ARML Lecture IV - Trigonometry

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It's time to return to that long lost subject that you studied back in the days of Algebra 2 or Precalculus, depending on where you were, or that subject that you haven't studied yet except as found and defined in right triangles as "SOHCAHTOA". Regardless of what you knew, you will learn something now...

1 Basic Facts

Of course, knowing facts will get you nowhere if you don't know any values to begin with. You should be able to regurgitate these in your sleep:

- $\sin(0^o) = \cos(90^o) = \tan(0^o) = 0$
- $\sin(30^o) = \cos(60^o) = \frac{1}{2}, \tan(30^o) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
- $\sin(45^0) = \cos(45^o) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \tan(45^o) = 1$
- $\sin(60^0) = \cos(30^o) = \frac{\sqrt{3}}{2}, \tan(60^o) = \sqrt{3}$
- $\sin(90^o) = \cos(0^o) = 1, \tan(90^o) =$undefined. ($\tan(x)$ approaches infinity as x approaches 90^o from below.)

These are some of the basic properties of sine and cosine that you will learn to recognize:

- $\sin(90^o - \theta) = \cos(\theta), \cos(90^o - \theta) = \sin(\theta)$
- $\sin(-\theta) = -\sin(\theta), \cos(-\theta) = \cos(\theta)$
- $\sin(180^o - \theta) = \sin(\theta), \cos(180^o - \theta) = -\cos(\theta)$
- $\sin^2(\theta) + \cos^2(\theta) = 1$ (Pythagorean Trig Identity)

2 Basic Trig

Of course, given the above information, you can only go so far as to compute trigonometry involving "nice" angles. To compute the sines, cosines, and tangents of other angles, you need the angle addition formulae:

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

You may wonder about angle subtraction, and what about still other angles? Well, angle subtraction can be derived by replacing β with $-\beta$ in the addition formulas. Once you've correctly derived the subtraction formulas, you might notice that replacing every $+$ with a $-$ and vice versa in the angle addition formulas yields the angle subtraction formulae.

Setting the two arguments equal yields the double angle formulae: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$, and finally, $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$. After finding the double angle formulae, setting θ to $\frac{\theta}{2}$ and solving backwards yields the half-angle formulae: $\cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$, where the sign depends on the quadrant of θ. We also find $\sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$. Substituting these into the tangent formula and simplifying that yields: $\tan(\frac{\theta}{2}) = \pm\frac{\sin(\theta)}{1+\cos(\theta)}$.

And now we come across the formulas that convert sums to products and vice versa. These are much more applicable than they might seem, so memorize them!

- $\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$
- $\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$
- $\sin(\alpha)\cos(\beta) = \frac{\sin(\beta+\alpha)-\sin(\beta-\alpha)}{2}$

Two big identities and a trigonometric form of a familiar beast lie ahead. The first two are important enough to be called laws, so misuse of them is breaking the law. You should already know the first two:

- **[Extended] Law of Sines** Let ABC be a triangle with sides a, b, and c, and of circumradius R. Then: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R$.
- **Law of Cosines** Given the previous triangle, $c^2 = a^2 + b^2 - 2ab\cos(C)$.
- **Trig Ceva** Let ABC be a triangle with points D, E, and F on sides BC, AC, and AB respectively of triangle ABC. Line segments AD, BE, and CF are concurrent if and only if $\frac{\sin(BAD)}{\sin(DAC)}\frac{\sin(ACF)}{\sin(FCB)}\frac{\sin(CBE)}{\sin(EBA)} = 1$.

3 Advanced Trig

Here lurk more precalculated trig values that double the number of accessible angles:

- $\sin(18^o) = \cos(72^o) = \frac{\sqrt{5}-1}{4}$
- $\sin(36^o) = \cos(54^o) = \sqrt{\frac{5-\sqrt{5}}{8}}$
- $\sin(54^o) = \cos(36^o) = \frac{\sqrt{5}+1}{4}$
- $\sin(72^o) = \cos(18^o) = \sqrt{\frac{5+\sqrt{5}}{8}}$

We previously learned about trig addition and subtraction, but now let us delve into the trig inverse addition and subtraction. Unfortunately, the only trig inverse expressions that are friendly are the arctan inverse expressions. The arcsine and arccosine functions boil down to radicals and case analysis. Without further adieu:

- $\tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}(\frac{a+b}{1-ab})$
- $\tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}(\frac{a-b}{1+ab})$

You probably noticed that these formulas resemble the tangent additon formulas from which they were derived. Here we present a very useful tool: the quadratic formula of trigonometry.

$$a\cos(\theta) + b\sin(\theta) = c \Longrightarrow \cos(\theta) = \frac{ac \pm b\sqrt{a^2+b^2-c^2}}{a^2+b^2}, \quad \sin(\theta) = \frac{bc \pm a\sqrt{a^2+b^2-c^2}}{a^2+b^2}$$

Of course, not all of the values work. You will need to check for extraneous solutions. Finally, we end with "Titu's Identity for Triangles", which leads to a family of related identities:

For any triangle ABC, $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$.

4 Applying Trig

The fun part of trig. More than anything, to be successful with trig type problems you must be creative. To demonstrate what this means, here is a fully solved problem from the 1996 USAMO.

[USAMO 96'] "Prove that the average of the numbers $n \sin n^o (n = 2, 4, 6, \ldots, 180)$ is $\cot(1^o)$."

Let $S = 2\sin(2^o) + 4\sin(4^o) + \ldots + 180\sin(180^o)$. Because $\sin(n^o) = \sin((180-n)^o)$, this expression reduces to $S = 180(\sin(2^o) + \sin(4^o) + \sin(6^o) + \ldots + \sin(88^o) + \sin(90^o)) - 90$. Now we multiply everything through by $\sin(1^o)$. We do this because we spy a difference of two degrees between each term in the series, and we know the sum to product formulas will probably do something. We get: $S\sin(1^o) = 180(\sin(2^o)\sin(1^o) + \sin(4^o)\sin(1^o) + \ldots + \sin(90^o)\sin(1^o)) - 90\sin(1^o) = 180(\frac{1}{2}(\cos(1^o) - \cos(3^o)) + \frac{1}{2}(\cos(3^o) - \cos(5^o)) + \ldots + \frac{1}{2}(\cos(89^o) - \cos(91^o))) - 90\sin(1^o)$. Hey, that telescopes nicely! We get: $S\sin(1^o) = 90(\cos(1^o) - \cos(91^o) - \sin(1^o)) = 90\cos(1^o)$. Dividing by $90\sin(1^o)$ gives us the desired result.

Don't worry about having to find such a clever trick though. Here is another problem from ARML 1988 that demonstrates that clever ways work, but brute-force will also do the job:

[ARML 88'] "If $0^o < x < 180^o$ and $\cos(x) + \sin(x) = \frac{1}{2}$, then find (p, q) such that $\tan(x) = -\frac{p+\sqrt{q}}{3}$."

METHOD I - Clever Solution: Square the expression and then simplify to get $\cos(x)\sin(x) = -\frac{3}{8}$. Multiply the original equation by $\cos(x) - \sin(x)$ to find that $\cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = \frac{1}{2}(\cos(x) - \sin(x))$. Adding half the original equation (flipped) gives us $2\cos^2(x) - \frac{3}{4} = \cos(x)$, a quadratic in $\cos(x)$ with a solution $\cos(x) = \frac{1-\sqrt{7}}{4}$. (We ignore the other solution because it is positive, and when multiplied by $\sin(x)$, which is positive for all x in the domain, the result must be $-\frac{3}{8}$.) Then we get $\cos^2(x) = \frac{4-\sqrt{7}}{8}$ and, after simplifying, $\frac{(\sin(x)\cos(x))}{\cos^2(x)} = \tan(x) = -\frac{4+\sqrt{7}}{3}$; so we get $(4, 7)$.

METHOD II - Straightforward Solution: Plug this mess into the quadratic formula of trigonometry, and what do we get? $\cos(x) = \frac{1*\frac{1}{2}\pm 1*\sqrt{1^2+1^2+(\frac{1}{2})^2}}{1^2+1^2} = \frac{1\pm\sqrt{7}}{4}$, $\sin(x) = \frac{1*\frac{1}{2}\pm 1*\sqrt{1^2+1^2+(\frac{1}{2})^2}}{1^2+1^2} = \frac{1\pm\sqrt{7}}{4}$. We examine the domain of x, and it is pretty obvious that the only solution is when $\cos(x)$ is negative, and $\sin(x)$ is positive. We plug in: $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{1+\sqrt{7}}{4}}{\frac{1-\sqrt{7}}{4}} = \frac{1+\sqrt{7}}{1-\sqrt{7}} = \frac{1+\sqrt{7}}{1-\sqrt{7}}\frac{-1-\sqrt{7}}{-1-\sqrt{7}} = \frac{-8-2\sqrt{7}}{6} = -\frac{4+\sqrt{7}}{3}$; and likewise we get $(4, 7)$.

Our last example is an amusing problem that illustrates a trick that shows up all too often:

[Eager, 2000] "Evaluate: $\cos(\frac{\pi}{7})\cos(\frac{2\pi}{7})\cos(\frac{3\pi}{7})\cos(\frac{4\pi}{7})\cos(\frac{5\pi}{7})\cos(\frac{6\pi}{7})$."

We set the product equal to P. We then multiply both sides of the expression by $\sin(\frac{\pi}{7})$. We use the fact that $\sin(2x) = 2\sin(x)\cos(x)$ a grand total of 6 times, and we use basic properties of sine to boil the entire mess on the right down leaving $P\sin(\frac{\pi}{7}) = -\frac{\sin(\frac{\pi}{7})}{64}$. Dividing out the catalyst gives us the answer $-\frac{1}{64}$.

5 Practice

All of the following problems can be solved with important ideas in trigonometry.

1. The angle α has the property that $\sin(\alpha) + \cos(\alpha) = \frac{2}{3}$. Compute $\sin(2\alpha)$.

2. Compute the exact numerical value of $\cos(\frac{\pi}{9})\cos(\frac{3\pi}{9})\cos(\frac{5\pi}{9})\cos(\frac{7\pi}{9})$.

3. If β is an angle for which $\cos(\beta)+\sin(\beta) = .2$, then what is the value of $\cos^4(\beta)+\sin^4(\beta)$?

4. Determine all real $0 \leq \gamma < 2\pi$ such that $1 + \sin(2\gamma) = \sin(\gamma + \frac{\pi}{4})$.

5. Determine the sum of the values of $\tan(\theta)$ for which $0 \leq \theta < \pi$ and $1 = 2004\cos(\theta) \cdot (\sin(\theta) - \cos(\theta))$.

6. $ABCDEFG$ is a regular heptagon inscribed in a unit circle. Compute the value of the expression
$$AB^2 + AC^2 + AD^2 + AE^2 + AF^2 + AG^2$$

7. Determine the smallest positive angle ξ for which $8\cos(\xi)\cos(3\xi)\cos(5\xi)\cos(7\xi) + \cos(12\xi) = \frac{1}{2}$, leaving your answer in radians.

8. In $\triangle ABC$, $\angle B \cong 3\angle C$. If $AB = 10$ and $AC = 15$, compute the length of BC.