Techniques in Sequences and Series

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1 Introduction

In this lecture I will outline several problem solving techniques in the general topic of sequences and series. I hope that today I will be able to cater to everyone's inner problem solver and I encourage you to work either in groups or alone to solve the provided challenge problems.

2 Techniques

2.1 Telescoping

Example (1): Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \ldots$

Hmph. This is definitely neither a geometric nor an arithmetic series, so how do we even solve this problem? The key to realizing the telescoping nature of this sum is through **partial fraction decompositions**. I quickly run by on the board how we may find that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

Thanks to this fact we have that: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1.$
Clearly, telescoping is quite the powerful technique and through employing clever factorization strategies tandem partial fraction decompositions many series problems may be solved almost immediately.

Example (2): Compute $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{n^4+4} = \frac{4}{1^4+4} + \frac{4(2)}{2^4+4} + \frac{4(3)}{3^4+4} + \frac{4(4)}{4^4+4} + \ldots$

I would like to highlight through this example how clever factorizations may reveal the telescoping nature of a series. Here, we use the **Sophie Germain Identity**: $n^4 + 4 = n^4 + 4n^2 + 4 - 4n^2 = (n+2)^2 - (2n)^2 = (n+2n+2)(n-2n+2)$. We note that $\frac{4n}{n^4+4} = \frac{1}{n^2-2n+2} - \frac{1}{n^2+2n+2}$. Using this fact, we telescope the sum and arrive at $\frac{3}{2}$ for our answer.

2.2 Arithmetico-Geometric

Arithmetico-Geometric series take the form where part of the sum is progressing arithmetically and the other is progressing geometrically. Most solutions incorporate clever manipulation of the original series itself.

Example (3): Compute $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2^n} = \frac{1}{2} - \frac{2}{4} + \frac{3}{8} - \frac{4}{16} + \dots$

The geometric nature of the denominator tempts us to divide by 2 and add back to our original series.

$$S = \tfrac{1}{2} - \tfrac{2}{4} + \tfrac{3}{8} - \tfrac{4}{16} + \tfrac{5}{32} + \dots$$

$$\tfrac{S}{2} = \qquad \tfrac{1}{4} - \tfrac{2}{8} + \tfrac{3}{16} - \tfrac{4}{32} + \dots$$

From this, we have that $\frac{3S}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} = \frac{1}{3}$. Therefore, $S = \frac{2}{9}$. *(Source: TJARML Practice)*

2.3 Combinatorial Sums

I claim a three step plan to evaluate any combinatorial sum:

1. Simplify through Identities
2. Apply Binomial Expansions
3. Work Backwards to the Combinatorics Problem

Identities: The following identities are extremely powerful in discovering the symmetry of a combinatorial sum and overall simplification.

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n$$

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$$

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

Binomial Expansions: Versatility with binomial expansions is crucial in some combinatorial sums. Be comfortable with the expansions of $(1 \pm i)^n$ and the binomial theorem generalized to non-integer powers:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots$$

Lastly, **Working Backwards to the Combinatorics Problem** is a powerful technique in translating from an algebraic sum to combinatorial arguments.

Example (4): Simplify $\displaystyle\sum_{i=1}^{n} i\binom{n}{i} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$

We apply one of the identies: $S = \binom{n}{n-1} + 2\binom{n}{n-2} + \cdots + (n-1)\binom{n}{1} + n\binom{n}{0}$. Adding back to the original sum, we have $2S = n(\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n})$. Therefore, $S = n2^{n-1}$.

2.4 Polynomial Sums

Polynomial sums are encountered quite frequently in the realm of contest math, but have you ever wondered how to arrive at the closed form of $\sum_{i=1}^{n} i^3$ or how to generalize to greater powers? **Discrete Calculus** definitely provides a powerful outlet in doing so, but today I would like to provide a simpler analysis bridging what we learned from Combinatorial Sums.

Example (5): Arrive at the closed form of $\displaystyle\sum_{i=1}^{n} i^3$

We will say that $i^3 = a_1\binom{i}{3} + a_2\binom{i}{2} + a_3\binom{i}{1}$, where a_1, a_2, and a_3 are coefficients we must determine. Through expanding the combinations, we realize that this boils down in the same we did partial fraction decompositions: $i^3 = a_1\left(\frac{i(i-1)(i-2)}{6}\right) + a_2\left(\frac{i(i-1)}{2}\right) + a_3 i$. By cleverly plugging in $i = 0, 1, 2$, we may determine $a_1 = 6$, $a_2 = 6$; $a_3 = 1$. This means that:

$\displaystyle\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n}\left(6\binom{i}{3} + 6\binom{i}{2} + \binom{i}{1}\right) = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} = \left(\frac{n(n+1)}{2}\right)^2$, as we learned from Combinatorial Sums. This powerful approach may be generalized to even further powers of the index.

2.5 Nested Radicals

Evaluation of nested radicals hinges on identifying the recursive nature of the sum at hand.

Example (6): Compute $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \ldots}}}}}$

We will define a function $f(x) = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{1 + \ldots}}}}$. We then have $f(x) = \sqrt{1 + xf(x+1)}$ or $[f(x)]^2 = 1 + xf(x+1)$. The left hand side is of order $2n$, whereas the right hand side is of order $n+1$, so we know that our solution is linear ($n = 1$). By plugging in $f(x) = ax + b$, we discover that the function $f(x) = x + 1$ satisfies this relation. Our sum is f(2), giving us a final answer of 3. *(Source: Ramanujan's Notebooks)*

2.6 Trignometric Sums

Often we must expose the telescoping nature of trignometric sums to get to the heart of the sum. The following are some helpful trignometric identities for telescoping.

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \tfrac{x-y}{1+xy} \qquad \bmod \pi$$

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$$

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin(\alpha)\sin(\beta)$$

Symmetry, like $\cos 180 - x = -\cos x$, is also another powerful tool in evaluating trignometric sums.

Example (7): Compute $\cos(\frac{\pi}{7}) - \cos(\frac{2\pi}{7}) + \cos(\frac{3\pi}{7})$

This is only three terms, but it's already looking nasty! Hoping things will telescope we look to the second identity, $S = \frac{2\sin\pi/7}{2\sin\pi/7}(\cos(\frac{\pi}{7}) - \cos(\frac{2\pi}{7}) + \cos(\frac{3\pi}{7})) = \frac{\sin(2\pi/7)+\sin(\pi/7)-\sin(3\pi/7)+\sin(4\pi/7)-\sin(2\pi/7)}{2\sin\pi/7}$. This simplifies to $\frac{\sin\pi/7}{2\sin\pi/7} = \frac{1}{2}$ and demonstrates telescoping is a powerful technique in evaluating trignometric series.

3 Problems

1. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$
2. In the sequence $\frac{1}{2}, \frac{5}{3}, \frac{11}{8}, \ldots$ the $(n+1)$st term is the sum of the numerator and the denominator of the nth term. The numerator of the $(n+1)$st term is the sum of the denominators of the $(n+1)$st term and the nth term. Find the limit of this sequence *(TJARML)*
3. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{F_n}{3^n}$, where $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. *(Mandelbrot)*
4. Compute the value of the infinite series $\displaystyle\sum_{n=2}^{\infty} \frac{n^4+3n^2+10n+10}{2^n(n^4+4)}$ *(HMMT 2006)*
5. Evaluate $\displaystyle\sum_{n=1}^{1994} \left[(-1)^n \frac{n^2+n+1}{n!}\right]$ *(Canada 1994)*
6. Evaluate $\displaystyle\sum_{n=1}^{2010} \left[(-1)^{n+1}\binom{2011}{2n}\right]$ *(TJML 2011)*
7. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(7n+32)3^n}{n(n+2)4^n}$ *(Mildorf)*
8. Evaluate $1\sin 2^\circ + 2\sin 4^\circ + 3\sin 6^\circ + \cdots + 90\sin 180^\circ$ *(TJML 2011)*
9. Prove that $\displaystyle\sum_{n=0}^{88} \left[\frac{1}{\cos(n)\cos(n+1)}\right] = \frac{\cos 1}{\sin^2 1}$ *(USAMO 1992)*
10. Where $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, prove that a quadrilateral $ABCD$ in the coordinate plane with vertices $A(F_n, F_{n-1})$,$B(F_{n+1}, F_n)$, $C(F_{n+2}, F_{n+1})$, $D(F_{n+3}, F_{n+2})$ always has an area of $\frac{1}{2}$ for all $n \geq 1$. *(BhandarkarA)*

11. Prove that for every positive integer n, and for every real number x not of the form $\frac{k\pi}{2^n}$, where $0 \leq t \leq n$ and k is an integer:

$$\sum_{a=1}^{n} \frac{1}{\sin(2^a x)} = \cot(x) - \cot(2^n x)$$

(IMO 1966)

12. Compute $\displaystyle\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$ *(Putnam 1977)*

13. Compute $\displaystyle\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$ *(HMMT 2008)*

14. Evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \left[\binom{2n}{n} \left(\frac{1}{5}\right)^n \right]$ *(HMMT 2008)*

15. Define the sequence $\{a_n\}$ by $a_0 = 1, a_1 = 1$, and $a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}}$ for $n \geq 2$ and $\{b_n\}$ by $b_0 = 1, b_1 = 3$, and $b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}}$ for $n \geq 2$. Prove that $\frac{b_n}{a_n} = \frac{(n+1)(n+2)}{2}$ for all $n \geq 0$ *(AIME 2008 extension)*