2013 ARML Advanced Polynomials

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1 Basics

DEFINITION: A polynomial is an expression involving the sum of powers of variables muliplied by a coefficient.

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

A useful thing to know is the **Fundamental Theorem of Algebra**, which tells us for some non-constant polynomial P with complex coefficients, there is at least one complex root. Or more simply, for a polynomial of order n, there are at most n roots that make $P(r) = 0$. Speaking of roots...

2 Root Them Out

The most typical thing to do when given a polynomial is to factor it and find it's roots. Here are some useful tools to use for our purposes:

2.1 Rational Root Theorem

The rational root theorem states that given a polynomial with all integer coefficients $P = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, all of its roots are in the form m/n where m and n are relatively prime and m and n are factors of a_0 and a_n, respectively.

Simple Excercise: Find the roots of $2x^n + 7x + 6$.
Solution: Using the rational root theorem we create a list of possible roots: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2}, \pm\frac{6}{2} \implies \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3, \pm 6$. Plugging in values and we find that the roots are $-\frac{3}{2}, -2$.

2.2 Descartes' Rule of Sign

Descartes' Rule of Sign allows us to count the number of positive and negative roots by counting the number of sign changes. Given the polynomial $P(x) = 2x^4 - 4x^3 - 26x^2 + 28x + 48$ written in descending order, we count the number of sign changes between consecutive terms to get 2 sign changes. So there are at most 2 positive roots. We find the maximum number of negative roots by plugging in $-x \implies P(-x) = 2x^4 + 4x^3 - 26x^2 - 28x + 48 \implies$ 2 roots. This rule further states that the actual number of positive/negative roots differ by an even number from its respective maximum.

2.3 Vieta's Formulas

Vieta has provided us a way to relate the sum and products of the roots of a polynomial to the polynomial's coefficients. For any polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$. We can then expand the right side to get:

$$P(x) = a_n x^n - a_n(r_1 + r_2 + \cdots + r_n)x^{n-1} + a_n(r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n)x^{n-2} + \cdots + (-1)^n a_n r_1 r_2 \cdots r_n.$$

Vieta's takes these equations and gives us the following:

$$\begin{aligned}
\frac{a_{n-1}}{a_n} &= r_1 + r_2 + \cdots + r_n \\
\frac{a_{n-2}}{a_n} &= r_1r_2 + r_1 + r_3 + \cdots + r_{n-1}r_n \\
&\vdots \\
\frac{a_0}{a_n} &= r_1r_2\cdots r_{n-1}r_n
\end{aligned}$$

2.4 Useful Factoring Identities

$$\begin{aligned}
a^2 - b^2 &= (a+b)(a-b) \\
a^3 \pm b^3 &= (a \pm b)(a^2 \mp ab + b^2) \\
(a+1)(b+1) &= ab + a + b + 1 \\
(a+b+c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\
a^3 + b^3 + c^3 - 3abc &= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)
\end{aligned}$$

3 More Advanced Stuff

3.1 Transforming Polynomials

Standard transformations that everyone should be used to seeing is translating the graph of a function h units to the right by replacing x with (x-h). But you can do more interesting things with polynomials. Let's say that we have the polynomial $P(x) = x^4 - 3x^2 + x - 9$ and we want to find the polynomial which has roots that are reciprocals to those of $P(x)$. We can use our superb factoring ability (or a calculator) to find the roots (1 pos, 1 neg, 2 imaginary) and take the reciprocals. Or we could transform the polynomial.

If we let the roots be r_1, r_2, r_3, and r_4, we know that $P(r_i) = P(\frac{1}{1/r_i}) = 0$. Thus the solutions of $P(1/x)$ are the reciprocals of the roots that we wanted. However, by definition $P(1/x)$ is not a polynomial. So we define a polynomial $G(x) = x^4P(1/x)$ to get $G(x) = x^4(\frac{1}{x}^4 - \frac{3}{x}^2 + \frac{1}{x} - 9) = -9x^4 + x^3 - 3x^2 + 1$, which has roots of $1/r_i$.

3.2 Newton's Sums

Newton's Sums gives us an interesting (and quick) way to compute the sum of the roots to a certain power. First let us define some variables: let n be the order of the polynomial, s_n denote the sum of the roots to the nth power, and a_i be the coefficient to the variable x^i. The general form for Newton sums is: $a_ns_k + a_{n-1}s_{k-1} + \cdots + ka_{n-k} = 0$. to be clear I've listed out the first few equations:

$$\begin{aligned}
0 &= a_ns_1 + a_{n-1} \\
0 &= a_ns_2 + a_{n-1}s_1 + 2a_{n-2} \\
0 &= a_ns_3 + a_{n-1}s_2 + a_{n-2}s_1 + 3a_{n-3}
\end{aligned}$$

Note how you have to use n equations to find s_n! Also note that n does not have to be larger than k (the order of the polynomial does limit the sums you can find). Let's look at an example:

Example: Find the sum of the cubes of the solutions of $x^2 + 2x + 1$.
Solution: We use Newton's sums to get the following equations:

$$\begin{aligned} 0 &= a_2 s_1 + a_1 = s_1 + 2 \\ 0 &= a_2 s_2 + a_1 s_1 + 2a_0 = s_2 + 2s_1 + 2 \\ 0 &= a_2 s_3 + a_1 s_2 + a_0 s_1 + 3a_{-1} = s_3 + 2s_2 + s_1 \end{aligned}$$

Solving down the line, we get $s_1 = -2, s_2 = -2 - 2(-2) = 2, s_3 = -(-2) - 2(2) = -2$. So our final answer is -2. This is easily checkable in this case, as the polynomial above was $(x+1)^2$ and so had double roots of $x = -1, (-1)^3 + (-1)^3 = -2$.

4 Polynomial Practice Problems

1. Show that $(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a)$ using Newton's Sums and Vieta's Formulas.
2. What is the sum of the coefficients of the polynomial which has roots that are twice those of $f(x) = x^4 - 3x^2 + x - 9$?
3. The roots of $f(x) = 3x^3 - 14x^2 + x + 62 = 0$ are a, b, and c. Find the value of $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$. (MAΘ)
4. Find $x^2 + y^2$ if x and y are positive integers such that
$$\begin{aligned} xy + x + y &= 71 \\ x^2 y + xy^2 &= 880. \end{aligned}$$
(AIME 1991, #1)
5. The parabola with equation $p(x) = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $q(x) = dx^2 + ex + f$. Find $a + b + c + d + e + f$ in terms of h and k. (AMC12 2001, #13)
6. Compute the sum of all the roots of $(2x+3)(x-4) + (2x+3)(x-6) = 0$.(AMC12 2002, #1)
7. Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. What is the number of possible values of k? (AMC12 2002, #12)
8. What is the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$? (AIME 1983, #3)
9. Find c if a, b, and c are positive integers which satisfy $c = (a+bi)^3 - 107i$, where $i^2 = -1$. (AIME 1985, #3)
10. Find the positive solution to $\frac{1}{x^2-10x-29} + \frac{1}{x^2-10x-45} - \frac{2}{x^2-10x-69} = 0$.(AIME 1990, #4)

Answers: 2)-147, 3) 83/74, 4) 146, 5) 2k, 6) 7/2, 7) 1, 8) 20, 9) 198, 10) 13.