}{\delta \Gamma^{\rho}{}_{\mu\sigma}} = 0$ + (∇ is torsion free) $\Longrightarrow$ ∇ is metric compatible

The second gives Levi-Civita's result (4) and together with the first gives Hilbert's result (7). So our three assumptions have been reduced to two.

III. PALATINI'S SECOND THEOREM

In (greek) curved space coordinates $\{\mu\}$ we have

$$\langle A, B\rangle_C = \langle A^\mu e_\mu, B^\nu e_\nu\rangle \tag{9a}$$
$$= A^\mu B^\nu \langle e_\mu, e_\nu\rangle \tag{9b}$$
$$= A^\mu B^\nu g_{\mu\nu} \tag{9c}$$

Say that I could choose some (latin) coordinates $\{a\}$, i.e.

$$A^a = e_\mu{}^a A^\mu \tag{10}$$
$$e_a = e^\mu{}_a e_\mu \tag{11}$$

s.t the metric is flat in these coordinates

$$\langle A, B\rangle_F = \langle A^a e_a, B^b e_b\rangle \tag{12a}$$
$$= A^a B^b \langle e_a, e_b\rangle \tag{12b}$$
$$= A^a B^b \eta_{ab} \tag{12c}$$

where η_{ab} is the Minkowski metric. The inner product is a scalar so it must be coordinate independent:

$$\langle A, B\rangle_C = \langle A, B\rangle_F \tag{13}$$

And therefore,

$$A^\mu B^\nu g_{\mu\nu} = A^a B^b \eta_{ab} \tag{14}$$

We can rewrite this using the vierbien.

$$A^\mu B^\nu g_{\mu\nu} = A^\mu B^\nu e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{15}$$

Since this holds for any vector we must have:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{16}$$

We have a relationship between the curved space metric and the flat space metric. We must now relate the covariant derivatives. In curved space

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\rho} A^\rho \tag{17}$$

The connection does not transform as a simple tensor. Therefore in the new coordinates, we define another connection, termed the spin connection, so that

$$\nabla_\mu A^a = \partial_\mu A^a + \omega_\mu{}^a{}_b A^b \tag{18}$$

Though the connection is not a tensor, the covariant derivative does transform as a tensor. In Appendix A, we relate the two covariant derivatives to obtain the result

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\rho{}_b \Gamma^\nu{}_{\mu\rho} - e^\rho{}_b \partial_\mu e_\rho{}^a \tag{19}$$

Which can be inverted to

$$\Gamma^\nu{}_{\mu\rho} = e^\nu{}_a \partial_\mu e_\rho{}^a + e^\nu{}_a e_\rho{}^b \omega_\mu{}^a{}_b \tag{20}$$

In these new coordinates, we can also use our connection to construct a new curvature tensor, called the spin curvature tensor, defined by:

$$\Omega^{ab}{}_{\mu\nu} \equiv \partial_\mu \omega^{ab}{}_\nu + \partial_\nu \omega^{ab}{}_\mu + [\omega_\mu, \omega_\nu]^{ab} \tag{21}$$

Transforming this to the new coordinate system we may use the definition (1) and the relation (24). It turns out that

$$R^{\rho\sigma}{}_{\mu\nu} = e^\rho{}_a e^\sigma{}_b \, \Omega^{ab}{}_{\mu\nu} \tag{22}$$

I'm not sure if it was Palatini, but someone realized something else fundamental. From the definition of the Ricci Scalar (3), we now obtain:

$$R = e^\mu{}_a e^\nu{}_b \, \Omega^{ab}{}_{\mu\nu} \tag{23}$$

Similar to Palatini's First Theorem, what happens if we assume

1. $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$
2. For $S = \int_M e^\mu{}_a e^\nu{}_b \, \Omega^{ab}{}_{\mu\nu}(\omega)$ that $\delta S = 0$

It turns out that:

- $\dfrac{\delta S}{\delta e_\mu{}^a} = 0 \Longrightarrow R_{\mu\nu} - g_{\mu\nu} R = 0$
- $\dfrac{\delta S}{\delta \omega_\mu{}^{ab}} = 0 \Longrightarrow \nabla$ is torsion free

Where did metric compatablity go? Being more careful, it follows from (19) that the vierbien is covariantly constant. Since in flat space we also assume that the metric is covariantly constant, we have the two results:

- $\nabla_\rho e^\mu{}_a = 0$
- $\nabla_\rho \eta_{ab} = 0$

This would then imply metric compatibility as:

$$\nabla_\rho \, g_{\mu\nu} = \nabla_\rho \left(e_\mu{}^a e_\nu{}^b \eta_{ab} \right) = 0 \tag{24}$$

We have that these two new assumptions are equivalent to the two assumption of Palatini's First Theorem.

IV. A NEW THEOREM

It turns out that the Ricci Scalar can be written solely in terms of the gamma matrices and the "spinor" connection so that

$$S = \int_M g_{\mu\nu} R^{\mu\nu}(\Gamma) \to \int_M \mathcal{L}(\gamma^\mu, \Pi_\mu) \tag{25}$$

By "spinor" connection and gamma matrices I mean those used in the covariant Dirac equation

$$\gamma^\mu D_\mu \psi = m\psi \tag{26}$$

Where the covariant derivative of spinors is defined by

$$D_\mu \equiv \partial_\mu - \Pi_\mu \tag{27}$$

This choice of covariant derivative is due to equation (55) of [1]. Similar to the previous two theorems, what happens if we assume that:

1. $\{\gamma^\mu, \gamma^\nu\} = 2 \, g^{\mu\nu} \, \mathrm{I}$
2. For $S = \int_M \mathcal{L}(\gamma^\mu, \Pi_\mu)$ that $\delta S = 0$

Let's work it out. First, we need to compute the action in terms of the gamma matrices and spin connection. In Appendix B, I find the expression for the action in terms of these objects alone. The result is:

$$R = -\frac{1}{4} \, \mathrm{tr}\{[\Phi_{\rho\lambda}, \gamma^\rho], \gamma^\lambda\} \tag{28}$$

where

$$\Phi_{\mu\nu} \equiv \partial_\nu \Pi_\mu - \partial_\mu \Pi_\nu + [\Pi_\mu, \Pi_\nu] \tag{29}$$

Taking the variation of that action will give the correct equations of motion. However, I'll present a different method. Since the action can be written solely in terms of the metric and affine connection, it suffices to just compute these objects as functions of the gamma matrices and spinor connection.

A. Re-expressing the Metric

From the anti-commutation relations, we have that

$$\{\gamma^\mu, \gamma^\nu\} = 2 \, g^{\mu\nu} \, \mathrm{I} \tag{30}$$

Hence

$$g^{\mu\nu} = \frac{1}{8} \, \mathrm{tr}\{\gamma^\mu, \gamma^\nu\} \tag{31}$$

B. Re-expressing the Affine Connection

We have that the gamma matrices are covariantly constant so that

$$\nabla_\nu \gamma^\mu = \partial_\nu \gamma^\mu + \Gamma^\mu{}_{\nu\ell} \gamma^\ell + [\gamma^\mu, \Pi_\nu] = 0 \tag{32}$$

This assertion and choice of covariant derivative is due to equation (8) of [1]. We can solve this for $\Gamma^\rho{}_{\mu\nu}$. Rearranging the relationship,

$$-\Gamma^\mu{}_{\nu\ell} \gamma^\ell = \partial_\nu \gamma^\mu + [\gamma^\mu, \Pi_\nu] \tag{33}$$

We can use the anti-commutation relation on the L.H.S.

$$\{\Gamma^{\mu}{}_{\nu\ell}\gamma^{\ell},\gamma^{\rho}\} = \Gamma^{\mu}{}_{\nu\ell}\{\gamma^{\ell},\gamma^{\rho}\} \tag{34a}$$

$$= 2\,\Gamma^{\mu}{}_{\nu\ell}\, g^{\ell\rho}\ \mathrm{I} \tag{34b}$$

Hence raising the index, we have that

$$\{\Gamma^{\mu}{}_{\nu\ell}\gamma^{\ell},\gamma^{\rho}\} = 2\,\Gamma^{\mu}{}_{\nu}{}^{\rho}\ \mathrm{I} \tag{35}$$

We can then use the commutation relation on the R.H.S. of (33). After taking the trace, we are left with:

$$\Gamma^{\mu}{}_{\nu}{}^{\rho} = -\frac{1}{8}\,\mathrm{tr}\{\partial_{\nu}\gamma^{\mu} + [\gamma^{\mu},\Pi_{\nu}]\,,\gamma^{\rho}\} \tag{36}$$

where the trace is over the spinor indices. Hence the metric and affine connection can be written as functions of the spinor connection and gamma matrices alone.

V. TAKING THE VARIATION

Second, we need to take the variation of

$$S = \int_M \mathcal{L}(\gamma^{\mu},\Pi_{\mu}) \tag{37}$$

where

$$\mathcal{L}(g_{\mu\nu},\Gamma) \to \mathcal{L}(\gamma^{\mu},\Pi_{\mu}) \tag{38}$$

In order for $\delta S = 0$, we must have that

$$\int d^4x\,[\frac{\delta S}{\delta\gamma^{\mu}}\delta\gamma^{\mu} + \frac{\delta S}{\delta\Pi_{\mu}}\delta\Pi_{\mu}] = 0 \tag{39}$$

Each term must vanish seperately so that

1. $\dfrac{\delta S}{\delta\gamma^{\mu}}\delta\gamma^{\mu} = 0$ for all variations $\delta\gamma^{\mu}$
2. $\dfrac{\delta S}{\delta\Pi_{\mu}}\delta\Pi_{\mu} = 0$ for all variations $\delta\Pi_{\mu}$

A. Variation w.r.t the Gamma Matrices

From above, since condition (1) must be true for any arbitrary variation of the gamma matrices, we must have:

$$\frac{\delta S}{\delta\gamma^{\mu}} = 0 \tag{40}$$

This can be expanded in terms of the metric and affine connection so that:

$$\frac{\delta S}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta\gamma^{\rho}} + \frac{\delta S}{\delta\Gamma^{\ell}{}_{\mu\nu}}\frac{\delta\Gamma^{\ell}{}_{\mu\nu}}{\delta\gamma^{\rho}} = 0 \tag{41}$$

after a change of indices $\mu \to \rho$ in the gamma matrices. We can do this chain rule expansion because the original Palatini Action could be written solely as a function of both the metric and the general affine connection.

The metric can then solely be written as a function of the gamma matrices by formula (31). Furthermore, since the spinor connection is fixed, the affine connection can also be written as a function solely of the gamma matrices using formula (36).

For the second term in (41), we have that

$$\frac{\delta S}{\delta\Gamma^{\ell}{}_{\mu\nu}} = \delta^{\nu}{}_{\ell}\nabla_{\lambda}\mathfrak{g}^{\mu\lambda} - \nabla_{\ell}\,\mathfrak{g}^{\mu\nu} \tag{42}$$

where we have abbreviated

$$\mathfrak{g}_{\mu\nu} \equiv g_{\mu\nu}\sqrt{g} \tag{43}$$

But since $g^{\mu\nu} = \dfrac{1}{8}\,\mathrm{tr}\{\gamma^{\mu},\gamma^{\nu}\}$ and $\nabla\gamma^{\mu} = 0$ we have that:

$$\nabla g_{\mu\nu} = 0 \tag{44}$$

And therefore

$$\frac{\delta S}{\delta\Gamma^{\ell}{}_{\mu\nu}} = 0 \tag{45}$$

Hence the second term in (41) is vanishing. For the first term in (41), we can compute the variation of the metric using equation (31). This is done in Appendix C. The result is:

$$\frac{\delta g^{\mu\nu}}{\delta\gamma^{\rho}} = \frac{1}{4}\,[\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]^{\mathrm{T}} \tag{46}$$

Inserting this into (41), we have

$$\frac{\delta S}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta\gamma^{\rho}} = \frac{\delta S}{\delta g_{\mu\nu}}[\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]^{\mathrm{T}}\,\frac{1}{4} \tag{47a}$$

$$= [R_{\mu\nu} - g_{\mu\nu}R\,][\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]^{\mathrm{T}}\,\frac{1}{4} \tag{47b}$$

where we have used Palatini's First Theorem to evaluate the first element. Now taking the transpose over the spinor indices:

$$\left(\frac{\delta S}{\delta g_{\mu\nu}}\frac{\delta g_{\mu\nu}}{\delta\gamma^{\rho}}\right)^{\mathrm{T}} = G_{\mu\nu}[\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]\,\frac{1}{4} \tag{48}$$

where we have abbreviated Einstein's tensor as:

$$G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R \tag{49}$$

We can take the anticommutator with γ^{λ}

$$\{\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}\ ,\ \gamma^{\lambda}\}$$

$$= \{\gamma^{\mu}\delta^{\nu}{}_{\rho},\gamma^{\lambda}\} + \{\gamma^{\nu}\delta^{\mu}{}_{\rho},\gamma^{\lambda}\} \tag{50a}$$

$$= \delta^{\nu}{}_{\rho}\{\gamma^{\mu},\gamma^{\lambda}\} + \delta^{\mu}{}_{\rho}\{\gamma^{\nu},\gamma^{\lambda}\} \tag{50b}$$

$$= 2\,[\delta^{\nu}{}_{\rho} g^{\mu\lambda}\mathrm{I} + \delta^{\mu}{}_{\rho} g^{\nu\lambda}\mathrm{I}] \tag{50c}$$

We can use this result in (48) so that

$$\{G_{\mu\nu}\,[\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]\ ,\ \gamma^{\lambda}\}$$

$$= G_{\mu\nu}\{[\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]\ ,\ \gamma^{\lambda}\} \tag{51a}$$

$$= 2\,G_{\mu\nu}[\delta^{\nu}{}_{\rho} g^{\mu\lambda}\mathrm{I} + \delta^{\mu}{}_{\rho} g^{\nu\lambda}\mathrm{I}] \tag{51b}$$

$$= 2\,[G_{\alpha\rho} + G_{\rho\alpha}]\,\mathrm{I} \tag{51c}$$

In the next section, we will show that the connection is symmetric and hence the Einstein tensor is symmetric so that this reduces to

$$\{G_{\mu\nu}[\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]\ ,\ \gamma^{\lambda}\} = 4\,G_{\alpha\rho}\,\mathrm{I} \tag{52}$$

Taking the trace over the spinor indices we then have

$$\frac{1}{16}\,\mathrm{tr}\{G_{\mu\nu}[\gamma^{\mu}\delta^{\nu}{}_{\rho} + \gamma^{\nu}\delta^{\mu}{}_{\rho}]\ ,\ \gamma^{\lambda}\} = G_{\alpha\rho} \tag{53}$$

From (41), (45), and (48) we are then left with:

$$G_{\alpha\rho} = 0 \tag{54}$$

B. Variation w.r.t the Spinor Connection

From before we have that

$$\frac{\delta S}{\delta \Pi_{\mu}}\delta\Pi_{\mu} = 0 \tag{55}$$

$\forall$ variations $\delta\Pi_{\mu}$

We can obtain a formula for $\delta\Pi_{\mu}$ in terms of the other variables. The γ are covariantly constant (32) so that:

$$\nabla_{\nu}\gamma^{\mu} = \partial_{\nu}\gamma^{\mu} + \Gamma^{\mu}{}_{\nu\ell}\gamma^{\ell} + [\gamma^{\mu}, \Pi^{\nu}] = 0 \tag{56}$$

Contract with γ_{μ}.

$$-\gamma_{\mu}\gamma^{\mu}\Pi_{\nu} = \gamma_{\mu}\partial_{\nu}\gamma^{\mu} + \gamma_{\mu}\Gamma^{\mu}{}_{\nu\ell}\gamma^{\ell} - \gamma_{\mu}\Pi_{\nu}\gamma^{\mu} \tag{57}$$

Using the vierbien, we can show that

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \{e^{\mu}{}_{a}\gamma^{a}, e^{\nu}{}_{b}\gamma^{b}\} \tag{58a}$$

$$= e^{\mu}{}_{a}e^{\nu}{}_{b}\{\gamma^{a}, \gamma^{b}\} \tag{58b}$$

Hence inverting the original commutation relation (30) we are left with

$$\{\gamma^{a}, \gamma^{b}\} = 2\,\eta^{ab}\mathrm{I} \tag{59}$$

Contracting over the latin indices, we will have that $\gamma^{a}\gamma_{a} = 4\mathrm{I}$. Since the inner product is a scalar, it will not depend on coordinates so that $\gamma^{\mu}\gamma_{\mu} = 4\mathrm{I}$. Using this fact in (57):

$$-4\Pi_{\nu} = \gamma_{\mu}\partial_{\nu}\gamma^{\mu} + \gamma_{\mu}\Gamma^{\mu}{}_{\nu l}\gamma^{l} - \gamma_{\mu}\Pi_{\nu}\gamma^{\mu} \tag{60}$$

In Appendix D, we show that the solution to the above equation is given by:

$$\Pi_{\nu} = -\frac{1}{4}\gamma_{\mu}[\partial_{\nu}\gamma^{\mu} + \Gamma^{\mu}{}_{\nu\ell}\gamma^{\ell}] - ieA_{\nu}\mathrm{I} \tag{61}$$

where A_{ν} is a four vector. Hence the spinor connection can be written solely as a function of the gamma matrices, the affine connection, and an arbitrary four vector. We can now rewrite the $\delta\Pi_{\mu}$ in (55) as

$$\delta\Pi_{\rho} = \frac{\delta\Pi_{\rho}}{\delta\gamma^{\mu}}\,\delta\gamma^{\mu} + \frac{\delta\Pi_{\rho}}{\delta\Gamma^{\lambda}{}_{\mu\nu}}\,\delta\Gamma^{\lambda}{}_{\mu\nu} + \frac{\delta\Pi_{\rho}}{\delta A_{\nu}}\,\delta A_{\nu} \tag{62}$$

after a change of indices $\mu \to \rho$ in the spinor connection. Since the affine connection can be written solely as a function of the vierbien and spin connection (20), we may express its variation as

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \frac{\delta\Gamma^{\lambda}{}_{\mu\nu}}{\delta e^{\sigma}{}_{b}}\,\delta e^{\sigma}{}_{b} + \frac{\delta\Gamma^{\lambda}{}_{\mu\nu}}{\delta\omega^{\sigma}{}_{ab}}\,\delta\omega^{\sigma}{}_{ab} \tag{63}$$

We can then use this to rewrite the variation of the spin connection. Using this in (62) we have that

$$\delta\Pi_{\rho} = \frac{\delta\Pi_{\rho}}{\delta\gamma^{\mu}}\,\delta\gamma^{\mu} + \frac{\delta\Pi_{\rho}}{\delta A_{\nu}}\,\delta A_{\nu} \tag{64a}$$

$$+\frac{\delta\Pi_{\rho}}{\delta\Gamma^{\lambda}{}_{\mu\nu}}\,[\frac{\delta\Gamma^{\lambda}{}_{\mu\nu}}{\delta e^{\sigma}{}_{b}}\,\delta e^{\sigma}{}_{b} + \frac{\delta\Gamma^{\lambda}{}_{\mu\nu}}{\delta\omega^{\sigma}{}_{ab}}\,\delta\omega^{\sigma}{}_{ab}] \tag{64b}$$

The curved space gamma matrices are being held fixed so that the first term vanishes. At this point we may also eliminate the variation in the vector potential. The reason for this is explained in Appendix E. We are left with:

$$\delta\Pi_{\rho} = \frac{\delta\Pi_{\rho}}{\delta\Gamma^{\lambda}{}_{\mu\nu}}\,[\frac{\delta\Gamma^{\lambda}{}_{\mu\nu}}{\delta e^{\sigma}{}_{b}}\,\delta e^{\sigma}{}_{b} + \frac{\delta\Gamma^{\lambda}{}_{\mu\nu}}{\delta\omega^{\sigma}{}_{ab}}\,\delta\omega^{\sigma}{}_{ab}] \tag{65}$$

In Appendix E, we also show that whenever the curved space gamma matrices are held fixed, the vierbien is fixed. Using this in the above equation, we have that:

$$\delta\Pi_{\rho} = \frac{\delta\Pi_{\rho}}{\delta\Gamma^{\lambda}{}_{\mu\nu}}\,[\frac{\delta\Gamma^{\lambda}{}_{\mu\nu}}{\delta\omega^{\sigma}{}_{ab}}\,\delta\omega^{\sigma}{}_{ab}] \tag{66}$$

The vector potential is fixed and the gamma matrices are fixed so the spinor connection is soleley a function of the affine connection. We can therefore eliminate the chain rule in this variable so that:

$$\delta\Pi_{\rho} = \frac{\delta\Pi_{\rho}}{\delta\omega^{\sigma}{}_{ab}}\,\delta\omega^{\sigma}{}_{ab} \tag{67}$$

Now that we have simplified the variation of the spinor connection, we can it into (55):

$$\frac{\delta S}{\delta \Pi_\rho}\frac{\delta \Pi_\rho}{\delta {\omega^\sigma}_{ab}}\,\delta {\omega^\sigma}_{ab} = 0 \tag{68}$$

Since this variation holds the curved space gamma matrices fixed, the action is solely a function of the spinor connection. We can therefore eliminate the chain rule in this variable so that:

$$\frac{\delta S}{\delta {\omega^\sigma}_{ab}}\,\delta {\omega^\sigma}_{ab} = 0 \tag{69}$$

$\forall$ variations $\delta {\omega^\sigma}_{ab}$ hence,

$$\frac{\delta S}{\delta {\omega^\sigma}_{ab}} = 0 \tag{70}$$

Since the vierbien is fixed, by Palatini's Second Theorem, we then have that the connection is symmetric. This along with (44) gives the Levi-Civita connection (4).

Using this method, I never actually had to compute the closed form of the action; I only used the chain rule to evaluate its variation. In Appendix B, I calculate the total action in full.

Appendix A: The relationship between the Spin and Affine Connections

In the curved basis, we have that

$$\nabla A = [\nabla_\mu A^\nu]\; dx^\mu \otimes \partial_\nu \tag{A1a}$$

$$= [\partial_\mu A^\nu + {\Gamma^\nu}_{\mu\rho} A^\rho]\; dx^\mu \otimes \partial_\nu \tag{A1b}$$

In the mixed basis we have,

$$\nabla A = [\nabla_\mu A^a]\; dx^\mu \otimes \partial_a \tag{A2a}$$

$$= [\partial_\mu A^a + {\omega_\mu}^a{}_b A^b]\; dx^\mu \otimes \partial_a \tag{A2b}$$

$$= [\partial_\mu ({e_\nu}^a A^\nu) + {\omega_\mu}^a{}_b {e_\rho}^b A^\rho]\; dx^\mu \otimes {e^\lambda}_a \partial_\lambda \tag{A2c}$$

$$= {e^\lambda}_a [\partial_\mu ({e_\nu}^a A^\nu) + {\omega_\mu}^a{}_b {e_\rho}^b A^\rho]\; dx^\mu \otimes \partial_\lambda \tag{A2d}$$

For the evaluation of the first term we can use the product rule so that

$$\nabla A = {e^\lambda}_a [{e_\nu}^a \partial_\mu A^\nu + A^\nu \partial_\mu {e_\nu}^a + {\omega_\mu}^a{}_b {e_\rho}^b A^\rho]\; dx^\mu \otimes \partial_\lambda \tag{A3}$$

We can simplify the above equation. Both λ and ν are dummy indices so set them equal.

$$\nabla A = [\partial_\mu A^\nu + {e^\nu}_a A^\nu \partial_\mu {e^a}_\nu + {e^\nu}_a {\omega_\mu}^a{}_b {e_\rho}^b A^\rho]\; dx^\mu \otimes \partial_\nu \tag{A4}$$

Switching ν to ρ in the middle term and rearranging

$$\nabla A = [\partial_\mu A^\nu + {e^\nu}_a (\partial_\mu {e^a}_\rho) A^\rho + {e^\nu}_a {e_\rho}^b {\omega_\mu}^a{}_b A^\rho]\; dx^\mu \otimes \partial_\nu \tag{A5}$$

Setting the curved and mixed results equal, we are left with

$${\Gamma^\nu}_{\mu\rho} A^\rho = {e^\nu}_a \partial_\mu {e_\rho}^a A^\rho + {e^\nu}_a {e_\rho}^b {\omega_\mu}^a{}_b A^\rho \tag{A6}$$

Since this must hold for any four vector we must have that:

$${\Gamma^\nu}_{\mu\rho} = {e^\nu}_a \partial_\mu {e_\rho}^a + {e^\nu}_a {e_\rho}^b {\omega_\mu}^a{}_b \tag{A7}$$

Which can be inverted to

$${\omega_\mu}^a{}_b = {e_\nu}^a {e^\rho}_b {\Gamma^\nu}_{\mu\rho} - {e^\rho}_b \partial_\mu {e_\rho}^a \tag{A8}$$

Appendix B: The Closed Form Action

Here, I'll derive the closed form of the Einstein-Hilbert Action in terms of the gamma matrices and spinor connection alone. The scalar density $\sqrt{g}$ can be written solely as a function of these variables using (31). So all we need to do is rewrite the Ricci Scalar.

1. The Spinor Curvature

We have that the covariant derivative of spinors is given by

$$\nabla_\mu \psi = \partial_\mu \psi - \Pi_\mu \psi \tag{B1}$$

We want to calculate the commutator of covariant derivatives: $[\nabla_\mu, \nabla_\nu]\,\psi$. This is analagous to how the Riemann curvature tensor is obtained. For the first term, we have

$$\nabla_\mu \nabla_\nu \psi = \nabla_\mu [\partial_\nu \psi - \Pi_\nu \psi] \tag{B2}$$

Expanding the covariant derivative,

$$\nabla_\mu \nabla_\nu \psi = \partial_\mu [\partial_\nu \psi - \Pi_\nu \psi] - {\Gamma_{\nu\mu}}^\rho [\partial_\rho \psi - \Pi_\rho \psi] \tag{B3}$$
$$- \Pi_\mu [\partial_\nu \psi - \Pi_\nu \psi]$$

Similarly we have

$$\nabla_\nu \nabla_\mu \psi = \partial_\nu [\partial_\mu \psi - \Pi_\mu \psi] - {\Gamma_{\mu\nu}}^\rho [\partial_\rho \psi - \Pi_\rho \psi] \tag{B4}$$
$$- \Pi_\nu [\partial_\mu \psi - \Pi_\mu \psi]$$

Subtracting these, we can obtain the commutator. The result is:

$$[\nabla_\mu, \nabla_\nu]\,\psi = \Phi_{\mu\nu} \psi + {T_{\mu\nu}}^\rho\, \nabla_\rho \psi \tag{B5}$$

where

$$\Phi_{\mu\nu} \equiv \partial_\nu \Pi_\mu - \partial_\mu \Pi_\nu + [\Pi_\mu\ ,\ \Pi_\nu] \tag{B6}$$

$${T_{\mu\nu}}^\rho \equiv {\Gamma_{\mu\nu}}^\rho - {\Gamma_{\nu\mu}}^\rho \tag{B7}$$

2. Two-Spinors

We can create a two spinor by contracting onto the gamma matrices. Choose some arbitrary four vector A_ρ and define

$$\sigma \equiv A_\rho \gamma^\rho \tag{B8}$$

This can be inverted as:

$$A^\rho = \frac{1}{8}\,\mathrm{tr}\{\sigma, \gamma^\rho\} \tag{B9}$$

We can follow suit with the previous section and take the commutator on this object $[\nabla_\mu, \nabla_\nu]\,\sigma$. Using (32), we have for the first term:

$$\nabla_\mu \nabla_\nu \sigma = \nabla_\mu \left(\partial_\nu \sigma + [\sigma, \Pi_\nu]\right) \tag{B10}$$

Expanding the covariant derivative,

$$\begin{aligned}\nabla_\mu \nabla_\nu \sigma = \partial_\mu \left(\partial_\nu \sigma + [\sigma, \Pi_\nu]\right) - \Gamma_{\nu\mu}{}^{\lambda} \left(\partial_\lambda \sigma + [\sigma, \Pi_\lambda]\right) \\ +[\left(\partial_\nu \sigma + [\sigma, \Pi_\nu]\right), \Pi_\mu]\end{aligned} \tag{B11}$$

Similarly,

$$\begin{aligned}\nabla_\nu \nabla_\mu \sigma = \partial_\nu \left(\partial_\mu \sigma + \left[\sigma, \Pi_\mu\right]\right) - \Gamma_{\mu\nu}{}^{\lambda} \left(\partial_\lambda \sigma + [\sigma, \Pi_\lambda]\right) \\ +[\left(\partial_\mu \sigma + \left[\sigma, \Pi_\mu\right]\right), \Pi_\nu]\end{aligned} \tag{B12}$$

Subtracting these, we arrive at the commutator:

$$[\nabla_\mu, \nabla_\nu]\,\sigma = [\Phi_{\mu\nu}, \sigma] + T_{\mu\nu}{}^{\lambda} \nabla_\lambda \sigma \tag{B13}$$

3. The Riemann Tensor in terms of the Gamma Matrices and Spinor Connection

We can use the result for two-spinors to help calculate the commutator on the gamma matrices. From (32) we have:

$$\nabla_\nu \gamma^\rho = \partial_\nu \gamma^\rho + \Gamma^\rho{}_{\nu\ell}\gamma^\ell + [\gamma^\rho, \Pi_\nu] = 0 \tag{B14}$$

So that the first term in the anti-commutator is given by

$$\nabla_\mu \nabla_\nu \gamma^\rho = \nabla_\mu \left(\partial_\nu \gamma^\rho + \Gamma^\rho{}_{\nu\ell}\gamma^\ell + [\gamma^\rho, \Pi_\nu]\right) \tag{B15}$$

We can expand out the covariant derivative:

$$\begin{aligned}\nabla_\mu \nabla_\nu \gamma^\rho = \partial_\mu \left(\partial_\nu \gamma^\rho + \Gamma^\rho{}_{\nu\ell}\gamma^\ell + [\gamma^\rho, \Pi_\nu]\right) \\ +\Gamma^\rho{}_{\mu\lambda} \left(\partial_\nu \gamma^\lambda + \Gamma^\lambda{}_{\nu\ell}\gamma^\ell + \left[\gamma^\lambda, \Pi_\nu\right]\right) \\ -\Gamma_{\nu\mu}{}^{\lambda} \left(\partial_\lambda \gamma^\rho + \Gamma^\rho{}_{\lambda\ell}\gamma^\ell + [\gamma^\rho, \Pi_\lambda]\right) \\ + \left[\left(\partial_\nu \gamma^\rho + \Gamma^\rho{}_{\nu\ell}\gamma^\ell + [\gamma^\rho, \Pi_\nu]\right), \Pi_\mu\right]\end{aligned} \tag{B16}$$

Taking the commutator and using (B13) we have:

$$[\nabla_\mu, \nabla_\nu]\,\gamma^\rho = [\Phi_{\mu\nu}, \gamma^\rho] + T_{\nu\mu}{}^{\lambda}\nabla_\lambda \gamma^\rho + R^\rho{}_{\ell\mu\nu}\gamma^\ell \tag{B17}$$

The gamma matrices are covariantly constant so that the entire L.H.S and the middle term on the R.H.S vanish. Hence we are left with:

$$0 = [\Phi_{\mu\nu}, \gamma^\rho] + R^\rho{}_{\ell\mu\nu}\gamma^\ell \tag{B18}$$

Rearranging,

$$R^\rho{}_{\ell\mu\nu}\gamma^\ell = -[\Phi_{\mu\nu}, \gamma^\rho] \tag{B19}$$

On the L.H.S of the above equation, we can take the anti-commutator with γ^λ

$$\{R^\rho{}_{\ell\mu\nu}\gamma^\ell, \gamma^\lambda\} \tag{B20a}$$

$$= R^\rho{}_{\ell\mu\nu}\{\gamma^\ell, \gamma^\lambda\} \tag{B20b}$$

$$= 2\,R^\rho{}_{\ell\mu\nu}\,g^{\ell\lambda}\,\mathrm{I} \tag{B20c}$$

$$= 2\,R^{\rho\lambda}{}_{\mu\nu}\,\mathrm{I} \tag{B20d}$$

Now we must take the anti-commutator on the R.H.S of (B19).

$$2\,R^{\rho\lambda}{}_{\mu\nu}\,\mathrm{I} = -\{[\Phi_{\mu\nu}, \gamma^\rho], \gamma^\lambda\} \tag{B21}$$

So that the Riemann Tensor

$$R^{\rho\lambda}{}_{\mu\nu} = -\frac{1}{8}\,\mathrm{tr}\{[\Phi_{\mu\nu}, \gamma^\rho], \gamma^\lambda\} \tag{B22}$$

the Ricci Tensor

$$R^\lambda{}_\nu \equiv R^{\rho\lambda}{}_{\rho\nu} = -\frac{1}{8}\,\{[\Phi_{\rho\nu}, \gamma^\rho], \gamma^\lambda\} \tag{B23}$$

and the Ricci Scalar

$$R \equiv R^\lambda{}_\lambda = -\frac{1}{8}\,\mathrm{tr}\{[\Phi_{\rho\lambda}, \gamma^\rho], \gamma^\lambda\} \tag{B24}$$

can all be written in terms of the gamma matrices and spinor connection alone.

Appendix C: The Variation of the Metric

The metric is related to the gamma matrices by the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}\,\mathrm{I} \tag{C1}$$

This can be inverted to

$$g^{\mu\nu} = \frac{1}{8}\,\mathrm{tr}\{\gamma^\mu, \gamma^\nu\} \tag{C2}$$

We can use this to take the variation of the metric:

$$\delta g^{\mu\nu} = \frac{1}{8}\,\delta\,\mathrm{tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] \tag{C3}$$

Use the fact that tr(AB) = tr(BA) so that

$$\delta g^{\mu\nu} = \frac{1}{4}\,\delta\,\mathrm{tr}[\gamma^\mu\gamma^\nu] \tag{C4}$$

The trace also commutes with the variation:

$$\delta g^{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,\delta[\gamma^\mu\gamma^\nu] \tag{C5}$$

Working out the parenthesis:

$$\mathrm{tr}\,\delta[\gamma^\mu\gamma^\nu] = \mathrm{tr}\,[\gamma^\mu\,(\delta\gamma^\nu) + (\delta\gamma^\mu)\,\gamma^\nu] \tag{C6a}$$

$$= \mathrm{tr}\,[\gamma^\mu\,(\delta\gamma^\nu) + \gamma^\nu\,(\delta\gamma^\mu)] \tag{C6b}$$

$$= \mathrm{tr}\,[\gamma^\mu\delta^\nu{}_\rho(\delta\gamma^\rho) + \gamma^\nu\delta^\mu{}_\rho(\delta\gamma^\rho)] \tag{C6c}$$

$$= \mathrm{tr}\,[\{\gamma^\mu\delta^\nu{}_\rho + \gamma^\nu\delta^\mu{}_\rho\}\,(\delta\gamma^\rho)] \tag{C6d}$$

We can use this in (C5) so that

$$\delta g^{\mu\nu} = \frac{1}{4}\,\mathrm{tr}\,[\{\gamma^\mu\delta^\nu{}_\rho + \gamma^\nu\delta^\mu{}_\rho\}\,(\delta\gamma^\rho)] \tag{C7a}$$

$$= \frac{1}{4}\,\mathrm{tr}\,[(\delta\gamma^\rho)\,\{\gamma^\mu\delta^\nu{}_\rho + \gamma^\nu\delta^\mu{}_\rho\}] \tag{C7b}$$

By the definition of the trace,

$$\delta g^{\mu\nu} = \frac{1}{4}\sum_{A,B}\,[(\delta\gamma^\rho)_{AB}\{\gamma^\mu\delta^\nu{}_\rho + \gamma^\nu\delta^\mu{}_\rho\}_{BA}] \tag{C8}$$

So that we can find the variation:

$$\frac{\delta g^{\mu\nu}}{\delta\gamma^\rho} = \frac{1}{4}\,[\gamma^\mu\delta^\nu{}_\rho + \gamma^\nu\delta^\mu{}_\rho]^{\mathrm{T}} \tag{C9}$$

Appendix D: Solving for the Spinor Connection

Given the differential equation (60)

$$-4\Pi_\nu = \gamma_\mu\partial_\nu\gamma^\mu + \gamma_\mu\Gamma^\mu{}_{\nu l}\gamma^l - \gamma_\mu\Pi_\nu\gamma^\mu \tag{D1}$$

we would like to determine the spinor connection. Begin with the ansatz

$$\Pi_\nu = W_\nu + Q_\nu \tag{D2}$$

where we have defined:

$$W_\nu \equiv -\frac{1}{4}\gamma_\mu[\partial_\nu\gamma^\mu + \Gamma^\mu{}_{\nu\ell}\gamma^\ell] \tag{D3}$$

and Q_ν is to be determined. Using the ansatz (D2) in the equation (D1) we are left with the condition:

$$4Q_\nu = \gamma_\mu[W_\nu + Q_\nu]\gamma^\mu \tag{D4}$$

At this point, we may observe that:

$$\gamma_\mu W_\nu\gamma^\mu = 0 \tag{D5}$$

This relation can be proven using the anti-commutation relation of the gamma matrices. Using the above in equation (D4), we find that the unknown must satisfy:

$$4Q_\nu = \gamma_\mu Q_\nu\gamma^\mu \tag{D6}$$

the solution to which is given by

$$Q_\nu = -ieA_\nu\mathrm{I} \tag{D7}$$

for an arbitrary four vector A_ν. So in total:

$$\Pi_\nu = -\frac{1}{4}\gamma_\mu[\partial_\nu\gamma^\mu + \Gamma^\mu{}_{\nu\ell}\gamma^\ell] - ieA_\nu\mathrm{I} \tag{D8}$$

Appendix E: Some comments on the Variation w.r.t the Spinor Connection

1. Fundamental Variables

Given the vierbien, the choice of metric for the curved space is completely unambigous. One simply takes the Minkowski metric and constructs it by projecting up with the vierbien:

$$g_{\mu\nu} \equiv e_\mu{}^a e_\nu{}^b\eta_{ab} \tag{E1}$$

Now, given the vierbien, it would seem ambiguous what to choose for the curved space gamma matrices. However, this ambiguity does not exist. In flat space, in order to solve the Dirac equation, one makes a choice of gamma matrices to use. From these, we can then construct the curved space gamma matrices by projecting up with the vierbien:

$$\gamma^\mu \equiv e^\mu{}_a\gamma^a \tag{E2}$$

The tetrad is the only variable.

2. A relation between the Gamma Matrices and Vierbien

We can now use the argument of the preceding section. I claim that whenever the curved space gamma matrices are fixed, the vierbien must also be fixed. Begin by taking the variation of (E2)

$$\delta\gamma^\mu = e^\mu{}_a\delta\gamma^a + \gamma^a\delta e^\mu{}_a \tag{E3}$$

If the curved space gamma matrices are held fixed then the L.H.S vanishes so that

$$0 = e^\mu{}_a\delta\gamma^a + \gamma^a\delta e^\mu{}_a \tag{E4}$$

Now, since the flat space gamma matrices are not variables, we have that

$$0 = \gamma^a\delta e^\mu{}_a \tag{E5}$$

To simplify the above equation, we may take the anti-commutator with γ^b:

$$0 = \{\gamma^a\delta e^\mu{}_a, \gamma^b\} \tag{E6a}$$

$$= (\delta e^\mu{}_a)\{\gamma^a, \gamma^b\} \tag{E6b}$$

$$= (\delta e^\mu{}_a)\,\eta^{ab}\,\mathrm{I} \tag{E6c}$$

We can take the trace over the spinor indices to eliminate the identity matrix. Finally, we can contract with the Minkowski metric to obtain:

$$0 = (\delta e_a{}^\mu)\,\eta^{ab}\eta_{bc} \tag{E7a}$$

$$= (\delta e^\mu{}_a)\,\delta^a{}_c \tag{E7b}$$

$$= \delta e^\mu{}_c \tag{E7c}$$

which is the desired result.

3. The Vector Potential

We can use the same sort of argument from the previous sections to say things about the vector potential.

The spinor connection is special in the sense that it depends on the object it is acting on. The affine and spin connection do not have this property. The covariant derivative of any four vector is independent of the four vector itself. The connection is always the same so that:

$$\nabla_\mu V^\sigma = \partial_\mu V^\sigma + \Gamma^\nu{}_{\mu\rho} V^\rho \tag{E8}$$

The same is true in flat space:

$$\nabla_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \tag{E9}$$

For two different Dirac spinors, however, they may be associated with different electromagnetic fields; and this argument will no longer hold. To illustrate this, we have that the connection for spinors is given by (D8):

$$\Pi_\mu = \pi_\mu - ieA_\mu \mathrm{I} \tag{E10}$$

where:

$$\pi_\mu \equiv -\frac{1}{4}\gamma_\nu [\partial_\mu \gamma^\nu + \Gamma^\nu{}_{\mu\ell}\gamma^\ell] \tag{E11}$$

For a spinor ψ associated with A_μ, the covariant derivative will be

$$D_\mu \psi = [\partial_\mu - \pi_\mu + ieA_\mu]\psi \tag{E12}$$

But for the spinor ψ' associated with B_μ, the covariant derivative will be

$$D_\mu \psi' = [\partial_\mu - \pi_\mu + ieB_\mu]\psi' \tag{E13}$$

The choice of covariant derivative seems to depend on the spinor itself. It would therefore seem ambigous, given a particular spinor, how to determine the nature of A_μ.

I claim, however, that this ambiguity does not exist. Given a particular choice of Dirac spinor, there is a fixed, predetermined four potential with which it is associated. It is precisely the four potential contained in the interaction term of its Lagrangian. From this, we can then construct the covariant derivative. There is no ambiguity in the vector potential as it is already specified. The variation of this object is therefore vanishing; it is not a variable.

On a closely related note, when you take the variation with respect to the affine or spin connections, there is only one. And hence, there is no choice that needs to be made. But when taking the variation with respect to the spinor connection, I must make a choice. Different spinors have different connections. So being more precise about the variation, I'm taking the variation of the Einstein-Hilbert Action with respect to a specific choice of spinor's connection.

ACKNOWLEDGMENTS

There is an immense body of work concerning the vierbien theory and Dirac equation in curved spacetime. Out of them all, I found an old paper by Schrödinger to be extremely helpful. It's titled *Diracsches Elektron im Schwerefeld I* (1932) and I cited it throughout this paper as [1]. Outside of research, Dirac's own book, *General Theory of Relativity*, proved to be useful for more historical definitions: such as those for the Riemann Tensor, Covariant Derviative, etc.

Those together provide the main source of reference for the current discourse. The proofs presented, however, are completely my own. Upon reflection, I came to believe that a symmetry must exist between the vierbien and spinor formalisms. This paper is the result of that conviction and serves as a proof of the after-mentioned correspondence.