TJUSAMO - Functional Equations

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Equations for unknown functions are called functional equations.

1 Introduction

1. A famous classical functional equation is $f(x+y) = f(x) + f(y)$.
2. Another classical equation is $f(x+y) = f(x)f(y)$.

Take a minute to try to prove the two examples above. These well-known and commonly used functional equations in two variables are Cauchy's functional equations.

Here are other functional equations that should be noted:
Jensen's functional equation: $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$
Alambert's functional equation: $f(x+y) + f(x-y) = 2f(x)f(y)$

Usually a functional equation has many solutions, and it is quite difficult to find all of them. However, these solution(s) can be found with greater efficiency by taking into account some additional properties, such as continuity, monotonicity, boundedness, or differentiability. What maybe even more useful to solve certain problems involving functional equations is utilizing mathematical induction or clever (sometimes, even simple) substitutions. The following practice problems will hopefully show what steps should be taken to solve questions dealing with functional equations.

2 Warm-Up Problems

1. Find all polynomials p satisfying $p(x+1) = p(x) + 2x + 1$.
2. Find all functions f which are defined for all real numbers x and for any x, y, satisfy

$$xf(y) + yf(x) = (x+y)f(x)f(y)$$

2. Find a function f defined for $x > 0$, so that $f(xy) = xf(y) + yf(x)$.
3. Find all continuous solutions of $f(x-y) = f(x)f(y) + g(x)g(y)$.
4. Find the function f which satisfies the functional equation

$$f(x) + f\left(\frac{1}{1-x}\right) = x$$

3 Intermediate-ish Problems

1. Find all functions defined by real numbers such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2$$

2. Find all continuous functions satisfying $3f(2x+1) = f(x) + 5x$.
3. Find all functions defined for all real numbers that satisfy

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$

   for all reals x and y.

4 Challenge Accepted?

1. (USAMO) Prove that there does not exist any real function such that

$$\tfrac{f(x)+f(y)}{2} \geq f\left(\tfrac{x+y}{2}\right) + \mid (x-y) \mid$$

2. (USAMO) Find all real functions such that $f(x^2 - y^2) = xf(x) - yf(y)$
3. Find all continuous functions from all reals to reals such that

$$f(x) + f(y) - f(x+y) = xy$$

4. Find all functions $f(x)$ from the positive integers to the positive integers such that

$$f^{f(n)}(n) = n + 1$$

5. Find all functions $f(x)$ defined by rational numbers such that $f(1) = 2$ and

$$f(xy) = f(x)f(y) - f(x+y) + 1$$