Novice Trigonometry

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1 Introduction

What is trig? According to the interwebs, "the branch of mathematics that deals with the relations between the sides and angles of plane or spherical triangles, and the calculations based on them." But what does this mean? Basically, we are going to relate sides of triangles with angles, and then manipulate the result.

2 Basics

The bare bones of trig rely on three major functions, sine, cosine, and tangent.

$$\begin{array}{ccc} \sin\theta = \frac{opposite}{hypotenuse} & \cos\theta = \frac{adjacent}{hypotenuse} & \tan\theta = \frac{opposite}{adjacent} \\ \csc\theta = \frac{1}{\sin\theta} & \sec\theta = \frac{1}{\cos\theta} & \cot\theta = \frac{1}{\tan\theta} \\ \sin(-\theta) = -\sin\theta & \cos(-\theta) = \cos\theta & \tan(-\theta) = -\tan\theta \end{array}$$

From here, we note three identities:

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$

3 Intermediary Concepts

We establish some formulae here that should be memorized:

- $\sin(a+b) = \sin a\cos b + \sin b\cos a$
- $\cos(a+b) = \cos a\cos b - \sin a\sin b$
- $\tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\tan b}$

From here we can derive double-angle formulae, which can be used to find half-angle formulae.

Another important series of formulae are know as sum to product formulae, from which we can derive the product to sum formulae:

- $\sin a + \sin b = 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})$
- $\sin a - \sin b = 2\sin(\frac{a-b}{2})\cos(\frac{a+b}{2})$
- $\cos a + \cos b = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$
- $\cos a - \cos b = -2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$

4 Advanced Tricks and Tips

Here is where the fun stuff begins:

4.1 Multi-Angle Formulas

- Euler's Formula states that $e^{i\theta} = \cos\theta + i\sin\theta$
- De Moivre's Theorem states that $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$

We can use these to derive multi-angle formulas!!!

4.2 Triangles

- Law of Sines: $\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R$, where R is the circumradius of the triangle
- Law of Cosines: $a^2 + b^2 = c^2 + 2ab\cos\angle C$
- Area of a Triangle: $\frac{1}{2}ab\sin\angle C$

5 Problems

1. Give the formula for $\cos(5\theta)$ in terms of x where $x = \cos\theta$.
2. Let x and y be real numbers such that $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$. The value of $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$. (Source: 2012 AIME II)
3. In Triangle ABC, D is the midpoint of BC and $\angle BAD = \frac{1}{3}\angle BAC$. If $\frac{AB}{AC} = \frac{3}{2}$, then what is the value of $\cos\angle DAC$? (Source: TJTST 3)
4. Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A)+\cos(3B)+\cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m. (Source: 2014 AIME II)