Homework 3

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2.2

$$
\begin{aligned}
e_x &= p_x(1 + e_{x+1}) \\
e_x &= \sum_{k=1}^{\infty} {}_kp_x \\
&= p_x + \sum_{k=2}^{\infty} p_x \times {}_{k-1}p_{k+1} \\
&= p_x(1 + \sum_{k=2}^{\infty} {}_{k-1}p_{k+1}) \\
&= p_x(1 + e_{x+1})
\end{aligned}
$$

Thus, by removing the first term from the sum and rewriting, we have found an alternate form to express e_x.

$$
\begin{aligned}
e_{60} &= p_{60}(1 + e_{61}) \\
p_x &= \frac{e_x}{(1 + e_{x+1})}
\end{aligned}
$$

$$
\begin{aligned}
p_{60} &= \frac{e_{60}}{(1 + e_{61})} = \frac{15.96}{16.27} \\
p_{61} &= \frac{e_{61}}{(1 + e_{62})} = \frac{15.27}{15.60} \\
p_{62} &= \frac{e_{62}}{(1 + e_{63})} = \frac{14.60}{14.94}
\end{aligned}
$$

$$
\begin{aligned}
{}_3p_{60} &= {}_1p_{60} \times {}_1p_{61} \times {}_1p_{62} \\
&= \frac{15.96}{16.27} \times \frac{15.27}{15.60} \times \frac{14.60}{14.94} \\
&= 0.93834
\end{aligned}
$$

We can find one-year survival probability values, ${}_1p_x$, by rearranging the above formula and plugging in our given e_x values. Then, we can express ${}_3p_x$ as the product of three sequential ${}_1p_x$ probabilities, arriving at ${}_3p_{60} = 0.93834$.

2.7

a.

The survival function $S_x(t)$ is defined as the complement of the distribution function F_xt. Thus, we can simply subtract from 1 to arrive at $S_x(t)$

$$\begin{aligned} S_0(t) &= 1 - F_0 t \\ &= 1 - (1 - e^{-\lambda t}) \\ &= e^{-\lambda t} \end{aligned}$$

b.

$$\begin{aligned} \mu_x &= \lim_{dx \to 0^+} \frac{F_0(x+dx) - F_0(x)}{dx(1 - F_0(x))} \\ &= \frac{F_0'(x)}{(1 - F_0(x))} \\ &= \frac{f_0(x)}{S_0(x)} \\ &= \frac{-S_0'(x)}{S_0(x)} \\ &= -\frac{d}{dx} ln(S_0(x)) \\ &= -\frac{d}{dt} ln(e^{-\lambda t}) \\ &= -\frac{d}{dt}(-\lambda t) \\ &= \lambda \end{aligned}$$

We can derive μ_x, the force of mortality, as the negative natural logarithm of the survival function, which is λ in this case.

b.

$$\begin{aligned} e_x &= \sum_{k=1}^{\infty} {}_k p_t \\ &= \sum_{k=1}^{\infty} S_x(k) \\ &= \sum_{k=1}^{\infty} e^{-\lambda k} \\ &= \frac{e^{-\lambda}}{1 - e^{-\lambda}} \\ &= \frac{1}{e^{\lambda} - 1} \end{aligned}$$

We can treat this expression as an infinite sum, and evaluate as such. The first term, when $k = 1$, is $e^{-\lambda}$, with a common ratio of $e^{-\lambda}$. For this sum to converge, $\mid e^{-\lambda k} \mid < 1$, which is achieved when $\lambda > 0$. Then, we can evaluate and arrive at $e_x = \frac{1}{e^{\lambda}-1}$.

d.

No, I do not believe this is reasonable to model mortality. The associated survival function initially sharply decreases, before gradually levelling off as age approaches infinity. This is due to the constate force of mortality $\mu_x = \lambda$.

However, this is contrary to the phenomenon of human mortality, where death does not begin to pick up at a significant rate until later in life. Thus, we would want to choose a survival function that more appropriately models the force of mortality as increasing over age, and thus would include x in the expression for μ_x such as the Gompertz or Makeham.

2.9

$$\frac{d}{dx}{}_tp_x = {}_tp_x(\mu_x - \mu_{x+t})$$

$$\begin{aligned}
\frac{d}{dx}{}_tp_x &= \frac{d}{dt}(S_x(t)) \\
&= \frac{d}{dt}exp(-\int_0^t \mu_{x+s}ds) \\
&= -(\mu_{x+t} - \mu_x) \times exp(-\int_0^t \mu_{x+s}ds) \\
&= (\mu_x - \mu_{x+t}) \times exp(-\int_0^t \mu_{x+s}ds) \\
&= (\mu_x - \mu_{x+t}) \times S_x(t) \\
&= {}_tp_x(\mu_x - \mu_{x+t})
\end{aligned}$$

If we evaluate the derivative using the chain rule, we see that the derivative of ${}_tp_x$ at a certain x is just ${}_tp_x$ scaled by a factor of $(\mu_x - \mu_{x+t})$. Writing these concepts as such provides more insight on the role of the force of mortality, μ, as the negated, unit-scaled, instantaneous rate of change in the survival function S_0 at a particular x.

2.18

a.

$${}_tp_x^* = ({}_tp_x)^2$$

$$\begin{aligned}
{}_tp_x^* &= exp(-\int_0^t \mu_{x+s}^*ds) \\
&= exp(-\int_0^t 2\mu_{x+s}ds) \\
&= exp(-2\int_0^t \mu_{x+s}ds) \\
&= \left(exp(-\int_0^t \mu_{x+s}ds)\right)^2 \\
&= ({}_tp_x)^2
\end{aligned}$$

We begin by substituting $\mu_{x+s}^* = 2\mu_{x+s}$ into our expression. From there, we can extract the factor of 2 and rearrange to where the exponentiated term is itself raised to the power of 2. From there, the definiton of μ_{x+s} follows and we see that the linear doubling of the force of mortality translates into a quadratic squaring of the survival function.