ARML Lecture V - The Geometry of Circles

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1 Angles

A central angle of a circle is an angle centered at the circle's center. If the angle's measure is θ, then we know by definition that the angle intercepts θ radians (out of 2π) of the circle's circumference. Likewise, an inscribed angle (one whose vertex is on the circle) of measure θ intercepts 2θ radians.

What if that angle has a vertex someplace else? If the vertex is outside the circle, and the angle subtends an outer arc of measure α and an inner arc of measure β, then the measure $\theta = \frac{\alpha-\beta}{2}$.

If the angle of measure θ has a vertex inside the circle, then it intercepts two arcs of (possibly the same) measure of α and β.

2 Power of a Point

The Power of a Point Theorem, which relates lengths formed by the intersections of lines and circles, is important enough to have a section of its own.

Specifically, if the point P is not on the circle, and point A is selected on the circle, we let B be the second intersection of line AP with the circle. The theorem says that the product AP * BP is constant as we move A. (The degenerate external case is the square of the length from P to A, since A and B are the same point.)

3 Cyclic Quadrilaterals

Inscribing geometric figures in circles often simplifies problems or creates interesting ones. Aside from triangles, the most common shapes found inscribed in circles are quadrilaterals, and these are called *cyclic quadrilaterals*.

Recalling that all triangles can be inscribed in a circle, we may be led astray into thinking that all quadrilaterals can be inscribed in circles. This however, is not true. Fortunately, there is a simple way to tell if a quadrilateral is special enough to be a cyclic quadrilateral:

Let ABCD be a quadrilateral. Then ABCD is cyclic iff angles A and C are supplementary (A + C = 180^o) or, equivalently, $\angle ABD \cong \angle ACD$.

The proof of this is simple, and again, relies on inscribed angles. Details are left to the reader, although there aren't many details. In fact, we could choose either pair of angles to add together.

An interesting result that is attributed to Ptolemy is often useful with cyclic quadrilaterals:

Ptolemy's Theorem Let cyclic quadrilateral ABCD have sides of (in rotational order) length a, b, c, and d, and diagonals of length e and f, then $ac + bd = ef$.

Another interesting theorem relates areas to cyclic-quads states:

Brahmagupta's Theorem Let a cyclic quadrilateral have sides of length a, b, c, and d. Let its area be K, and let s be the semiperimeter, so $s = \frac{a+b+c+d}{2}$. Then $K^2 = (s-a)(s-b)(s-c)(s-d)$ or equivalently, $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. This may resemble Heron's formula, and in fact, collapsing a side down to 0 yields an inscribed triangle and Heron's formula.

Problems involving cyclic quadrilaterals almost always involve similar triangles and power of a point; be on the lookout for these as you work with circles.

Finally, some problems exclusively comprised of circles. When this happens, you'll probably be happy to know the D.C.T.:

Descartes Circle Theorem Four circles are mutually tangent. Let their radii be r_1, r_2, r_3, and r_4. Let $s_1 = \pm r_1^{-1}$, and define s_2, s_3, s_4 similarly. If the circles are all externally tangent, all of the s_i should have the same sign. If three of the circles are contained in a larger circle, the s_k corresponding to the largest circle should have a different sign from the others. Finally, $2(s_1^2 + s_2^2 + s_3^2 + s_4^2) = (s_1 + s_2 + s_3 + s_4)^2$

4 Examples

After learning the tools, we are left to contemplate how to use them. Here are several fully solved problems:

"In cyclic quadrilateral ABCD, AB = 10, CD = 7, AD = 14. The diagonals AC and BD intersect at P, and are divided such that AP = 6, PC = 8, and PB = 4. Find the area of

ABCD."

By Power of a Point from P, AP * PC = BP * PD, 48 = 4PD, PD = 12. We use Ptolemy's theorem to obtain AB * CD + BC * DA = AC * BD, 70 + 14BC = 224, BC = 11. Now that we know all the sides, we find that the semiperimeter is 21, and by Brahmagupta, the area K of the cyclic-quad is $\sqrt{(21-10)(21-11)(21-7)(21-14)} = \sqrt{11 * 10 * 14 * 7} = \sqrt{14^2 * 55} = 14\sqrt{55}$, and we are done.

"ABCD is inscribed in circle O. The diagonals of the quadrilateral intersect at P. C is the midpoint of arc BD, PC = 4, BC = 6, and PB = 3. What is the perimeter of ABCD?"

Because C is the midpoint, BC = DC, so DC = 6. Also, angles DAC and CAB are congruent because they intercept segments of equal length from the same side of the segments. Since we are in a circle, angles CDB and CAB are congruent because they intercept the same segment from the same side. We have angle-angle similarity between triangles CPD and CDA. This yields $\frac{AC}{DC} = \frac{DC}{PC}$, from which we find PA = 5. Power of a point from P yields DP = $\frac{20}{3}$. By similar triangles, AD = 10. We also find AB = $\frac{9}{2}$. Adding everything up, we get $\frac{53}{2}$.

"Lines l_1 and l_2 form a 17 degree angle, and ω_1 and ω_2 are circles centered at O_1 and O_2 respectively. ω_1 is tangent to l_1 at A. l_2 intersects ω_1 at B and C. l_3 is a line parallel to l_1 that passes through B. l_3 intersects ω_1 again at D. Ray O_1D intersects ω_2 at E and F with E on segment O_1F. Ray O_1C intersects ω_2 at G and H with G on segment O_1H. I is selected on minor arc GH, and J is selected on major arc GFH such that H is the midpoint of minor arc IJ. Lines GI and EH intersect at K, and the measure of angle IKH is 13 degrees. What is the measure of angle JO_2F?"

Because l_1 and l_3 are parallel, the measure of angle CBD is 17 degrees. This is an inscribed angle, so the measure of arc CD is 34 degrees, and the measure of angle $C0_1$D is 34^o. Another expression for the measure of this angle is $\frac{mFH-mGE}{2}$. We also know that $13^o = mIKH = mGKE = \frac{mGE-mHI}{2}$. Adding these two expressions produces $47^o = \frac{mFH-mHI}{2}$. Because H is the midpoint of arc IJ, $mHI = mHJ$. Substituting this, we get $47^o = \frac{mFH-mHJ}{2}$, which yields $mJF = 94^o$. The central angle that intercepts this arc has the same measure.

5 Practice

All of the following problems can be solved with important geometric techniques.

1. In triangle ABC, $AB = 8$, $BC = 12$, and $CA = 16$. D is the midpoint of $\overline{AB}$ and E is the point on $\overline{AC}$ such that $m\angle BCE + m\angle EDB = \pi$. Compute the area of $BCED$.

2. (ARML, 2002 #8 (sort of)) $ABCDEFG$ is a regular heptagon with perimeter 7007. Compute the value of the expression $AD \cdot BE - CG \cdot DF$.

3. (AMC 12, 2004 #19) Circles A, B, C, and D are mutually tangent. The radius of D is 2 and the radius of A is 1. If D contains the other thre circles, which are externally tangent to each other, and $B \cong C$, then what is the radius of circle B?

4. In triangle ABC, $AB : BC : CA = 2 : 3 : 4$. P is on arc AC of triangle ABC such that $PA = 3$ and $PB = 7$. Compute PC.

5. $ABCD$ is a convex quadrilateral in which $AC = AD$, $AB = 3$, $BD = 8$. Compute the area of $ABCD$ if $m\angle DAC = m\angle ABD = 60^o$.

6. In triangle ABC, $AB = 4$, $BC = 9$, and $CA = 6$. Circle ω_1 contains B and is tangent to $\overline{AC}$ at A. ω_1 passes through BC at D. Circle ω_2 also contains B, but is tangent to $\overline{AC}$ at C. AB is extended beyond B to E on ω_2. Compute the area of $\triangle CDE$.

7. (AMC 12, 2000 #24) A is the center of circle ω_1 and B is the center of circle ω_2. A is on ω_2 and B is on ω_1. Let C be a point intersection of ω_1 and ω_2. Ω is the circle that is internally tangent to ω_1 and ω_2 and also $\overline{AB}$. If the length of minor arc AC is 12, then what is the circumference of Ω?

8. $ABCDE$ is a cyclic pentagon with $AB = BC = CD = 2$ and $DE = EA = 3$. Compute the length of AD.