Geometry 2 – Collinearity and Concurrency

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We say that three lines *concur* if they meet at a point, and that three points are *collinear* if they lie on a line.

1 Some Common Helpful Theorems

1.1 Menelaus' Theorem

Given a triangle ABC and points D, E, and F on lines BC, AC, and AB, respectively, points D, E, and F are collinear if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$.

1.2 Ceva's Theorem

Given a triangle ABC and points D, E, and F on lines BC, AC, and AB, respectively, lines AD, BE, and CF are concurrent if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.

Trig Ceva: Lines AD, BE, and CF are concurrent if and only if $\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin CBE}{\sin EBA} \cdot \frac{\sin ACF}{\sin FCB} = 1$.

1.3 Simson's Theorem

Still remember from the last geo lecture?

Now try proving Ceva's and Menelaus' theorems.

2 Some Less-Common Helpful Theorems

2.1 Pappus' Theorem

Let points A, C, and E be on l_1, and points B, D, and F on l_2. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

2.2 Pascal's Theorem

Let points A, B, C, D, E, and F lie on a circle. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

2.3 Brianchon's Theorem

If a hexagon $ABCDEF$ can be circumscribed about a circle, then AD, BE, and CF are concurrent.

3 Radical Axes

The power of a point P with respect to a circle with center O and radius r is defined to be $d^2 - r^2$ where d is the distance from P to O. The radical axis of two circles is the locus of points that have the same power with respect to both circles. Prove that this axis always happens to be a line. If the two circles we're considering intersect, then what is their radical axis? What if they don't intersect?

3.1 The Radical Axis Theorem

Prove that the three radical axes determined by three distinct circles concur at a point.

4 Important Conclusion

We're teaching a lot of theorems in this lecture, but the important part is figuring out how to use them, not just memorizing them. Also, using some fancy theorem is not nearly the only method to solve a collinearity/concurrency problem; don't forget the basics!

5 Problems

1. Given triangle ABC and its circumcircle O, the tangent to the circle at A intersects line BC at X. Similarly, the tangents to the circle at B and C intersect lines AC and AB at Y and Z, respectively. Prove that X, Y, and Z are collinear.

2. The isogonal conjugate Y of point X with respect to a triangle ABC is the point obtained by reflecting the lines AX, BX, and CX about the angle bisectors of angles A, B, and C, respectively. Prove that these three lines do indeed concur at a point.

3. Circles C_1, C_2, and C_3 lie in a plane. The common external tangents of C_1 and C_2 intersect at X. Similarly, points Y and Z are the intersections of the common external tangents of C_2 and C_3, and C_3 and C_1, respectively. Prove that X, Y, and Z are collinear.

4. Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let G and O denote the centroid and circumcenter of the triangle ABC. Prove that D, O, and G are collinear. (**Hint:** Construct the following points: the midpoint of AB, the midpoint of BC, $AB \cap CD$, and $DA \cap BC$.)

5. Let A, B, C, D be four distinct point on a line, in that order. The circles with dismaters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and then line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent.

6. In quadrilateral $ABCD$, AC bisects $\angle BAD$. A point E is chosen on CD, and BE intersects segment AC at F. DF intersects BC at G. Prove that $\angle GAC = \angle CAE$.

7. Let O be a point inside equilateral traingle ABC. Let P, Q, and R be the intersections of AO with BC, BO with CA, and CO with AB, respectively. Prove that $\frac{PQ}{AR} \cdot \frac{QR}{BP} \cdot \frac{RP}{CQ} \geq 1$.