3D Geometry

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3D Geometry is often considered one of the hardest branches of geometry. This is because nobody is born with the ability to mentally manipulate 3D objects. Additionally, most people only practice geometry involving 2D shapes. Therefore, the easiest way to master 3D Geometry is to learn how to break apart 3D problems into multiple 2D problems. If you can divide 3D objects into many 2D pieces, you can effectively turn an AIME level problem into a SAT level Geometry problem.

Just like most other Geometry problems, 3D Geometry problems can oftentimes be solved with theorems that are taught in high school math classees (Alg1, Geo, Alg2, Trig). Subsequently, this lecture will concentrate on presenting advanced 3D Geometry problems and studying how they can be turned into simple 2D problems.

Problem: A circular torus with inner radius r and outer radius R is sitting on a plane. If a sphere is placed on the torus so that it is tagent to both the torus and the plane at an infinite number of points, what is the radius of the sphere?

Problem: A cube with side length 10 is suspended above a plane. The vertex closest to the plane is labeled A. The three vertices adjacent to vertex A are at heights 10, 11, and 12 above the plane. Find the distance from vertex A to the plane. (AIME 2011)

Problem: A square pyramid with base $ABCD$ and vertex E has eight edges of length 4. A plane passes through the midpoints of AE, BC and CD. Find the area of the plane's intersection with the pyramid. (AIME 2007)

Extra Practice: The problems below are of moderate/easy difficulty. If you want more practice with really hard 3D Geometry problems, google "aops aime". There are one or two 3D Geometry problem of hard difficulty in most of the AIMEs (they're usually numbers 13, 14 or 15).

- In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15cm^2 and the area of face ABD is 12cm^2. These two faces meet each other at a 30 angle. Find the volume of the tetrahedron. (AIME 1984)
- Let A, B, C and D be the vertices of a regular tetrahedron each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = \frac{n}{729}$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.