Expected Value

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1 Definition

Let X be a random discrete variable. Then,

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

where n is the total number of possible outcomes, x_i denotes the value (weight) of a certain outcome, and p_i denotes that outcome's respective probability. Note that expected value has nothing to do with most *probable* value. Rather, it gives a "weighted average" of the possible outcomes.

2 Properties

For contest purposes, these may be useful in expected value calculations.

- $E(X + c) = E(X) + c$
- $E(X + Y) = E(X) + E(Y)$
- $E(cX) = cE(X)$
- $E(XY) = E(X)E(Y)$ if and only if X and Y are uncorrelated

3 Examples

3.1 Easy

A six-sided die with numbers 1-6 has probabilities in the ratio $1 : 2 : 3 : 4 : 5 : 6$ for landing with sides $1, 2, 3, 4, 5, 6$ facing up, respectively. Find the expected roll value.

Solution: Based on the definition, our expected roll value is equal to the sum of each possible roll multiplied by its respective probability. Thus, $E(X) = \frac{1}{21} \cdot 1 + \frac{2}{21} \cdot 2 + \frac{3}{21} \cdot 3 + \frac{4}{21} \cdot 4 + \frac{5}{21} \cdot 5 + \frac{6}{21} \cdot 6 = \boxed{4.\overline{3}}$

3.2 Medium

(SKim) 20 math teamers are trying to form 10 pairs. If there are 5 students of each grade (5 freshmen, 5 sophomores, etc.), then what is the expected number of pairs with students of different grades?

Solution: Let us consider each pair separately. For one pair, there is $\frac{20}{20} \cdot \frac{15}{19}$ chance of having two students of different grades. The expected number of such pairs in one pair is $0 \cdot \frac{4}{19} + 1 \cdot \frac{15}{19}$. Then, by linearity of expected value, the answer is $10 \cdot \frac{15}{19} = \boxed{\frac{150}{19}}$.

3.3 Hard

(CLRS) We have 5 balls. Suppose that balls are tossed into 4 bins. Each toss is independent, and each ball is equally likely to end up in any bin. What is the expected number of ball tosses before at least one of the bins contains two balls?

Solution: By linearity of expected value, we can consider each ball's expected value of being tossed (in this case, equal to the ball's probability of being used) and then sum up those five values. Without loss of generality, we can order the five balls. (A different ordering would still yield the same sum of expected values.)

There is no way we can use less than 2 balls, so the first and the second ball each have expected value of 1. The third ball is only used if the first two balls landed on different bins, so its expected value is $\frac{3}{4}$. The fourth ball is only used when the previous three landed on different bins, so its expected value is $\frac{3}{4} \cdot \frac{2}{4}$. Similarly, the fifth ball's expected value is $\frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4}$. Now, by summing all 5 values, the answer is $1 + 1 + \frac{3}{4} + \frac{3}{8} + \frac{3}{32} = \boxed{\frac{103}{32}}$.

4 Problems

1. Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. The row $GBBGGGBGBGGGBGBGGBGG$, for example, has $S = 12$. What is the value of $E(S)$ if we can randomly choose any arrangement of 20 people?

2. What is the expected number of ocurrences of the string TJ in a permutation of the string "TJTJTJTJTJTJ"?

3. Brian, Kee Young, Dan, and Greyson are playing tuolaji, which is played with two decks of 54 cards each (the standard 52 plus two jokers). Akshar, who doesn't understand the game at all but is watching anyway, counts the number of cards that show up before each joker (including cards before a previous joker and previous jokers, but not the current joker). What is the expected sum of the four numbers that he will obtain?

4. Olivia knits one sock every four seconds. Every fifth sock has a Mandelbrot pattern and all other socks have distinct colors. After each sock is made, Olivia flips a fair coin and gives the sock to Diana if the coin lands heads. What is the expected amount of time Diana has to wait to get a matching pair of Mandelbrot socks?

5. A coin is flipped 100 times. What is the expected number of runs of 4 or more heads ina row? Count a run of 5 heads (for example) as just one run.

Sources: Sin Kim, Brian Hamrick, Andre Kessler, Adam Hood, AHSME