Counting

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1 The Pigeonhole Principle

Formally, the Pigeonhole Principle states that if we place n balls into k boxes, where $n > k$, then at least one box must contain more than 1 ball.

More generally, if we have n items to be placed in k boxes, then at least one box must contain at least $\lfloor \frac{n-1}{k} \rfloor + 1$ items.

Concept: Whenever we have to show that "a pair" of objects or "at least 2" objects share some property, that is the cue to think about the Pigeonhole Principle.

Examples:

- Given a unit square and 5 points in the square, show that there must exist a pair of these points that are at most $\sqrt{2}/2$ distance apart.
- A group of 15 friends has \$100 among them, and each person has an integer number of dollars. Prove that two of them must have the same amount.

2 Counting with Symmetry: reflection/rotation of arrays

Examples:

- You have 2003 switches, numbered from 1 to 2003, arranged in a circle. Initially, each switch is either ON or OFF, and all configurations of switches are equally likely. You perform the following operation: for each switch S, if the two switches next to S were initially in the same position, then you set S to ON; otherwise, you set S to OFF. What is the probability that all switches will now be ON?
- How many different 4×4 arrays whose entries are all 1's and -1's have the property that the sum of the entries in each row is 0 and the sum of the entries in each column is 0?

3 Fibonacci Numbers

The *Fibonacci numbers* are defined by $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all positive integers $n > 2$. The sequence of numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

A closed-form formula (not very useful in problem-solving):

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Many counting problems have the Fibonacci numbers as their solution, especially those in which we can relate a problem of size n to the same problem of size $n-1$ and $n-2$.

Example:

- For any positive integer n, determine the number of ordered sums of positive integers greater than 1 summing to n. (For example, if $n = 6$, then the sums are $6, 4+2, 2+4, 3+3$ and $2+2+2$.)

4 Problems

1. What is the maximum number of kings that we can place on an 8×8 chessboard, such that no two kings are adjacent (including diagonally)?

2. Prove that for every prime number p except 2 and 5, there is a power of p that ends with digits 0001.

3. A *proportion* is created by filling the four blank spaces of $\_\_ : \_\_ = \_\_ : \_\_$ with numbers. Given four distinct numbers a, b, c, and d, how many ways are there to use all four numbers to fill the blanks? Assuming that at least one of these permutations does produce a correct proportion, what is the probability that a *random* permutation of a, b, c, and d will produce a correct proportion?

4. Simplify the product
$$\prod_{k=2}^{100} \left(\frac{F_k}{F_{k-1}} - \frac{F_k}{F_{k+1}} \right)$$
where $\{F_k\}$ are Fibonacci numbers.

5. Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

6. A large elementary school class goes on a field trip to see a play. The front row of the theater has 11 seats. No boy wants to sit between 2 girls or sit at the end of the row next to a girl, and no girl wants to sit between two boys or sit at the end of the row next to a boy. In how many ways can the row of seats be assigned to boys and girls?