# Polynomials

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Many algebra problems involve working with polynomials. Some basic things that you should know include the Fundamental Theorem of Algebra, the Remainder Theorem, the Rational Root Theorem, and of course, Vieta's Formulas. The key to solving polynomials is to be able to apply these, along with other various techniques.

## 1 A Few Factorizations to Keep in Mind

$$\begin{aligned}
x^2 + y^2 + z^2 &= (x + y + z)^2 - 2(xy + yz + zx) \\
x^2 + y^2 + z^2 - xy - yz - zx &= \frac{(x - y)^2 + (y - z)^2 + (z - x)^2}{2} \\
(x + y + z)(xy + yz + zx) &= (x + y)(y + z)(z + x) + xyz \\
x^3 + y^3 + z^3 - 3xyz &= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \\
x^3(y - z) + y^3(z - x) + z^3(x - y) &= -(x - y)(y - z)(z - x)(x + y + z) \\
(x^7 + 1) - (x + 1)^7 &= -7x(x + 1)(x^4 + 2x^3 + 3x^2 + 2x + 1)
\end{aligned}$$

## 2 Symmetry

Symmetric sums and symmetric polynomials show up often.
The most common symmetric sums are Vieta's formulas. Given a problem like
Solve the system of equations:

$$\begin{aligned}
x + y + z &= 2 \\
xy + yz + zx &= -19 \\
xyz &= -20
\end{aligned}$$

we can simply solve the monic cubic polynomial with $x$, $y$, and $z$ as its roots.

Symmetric polynomials are defined such that any variable is interchangeable with any other variable. For a symmetric polynomial of two variables, f(x,y) = f(y, x) for all x and y.

Substitions often come in handy for symmetric polynomials of one variable.
For example, to find all real solutions to $x^4 + (2-x)^4 = 34$, a substitution like $y = 1 - x$ would simplify the problem.

## 3 Inverse Roots

Given that $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, we could try

$$P\left(\frac{1}{x}\right) = a_n\left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + ... + a_0 = \frac{a_n + a_{n-1}x + ... + a_0x^n}{x^n}$$

to get the polynomial with inverse roots. But then $P(\frac{1}{x})$ isn't a polynomial, so we would have to multiply it by $x^n$ to get the polynomial with inverse roots:

$$Q(x) = x^n P\left(\frac{1}{x}\right) = a_n + a_{n-1}x + ... + a_0x^n,$$

or just $P(x)$ with the coefficients in reverse order.

## 4 Multiplied Roots

These work in the same way as polynomials with inverse roots, except that the roots of $Q(x)$ are m times the roots of $P(x)$. Then $Q(x) = m^n P\left(\frac{x}{m}\right)$.

## 5 Constructing Polynomials from Roots

Here's a classic example:
If $P(x)$ denotes a polynomial of degree $n$ such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, ..., n$, determine $P(n+1)$. *[USAMO]*

We can't set $Q(x) = P(x) - \frac{x}{x+1}$ since Q(x) isn't a polynomial.

$$P(x) - \frac{x}{x+1} = \frac{(x+1)P(x) - x}{x+1},$$

so we can set $R(x) = (x+1)P(x) - x$ instead. Plugging in $x = -1$ gives $R(-1) = 1$.
$P(x)$ is a polynomial of degree $n$, so $R(x)$ must be of degree $n+1$. Then we can also set

$$R(x) = cx(x-1)(x-2)...(x-n).$$

Then

$$1 = R(-1) = c(-1)(-2)(-3)...(-n-1) = c(-1)^{n+1}(n+1)!,$$

so that

$$c = \frac{1}{(-1)^{n+1}(n+1)!} = \frac{(-1)^{n+1}}{(n+1)!}.$$

Since we want to find $P(n+1)$, we can now write out

$$R(n+1) = \frac{(-1)^{n+1}}{(n+1)!}(n+1)(n)(n-1)...(2)(1) = (-1)^{n+1}.$$

From $R(x) = (x+1)P(x) - x$, $P(x) = \frac{R(x)+x}{x+1}$.

So $P(n+1) = \frac{R(n+1)+n+1}{n+2} = \frac{(-1)^{n+1}+n+1}{n+2} = \begin{cases} \frac{n}{n+2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

# 6 Problems

1. The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $m/n$, where $m$ and $n$ are relatively prime positive integers, find $m+n$. *[Problem 8, 2005 AIME]*

2. Find all ordered triples $(x, y, z)$ that satisfy the system
$$\begin{aligned} \sqrt{x} + \sqrt{y} + \sqrt{z} &= 10 \\ x + y + z &= 38 \\ \sqrt{xy} + \sqrt{yz} + \sqrt{zx} &= 30 \end{aligned}$$

3. Find all solutions to the equation $(z^2 - 3z + 1)^2 - 3(z^2 - 3z + 1) + 1 = z$.

4. Find all real solutions of $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$. *[AHSME]*

5. Find all solutions to the system
$$\begin{aligned} a + b + c &= 1 \\ a^2 + b^2 + c^2 &= 2 \\ a^4 + b^4 + c^4 &= 3 \end{aligned}$$

6. Let $p(x)$ be a degree 2 polynomial such that $p(1) = 1, p(2) = 3$, and $p(3) = 2$. Then $p(p(x)) = x$ has four real solutions. Find the only such solution that is not an integer. *[Mandelbrot]*

7. The polynomial $P(x)$ is cubic. What is the largest value of $k$ for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$? *[Problem 8, 2007 AIME]*

8. If $f(x)$ is a ninth degree polynomial such that $f(n) = \frac{n^2+1}{n}$ for $n = 1, 2, 3, ..., 10$, then find the value of $f(11)$.

9. A polynomial $P(x)$ of degree $n$ satisfies $P(k) = \frac{1}{k}$ for $k = 1, 2, 4, 8, ..., 2^n$. Find $P(0)$.