ARML - Combo Practice 2

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1 Review from Combo 1

1. **Forming Groups:** (AIME) The positive odd integers x_1, x_2, x_3, and x_4 sum to 98. How many ordered quadruples (x_1, x_2, x_3, x_4) are possible?

2. **Sets:** A moose, a penguin, and a llama all took the same 10-question true-false test. Given that each of the 10 questions was correctly answered by at least one of the three, what is the probability that all three animals received full scores?

2 Mapping to Sequences

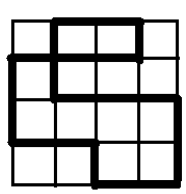
Basic: How many possible ways are there to rearrange the letters in IVYISPROBABLYLAUGHINGASSH-EREADSTHIS? (Hint: There are seven letters that appear only once, six that appear twice, one that appears three times, two that appear four times, and one that appears five times, for a total of 35 letters.)

1. In how many ways can an ant travel from $(0, 0)$ to $(6, 6)$ if he may only move one unit up or one unit to the right at any given step? What if he must pass through the point $(2, 4)$?

2. A spider needs to put on eight shoes and eight socks, but on each leg the sock must go on before the shoe. In how many different orders can the spider do this?

3. A turkey is a navigating the edges of a cube, starting from Vertex Wild. Adjacent to Vertex Wild are Vertices GroceryStore, MeatHouse, and DinnerTable. If the turkey blindly runs along one edge of the cube at each of seven steps, picking one of the three edges adjacent to the vertex it is currently on at each step to traverse (with equal probability), what is the probability that after seven steps the turkey ends at one of GroceryStore, MeatHouse, or DinnerTable?

4. **Practice:** The Gingerbread Man is caught by a mathematics-loving cat. After much pleading by the Gingerbread Man, the cat agrees that if the Gingerbread Man can tell it how many ways a continuous path (loop) can be drawn using the lines of a grid of nxn lines such that
   1) the path contains exactly $2n$ line segments, and
   2) each vertical and horizontal line of the grid contains exactly one segment of the path,
   then the cat will spare the Gingerbread Man. Please help.
   For example, the following is a valid path for $n = 5$:



3 Exploiting Symmetry/Pairings

Basic: David and Archis play a very complicated game that lasts a total of 1000 turns. First they flip a fair coin to determine who goes first. Afterwards, they alternate turns, and on turn i, $1 \leq i \leq 1000$, the player whose turn it is picks an integer divisor of i, computes its 529th power, and adds that to a cumulative sum. If the final total after the 1000th turn is an integer multiple of 9001, then the player that went first wins, otherwise the other player wins. Compute the probability that David wins.

0. There are many ways to split the nine integers $1, 2, ..., 9$ into three disjoint groups of three. Compute the probability that if one such splitting is random chosen, the numbers 2 and 4 will be in the same group.

1. Compute the number of subsets S of $\{1, 2, ..., 10\}$ such that the sum of the elements of S is less than 28.

2. The 625 runners in a race are assigned the ID numbers 1,2,...,625. Given that no two runners tied, what is the probability that the numbers of the first place, second place, and third place runners are strictly in increasing order?

3. (AIME) Let $n = 2^{31} \cdot 3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

4. **Practice:** (AIME) For $\{1, 2, ..., n\}$ and each of its nonempty subsets a unique *alternating sum* is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 4, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$ and for $\{5\}$ it is simply 5.) Find the sum of all such alternating sums for $n = 7$.

4 PIE (Not the Food)

Basic: What is the probability that a randomly-selected integer from 1 to 100, inclusive, is a multiple of either 3 or 5 but not both?

1. **Practice:** Atthew, Aniel, Im, Ara, and Inston stand in a line in that order and each hold up a single random digit from 0-9, inclusive. Compute the probability that at least three of them, standing in three consecutive positions, hold up the same digit.

2. **Practice:** (AIME) Two of the squares of a 7 x 7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

5 Additional Problems

1. (AMC) Real numbers x, y, and z are chosen independently and at random from the interval $[0, n]$ for some positive integer n. The probability that no two of x, y, and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n?

Check out the VMT Wiki or bother Victoria if you'd like more practice problems. Remember that the point of these lectures was to expose you to different techniques you may or may not have seen before; it takes lots and lots of practice to really learn how to use them.