Inequalities

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1 Inequalities

Although inequalities do not appear on every single competition, they are not uncommon on proof-based olympiads such as the USA(J)MO. It is difficult to solve these kinds of problems if one has never solved one, so that purpose of this lecture is to introduce the common theorems and some useful tips to solving inequalities.

2 Theorems

2.1 Trivial Inequality

$x^2 \geq 0$, for all real numbers x. This is a very basic inequality that should be kept in mind while solving inequalities.

2.2 AM-GM Inequality

The Arithmetic Mean-Geometric Mean Inequality, also known as AM-GM, is one of the most frequently used theorems.

It states that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

for positive real numbers $a_1, a_2, \cdots, a_n$ with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Sometimes, you will have to use **Weighted AM-GM**, which states that

$$\omega_1 a_1 + \omega_2 a_2 + \cdots + \omega_n a_n \geq a_1^{\omega_1} a_2^{\omega_2} \cdots a_n^{\omega_n}$$

for positive real numbers $a_1, a_2, \cdots, a_n, \omega_1, \omega_2, \cdots, \omega_n$ with $\omega_1 + \omega_2 + \cdots + \omega_n = 1$.

2.3 Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality is not as common as AM-GM, but it is also widely used.

It states that

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2$$

for all real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ with equality if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

2.4 A Helpful Inequality

If a, b, x, y are real numbers and $x, y > 0$, then the following inequality holds:

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}.$$

The proof for this is quite simple. Clearing out the denominators, we can change the inequality to

$$a^2 y(x+y) + b^2 x(x+y) \geq (a+b)^2 xy,$$

which becomes $(ay - bx)^2 \geq 0$, which is obvious from the Trivial Inequality.

A different method of proving this inequality is by using the Cauchy-Schwarz Inequality in the following way:

$$(a+b)^2 = \left(\frac{a}{\sqrt{x}}\sqrt{x} + \frac{b}{\sqrt{y}}\sqrt{y}\right)^2 \leq \left(\frac{a^2}{x} + \frac{b^2}{y}\right)(x+y).$$

By a simple inductive argument, we can show that

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_n} + \cdots + \frac{a_n^2}{x_n} \geq \frac{\left(a_1 + a_2 + \cdots + a_n\right)^2}{x_1 + x_2 + \cdots + x_n}$$

for all real numbers $a_1, a_2, \cdots, a_n$ and $x_1, x_2, \cdots, x_n > 0$, with equality if and only if

$$\frac{a_1}{x_1} = \frac{a_2}{x_2} = \cdots = \frac{a_n}{x_n}.$$

3 Techniques

3.1 Substitution

Substitutions are mainly used in two cases: (a) When some expression appears multiple times in the inequality; (b) When there are constraints on the variables in the inequality.

Substitution is generally used to make a complicated inequality simpler and easier to solve.

3.2 Find the Equality Cases!

Most of the inequality problems have $\geq$ or $\leq$ rather than $>$ or $<$. It is a good idea to find a case where the two sides of the inequality are equal.