ARML Lecture VI - Root Equations and Recursion

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1 Root Equations

1.1 The Concept

Getting anywhere without the basic concept is difficult.

$$x = f(x) \Longrightarrow x = f(f(x)) \text{ and } x = f(f(f(\ldots f(x)\ldots)))$$

The applications of this include solving equations that are apparently generated, both finite and infinite recursion. The idea is that if x is a solution to the root equation, it is certainly a solution to the recursively generated equation.

1.2 Usage 1

Some recursive problems are entirely root equations. Typically, these are the equations that have root equations with only one solution for x. Consider solving for all real x in the following equation:

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + x}}}}$$

This would be difficult to solve using conventional algebra as squaring four times will lead to an expression of a high degree. We could try guess and check, but that won't teach us anything. Instead, we can recognize the root equation that generated this expression:

$$x = \sqrt{2 + x}$$
$$x = \sqrt{2 + \sqrt{2 + x}}$$
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$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + x}}}}$$

Thus, we need only to solve the root equation for x. This is a managable task:

$$\begin{aligned} x &= \sqrt{2+x} \\ x^2 &= 2+x \\ x^2-x-2 &= 0 \\ x &= \tfrac{1\pm 3}{2} = -1, 2 \end{aligned}$$

In solving the root equation, we squared x, which was originally equal to the square root of an expression, and as such it cannot be negative. This eliminates -1 as a solution, and we are left with 2. Substituting into the original equation, we can see that x = 2 causes all of the square roots to dissappear nicely, and we are done.

1.3 Usage 2

Some recursive equations are not completely solved by root equations, but are simplified by solutions obtained from the root equation. Consider solving for all real x in the following equation:

$$x = \tfrac{2}{\frac{2}{x}+2+\frac{x}{2}} + 2 + \tfrac{\frac{2}{x}+2+\frac{x}{2}}{2}$$

This equation can be simplified with conventional algebra like so:

$$\begin{aligned} x &= \tfrac{4x}{x^2+4x+4} + 2 + \tfrac{x^2+4x+4}{4x} \\ x(4x)(x^2+4x+4) &= (4x)^2 + 2(4x)(x^2+4x+4) + (x^2+4x+4)^2 \\ 4x^4+16x^3+16x^2 &= x^4+16x^3+72x^2+64x+16 \\ 3x^4+0x^3-56x^2-64x-16 &= 0 \end{aligned}$$

This being a quartic, we are forced to stop here until we know some factors unless we are willing to do guesswork. Upon inspection of the original equation, we discover a root equation, and solve it:

$$\begin{aligned} x &= \tfrac{2}{x} + 2 + \tfrac{x}{2} \\ 2x^2 &= 4+4x+x^2 \\ x^2-4x-4 &= 0 \\ x &= \tfrac{4\pm\sqrt{32}}{2} = 2\pm 2\sqrt{2} \end{aligned}$$

Since x = 0 is not a solution, multiplying by 2x was valid, so we have two factors of the quartic we previously obtained. Using the fact that x^2-4x-4 is a factor of the quartic, we can continue:

$$\begin{aligned} (x^2-4x-4)(3x^2+12x+4) &= 0 \\ 3x^2+12x+4 &= 0 \\ x &= \tfrac{-12\pm\sqrt{96}}{6} = -2\pm\tfrac{2\sqrt{6}}{3} \end{aligned}$$

Combining these two additional roots, we find that $x = 2\pm 2\sqrt{2}$ or $-2\pm\frac{2\sqrt{6}}{3}$

1.4 Composition of Functions

Related is the algebra behind the general case of functions of functions. For example, consider solving for all values of x that satisfy

$$f(g(h(x))) = 0$$

where $f(x) = x^2 - 5x - 14$, $g(x) = x^3 + 6$, and $h(x) = 2x + 1$. To do this, find all values of x such that $f(x) = 0$, those values of x, say a and b, correspond to values of $g(h(x))$. The equations $g(x) = a$ and $g(x) = b$ can in turn be solved for values of $h(x)$. Finally, $h(x)$ can be solved for these values, yielding the solutions x to $f(g(h(x))) = 0$.

2 Other Recursion

2.1 Applications

Recursion, although tedious and very slow (On the order of $O(k^N)$), can be used to solve problems involving reasonably small n. You should consider using it if there is a problem in which you can find the first couple terms easily based on a simple function. Consider the following problem:

How many positive integers satisfy satisfy both of the following?

i) All of the number's digits are either 1, 2, or 3.
ii) The sum of all of the digits is 10.

We attack this problem by defining a function f on the integers, such that $f(N) =$ the number of positive integers consisting of only 1, 2, and 3 whose digits sum to N. The answer will be $f(10)$. We can find small values of f by scribbling down a few numbers. We find $f(1) = 1, f(2) = 2$, and $f(3) = 4$. Based on the stipulation that all of the digits must be 1, 2, or 3, we also define $f(N+3) = f(N+2) + f(N+1) + f(N)$, where $N \geq 1$. Using this formula, we can find $f(10)$ like so:

$$\begin{gathered}
f(4) = 4 + 2 + 1 = 7 \\
f(5) = 7 + 4 + 2 = 13 \\
f(6) = 13 + 7 + 4 = 24 \\
f(7) = 24 + 13 + 7 = 44 \\
f(8) = 44 + 24 + 13 = 81 \\
f(9) = 81 + 44 + 24 = 149 \\
f(10) = 149 + 81 + 44 = 274
\end{gathered}$$

So, $f(10) = 274$, and we are done.

2.2 Drawbacks to Recursion Solutions

The biggest problem with recursion is that it is a slow process to use it to find a large value of a function. For example, consider the Fibonacci sequence. Finding a_{100000} recursively, even with a calculator, would take an eternity!

Another problem is that in using recursion, we perform a lot of simple arithmetic and are more likely to make a trivial mistake.

To get around both of these problems, we consider the use of:

Conversion - One can convert a recursive definition into an explicit formula or vice versa using the property that: A sequence $\{X\}$ defined by $a_k X_n + a_{k-1} X + \cdots + a_0 X_{n-k} = 0$ and some initial values $X_1, \ldots X_k$ can be written explicitly as $X_n = \omega_1 r_1^n + \cdots + \omega_k r_n^k$, where r_i are the k distinct (possibly complex) roots of the polynomial $a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$ and ω_i are chosen according to the values of $X_1, \ldots X_k$.

By applying the above method, it can be shown that $a_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\varphi = \frac{1-\sqrt{5}}{2}$.

3 Practice

Use your newlearned skills to show that the answers in section 3.2 accurately represent the answers to the questions in 3.1.

3.1 Problems

These problems are all solveable by methods shown in this lecture.

1. Solve for all real x that satisfy the following equation:

$$x = 54 - \cfrac{473}{54 - \cfrac{473}{54 - \cfrac{474}{54 - \cfrac{473}{54 - \cfrac{473}{x}}}}}$$

2. Solve for all real x that satisfy the following equation:

$$(x^2 + x - 2)^2 + (x^2 + x - 2) - 2 = x$$

3. Evaluate the following infinite expression:

$$\sqrt{2003 + \sqrt{2003 + \sqrt{2003 + \cdots}}}$$

4. Solve for all real x that satisfy the following equation:

$$x = \frac{-1 + 5\left(\frac{1-5x^2}{2}\right)^2}{2}$$

5. Solve for all real x that satisfy the infinite equation:

$$2x = \frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{\cdots}}}}$$

6. The sequence a_N satisfies the recursion $a_n = 3a_{n-1} + 28a_{n-2}$ for integers $n > 2$. If $a_1 = 3$ and $a_2 = 65$, write an explicit formula for a_n in terms of n.

7. The sequence Ω_N obeys $\Omega_n = 1492 \cdot 2^n + 1776 \cdot 1002^n$. There exist two real numbers a and b such that the recursion $\Omega_n = a\Omega_{n-1} + b\Omega_{n-2}$ holds for all integers $n > 2$. Determine the ordered pair (a, b).

8. The numbers in the sequence S_N satisfy the recursion $S_{n+1} = 2S_n + 1$ for any positive integers n. If $S_0 = 1$ and $S_1 = 16$, write an equation expressing S_n explicitly in terms of n.
Zeta

3.2 Answers

These may or may not be the correct answers. That's for you to determine...

1. $x = 43$ or 11.
2. $x = \pm\sqrt{2}, 0,$ or -2.
3. $\frac{1+\sqrt{8013}}{2}$
4. $x = \frac{-1\pm\sqrt{5}}{5}$ or $\frac{1\pm\sqrt{6}}{6}$
5. $x = \pm\frac{1}{\sqrt{6}} = \pm\frac{\sqrt{6}}{6}$.
6. $a_n = 7^n + (-4)^n$.
7. $(1004, -2004)$.
8. $S_n = 15 \cdot 2^n - 14$.