Trigonometric Identities

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1 It's all about the identities!

Trigonometry is one of the most formula-intensive topics in all of contest math. Nearly all trigonometry problems can be solved with a relatively small set of identities. This means that learning to solve trigonometric problems takes less time than learning other topics. Remember: identities are the tools we need to solve these problems.

2 Theorem: $e^{i\theta} = \cos\theta + i\sin\theta$

Proof: There are two common proofs for this. First, $f(x) = e^{ix}$ and $f(x) = \cos x + i\sin x$ have the same Taylor Series. Second, both $f(x) = e^{ix}$ and $f(x) = \cos x + i\sin x$ are solutions to the differential equation $\frac{dy}{dx} = iy$. Because the solution to a differential equation is unique, they must be equal.

3 Angle Addition and Subtraction Identities

$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)} = e^{i\alpha} \times e^{i\beta} = (\cos\alpha + i\sin\alpha) \times (\cos\beta + i\sin\beta)$. By equating the real and imaginary parts after expanding, we arrive at the following formulae:

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

After using $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and simplifying, we have:

- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

- $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

Exercise 1 : Simplify $\frac{\sqrt{3}\sin(x+30^\circ) - \cos(x+30^\circ)}{4\cos x \sin(x+30^\circ) - 4\sin x \cos(x+30^\circ)}$. (Source: Mandelbrot)

Exercise 2 : Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$ What is $\cos(a-b)$? (Source: AMC)

4 Multiple Angle Identities

Double-angle and triple-angle Identities can be derived from the angle addition identities. With $\sin(2\theta) = \sin(\theta + \theta)$, $\cos(2\theta) = \cos(\theta + \theta)$, and $\tan(2\theta) = \tan(\theta + \theta)$, we draw the following conclusions:

- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$

In order to derive an expression for $\sin(3\theta)$, $\cos(3\theta)$, and $\cos(3\theta)$, we use $\sin(3\theta) = \sin(2\theta + \theta)$, $\cos(3\theta) = \cos(2\theta + \theta)$, $\tan(3\theta) = \tan(2\theta + \theta)$, and simplify:

- $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$
- $\cos(3\theta) = -3\cos\theta + 4\cos^3\theta$
- $\tan(3\theta) = \frac{3\tan(\theta) - \tan^3(\theta)}{1 - 3\tan^2(\theta)}$

For any n, n-tuple angle identities can be found using $e^{in\theta} = \cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$ and equating the real and imaginary parts.

To find the half-angle identities, we take the double-angle identities for cosine and solve for the half-angle:

- $\sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$
- $\cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$

- $\tan(\frac{\theta}{2}) = \pm\frac{\sin(\frac{\theta}{2})}{\cos(\frac{\theta}{2})} = \pm\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} = \pm\frac{\sin(\theta)}{1+\cos(\theta)} = \pm\frac{1-\cos(\theta)}{\sin(\theta)}$

Note the $\pm$. The sign depends on the angle in question.

Exercise 1 : Find $\cos(\frac{\pi}{7})\cos(\frac{2\pi}{7})\cos(\frac{4\pi}{7})$.

Exercise 2 : Compute the number of degrees in the smallest positive angle x that satisfies the equation $8\sin(x)\cos^5(x) - 8\sin^5(x)\cos(x) = 1$.

5 Sum-to-Product and Product-to-Sum Identities

The sum-to-product and product-to-sum identities are rearrangements of the angle addition identities. Instead of taking up (a lot of) space in these notes, I will derive these expressions during my lecture.

Sum-to-product:

- $\cos(\alpha) + \cos(\beta) = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
- $\cos(\alpha) - \cos(\beta) = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$
- $\sin(\alpha) + \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
- $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

Product-to-sum:

- $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$
- $\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$

Exercise 1 : Compute $\frac{\sin 13^\circ + \sin 47^\circ + \sin 73^\circ + \sin 107^\circ}{\cos 17^\circ}$. (Source: ARML)

Exercise 2 : Evaluate $\frac{\cos(1^\circ)+\cos(2^\circ)+\cos(3^\circ)+...+\cos(43^\circ)+\cos(44^\circ)}{\sin(1^\circ)+\sin(2^\circ)+\sin(3^\circ)+...+\sin(43^\circ)+\sin(44^\circ)}$. (Source: AIME)