TJUSAMO - Pigeonhole Principle

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The pigeonhole content of this lecture is adapted largely from Thomas Mildorf's 2007 Pigeonhole lecture. Credits are also owed to various MOP instructors (Po-Shen Loh, Zuming Feng) and previous TJUSAMO lecturers (05PDiao).

1 Problems from Previous Weeks

1. Let p be a prime with $p > 5$. Prove that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$.
2. Consider the sequence $a_1, a_2, \ldots$ defined by $a_n = 2^n + 3^n + 6^n - 1$ for all positive integers n. Determine all positive integers that are relatively prime to every term of the sequence.
3. Determine if it is possible to arrange $1, 2, \ldots, 1000$ in a row such that the average of any pair of distinct numbers is not located in between the numbers.
4. Prove that for any integer $a \geq 2$, $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$.
5. Let ABC be an acute triangle. A circle going through B and the triangle's circumcenter O intersects BC and BA at points P and Q respectively. Prove that the intersection of the heights of the triangle POQ lies on the line AC.
6. A bookshelf contains n volumes, labeled 1 to n, in some order. The librarian wishes to put them in the correct order as follows: The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it in the k-th position. For example, if the bookshelf contains the volumes $1, 3, 2, 4$ in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order $1, 2, 3, 4$.

   (a) Show that if this process is repeated, then, regardless of how the librarian selects which books to move (as long as the rules are obeyed), all the volumes will eventually be in the correct order.
   (b) What is the largest number of steps that this process can take (for all the books to end up in the correct order)?

2 Pigeonhole Problems

The Pigeonhole Principle states that if m pigeons are placed into n pigeonholes, then there must be a pigeonhole with at least $\lceil \frac{m}{n} \rceil$ pigeons in it. Sounds simple enough right? What makes pigeonhole problems tricky is twofold: 1) Identifying the problem as requiring pigeonhole, and 2) figuring out what should be the pigeons and what should be the holes.

Keep in the mind that pigeonhole is often the most powerful when m is just more than a multiple of n. Consider reducing n if this is not the case.

1. Given integers $a_1, a_2, ..., a_n$, show that some non-empty subset of these integers has a sum divisible by n.

2. Let n be a positive integer. Show that if S is a subset of $\{1, 2, ..., 2n\}$ containing $n+1$ elements, then (a) there are two distinct coprime elements of S; (b) there are two distinct elements of S such that one divides the other.

3. 61 points are chosen inside a circle of radius 4. Show that among them there are two that are distance at most $\sqrt{2}$ from each other.

4. In a square grid of 169 points, 53 are marked. Prove that some 4 marked points form a rectangle with sides parallel to the grid lines.

5. The squares of an 8x8 checkerboard are filled with the numbers $\{1, 2, ..., 64\}$. Prove that some two adjacent squares (sharing a side) contain numbers differing by at least 5.

6. Let set S contain five integers each greater than 1 and less than 120. Show that S either contains a prime, or two elements of S share a prime divisor, or both.

7. Show that any convex polyhedron has two faces with the same number of edges.

8. Let $a_1, ..., a_{100}$ and $b_1, ..., b_{100}$ be two permutations of the integers from 1 to 100. Prove that, among the products $a_1b_1, a_2b_2, ..., a_{100}b_{100}$, there are two with the same remainder upon division by 100.

9. Let n be a positive integer that is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

10. The set $\{1, 2, ..., 16\}$ is partitioned into three sets. Prove that one of the subsets contains some numbers x, y, z (not necessarily distinct) such that $x + y = z$.

11. A 6x6 rectangular grid is tiled with non-overlapping 1x2 rectangles. Must there be a line through the interior of the grid that does not pass through a rectangle?

12. The *Ramsey Number* $R(s,t)$ is the mininum integer n for which every red-blue coloring of the edges of K_n contains a completely red K_s or a completely blue K_t. Prove that
$$R(s,t) \leq \binom{s+t-2}{s-1}$$
.

13. License plates in a certain province have six digits from 0 to 9 and may have leading zeroes. If two plates must always differ in at least two places, what is the largest number of plates that is possible?

14. A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.