TJAIME: Triangle Geometry I

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1 Area

It is hard to imagine a world of geometry without triangles. One of the most important things we'll deduce from triangles is area. Here are some common ways to find a triangle.
If K is the area of triangle ABC with side lengths a, b, and c,

- $K = \frac{1}{2}ah_a$, where h_a is the height to side a. You should already know this one...
- $K = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$, the semiperimeter. This is *Heron's Formula.*
- $K = \frac{1}{2}ab\sin(\angle C)$.

Two other special area formulas result from use of the *inradius* and *circumradius*.

- $K = rs$, where r is the inradius and s is the semiperimeter.
- $K = \frac{abc}{4R}$, where R is the circumradius.

Notes/Diagram:

Challenge: Can you prove why $K = rs$ is true?

2 Special Points

Oftentimes, geometry problems with triangles will involve special points in a triangle. These are:

- **Centroid:** The intersection of the *medians* of the triangle. The distance from the centroid to a vertex is always twice its distance to the opposite side.
- **Incenter:** The intersection of the triangle's three *angle bisectors*; this point is the center of the triangle's incircle.

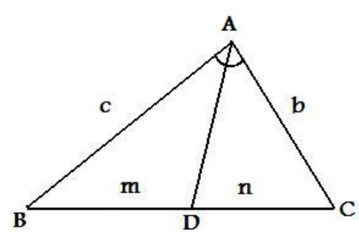
- **Circumcenter:** The intersection of the *perpendicular bisectors* of each side; the center of the circumcircle is located here.
- **Orthocenter:** The intersection of the triangle's altitudes.

Notes/Diagrams:

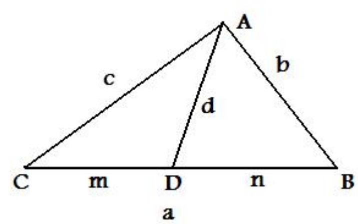
Challenge: Can you prove why the intersection of a triangle's angle bisectors results in the incenter? Why does the intersection of a triangle's perpendicular bisectors result in the circumcenter?

3 Useful Theorems

- **Angle Bisector Theorem:** Given $\triangle ABC$ and angle bisector AD, where D is on side BC, then $\frac{c}{m} = \frac{b}{n}$.



- **Stewart's Theorem:** Given triangle $\triangle ABC$ with sides of length a, b, c opposite vertices A, B, C, respectively, if *cevian* AD is drawn so that $BD = m, DC = n$ and $AD = d$, we have that $bmb + cnc = man + dad$. "A man and his dad put a bomb in the sink."

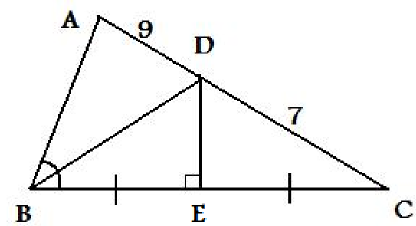


4 Example Problem

2002 AMC 12 Problem # 23:
In triangle ABC, side AC and the perpendicular bisector of BC meet in point D, and BD bisects

$\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle ABD?

Solution:
The first step in any geometry problem is to DRAW A DIAGRAM:



To find the area of $\triangle ABD$, we'd be best off finding the three side lengths and applying Heron's Formula. We already know that $AD = 9$. Let point E be the intersection of perpendicular bisector DK with side BC. By SAS, we have $\triangle BDE \cong \triangle CDE$, giving $BD = 7$. We last seek the length of AB. By the Angle Bisector Theorem, $\frac{AB}{9} = \frac{BC}{7}$. By substituting into Stewart's Theorem, $7AB^2 + 9BC^2 = 49 * 16 + 63 * 16$. We solve to get $AB = 12$. The semiperimeter of $\triangle ABD$ equals $\frac{9+7+12}{2} = 7$, so its area can now be found using Heron's Formula, and equals $\sqrt{14(14-9)(14-7)(14-12)} = \boxed{14\sqrt{5}}$.
Note: The above problem could have been solved in a variety of ways. Here we only showed one way to solve it.

5 Problems

1. Triangle $\triangle ABC$ has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

2. In $\triangle ABC$, $\cos(2A - B) + \sin(A + B) = 2$ and $AB = 4$. What is BC?

3. Triangle ABC has side-lengths $AB = 12$, $BC = 24$, and $AC = 18$. The line through the incenter of $\triangle ABC$ parallel to $\overline{BC}$ intersects $\overline{AB}$ at M and $\overline{AC}$ at N. What is the perimeter of $\triangle AMN$?

4. Triangle ABC has $\angle BAC = 60°$, $\angle CBA \leq 90°$, $BC = 1$, and $AC \geq AB$. Let H, I, and O be the orthocenter, incenter, and circumcenter of $\angle ABC$, respectively. Assume that the area of the pentagon $BCOIH$ is the maximum possible. What is $\angle CBA$?

5. Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and CL is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, line LP.

6. Triangle ABC has right angle at B, and contains a point P for which $PA = 10, PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.

Problem Sources: AHSME, AMC12, AIME
Lecture material adapted from ARML 2002 Triangle Geometry Lecture