ARML Lecture I - Sequences and Series

VMT Math Team

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1 The Basics

A sequence is a list of numbers, typically denoted $a_1, a_2, \ldots$ Generally, in contests, we encounter two special types of sequences. *Arithmetic* sequences are sequences in which two consecutive terms a_n and a_{n-1} have a common difference, $a_n - a_{n-1} = d$. The other type, a *geometric* sequence has the characteristic ratio r between any two terms, with $\frac{a_n}{a_{n-1}} = r$.

Occasionally, we encounter a recursively defined sequence. That is, defining a general term as a function of a previous term. Perhaps the most common recursive sequence is the Fibonacci sequence, $1, 1, 2, 3, 5, 8, 13, \ldots$ defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for integers $n > 2$. Explicitly, we have $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$.

There are many notable facts about sequences:

- For an arithmetic sequence, we have $a_1 + a_2 + \cdots + a_n = (n)(\frac{a_1+a_n}{2})$. To verify this, consider pairing up the first and last terms, the second and penultimate terms, etc.

- For a geometric sequence with common ratio r, $a_1 + a_2 + \cdots + a_n = a_1(\frac{1-r^n}{1-r})$. To prove this, take advantage of the common ratio r and confirm that $(1-r)(a_1 + \cdots + a_n) = a_1 - r \cdot a_n = a_1(1-r^n)$.

- For the geometric sequence begining with a that has common ratio r such that $|r| < 1$, $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$. If $|r| < 1$, then what happens to r^n for a large value of n?

- $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. This is a special case of the general arithmetic sequence sum.

- $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. To verify the "sum of squares" and "sum of cubes" identities, consider induction on n over the positive integers.

- $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

- $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$. Consider a polynomial equivalent to $\frac{\sin \pi z}{z}$ and comparing the expansion of the first two terms with the corresponding power series.

2 Techniques

Given these facts we can compute a number of expressions. We know that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$ by an application of the infinite geometric sum formula with $a=r=\frac{1}{2}$. We can easily compute $1+2+3+\cdots+100=5050$. But what if an expression doesn't fit any of the known formulas? We consider two powerful techniques.

Algebra - It often helps to set $S_n=a_1+a_2+\cdots+a_n$. This was the major step in solving a difficult problem from the Duke Math Meet of 2002. Compute a_{16} given $a_1, a_2, \ldots, a_{50}$ with the property that for positive integers $n \leq 50$,

$$n(a_1+a_2+\cdots+a_n)=1+(a_{n+1}+a_{n+2}+\cdots+a_{50})$$

Setting $S_n=a_1+a_2+\cdots+a_n$, we have $nS_n=1+S_{50}-S_n$ from which $S_n=\frac{1+S_{50}}{n+1}$. Setting $n=50$ gives $S_{50}=\frac{1}{50}$, so $S_n=\frac{51}{50(n+1)}$. $a_{16}=S_{16}-S_{15}=\frac{51}{50}(\frac{1}{17}-\frac{1}{16})=\frac{-3}{800}$.

We can manipulate sums by subtracting multiples from each other. Example:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots=S$$

The numerators form an arithmetic sequence, so if we could subtract adjacent numerators somehow, we would get an infinite geometric sum... We can do this! We see that $2S=\frac{1}{1}+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\cdots$, and consequently $S=2S-S=\frac{(1-0)}{1}+\frac{(2-1)}{2}+\frac{(3-2)}{4}+\frac{(4-3)}{8}+\cdots=\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$.

Telescoping - Generically, this is the fact that $\sum_{i=1}^{n} f(i)-f(i-1)=(f(1)-f(0))+(f(2)-f(1))+(f(3)-f(2))+\cdots+(f(n)-f(n-1))=f(n)-f(0)$. This can be applied in a number of ways, with the goal of reducing a lot of terms to computably few. Almost anything can be forced to telescope. We will show how to apply telescoping to a number of problems.[1]

$$\begin{aligned} Ex.1: \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots &= \\ (1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+\cdots &= 1 \end{aligned}$$

$$\begin{aligned} Ex.2: \frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}} &= \\ (\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{100}-\sqrt{99}) &= 9 \end{aligned}$$

[1] Adjacent terms might not always cancel out. Related to example 2 is the sum: $\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{2}+\sqrt{4}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\cdots+\frac{1}{\sqrt{97}+\sqrt{99}}+\frac{1}{\sqrt{98}+\sqrt{100}}=\frac{1}{2}((\sqrt{3}-\sqrt{1})+(\sqrt{4}-\sqrt{2})+(\sqrt{5}-\sqrt{3})+\cdots+(\sqrt{99}-\sqrt{97})+(\sqrt{100}-\sqrt{98}))=\frac{1}{2}(10+3\sqrt{11}-\sqrt{2}-1)=\frac{9+3\sqrt{11}-\sqrt{2}}{2}$

$$\begin{aligned} Ex.3 : \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{2k^2}\right) &= \sum_{k=1}^{\infty} \tan^{-1}(2k+1) - \tan^{-1}(2k-1) \\ &= \tan^{-1}(\infty) - \tan^{-1}(1) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} Ex.4 : \sum_{k=1}^{99}(k^2+k+1)k! &= \sum_{k=1}^{99}\left((k+1)\cdot(k+1)! - k\cdot k!\right) \\ &= 100\cdot 100! - 1 \end{aligned}$$

$$\begin{aligned} Ex.5 : \sum_{k=1}^{99}(k^4+2k^3+2k^2) &= \frac{1}{5}\sum_{k=1}^{99}(5k^4+10k^3+10k^2) \\ &= \frac{1}{5}\sum_{k=1}^{99}\big((k+1)^5 - k^5 - 5k - 1\big) \\ &= \frac{1}{5}\left(100^5 - 1^5 - \sum_{k=1}^{99}(5k+1)\right) \\ &= \frac{1}{5}(999999999 - 99\cdot 250 - 99) = 199995030 \end{aligned}$$

3 Recursive Sequences

When solving problems involving recursive sequences, it is often important to find repetition somewhere. In most cases, this can be done by listing a few terms, but occasionally it is necessary to employ other tactics. We summarize two below:

- *Modulos* - Consider the sequence in various modulos. Often it is asked "What is the last digit of...?" which should immediately bring modulo 10 to mind.
- *Conversion* - The recursive definition can be converted to an explicit formula or vice versa. A sequence $\{X\}$ defined by $a_k X_n + a_{k-1}X + \cdots + a_0 X_{n-k} = 0$ and some initial values $X_1, \ldots X_k$ can be written explicitly as

  $$X_n = \omega_1 r_1^n + \cdots + \omega_k r_k^n$$

  where r_i are the k distinct (possibly complex) roots of the polynomial $a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0$ and ω_i are chosen according to the values of $X_1, \ldots, X_k$.[2] Moreover, if $r_1, \ldots, r_m$ are the solutions of $a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0 = 0$ where r_i appears d_i times, then

  $$X_n = \sum_{i=1}^{m} P_i(n) r_i^n$$

[2] So that the linear system $\left(\sum_{i=1}^{k}\omega_i r_i^m\right) = X_m$ is satisfied for $\forall m \in \{1, 2, \ldots, k\}$.

where each $P_i(n)$ is a polynomial in n with degree $(d_i - 1)$.

4 Practice

The following problems can be solved with important sequence ideas:

1. Determine the exact value of $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots$

2. Compute the exact value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

3. Determine the value of $\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + \cdots$

4. Compute the smallest positive angle x such that $\sin x + \sin 3x + \cdots + \sin 99x = \csc x$, expressing your answer in radians.

5. $a_1, a_2, a_3, \ldots, a_{100}$ are real numbers satisfying

$$a_1 + \cdots + a_n = n(1 + a_{n+1} + \cdots + a_{100})$$

for all integers $1 \leq n \leq 100$. What is the value of a_{13}?

6. Josh is walking from his house to ARML practice, but instead of taking the most direct route, he decides to take a more interesting route. Starting from his house, he walks 5 miles due east. He then turns to his left and walks directly to a place that is 2 miles further east and 3 miles north. For the third leg of his journey, he turns the same amount to his left and walks for the same fraction of the previous leg of his journey. He continues this process and eventually converges upon the ARML practice. How far would he have to walk to go straight from his house to ARML practice?

7. $a_1, a_2, a_3, \ldots, a_{100}$ are real numbers satisfying

$$1 \cdot a_1 + \cdots + k \cdot a_k + \cdots + n \cdot a_n = n \cdot (1 + a_{n+1} + \cdots + a_{100})$$

for positive integers $n \leq 100$. Compute the value of $\frac{a_9}{a_{11}}$.

8. (41st AHSME) Let $R_n = \frac{1}{2}(x^n + y^n)$, where $x = 3 + 2\sqrt{2}$ and $y = 3 - 2\sqrt{2}$. Compute the units digit of R_{12345}.