Geometry 1 – The Basics

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1 Cyclic Quads

We know that all triangles can be inscribed in a circle, but what about quadrilaterals? The answer is no (consider a very skinny rhombus, for example), but those quadrilaterals that can be inscribed in circles have some nice properties and are very, very useful. Let's call them cyclic quads.

1.1 Properties of Cyclic Quads/Determining if a Quad is Cyclic

A quadrilateral $ABCD$ is cyclic iff:

- $\angle ABD = \angle ACD$
- $\angle ABC + \angle CDA = 180°$
- (Ptolemy's Theorem) $AB * CD + AD * BC = AC * BD$
- (Power of a Point) $AP * PC = BP * PD$ where P is the intersection of AC and BD

Note: Lots of right angles means lots of cyclic quads.

2 Similar Triangles

Ways to prove that triangles are similar: AA, SAS (not SSA), SSS.

3 Areas

Area of a triangle $= \frac{1}{2}ah_a = \frac{1}{2}ab\sin C = rs = \frac{abc}{4R}$.

Areas are useful for equating things, but are possibly even more useful for determining ratios, which makes them very powerful when used in conjunction with similar triangles and cyclic quads.

4 General tips for geometry problems

- Neat diagrams let you see things.
- Make conjectures (and verify them before proving).
- Work both ways.
- Don't just stare at a diagram.

- Try making various constructions (parallel lines, extensions, reflections,...) if you feel that you don't have enough to work with.
- If the diagram is extremely overwhelming, try drawing multiple diagrams that each show only a portion of the problem so it's more manageable.

5 Problems

1. Segment AB is rotated an angle of θ around point P to segment A'B'. What is the angle formed between lines AB and A'B'?
2. Triangle ABC has orthocenter H. The line AH intersects BC at D and the circumcircle of triangle ABC at A'. Prove that HD = DA'.
3. (Simson's Theorem) Given a triangle ABC and a point P, let the feet of the perpendiculars from P to lines AB, BC, and CA be D, E, and F, respectively. Prove that points D, E, and F are collinear if and only if P lies on the circumcircle of triangle ABC.
4. (Nine-Point Circle) Given a triangle ABC with orthocenter H, prove that the three midpoints of the sides of the triangle, the three feet of the altitudes of the triangle, and the three midpoints of segments AH, BH, and CH all lie on a circle.
5. The incircle of triangle ABC has center O and is tangent to sides AB, BC, and CA at points X, Y, and Z, respectively. BZ intersects the circle again at P. M is the midpoint of XY. Show that APMO is cyclic.
6. Let ABCD be a convex quadrilateral. Diagonals AC and BD meet at O. Circumcircles of triangles BCO and ADO meet at M (other than O). Line OM meet the circumcircles of triangles OAB and OCD at T and S, respectively. Prove that M is the midpoint of TS.
7. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. Let M be the pont of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.
8. The diagonals of cyclic hexagon $A_1A_2A_3A_4A_5A_6$ concur at a point. Given that $A_1A_2 = A_3A_4 = A_5A_6$ and that A_3A_5 and A_1A_4 intersect at P, prove that $\frac{A_3P}{PA_5} = (\frac{A_1A_3}{A_3A_5})^2$.
9. In convex pentagon ABCDE, [ABC] = [BCD] = [CDE] = [DEA] = [EAB] = 1, where [ABC] denotes the area of triangle ABC. Compute the area of the pentagon. Futhermore, show that there are infinitely many non-congruent pentagons having the above property.
10. A point P lies on the circumcircle of triangle ABC. Lines AB and CP meet at E, and lines AC and BP meet at F. The perpendicular bisector of line segment AB meets line segment AC at K, and the perpendicular bisector of line segment AC meets line segment AB at J. Prove that $(\frac{CE}{BF})^2 = \frac{AJ \cdot JE}{AK \cdot KF}$.
11. The diagonals of quadrilateral ABCD intersect at O. Let M and N be the midpoints of sides AD and BC, respectively. Distinct points P and Q are the orthocenters of triangles ABO and CDO, respectively. Prove that PQ is perpendicular to MN.