Holiday Contest 2012 - A Block Solutions

December 19, 2012

Set 1

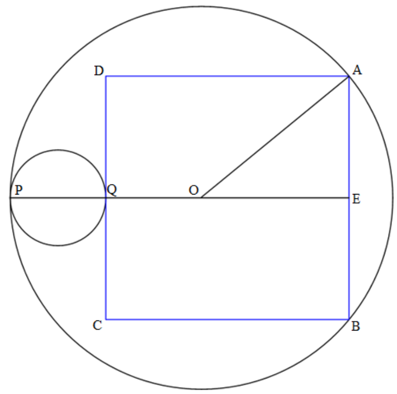
1. Come holiday time, three friends were boasting about how many presents they each had received. Kristina claimed she had gotten 8^{10} presents. Not to be outdone, Saketh said he had 9^9. In order to one-up Saketh, Billy boasted that he had gotten 10^8 presents. Who amongst the Tired Trio truly received the most presents?
   Answer: We note that the difference between the bases are so small that the exponents have a larger impact. Thus $\boxed{Kristina}$ has the most presents.

2. Lemy Ree was decorating his favorite tree and asked his best friend Warker Pon to hand him some ornaments from a box with 32 ornaments in it: 6 blue, 8 red, 4 yellow, and 14 white. Lemy had severe allergic reactions anytime he saw the color white (like this page) and as a result, he asked Warker to hand him some non-white ornaments. Unfortunately, Warker did not hear Lemy and the unknowing Mr. Pon decided to pull out three ornaments all at once. What is the probability that it was not Lemys Last Christmas?
   Answer: There are a total of 18 ornaments that are non-lethal to Lemy. The number of combinations of three non-lethal ornaments is thus $\binom{18}{3}$ while the total number of possible 3-ornaments combinations is $\binom{32}{3}$. Thus the probability of Lemy not dieing is $\frac{18*17*16}{32*31*30} = \boxed{\frac{51}{310}}$

3. Kiffa, who is visiting from China, decided to celebrate the holidays in true VMT style. He called up his trusty sidekick Ritwik and asked him for the most challenging problem Ritwik had ever heard. Ritwik responded with the following question: What is the smallest positive integer k that such that $2 * k$ is a perfect square and $3 * k$ is a perfect cube? What is the answer that Lil Kiffer gave his secondhand sidekick?
   Answer: To satisfy the first requirement, we know that k has to have an odd factor of 2. Since $3 * k$ is a cube, we know that there is a minimum of three factors of 2. Using a similar process, we know that k has to have at least two factors of 3, also satisifying that $2 * k$ is a perfect square. Thus the minimum value of k is $2 * 2 * 2 * 3 * 3 = \boxed{72}$

Set 2

4. Katie and Julie are baking Holiday cookies for a bake sale. They flatten out a circle of dough with radius 20. One of the cookie cutters is a circle of radius 5 and another is a square with x edge length. The square cookie can be cut out tangent to the circle and has two corners on the edge of the dough. What is the length x of the cookie cutter? Answer:

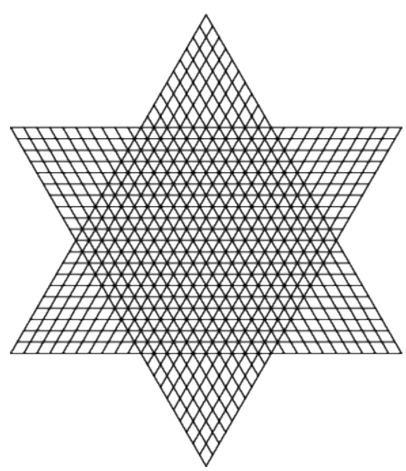


Call the center of the larger circle O. Extend the diameter $\overline{PQ}$ to the other side of the square (at point E), and draw $\overline{AO}$. We now have a right triangle, with hypotenuse of length 20. Since $OQ = OP - PQ = 20 - 10 = 10$, we know that $OE = AB - OQ = AB - 10$. The other leg, AE, is just $\frac{1}{2}AB$. Apply the Pythagorean Theorem:

$(AB - 10)^2 + \left(\frac{1}{2}AB\right)^2 = 20^2$
$AB^2 - 20AB + 100 + \frac{1}{4}AB^2 - 400 = 0$
$AB^2 - 16AB - 240 = 0$

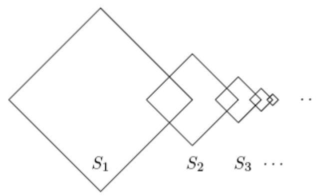
The quadratic formula shows that the answer is $\frac{16 \pm \sqrt{16^2 + 4 \cdot 240}}{2} = 8 \pm \sqrt{304}$. Discard the negative root, so our answer is $\boxed{8 + \sqrt{304}}$.

5. Joe Park is skating one day in a private rink. He, feeling very festive, imagines the center of the rink to be $(0, 0)$ on a coordinate system. He graphs the equations $y = k$, $y = \sqrt{3} * x + 2 * k$, and $y = -\sqrt{3} * x + 2k$ for $k = -10, -9, -8, \ldots, 9, 10$. How many equilateral triangles of side $\frac{2}{\sqrt{3}}$ did he make during his ice skating adventures?
Answer: The above equations yield the figure below:



Solving the above equations for $k = \pm 10$, we see that the hexagon in question is regular, with side length $\frac{20}{\sqrt{3}}$. Then, the number of triangles within the hexagon is simply the ratio of the area of the hexagon to the area of a regular triangle. Since the ratio of the area of two similar figures is the square of the ratio of their side lengths, we see that the ratio of the area of one of the six equilateral triangles composing the regular hexagon to the area of a unit regular triangle is just $\left(\frac{20/\sqrt{3}}{2/\sqrt{3}}\right)^2 = 100$. Thus, the total number of unit triangles is $6 \times 100 = 600$. There are 6×10 equilateral triangles formed by lines on the edges of the hexagon. Thus, our answer is $600 + 60 = \boxed{660}$.

6. Kristina is hosting a holiday party at her house. She made 5 square $2D$ cookies (S_1, S_2, S_3, S_4, and S_5), the first of which is 1x1, and the subsequent cookies have $\frac{1}{4}$ the area of the preceding cookie. She lays them out so that the two adjacent sides of the square S_i are the perpendicular bisectors of two adjacent sides of square S_{i+1}. What is the total area enclosed by the squares?



Answer: As the area differ by a factor of $\frac{1}{4}$, we know that the ratio between the sides is $\frac{1}{2}$. In addition, we notice that the sum of the areas of the squares (including overlap) is a geometric sequence: $1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2$. We should then subtract the areas of the intersections, which is $\left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{32}\right)^2$: $1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 - \left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{1}{32}\right)^2\right] = 1 + (\frac{1}{2})^2 - (\frac{1}{32})^2$ We notice that the majority of the terms cancel, leaving $1 + \frac{1}{4} - \frac{1}{1024}$, which simplifies down to $\frac{1024+(256-1)}{1024}$. Thus, our answer is $\boxed{\frac{1279}{1024}}$. Alternatively, we can take the area of the first square and add on $\frac{3}{4}$ of the areas of the remaining squares, which produces the same answer.

Set 3

7. A circular winter storm of 50 miles in diameter starts off on a coordinate plane at $(0, 110)$ and travels SE at a speed of $\frac{1}{2}\sqrt{2}$ miles per minute. Rudolph sees this storm and starts dashing through the snow towards his home far down the positive x-axis (East) at a speed of $\frac{2}{3}$ miles per minute. Unfortunately, Rudolph enters the storm at time t_1 along his way back, but leaves the storm circle later at time t_2. What is $\frac{t_2+t_1}{2}$? Answer: As the storm is traveling SE, its x and y speed is the same $(45-45-90$ triangle) and is $\frac{1}{2}$ miles per minute. Using coordinates, we see that at time t the location of Rudolph is $(\frac{2}{3}t, 0)$ and the location of the storm is $(\frac{1}{2}t, 110 - \frac{1}{2}t)$. Thus, by the distance formula, we know that the Rudolph enters the storm or leaves it when $\left(\frac{1}{6}t\right)^2 + \left(110 - \frac{1}{2}t\right)^2 = 25^2$. Expanding gives us $\frac{5}{18}t^2 - 110t + 11475 = 0$. The times t_1 and t_2 therefore are just the roots of the quadratic. Through vieta we can find the sum of the roots, which gives us $t_1 + t_2 = \frac{110}{\frac{5}{18}} = 396 \implies \frac{t_1+t_2}{2} = \boxed{198}$.

8. After the storm blew past Rudolph, it hit a town of gingerbread houses. Thus it was snowing heavily outside when Johnny was heading home from school. Although Johnny could now barely see his own hand in front of his face, he still has a good sense of where his house is. However, after walking for a certain distance he figures that he has drifted off track, and will stop, turn 90 degrees clockwise, and continue walking until he thinks he is lost again. If Johnny first walks from the origin (school) 1 mile North, then East for a $\frac{1}{2}$ mile, then South for a $\frac{1}{4}$ mile, and so on, going $\frac{1}{2}$ the distance gone before, what coordinate will he end up at relative to the school?

Answer: Once Johnny travels a distance l in one direction, the next time he faces the same direction he will travel $l * (\frac{1}{2})^4 = \frac{1}{16}l$. Thus the total distance traveled in each direction is the sum of an infinite geometric series with first terms of $1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$ with a common ratio of $\frac{1}{16}$. Thus the x-coordinate is $1*(\frac{1}{1-\frac{1}{16}}) - \frac{1}{4}*(\frac{1}{1-\frac{1}{16}}) = \frac{3}{4}*\frac{16}{15} = \frac{4}{5}$. Similarly, the y-coordinate is $\frac{1}{2}*(\frac{1}{1-\frac{1}{16}}) - \frac{1}{8}*(\frac{1}{1-\frac{1}{16}}) = \frac{3}{8}*\frac{16}{15} = \frac{2}{5}$. Thus Johnny ends up at the coordinate $\boxed{(\frac{4}{5}, \frac{2}{5})}$.

9. The increasing sequence $1, 3, 4, 9, 10, 12, 13, ...$ consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence. Answer: As we are only allowed to use distinct powers of 3, we can represent all numbers in base 3 using 0s and 1s. Notice that as we are only using 0s and 1s that each term in the sequence in base 3 is a number in base 2. So the 100th number is simply 100 in base 2 $\implies 1100100_2$. Thus $1100100_3 = 729 + 243 + 9 = \boxed{981}$

Set 4

10. Santa Clause has recently called for a round table meeting between 5 elves, 5 reindeer, and 5 trolls. At this meeting, the fifteen individuals will sit in a circular table ordered from 1 to 15 in clockwise order. Santas orders are that (1) An elf must sit in chair 1, (2) A reindeer must sit in chair 15, and (3) No elf can sit immediately to the left of a reindeer, no reindeer can sit immediately to the left of a troll, and no troll can sit immediately to left of an elf. Given that everyone follows Santas orders (lest they be placed on the naughty list) the number of possible arrangements is $N * (5!)^3$. Find N.
Answer: We see the number of ways to seat people with only regard to race is equal to N. Starting with chair 1 going clockwise, we see that the seating must go as follows: group of Martians, group of Venusians, group of earthlings, group of Martians, etc. We now divide this into cases based upon how many groups of each there are.
Case 1: 1 group of each race. Each group must be the 5 people from each race sitting together. Thus, there is only 1 arrangement in this case.
Case 2: 2 groups of each race. Since each group has at least one person, give each group one person from the given race. There are 3 people left from each race to put into the groups. With balls and urns(3 balls and 1 urn), we find that for each race, there are $\binom{4}{1} = 4$ ways to place the people into groups. Thus, there are $4^3 = 64$ arrangements in this case.
Case 3: 3 groups of each race. Similar to above. There are $6^3 = 216$ arrangements.
Case 4: 4 groups of each race. Similar to above. There are $4^3 = 64$ arrangements in this case
Case 5: 5 groups of each race. Give each group one person. There are no people left, so there is only 1 in this case. Thus, there are $1 + 64 + 216 + 64 + 1 = \boxed{346}$ seating arrangements

11. Frobenius the Snowman (Frostys mathematically inclined cousin) was interested in finding the number of positive integers with three not ncessarily distinct digits, abc, with $a \neq 0$, $c \neq 0$, such that both abc and cba are divisible by 4. Please help Frobenius find the answer to this question.
Answer: A number is divisible by four if its last two digits are divisible by 4. Thus, we require that $10b + a$ and $10b + c$ are both divisible by 4. If b is odd, then a and c must both be $2 \pmod 4$ meaning that a and c are $2 or 6$. If b is even, then a and c must be $0 \pmod 4$ meaning that a and c are 4 or 8. For each choice of b there are 2 choices for a and 2 for c for a total of $10 \cdot 2 \cdot 2 = \boxed{40}$ numbers.

12. John Jacob Jingleheimer Schmidt recently realized that y, the number of candy canes he will receive this winter break, follows the form $y = \frac{9x^2 sin^2 x + 4}{sin x}$. Find the minimum number of candy canes John will receive given that $0 < x < \pi$.
Answer:Let $y = x \sin x$. We can then rewrite the expression as $\frac{9y^2+4}{y} = 9y + \frac{4}{y}$. Since $x > 0$ and $\sin x > 0$ because $0 < x < \pi$, we have $y > 0$. This means we can apply AM-GM: $\frac{9y+\frac{4}{y}}{2} \geq \sqrt{9y \cdot \frac{4}{y}} =$

$6 \implies 9y + \frac{4}{y} \geq 12$. We see that the equality holds when $9y = \frac{4}{y} \iff y^2 = \frac{4}{9} \iff y = \frac{2}{3}$, which occurs when $when x \sin x = \frac{2}{3}$. Therefore, the minimum value is $\boxed{12}$.

Set 5

13. Define a positive integer n to be a factorial tail if there is some positive integer m such that the decimal representation of $m!$ ends with exactly n zeroes. How many positive integers less than 1992 are not factorial tails?
Answer: Because there area always enough powers of two, we need only consider the multiples of 5. Notice that for every multiple of $25, 125, 625$, etc, we jump a power of 10. For example, 24! has 4 zeroes (from $5, 10, 15$, and 20), but 25! has 6 zeroes, of which two 5s are contrributed from 25. Thus we need to only count the number of multiples of $25, 125, 625$, etc. that are less than x, where $x!$ has 1992 zeroes. It happens that $x = 7980$, and there are 1596 multiples of 5, including 319 multiples of 25, 63 multiples of 125, 12 multiples of 625, and 2 multiples of 3125. Hence, the total number of numbers skipped is $319 + 63 + 12 + 2 = \boxed{396}$.

14. Katie and Sarthak, little elves living in the North Pole, are walking in the same direction, Katie at 3 feet per second and Sarthak at 1 foot per second, on parallel paths that are 200 feet apart. The North Pole 100 feet in diameter is centered midway between the paths. At the instant when the building first blocks the line of sight between Sarthak and Katie, they are 200 feet apart. What is the amount of time, in seconds, before Sarthak and Katie can see each other again?
Answer:
Let S_1 and K_1 be Sarthak's and Katie's starting point respectively and S_2 and K_2 be their ending points. Draw the radius from the center of the circle, lets call it O, to S_2K_2, and another parallel to S_1K_2. Call the intersections of the two radii with S_2K_2, L_1 and L_2 respectively. Drop a perpendicular from S_1 onto K_1K_2. Call the intersection of this perpendicular with K_1K_2, L_3. Draw the radius from O to SK. Call the intersection with S_1K_1, L_4. We have $\triangle OL_1L_2 \sim \triangle S_2L_3K_2$. If we let $S_1S_2 = x$ then it is easy to see that $K_1K_2 = 3x$ and $L_3K_2 = 2x$. If we let $L_1L_2 = m$, we have $\frac{L_1L_2}{OL_2} = \frac{K_2L3}{L_3S_2} = \frac{2x}{200} = \frac{m}{50} \Rightarrow m = \frac{x}{2}$. Applying the Pythagorean Theorem yields $OL_2 = \sqrt{50^2 + \frac{x^2}{4}}$. Since L_2L_4 is midway between S_1S_2 and K_1K_2 we have $\frac{3x+x}{2} = L_2L_4 = 50 + \sqrt{50^2 + \frac{x^2}{4}} \Rightarrow x = \boxed{\frac{160}{3}}$

15. In $\triangle ABC, AB = 360, BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects angle ABC. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. The ratio $DE : EF$ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.
Answer: As we are dealing with ratios, we can rescale the triangle for ease of computation. Consider the triangle as above except $AB = 120, BC = 169, CA = 260$. In the following, let the name of a point represent the mass located there. By the angle bisector theorem, we can place mass points on C,D,A of $120, 289, 169$ respectively. Thus, a mass of $\frac{289}{2}$ belongs at F (seen by reflecting F across BD, to an image which lies on AB). Having determined CB/CF, we reassign mass points to determine FE/FD. This setup involves $\triangle CFD$ and transversal MEB. For simplicity, put a masses of 240,289 at C,F. To find the mass we should put at D, we compute CM/MD: applying the angle bisector theorem again and using the fact M is a midpoint, we find $\frac{CM}{MD} = \frac{169 \cdot \frac{260}{289} - 130}{130} = \frac{49}{289}$. At this point we could find the mass at D but it's unnecessary. So, $\frac{DE}{EF} = \frac{F}{D} = \frac{F}{C}\frac{C}{D} = \frac{289}{240}\frac{49}{289} = \frac{49}{240} \implies 49 + 240 = \boxed{289}$.