ST 501 HW3

Evan Whitfield

2.5) Assume $u < v$.

a) $P(u < X \leq v) = \sum_{x=u+1}^{v} P_X(x)$ for discrete RV X.

Assume a is the least possible realization of X.
Then,
$$\sum_{x=u+1}^{v} P_X(x) = \sum_{x=a}^{v} P_X(x) - \sum_{x=a}^{u} P_X(x)$$
$$= P(X \leq v) - P(X \leq u)$$
$$= F_X(v) - F_X(u)$$

b) $P(u < X \leq v) = \int_u^v f_X(x)\,dx$ for continuous RV X.
$$= \int_{-\infty}^{v} f_X(x)\,dx - \int_{-\infty}^{u} f_X(x)\,dx$$
$$= F_X(v) - F_X(u)$$

2.7)
$$p_Y(y) = \begin{cases} p & \text{if } y = 1 \\ 1-p & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$F_Y(0) = P(Y \leq 0) = p_Y(0) = 1-p$

$F_Y(1) = P(Y \leq 1) = p_Y(0) + p_Y(1) = p + (1-p) = 1$

$$\boxed{F_Y(y) = \begin{cases} 0 & y < 0 \\ 1-p & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}}$$

2.31) Define Y = # of phone calls in an hour, then $Y \sim Poi(2)$, therefore

$$p_Y(y) = \frac{(2)^y e^{-2}}{y!} \quad \text{for } y = 0, 1, 2, \ldots$$

a) Define X = # of phone calls in a 10 min period, then $X \sim Poi(\frac{2}{6})$, therefore

$$p_X(x) = \frac{(\frac{2}{6})^x e^{-2/6}}{x!} \quad \text{for } x = 0, 1, 2 \ldots$$

$$P(\text{Phone rings}) = 1 - P(X=0)$$

$$= 1 - \frac{(\frac{2}{6})^0 e^{-2/6}}{0!} = \boxed{1 - e^{-2/6} \approx 0.283}$$

b) Define A = # of phone calls in a "b" minute period. then $A \sim Poi(\frac{2b}{60})$, therefore

$$P(A=0) = 0.5$$

$$\frac{(\frac{2b}{60})^0 e^{-2b/60}}{0!} = 0.5$$

$$e^{-2b/60} = 0.5 \Rightarrow \frac{-2b}{60} = \ln(0.5)$$

$$\Rightarrow \boxed{b = -30 \ln(0.5) \approx 20.8 \text{ min}}$$

or

$$b = 30 \ln(2)$$

2.33) Let $F(x) = 1 - \exp(-\alpha x^{\beta})$ for $x \geq 0$, $a > 0$, $\beta > 0$
and $F(x) = 0$ for $x < 0$.

$$F(x) = 1 - \exp(-\alpha x^{\beta}) = 1 - e^{-\alpha x^{\beta}}$$

$$f(x) = \frac{d}{dx} F(x) = \boxed{\begin{array}{c} \alpha \beta x^{\beta-1} e^{-\alpha x^{\beta}} \\ \text{or} \\ \alpha \beta x^{\beta-1} \exp(-\alpha x^{\beta}) \end{array}}$$

2.40) $f(x) = cx^2$ for $0 \leq x \leq 1$ +
$f(x) = 0$ otherwise

a) $\int_0^1 cx^2\,dx = 1 \Rightarrow \frac{c}{3} x^3 \Big|_0^1 = 1 \Rightarrow \frac{c}{3} = 1 \Rightarrow \boxed{c = 3}$

b)
$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

c) $P(.1 \leq X < .5) = F(.5) - F(.1)$

$$= (.5)^3 - (.1)^3 = \boxed{0.124}$$

Other Problems

(1) Yes, if $F_X(x)$ is a CDF, then $[F_X(x)^\alpha]$ will also be a CDF for a given $\alpha > 0$.

Looking at CDF Theorem

a) If $\lim_{x \to -\infty} F_X(x) = 0$, then $\lim_{x \to -\infty} (F_X(x))^\alpha = 0$

If $\lim_{x \to \infty} F_X(x) = 1$, then $\lim_{x \to \infty} (F_X(x))^\alpha = 1$

b) If x_1, x_2 where $x_1 \leq x_2$, then if

$F_X(x_1) \leq F_X(x_2)$ then $(F_X(x_1))^\alpha \leq (F_X(x_2))^\alpha$

So $(F_X(x))^\alpha$ is nondecreasing

c) If $F_X(x)$ is right continuous, then $(F_X(x))^\alpha$ would also be right continuous.

Since $(F_X(x))^\alpha$ follows all three parts of the theorem, then it must be a valid CDF.

(2) To find the probability from a PDF from a to b, we look at the area under the curve from a to b. This often means taking the integral of the PDF from point a to point b

To find the probability from a PMF from a to b, we sum the probabilities of the values between a + b. We need to check if either a or b should be included. This can often be done by summing the heights of the included bars from the PMF.

(3) a)

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 0.6 & \text{if } 0 \le y < 1 \\ 0.8 & \text{if } 1 \le y < 2 \\ 0.9 & \text{if } 2 \le y < 3 \\ 0.98 & \text{if } 3 \le y < 4 \\ 1 & \text{if } y \ge 4 \end{cases}$$

b) $P(Y \ge 2) = P(Y=2) + P(Y=3) + P(Y=4)$

$= 0.1 + 0.08 + 0.02$

$= \boxed{0.2}$

c) The 0.85 quantile, or $y_{0.85}$, would be $\boxed{\text{2 stings}}$.

There is an 85% chance that a camper will have two or fewer stings.

(4) a) $\int_1^{10} ce^{-y}\,dy = 1$

$-ce^{-y}\Big|_1^{10} = 1$

$(-ce^{-10}) - (-ce^{-1}) = 1$

$c(e^{-1}-e^{-10}) = 1 \Rightarrow \boxed{c = \frac{1}{e^{-1}-e^{-10}} \approx 2.719}$

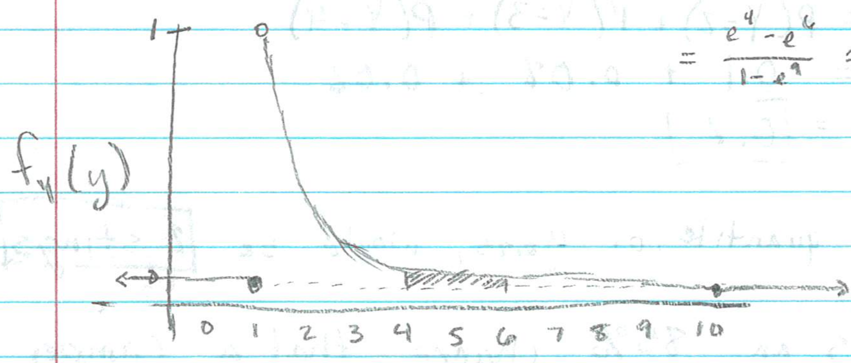
*alternatively: $\frac{e^{10}}{e^{9}-1}$

b) $F_Y(y) = \frac{-e^{10}}{e^{9}-1}\cdot e^{-y} + 1 = \frac{e^{10-y}}{1-e^{9}} + 1$ from $1 < y < 10$

so $F_Y(y) = \begin{cases} 0 & \text{if } y \leq 1 \\ \frac{e^{10-y}}{1-e^{9}} + 1 & \text{if } 1 < y < 10 \\ 1 & \text{if } y \geq 10 \end{cases}$

c) $P(4<Y<6) = F_Y(6) - F_Y(4) = \left(\frac{e^{4}}{1-e^{9}} + 1\right) - \left(\frac{e^{6}}{1-e^{9}} + 1\right)$

$= \frac{e^{4}-e^{6}}{1-e^{9}} \approx 0.043$



d) $F_Y(y) = \dfrac{e^{10-y_{0.5}}}{1-e^{9}} + 1 = 0.5$

$$\Rightarrow e^{10-y_{0.5}} = 0.5(e^{9}-1)$$

$$\Rightarrow 10-y_{0.5} = \ln(0.5e^{9}-0.5)$$

$$\Rightarrow \boxed{\begin{array}{l} y_{0.5} = 10-\ln(0.5e^{9}-0.5) \\ \Rightarrow y_{0.5} \approx 1.693 \end{array}}$$