ARML Lecture XIII.V - Geometry Revisited, Revisited

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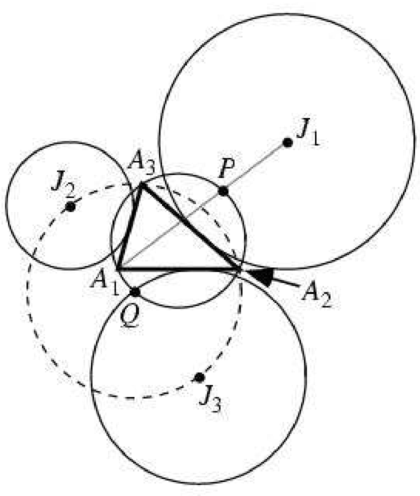
~~May 20, 2004~~ May 27, 2004

"When everything else is the same, the simplest explanation is the best." - *Occam's Razor*

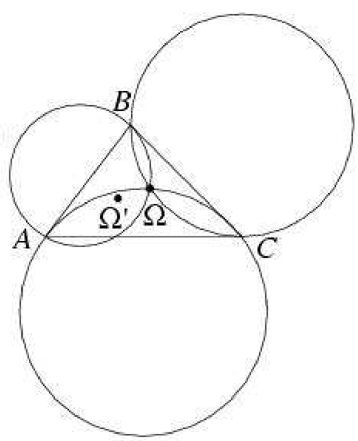
Here we will present a thorough discussion of the lesser known but elegant theorems in Euclidean geometry. Although most problems will not require them, never underestimate the ability to conclude that a strange problem is a direct consequence of the Archimedes Midpoint Theorem.

1 Triangles Revisited

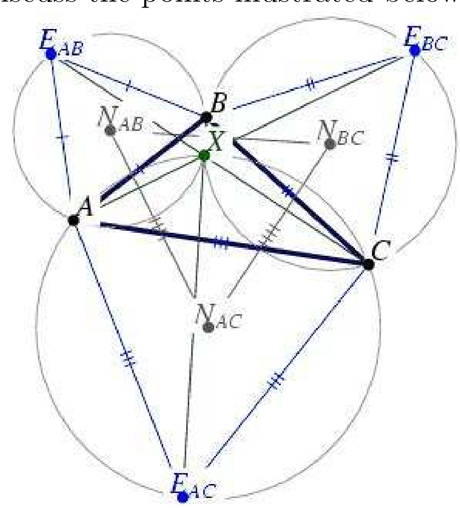
Triangles are the most common shape in contest geometry problems, hence, you should be keenly familiar with a wide range of triangle geometry. You should already know much about the *incenter, circumcenter, orthocenter*, and *centroid* of a triangle (usually denoted I, O, H, and G respectively). You should know that the length of the median $\overline{AM_A}$ satisfies $AM_A^2 = \frac{1}{4} \cdot (2AB^2 + 2AC^2 - BC^2)$. Perhaps you didn't know that. In any case you do now. Euler showed that the incenter, centroid, and circumcenter are collinear, in that order, with $2OG = GH$. He also proved that $OI = \sqrt{R \cdot (R - 2r)}$. There are other special points that you should know a little bit about. We will discuss the points illustrated below.



Excenter



Brocard Point



Fermat Point

- **Excenter** - In $\triangle A_1A_2A_3$, the name of the center of the circle that is tangent to $\overline{A_2A_3}$ and the extensions of $\overline{A_1A_2}$ and $\overline{A_1A_3}$ beyond A_2 and A_3 respectively. The radius of this circle is denoted r_1. There are three excenters, the other two are defined analogously. Explicitly, $r_1 = \frac{K}{s-a}$. If we let J_1 denote the excenter opposite A, we have $m\angle IA_2J_1 = m\angle J_1A_3I = \frac{\pi}{2}$, a result that follows easily by an angle bisector argument. This identity establishes that $\overline{IJ_1}$ is a diameter of the circumcircle of quadrilateral $IA_2J_1A_3$.

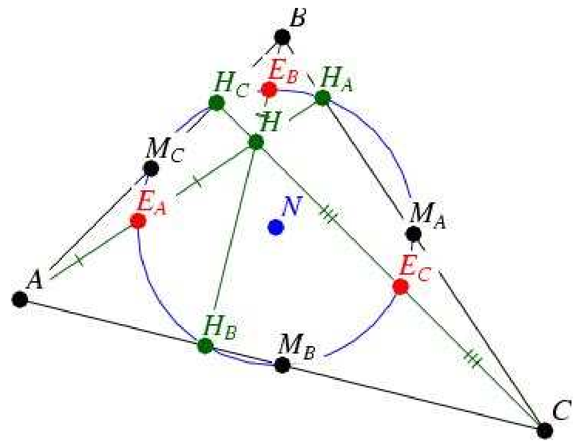
- **Brocard Point** - This is the name given to a point Ω inside triangle ABC that satisfies $\angle\Omega AB \cong \angle\Omega BC \cong \angle\Omega CA$. (There are two distinct Brocard points for any non-equilateral triangle, where the other Ω' satisfies $\angle\Omega' BA \cong \angle\Omega' AC \cong \angle\Omega' CB$.) Let $\theta = m\angle\Omega AB$. Then a symmetric argument can verify that $\cot\theta = \frac{a^2+b^2+c^2}{4K}$. Another way of thinking about a Brocard point is the intersection of the circles ω_1, ω_2, ω_3 where ω_1 is tangent to $\overline{BC}$ at B and contains A, ω_2 is tangent to $\overline{CA}$ at C and contains B, and ω_3 is tangent to $\overline{AB}$ at A and contains C.

- **Fermat Point** - This is the unique point (often denoted F) in space that corresponds to triangle ABC so that $\angle CFB \cong \angle AFC \cong \angle BFA$. Obviously, all three angles have a measure of $\frac{2\pi}{3} = 120^o$. Let A' be the point on the opposite side of $\overline{BC}$ from A such that triangle $A'BC$ is equilateral, and define B' and C' analogously. Then $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ concur at F. Furthermore, the quadrilaterals $FBA'C$, $FAC'B$, and $FCB'A$ are all cyclic. A symmetric Ptolemy argument yields the formula $FA = \frac{FB'+FC'-FA'}{2}$. The Fermat point is the unique point X that minimizes $AX + BX + CX$ in an acute triangle.

As if that weren't enough, there are many elegant identities that enable more thorough algebraic manipulation of triangular phenomena. A few are listed below. For any triangle ABC:

1. $\tan(A) + \tan(B) + \tan(C) = \tan(A)\tan(B)\tan(C)$.
2. $\cos^2(A) + \cos^2(B) + \cos^2(C) + 2\cos(A)\cos(B)\cos(C) = 1$.
3. $\sin(2A) + \sin(2B) + \sin(2C) = 4\sin(A)\sin(B)\sin(C)$.
4. $AH = 2R\cos(A)$.
5. $1 + \frac{r}{R} = \cos(A) + \cos(B) + \cos(C)$.
6. $4R + r = r_1 + r_2 + r_3$.
7. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.
8. $a \cdot IA^2 + b \cdot IB^2 + c \cdot IC^2 = abc$.

Often we will be interested in computing ratios between side lengths. There are two theorems related to this, but the conglomeration of the two is neatly summarized by a general result known as **Mass Points**. Let ABC be a triangle and points D, E, and F points on lines BC, AC, and AB respectively, such that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at P. Consider putting weights ω_1, ω_2, and ω_3 on A, B, and C respectively. If any two of the following are true, then all six are true:[1]



- $\omega_1 \cdot AE = \omega_3 \cdot EC$
- $\omega_3 \cdot CD = \omega_2 \cdot DB$
- $\omega_2 \cdot BF = \omega_1 \cdot FA$
- $\omega_1 \cdot AP = (\omega_2 + \omega_3) \cdot PD$
- $\omega_2 \cdot BP = (\omega_3 + \omega_1) \cdot PE$
- $\omega_3 \cdot CP = (\omega_1 + \omega_2) \cdot PF$

A logical bridge for the step from triangles to circles is a final theorem that summarizes the geometry of the **nine-point circle** also called *Euler's circle* or the *Feuerbach circle*. Let ABC be a triangle. Let M_A be the midpoint of $\overline{BC}$, and define M_B and M_C analogously. Let H_A be the foot of the altitude from A, and define H_B and H_C analogously. Where H denotes the orthocenter, define E_A to be the midpoint of $\overline{HA}$, and define E_B and E_C similarly. The points $M_A, M_B, M_C, H_A, H_B, H_C, E_A, E_B$ and E_C all lie on a common circle called the nine-point circle. $\overline{M_A E_A}$ is a diameter of the circle, and all angles of the form $\angle M_A X E_A$ where X is a point on the nine-point circle are right angles. The center of the nine-point circle is usually denoted N, and its radius R_N is given by $R_N = \frac{R}{2}$. An interesting result is that the nine-point circle bisects any segment $\overline{HX}$, where X is a point on the circumcircle of ABC, not merely when X is A, B, or C.

2 Everything Else about Circles

Next to triangles, circles are the most common finding in geometry problems. You should already be familiar with Power of a Point, Ptolemy, and Brahmagupta, in addition to the various angle chasing theorems. We shall now discuss some of the lesser known properties of circles.

- **Eight-point circle** - In any quadrilateral $ABCD$, the midpoints M, O, S, and P of sides $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$ respectively, form a parallelogram.[2] If quadrilateral

[1]Under the constraint of directed distances; i.e. there will be negative weights involved if any of D, E, and F lie outside of $\triangle ABC$. The related theorems are **Ceva**, which says that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ concur iff $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, and **Menelaus**, which states that D, E, and F are collinear iff $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$.

[2]If $ABCD$ is convex, then parallelogram $MOSP$ has $\frac{1}{2}$ of the area of $ABCD$. (Why?) Another potentially useful fact is that for any parallelogram $ABCD$, $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$.

$ABCD$ is convex and has perpendicular diagonals, then $MOSP$ is a rectangle. Let M' be the foot of the altitude from M to $\overline{CD}$, and define O', S', and P' analogously. Then M, O, S, P, M', O', S', and P' lie on a common circle in which $\overline{MS}$ and $\overline{OP}$ are diameters.

- **Descartes Circle Theorem** - Suppose that circles A, B, C, and D are externally tangent and have radii r_A, r_B, r_C, and r_D respectively.[3] Define $\kappa_A = \pm\frac{1}{r_A}$, and define κ_B, κ_C, and κ_D analogously. If circle X contains the others, then κ_X should have the opposite sign from the others. Then

$$2 \cdot (\kappa_A^2 + \kappa_B^2 + \kappa_C^2 + \kappa_D^2) = (\kappa_A + \kappa_B + \kappa_C + \kappa_D)^2$$

  This was extended to spheres by Soddy and later to n-space, where there are $n+2$ tangent hyperspheres that obey

$$n \cdot \left(\sum_{i=1}^{n+2} \kappa_i^2\right) = \left(\sum_{i=1}^{n+2} \kappa_i\right)^2$$

- **Archimedes Midpoint Theorem** - Let A and B be points on circle ω, and let C be the midpoint of major arc ACB. If D is any point on minor arc BC and E is the foot of the altitude from C to $\overline{AD}$, then $AE = ED + DB$.

- **Fuhrmann's Theorem** - Let $ABCDEF$ be a convex cyclic hexagon. Then

$$AD{\cdot}BE{\cdot}CF = AB{\cdot}FC{\cdot}ED + BC{\cdot}AD{\cdot}FE + CD{\cdot}BE{\cdot}AF + AB{\cdot}CD{\cdot}EF + BC{\cdot}DE{\cdot}FA$$

- **Apollonius Circle** - Let A and B be two points separated by a distance d. Let ω be the locus of all points P such that $\frac{AP}{BP} = r$. Then ω is a circle of radius $R_\omega = \frac{rd}{1-r^2}$. Moreover, if $r < 1$, then ω is centered at O on the extension of $\overline{AB}$ past A such that $OA = \frac{r^2 d}{1-r^2}$.

- **The N-gon**[4] - Let $A_1A_2\ldots A_n$ be a regular n-sided polygon with a circumradius of R. Let X be any point on the circumcircle of $A_1\ldots A_n$. Then

$$\begin{aligned}\sum_{i=1}^{n} XA_i^2 &= 2R^2 n \\ \prod_{i=2}^{n} A_1A_i &= nR^{n-1}\end{aligned}$$

[3] One "circle" may actually be a line of radius ∞.

[4] There are closely related trig identities. The sine product: for any positive integer n, $\prod_{i=1}^{n-1} \sin(\frac{i\pi}{n}) = \frac{n}{2^{n-1}}$; and the cosine product: for any *odd* positive integer n, $\left|\prod_{i=1}^{n-1} \cos(\frac{i\pi}{n})\right| = \frac{1}{2^n}$. (Where simple parity analysis can determine the sign. What is the cosine product for even n?)

In addition to these, there are other conceptual tricks that may be of use when solving circle geometry. One such example is "dual normality." If there are 2 circles ω_1 and ω_2 centered at O_1 and O_2 respectively such that they have a common point of tangency T, then O_1, O_2, and T are collinear. This generalizes to n circles where T is a common tangent to all circles. In this case, $O_1, \ldots, O_n$, and T are collinear.

3 The Next Dimension

3-space is perhaps the most neglected topic in geometry. Most know only the volume and surface area of spheres and rectangular parallelpipeds (a.k.a. boxes).[5] Here we shall present other formulae, all of which can save a lot of work. First, we consider the 3-D analog of an equilateral triangle.

- **Regular Tetrahedron** - Suppose that T is a regular tetrahedron of edge length s. It has a spatial-altitude of length $h = s\frac{\sqrt{6}}{3}$. Given this expression for h, it is then easy to verify that the volume of T is given by $V = \frac{s^3\sqrt{2}}{12}$ and its surface area is given by $A = s^2\sqrt{3}$.

For the first time,[6] coordinates actually yield elegant results. First we revisit two vector operators, the *dot product* and the *cross product*. Suppose $u = \ < u_1, u_2, u_3 >$ and $v = \ < v_1, v_2, v_3 >$. Let $|u|$ denote the magnitude of u, given by $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$, and suppose that θ is the angle between u and v. Then:

$$\begin{aligned} u \cdot v &= u_1v_1 + u_2v_2 + u_3v_3 = |u||v|\cos(\theta) \\ u \times v &= \ < u_2v_2 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 > \ = |u||v|\sin(\theta) \end{aligned}$$

With these operators, coordinate geometry becomes an elegant approach in three-space. Here are some note-worthy applications:

- *Computing area of a sloped surface* - Consider triangle ABC with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$. Let $u = \ < x_2 - x_1, y_2 - y_1, z_2 - z_1 >$ and $v = \ < x_3 - x_1, y_3 - y_1, z_3 - z_1 >$. Then the area of $\triangle ABC$ is given by:

$$A = \frac{1}{2}|u \times v|$$

[5] A sphere of radius R has volume $\frac{4}{3}\pi R^3$ and surface area $4\pi R^2$. A box of dimensions l, w, and d has volume lwd and surface area $2(lw + ld + wd)$. A cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$ and surface area $\pi r(r + \sqrt{r^2 + h^2})$.

[6] Not really. The best proof of the **British Flag Theorem** uses coordinates. It states that if $ABCD$ is a rectangle and P is any point in space, then $AP^2 + CP^2 = BP^2 + DP^2$.

- *Computing volume*[7] - Consider a tetrahedron $ABCD$ with vertices A, B, and C defined as above, and $D(x_4, y_4, x_4)$. Let u and v be defined as above, and define $w = < x_4 - x_1, y_4 - y_1, z_4 - z_1 >$. Then the volume of $ABCD$ is given by

$$V = \frac{1}{6}(u \times v) \cdot w$$

Related is the volume of the parallelpiped determined by u, v, and w, which is given by $(u \times v) \cdot w$.

Other important 3-space coordinate-geometry ideas include projections and distance between lines. These can be solved with simple trigonometry and vector operations. But what about the special points, the circumcenter, incenter, orthocenter, and centroid? How do they change when moved into three space? We shall provide a cursory analysis of these points in a general tetrahedron.

- **Circumcenter** - A general tetrahedron has a circumcenter. It can be found be intersecting the normals of each face that pass through the circumcenter of each face.

- **Incenter** - The incenter does exist for a general tetrahedron, but it is rather difficult to compute the location of I. However, it is easily shown that the inradius r is given by $r = \frac{3V}{A}$, where A is the surface area of the tetrahedron and V is the volume.

- **Orthocenter** - The orthocenter does *not* exist for a general tetrahedron.

- **Centroid** - The centroid of a general tetrahderon does exist. If three linearly independent vectors u, v, and w emenate from the same point A to vertices B, C, and D, then the centroid of tetrahedron $ABCD$ is given by a shift of $\frac{1}{4}(u + v + w)$ from A. Let G_A denote the centroid of $\triangle BCD$. Then A, G, and G_A are collinear with $AG : GG_A = 3 : 1$.

We end our excursion to three-space with an elegant theorem. Let F denotes the number of faces, V the vertices, and E the number of edges for some polyhedron,[8] **Euler's Theorem** provides the relationship

$$F + V = E + 2$$

[7]The **Theorem of Pappus** is an elegant theorem that yields the volume of a solid formed by rotation. If G is the centroid of some planar region S and l is some line that does not pass through S, the volume of the solid formed by rotating S about l is $V = 2\pi r A$, where r is the distance from G to l, and A is the area of S.

[8]Some common polyhedra have (F,V,E) as follows: Parallelpiped(6,8,12), Tetrahedron(4,4,6), Octahedron(8,6,12), Dodecahedron(12,20,30), Icosahedron(20,12,30).

4 Pulling the Rabbit out of the Hat

Now that you have a thorough knowledge of contest-type theorems in geometry, you should be able to solve most any problem. Occasionally, however, one stumbles upon a problem that simply doesn't crumble under any such theorems. This is when it becomes important to have a good sense of geometric constructions. Before starting algebraic work on any problem, scan for qualitative facts. Things to look of for include but are not limited to:

- **Congruency** - It is often not inherently obvious based on numbers that two shapes are congruent.
- **Nice Points** - See if adding another point to a digram simplifies things nicely. Can you obtain a lot of isosceles triangles by drawing in a point?
- **Angles** - These often lead directly to circles, which are good in the sense that they imply many other facts. If one angle is twice another, there is a chance that the larger angle is at the center of a circle that contains the other.
- **Exchanging Sides** - A useful trick that shows up in circle geometry. For an inscribed polygon, any permutation of its sides forms another polygon that fits in a circle of the same radius.
- **Extra Freedom** - Consider degrees of freedom, is there any reason why one cannot assert that an angle is right or has some other nice measure? Can we assert that those lines are parallel? If so, then **Crossed Ladders Theorem**[9] and **Stengel's Theorem**[10] might apply.
- **Affine Transformation** - If all of the given information consists of areas and / or ratios of the lengths of collinear segments, then one may consider *stretching* space to get additional information such as a "bonus" right angle.

5 Practice

All of the following problems can be solved with important methodology in geometry.

1. $ABCD$ is a convex quadrilateral. M, O, S, and P are the midpoints of $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$ respectively. If $PO^2 = 32$, $OM^2 = 25$, and $MS^2 = 36$, then what is the area of $ABCD$?

[9] Let A, B, C, and D be points such that $\overline{AB} \parallel \overline{CD}$ with B and D on the same side of the line through A and C. Let E denote the intersection of $\overline{AD}$ and $\overline{BC}$. Let F be the unique point on $\overline{AC}$ such that $\overline{EF} \parallel \overline{AB}$. Then $\frac{1}{AB} + \frac{1}{CD} = \frac{1}{EF}$.

[10] Let ABC be a triangle, and D and E points on $\overline{BC}$ and $\overline{AB}$ respectively. Let F denote the intersection of $\overline{AD}$ and $\overline{CE}$. If G, H, I, and J are points on $\overline{AC}$ such that $\overline{BG} \parallel \overline{DH} \parallel \overline{EI} \parallel \overline{FJ}$, then $\frac{1}{EI} + \frac{1}{DH} = \frac{1}{FJ} + \frac{1}{BG}$.

2. Points A, B, C, and D lie on a circle such that B is the midpoint on major arc ABC, and D lies on minor arc AB. Let E be the foot of the altitude from B to $\overline{CD}$. If $AB = 13$, $BD = 9$, and $BE = 5$, then determine the exact length of AB.

3. (HMMT 2003, Geometry #10) In convex quadrilateral $ABCD$, $\frac{AD}{AC} = \frac{3}{4}$, and $\angle BCA \cong \angle ABC \cong \angle BDA$. M is the midpoint of $\overline{AB}$, and N is a point on $\overline{CD}$ such that lines AC, BD, and MN are concurrent. Compute $\frac{DN}{NC}$ if $m\angle BDA = 60^o$.

4. What is the volume of the tetrahedron $ABCD$ with vertices at $A(0, 0, 0)$, $B(1, 1, 2)$, $C(2, 0, 1)$, and $D(3, 3, 0)$?

5. In quadrilateral $ABCD$, $AB = BC = 3$ and $CD = 7$. What is the greatest area that $ABCD$ could have?

6. A and B are points on the positive x and y axes respectively. C is a point in the interior of the triangle formed by A, B, and the origin. Beams of light are shined from A and B through C, and intersect the y and x axes at D and E respectively. The distances from C, D, and E to $\overline{AB}$ are 8, 10, and 12 respectively. Determine the distance from $\overline{AB}$ to the origin.

7. ω is a minor circular arc of curvature[11] $\kappa = \frac{1}{25}$ that connects points A and B. C is a point on ω. When reflected over $\overline{AB}$, ω maps to ω' and C maps to C' on ω'. Ω is the circle tangent to all three sides of the smaller region bounded by ω, ω', and $\overline{CC'}$. If $CC' = 26$ and $AB = 48$, then determine the radius of Ω.

8. $ABCDE$ is a cyclic pentagon such that $AB = BC = CD = 2$ and $DE = EA = 3$. Determine the length of $\overline{AD}$.

9. $\triangle ABC$ is an isosceles triangle in which $AC = BC$ and $m\angle C = 20^o$. D is on $\overline{BC}$ such that $AD = CD$, and E is on $\overline{AC}$ such that $AB = AE$. Determine the degree measure of $\angle ADE$.

10. Points M, O, S, and P lie on circle ω, in that order, such that $\overline{OP}$ bisects $\overline{MS}$. Let l_1 be the tangent to ω at O, and l_2 the tangent at P. The line through M and S intersects l_1 and l_2 at A and B respectively. If $AM = 12$ and $MS = 155$, then determine the value of PB^2.

6 Help!

A mere list of solutions does not teach much intuition, which is the ultimate goal behind practice. Therefore, only hints are provided here.

[11] A measure of how quickly a curve changes its direction as it is traversed; explicitly defined point-wise by $\kappa := \frac{1}{R}$, where R is the radius of the circle that exhibits the same bend at a point. In this case, the radius of the circle that contains ω is 25.

1. Recall that the midpoints of the sides of any quadrilateral form a parallelogram.

2. Notice that all of the values except one can be obtained quickly and that there is a theorem in this lecture that obtains the desired length.

3. Verify that $ABCD$ is cyclic. A number of approaches can be used. Extend some rays and apply ratio theorems from this lecture or try similar triangles.

4. Use some vector identities, this problem is straightforward.

5. Argue that $\overline{AD}$ is a diameter of the circumcircle of $ABCD$. Then exchange BC and CD to make it easier to calculate the length of this diameter and the area.

6. The distances from $C, D, E,$ and O to $\overline{AB}$ are determined by the lengths of the perpendiculars from them to $\overline{AB}$. These perpendiculars are all parallel.

7. Draw the remainder of ω and then use the principle of dual normality and Pythagoras to obtain an expression for R_Ω.

8. This arrangement of sides really isn't that useful. Wouldn't we rather have $AB = BC = DE = 2$ and $CD = EA = 3$ to get at least one of those diagonals?

9. Draw point F on $\overline{AC}$ so that $\overline{DF} \parallel \overline{AB}$ to supplement the angle chasing.

10. Consider the center of ω and the angles it forms with various combinations of given points.