# Geometry Performance Contest - Solutions

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1. **Problem:** Given regular pentagon $ABCDE$, point $F$ is inside the pentagon such that $ABF$ is an equilateral triangle. Find the measure of $\angle FCD$ in degrees.

   **Solution:** Each interior angle of a regular pentagon is $108^\circ$ and $60^\circ$ for an equilateral triangle. Thus, $\angle FBC = 48^\circ$. Because $BF = BC$, $\triangle FBC$ is isosceles, so $\angle BFC = \angle BCF = 66^\circ$. We then have $\angle FCD = 108^\circ - \angle BCF = \boxed{42^\circ}$.

2. **Problem:** In right triangle $ABC$, $\angle B = 90^\circ$, $M$ is the midpoint of $AC$, and points $X$ and $Y$ are on sides $AB$ and $BC$ respectively. Given that $\angle XMY = 90^\circ$, if $BX > AX$, $AX = 6$, and $CY = 9$, find $XY$.

   **Solution:** Construct a line through $C$ parallel to $AB$ such that this line intersects the extension of $XM$ at $X'$. From vertical angles $\angle AMX = \angle CMX'$ and from parallel lines $\angle XAM = \angle X'CM$, and because $M$ is the midpoint of $AC$, $AM = MC$. Therefore, $\triangle AMX$ is congruent to $\triangle CMX'$, so $X'C = AX = 6$ and $XM = MX'$. Because of the latter, $XY = X'Y$. From the Pythagorean Theorem, $X'Y = \sqrt{CX'^2 + CY^2} = \boxed{3\sqrt{13}}$.

3. **Problem:** In $\triangle ABC$, $AB = 13, BC = 14$, and $CA = 15$. Point $D$ lies on side $BC$ such that triangles $ABD$ and $ACD$ have equal inradii of length $6 - 2\sqrt{3}$. Find the length of $AD$.

   **Solution:** The area of $\triangle ABC$ is 84 from Heron's formula (or by noticing that the altitude to $BC$ has length 12). Let $r = 6 - 2\sqrt{3}$. Because $[ABD] = \frac{1}{2}r(AB + BD + DA) = \frac{1}{2}r(13 + BD + DA)$ and $[ACD] = \frac{1}{2}r(AC + CD + DA) = \frac{1}{2}r(15 + CD + DA)$, we have

   $$84 = \frac{1}{2}r(13+BD+DA)+\frac{1}{2}r(15+CD+DA) = \frac{1}{2}r(28+(BD+CD)+2DA) = \frac{1}{2}r(28+14+2DA) = r(21+DA)$$

   so $DA = \frac{84}{r} - 21 = (21 + 7\sqrt{3}) - 21 = \boxed{7\sqrt{3}}$.

4. **Problem:** Let $[ABC]$ denote the area of $\triangle ABC$. Given rectangle $MNPQ$, $X$ and $Y$ are on $PQ$ and $NP$, respectively, such that $[MNY] = 12$ and $[YPX] = [XQM] = 8$. If $MN = x$ where $x = \frac{7\pi\sqrt{3}}{5}$, find $[MXY]$.

   **Solution:** Let $MQ = y$. We have $NY = \frac{24}{x}$ and $QX = \frac{16}{y}$. Therefore, $YP = y - \frac{24}{x}$ and $XP = x - \frac{16}{y}$. We now have the following equation from $\triangle$XPY:

   $$16 = \left(y - \frac{24}{x}\right)\left(x - \frac{16}{y}\right) \Rightarrow xy + \frac{16 \cdot 24}{xy} - 24 - 16 = 16 \Rightarrow (xy)^2 - 56xy + 16 \cdot 24 = 0 \Rightarrow (xy-48)(xy-8) = 0$$

   Since $xy$ represents the area of rectangle $MNPQ$, it must be larger than 8, so we find $xy = 48$. Thus, $[MXY] = xy - [MNY] - [YPX] - [XQM] = 48 - 12 - 8 - 8 = \boxed{20}$.

5. **Problem:** Quadrilateral $ABCD$ is inscribed in a circle such that $AB = BC = 6$, $AD = 4$, and $\angle CDA = 120°$. Find $BD$.

   **Solution:** Because $ABCD$ is inscribed in a circle and $\angle CDA = 120°$, $\angle ABC = 60°$. Because of this and the fact that $\triangle ABC$ is isosceles, $\triangle ABC$ is in fact equilateral so $AC = 6$. Using the law of cosines on $\triangle ACD$, $6^2 = 16 + CD^2 + 4CD \Rightarrow CD^2 + 4CD - 20 = 0$. Solving for $CD$, $CD = 2\sqrt{6} - 2$. From Ptolemy's Theorem, $6BD = 6(2\sqrt{6} - 2) + 6(4)$ so $BD = \boxed{2\sqrt{6} + 2}$.

6. **Problem:** In triangle $PQR$, the circle with diameter $PR$ intersects *segments* $PQ$ and $QR$ at $M$ and $N$ respectively, such that $PM < RN$. If $[PQR] = 4[QMN]$ and it is given that one of the angles in $\triangle PQR$ is 1.4 times another angle, find the maximum possible value of $\angle QPR$ in degrees.

   **Solution:** $\angle QMN = 180° - \angle NMP = \angle PRN$. Therefore, triangles $QMN$ and $QRP$ are similar, so $\frac{QM}{QR} = \frac{QN}{QP}$. (This fact can also be deduced using Power of a Point.) From the fact that $\frac{[QMN]}{[PQR]} = \frac{1}{4}$, we have $\frac{QM*QN}{QP*QR} = \frac{1}{4} \Rightarrow \frac{QN}{QP} = \frac{1}{2}$. Because $PR$ is a diameter, $\angle PNR = \angle PNQ = 90°$. $\angle PQN$ is therefore $\arccos(\frac{1}{2}) = 60°$. Using the fact that one angle is 1.4 times another, there are only three possible cases to check for the remaining angles and we find that the one that yields the greatest value of $\angle QPR$ is when $\angle QPR = \boxed{84°}$, $\angle PQR = 60°$, and $\angle QRP = 36°$.