TJUSAMO - Graph Theory

14VXia

February 11th, 2013

1 Review from Last Week

1. Let O be a point inside equilateral traingle ABC. Let P, Q, and R be the intersections of AO with BC, BO with CA, and CO with AB, respectively. Prove that $\frac{PQ}{AR} \cdot \frac{QR}{BP} \cdot \frac{RP}{CQ} \geq 1$.

2. Let P be a point in the interior of a triangle ABC, and let D, E, F be the point of intersection of the line AP and the side BC of the triangle, of the line BP and the side CA, and of the line CP and the side AB, respectively. Prove that the area of triangle ABC must be 6 if the area of each of the triangles PFA, PDB, and PEC is 1.

2 Graph Theory Definitions

- A *simple* graph is one in which each edge connects a distinct pair of vertices.
- connected, degree $d(v)$, tree
- A graph is *bipartite* if its vertices can be split into two disjoint subsets such that edges run only between vertices in different subsets.
- The complete graph with n vertices, K_n, is the simple graph that has all possible edges.
- paths, cycles, walks, circuits
- Hamiltonian path/cycle, Eulerian circuit
- A graph is *planar* if it can be drawn in a plane such that none of its edges intersect (except at vertices). Planar graphs have faces. (Euler Characteristic: $V - E + F = 2$.)

3 Problems

1. Show that every graph has at least two vertices with equal degree.

2. Every connected graph with all degrees even has an Eulerian circuit.

3. Prove that every finite, simple, planar graph has an orientation such that every vertex has outdegree at most 3.

4. In a tournament with n players ($n \geq 4$), each player plays against every other player once, and there are no draws. Suppose that there are no four players (A, B, C, D) such that A beats B, B beats C, C beats D and D beats A. Determine, as a function of n, the largest possible number of unordered triples of players (A, B, C) such that A beats B, B beats C and C beats A.

5. (Dirac's Theorem on Hamiltonian cycles) Let G be a graph on n vertices with all degrees at least $n/2$. Show that G has a Hamiltonian cycle.

6. Find the smallest positive integer n such that for any n points $(A_1, A_2, ...A_n)$ chosen on a circle with center O, out of the nC2 angles of the form $\angle A_i O A_j$, at least 1000 of these angles have measures no greater than $120°$.

7. There are 8 persons at a party. Suppose that for any 5 persons, there are 3 of them who know each other. Show that there are 4 persons who know each other.