

Homework 3

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22 February 2022

2.2

$$\begin{aligned}
 e_x &= p_x(1 + e_{x+1}) \\
 e_x &= \sum_{k=1}^{\infty} {}_k p_t \\
 &= p_x + \sum_{k=2}^{\infty} p_x \times {}_{k-1} p_{k+1} \\
 &= p_x + \sum_{k=2}^{\infty} p_x \times {}_{k-1} p_{k+1} \\
 &= p_x(1 + \sum_{k=2}^{\infty} {}_{k-1} p_{k+1}) \\
 &= p_x(1 + e_{x+1})
 \end{aligned}$$

Thus, by removing the first term from the sum and rewriting, we have found an alternate form to express e_x .

$$\begin{aligned}
 e_{60} &= p_{60}(1 + e_{61}) \\
 p_x &= \frac{e_x}{(1 + e_{x+1})}
 \end{aligned}$$

$$\begin{aligned}
 p_{60} &= \frac{e_{60}}{(1 + e_{61})} = \frac{15.96}{16.27} \\
 p_{61} &= \frac{e_{61}}{(1 + e_{62})} = \frac{15.27}{15.60} \\
 p_{62} &= \frac{e_{62}}{(1 + e_{63})} = \frac{14.60}{14.94}
 \end{aligned}$$

$$\begin{aligned}
 {}_3 p_{60} &= {}_1 p_{60} \times {}_1 p_{61} \times {}_1 p_{62} \\
 &= \frac{15.96}{16.27} \times \frac{15.27}{15.60} \times \frac{14.60}{14.94} \\
 &= 0.93834
 \end{aligned}$$

We can find one-year survival probability values, ${}_1 p_x$, by rearranging the above formula and plugging in our given e_x values. Then, we can express ${}_3 p_x$ as the product of three sequential ${}_1 p_x$ probabilities, arriving at ${}_3 p_{60} = 0.93834$.

2.7

a.

The survival function $S_x(t)$ is defined as the complement of the distribution function $F_x t$. Thus, we can simply subtract from 1 to arrive at $S_x(t)$

$$\begin{aligned} S_0(t) &= 1 - F_0 t \\ &= 1 - (1 - e^{-\lambda t}) \\ &= e^{-\lambda t} \end{aligned}$$

b.

$$\begin{aligned} \mu_x &= \lim_{dx \rightarrow 0^+} \frac{F_0(x + dx) - F_0(x)}{dx(1 - F_0(x))} \\ &= \frac{F'_0(x)}{(1 - F_0(x))} \\ &= \frac{f_0(x)}{S_0(x)} \\ &= \frac{-S'_0(x)}{S_0(x)} \\ &= -\frac{d}{dx} \ln(S_0(x)) \\ &= -\frac{d}{dt} \ln(e^{-\lambda t}) \\ &= -\frac{d}{dt} (-\lambda t) \\ &= \lambda \end{aligned}$$

We can derive μ_x , the force of mortality, as the negative natural logarithm of the survival function, which is λ in this case.

b.

$$\begin{aligned} e_x &= \sum_{k=1}^{\infty} k p_t \\ &= \sum_{k=1}^{\infty} S_x(k) \\ &= \sum_{k=1}^{\infty} e^{-\lambda k} \\ &= \frac{e^{-\lambda}}{1 - e^{-\lambda}} \\ &= \frac{1}{e^{\lambda} - 1} \end{aligned}$$

We can treat this expression as an infinite sum, and evaluate as such. The first term, when $k = 1$, is $e^{-\lambda}$, with a common ratio of $e^{-\lambda}$. For this sum to converge, $|e^{-\lambda k}| < 1$, which is achieved when $\lambda > 0$. Then, we can evaluate and arrive at $e_x = \frac{1}{e^{\lambda} - 1}$.

d.

No, I do not believe this is reasonable to model mortality. The associated survival function initially sharply decreases, before gradually levelling off as age approaches infinity. This is due to the constate force of mortality $\mu_x = \lambda$.

However, this is contrary to the phenomenon of human mortality, where death does not begin to pick up at a significant rate until later in life. Thus, we would want to choose a survival function that more appropriately models the force of mortality as increasing over age, and thus would include x in the expression for μ_x such as the Gompertz or Makeham.

2.9

$$\begin{aligned}
\frac{d}{dx} {}_t p_x &= {}_t p_x (\mu_x - \mu_{x+t}) \\
\frac{d}{dx} {}_t p_x &= \frac{d}{dt} (S_x(t)) \\
&= \frac{d}{dt} \exp\left(-\int_0^t \mu_{x+s} ds\right) \\
&= -(\mu_{x+t} - \mu_x) \times \exp\left(-\int_0^t \mu_{x+s} ds\right) \\
&= (\mu_x - \mu_{x+t}) \times \exp\left(-\int_0^t \mu_{x+s} ds\right) \\
&= (\mu_x - \mu_{x+t}) \times S_x(t) \\
&= {}_t p_x (\mu_x - \mu_{x+t})
\end{aligned}$$

If we evaluate the derivative using the chain rule, we see that the derivative of ${}_t p_x$ at a certain x is scaled by a factor of $(\mu_x - \mu_{x+t})$. Writing these concepts as such provides more insight on the role of the force of mortality, μ , as the negative, scaled, instantaneous rate of change in the survival function S_0 at a particular x .

2.18a