Combo Practice

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There are certain types of problems you absolutely need to know how to do. Once you have those, 95% of high school combo is just a matter of turning a new problem into one of those basic ones.

1 Things to Watch Out For

- Read the question. Probabilities are never greater than one. Grr.
- **Example (Trap!):** There are many ways to split the nine integers $1, 2, ..., 9$ into three disjoint groups of three. Compute the probability that if one such splitting is random chosen, the numbers 2 and 4 will be in the same group.
- Make sure for counting problems you know whether order matters or not and whether objects are distinguishable or not. For probability questions, it often does not make a difference, but you have to be consistent with yourself as you solve the problem.
- Don't just sit and stare at a problem because it contains a large number; try small cases.
- Count starting from the most restrictive condition first; alternatively, exploit symmetry.
- It might be easier to count the compliment.

2 Forming Groups

Basic: How many triples of nonnegative integers (a_1, a_2, a_3) are there such that $a_1 + a_2 + a_3 = n$?

1. Julie's buying frozen yogurt for seven of her friends at Menchie's. If the flavors she's considering are banana, carrot cake, taro, and white chocolate mousse, how many different orders might she place?
2. (102CP) Determine the number of ways to choose five numbers from the first 18 positive integers such that any two chosen numbers differ by at least 2.
3. Barry is buying fruits. If bananas are sold in clusters of five (Barry cannot buy part of a cluster.), pears must be bought in packages of two, and Barry can neither buy more than four apples nor more than one mangosteen, how many distinct ways can Barry buy a total of thirty pieces of fruit? (For example, one possible way is for him to purchase 10 bananas, 18 pears, 1 apple, and 1 mangosteen.)
4. **Practice:** (AIME) The positive odd integers x_1, x_2, x_3, and x_4 sum to 98. How many ordered quadruples (x_1, x_2, x_3, x_4) are possible?

3 Sets

Basic: How many (possibly empty) subsets are there of an n-element set?

1. In how many ways can we form two disjoint subsets from an n-element set? How about k disjoint subsets?

2. Given an n-element set S and a positive integer k, how many sequences of subsets $(T_1, T_2, ..., T_k)$, where each T_i is a subset of S, are there such that $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$?

3. **Practice:** A moose, a penguin, and a llama all took the same 10-question true-false test. Given that each of the 10 questions was correctly answered by at least one of the three, what is the probability that all three animals received full scores?

4 Mapping to Sequences

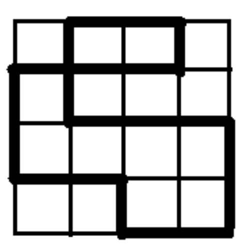
Basic: How many possible ways are there to rearrange the letters in IVYISPROBABLYLAUGHINGASSHEREADSTHIS? (Hint: There are seven letters that appear only once, six that appear twice, one that appears three times, two that appear four times, and one that appears five times, for a total of 35 letters.)

1. In how many ways can an ant travel from $(0, 0)$ to $(6, 6)$ if he may only move one unit up or one unit to the right at any given step? What if he must pass through the point $(2, 4)$?

2. A spider needs to put on eight shoes and eight socks, but on each leg the sock must go on before the shoe. In how many different orders can the spider do this?

3. A turkey is a navigating the edges of a cube, starting from Vertex Wild. Adjacent to Vertex Wild are Vertices GroceryStore, MeatHouse, and DinnerTable. If the turkey blindly runs along one edge of the cube at each of seven steps, picking one of the three edges adjacent to the vertex it is currently on at each step to traverse (with equal probability), what is the probability that after seven steps the turkey ends at one of GroceryStore, MeatHouse, or DinnerTable?

4. **Practice:** The Gingerbread Man is caught by a mathematics-loving cat. After much pleading by the Gingerbread Man, the cat agrees that if the Gingerbread Man can tell it how many ways a continuous path (loop) can be drawn using the lines of a grid of nxn lines such that
   1) the path contains exactly $2n$ line segments, and
   2) each vertical and horizontal line of the grid contains exactly one segment of the path,
   then the cat will spare the Gingerbread Man. Please help.
   For example, the following is a valid path for $n = 5$:



5 Different Bases

1. (102CP) The increasing sequence $1, 3, 4, 9, 10, 12, 13, ...$ consists of all positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence.