Trigonometry

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A lot of trig problems only require manipulation of different equations and properties. Once you know these, you'll be able to change many trig problems into relatively simple algebraic problems.

1 Basic Trig Definitions

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \quad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
$\sin\left(-\theta\right)=-\sin\theta \quad \cos\left(-\theta\right)=\cos\theta$
$\sin\left(\pi-\theta\right)=\sin\theta \quad \cos\left(\pi-\theta\right)=-\cos\theta$
$\sin^2\theta+\cos^2\theta=1$
From the Pythagorean theorem, we can prove these three neat results:
$\sin^2(A)+\cos^2(A)=1, \qquad 1+\tan^2(A)=\sec^2(A), \qquad 1+\cot^2(A)=\csc^2(A)$

2 Basic Trig Laws

There are two basic equations that you should know and be ready to utilize in many trig problems: the Law of Sines and Law of Cosines

2.1 Law of Sines (Extended)

$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}=2R$, where R is the circmuradius of $\triangle ABC$.

2.2 Law of Cosines

$c^2=a^2+b^2-2ab\cos\theta$, where θ is the angle opposite the side with length c.

3 Other Properties/Formulae

These also appear often in trig problems, so it's worth it to memorize them (or know how to derive them).

For angles larger than 90°, we start from the new point C and draw a perpendicular to the x-axis. Knowing this, we can see why these properties are true:

$$\sin(A) = \sin(180° - A) = -\sin(-A) = \sin(360° + A) = \cos(90° - A)$$

$$\cos(A) = -\cos(180° - A) = \cos(-A) = \cos(360° + A) = \sin(90° - A)$$

$$\tan(A) = -\tan(-A) = \tan(180° + A)$$

Angle sum formulas are also useful. From these, we can also get double angle formulas:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(A)$$

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$$

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

We can also get the half-angle formulas:
$\sin(A/2) = \pm\sqrt{\frac{1-\cos(A)}{2}}$, $\quad\cos(A/2) = \pm\sqrt{\frac{1+\cos(A)}{2}}$, $\quad\tan(A/2) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}} = \pm\frac{\sin A}{1+\cos A}$
Product-Sum formulas sometimes come in handy.
$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$, $\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$, $\sin(\alpha)\cos(\beta) = \frac{\sin(\beta+\alpha)-\sin(\beta-\alpha)}{2}$

4 Practice Problems

1. [1995 AIME #7] Given that $(1 + \sin t)(1 + \cos t) = \frac{5}{4}$, compute $(1 - \sin t)(1 - \cos t)$.

2. If $\cos x + \sin x = 0.2$, compute $\cos^4 x + \sin^4 x$.

3. Compute $\sin 18°$.

4. [2000 AIME II #15] Find the least positive integer n such that $\frac{1}{\sin 45° \sin 46°} + \frac{1}{\sin 47° \sin 48°} + \cdots + \frac{1}{\sin 133° \sin 134°} = \frac{1}{\sin n°}$.

5. [2006 AIME I #12] Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

6. [2003 AIME I #10] Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106°$. Point M is in the interior of the triangle so that $\angle MAC = 7°$ and $\angle MCA = 23°$. Find the number of degrees in $\angle CMB$.