2012-13 TJUSAMO Practice Olympiad 1

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Hi guys, sorry Sohail and I couldn't be here today (Soahil is currently in Mexico taking the Mexican Math Olympiad because he's cool like that, and I'm working with the PUMaC team trying to understand algebriac integers.), but thanks for coming! :) We realize that we really haven't given many lectures so far so it seems a bit early for the practice contest, but since there was no one to lecture we had no choice. Taking that fact into consideration, feel free to treat this practice contest as a sort of pre-test if you'd prefer; we're trying to get a sense of where you stand in terms of proof-writing through this exercise, so definitely shoot for partial credit and turn stuff in!

When writing your solutions, start each new question on a new page. Put your name, the problem number, and the page number of your solution out of the total for that problem in the upper-right corner of each page you write on. (For example, "Victoria Xia, Problem 2, Page 2/3" means that you are impersonating me, writing a solution for problem 2, and your solution to problem 2 is three pages total and I'm looking at the second of those three.) Try to leave margins so I can give you feedback and only write on one side of each sheet.

If you do not understand any of the problems or have other questions, feel free to find me in the PUMaC practice room (one of these trailers; ask whomever is proctoring) and ask. Good luck and enjoy!

1. Let m and n be nonnegative integers. Prove that $\frac{(2m)!(2n)!}{m!n!(m+n)!}$ is always an integer.

2. Let ABC be an acute triangle. A circle going through B and the triangle's circumcenter O intersects BC and BA at points P and Q respectively. Prove that the intersection of the heights of the triangle POQ lies on the line AC.

3. A bookshelf contains n volumes, labeled 1 to n, in some order. The librarian wishes to put them in the correct order as follows: The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it in the k-th position. For example, if the bookshelf contains the volumes $1, 3, 2, 4$ in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order $1, 2, 3, 4$.

   (a) Show that if this process is repeated, then, regardless of how the librarian selects which books to move (as long as the rules are obeyed), all the volumes will eventually be in the correct order.
   (b) What is the largest number of steps that this process can take (for all the books to end up in the correct order)?