Inequalities

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April 25, 2013

1 Basic Inequalities

1.1 Trivial Inequality

The Trivial Inequality states that the square of any real number is nonnegative:

$$x^2 \geq 0 \ \forall \ x \in \mathbb{R}.$$

Simple? Yes, but it's at the heart of countless problems.

1.2 QM-AM-GM-HM

The Quadratic Mean (Root Mean Square) - Arithmetic Mean - Geometric Mean - Harmonic Mean Inequality states that, for positive reals $x_1, x_2, \ldots, x_n$,

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

You are probably familiar with the AM-GM Inequality, but it is important to acquaint yourself with the Harmonic Mean. This is especially useful when the denominators of an expression are more complicated than the numerators. Note that equality is reached when all x_i are equal. The generalization of the AM-GM inequality is the Power Mean Inequality. It states that, for real numbers k_1, k_2 and $k_1 \geq k_2$,

$$\left(\frac{\sum_{i=1}^{n} x_i^{k_1}}{n}\right)^{\frac{1}{k_1}} \geq \left(\frac{\sum_{i=1}^{n} x_i^{k_2}}{n}\right)^{\frac{1}{k_2}}.$$

1.3 Cauchy-Schwarz Inequality

The elementary form of the Cauchy-Schwarz Inequality states

$$(x_1y_1 + \cdots + x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Note that equality is reached when, for some constant k, $x_i = ky_i\ \forall\ i$.

1.4 Rearrangement Inequality

The Rearrangement Inequality is an application of the Greedy Algorithm. Define $A = \{a_1, a_2, \cdots, a_n\}$ as any permutation of a set of real numbers. Similarly, define $B = \{b_1, b_2, \cdots, b_n\}$ as any permutation of another set of real numbers. Let $s = a_1b_1 + a_2b_2 + \cdots + a_nb_n$. The Rearrangement Inequality states that s is maximal when A and B are sorted similarly, e.g. $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. It also states that s is minimal when they are sorted oppositely.

2 Common Techniques

2.1 Substitution

Expressing the inequality in terms of other variables, i.e. substituting $x = a + b, y = b + c, z = c + a$, can often simplify the problem, especially when denominators are complicated and we can afford to make numerators slightly more complicated.

2.2 Symmetry and Homogeneity

When the inequality is symmetric, try assuming WLOG $a \leq b \leq c$, etc. This is helpful when one side of the inequality has max or min operators. We can often try proving that each term in the LHS is greater than a corresponding term in the RHS, i.e. if $a > x, b > y, c > z$, then $a + b + c > x + y + z$. When all terms in an inequality are of the same degree, we can assume one of the variables to be 1. Often, a better substitution is $a + b + c = 1$ or $abc = 1$, as this preserves symmetry.

2.3 Applying Weights

Applying weights to AM-GM eliminates undesirable variables in the Geometric Mean.
Example. Prove $2x + \frac{1}{x^2} \geq 3 \; \forall \; x \in \mathbb{R}^+$.
Directly applying AM-GM yields $2x + \frac{1}{x^2} \geq 2\sqrt{\frac{2}{x}}$?! However, splitting $2x$ into $x + x$ before applying AM-GM achieves the desired result.

2.4 The Cauchy Reverse Technique

The Cauchy Reverse Technique helps deal with reversed signs in expressions.
Example. (Bulgaria TST 2003) Let a, b, c be positive real numbers with sum 3. Prove that

$$\frac{a}{1+b^2} + \frac{b}{1+c^2} + \frac{c}{1+a^2} \geq \frac{3}{2}.$$

It's impossible to directly use AM-GM for the denominators because the sign will be reversed

$$\frac{a}{1+b^2} + \frac{b}{1+c^2} + \frac{c}{1+a^2} \leq \frac{a}{2b} + \frac{b}{2c} + \frac{c}{2a} \geq \frac{3}{2}?!$$

However, we can switch the signs by dividing the denominator into the numerator:

$$\frac{a}{1+b^2} = a - \frac{ab^2}{1+b^2} \geq a - \frac{ab^2}{2b} = a - \frac{ab}{2}.$$

The inequality becomes

$$\sum_{cyc} \frac{a}{1+b^2} \geq \sum_{cyc} a - \frac{1}{2}\sum_{cyc} ab \geq \frac{3}{2}$$

since $3(\sum ab) \leq (\sum a)^2 = 9$.
This concludes the proof. Equality holds for $a = b = c = 1$.

3 Problems

1. Let a, b, c be positive real numbers. Prove the inequality

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \leq \frac{a+b+c}{4}.$$

2. Let a, b, c, d be positive real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove the inequality

$$a + b + c + d \geq ab + bc + cd + da.$$

3. (USAMO 2003) Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

4. (Russia MO 2004) Let a, b, c be positive real numbers with sum 3. Prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca.$$

5. Let a, b, c, d be positive real numbers. Prove that

$$16(abc + bcd + cda + dab) \leq (a + b + c + d)^3.$$

6. Prove that $x_1 x_2 \cdots x_n \geq (n-1)^n$, if $x_1, x_2, \cdots, x_n > 0$ satisfy

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} = 1.$$

7. (Mathlinks Contest) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{\frac{a+b}{a+1}} + \sqrt{\frac{b+c}{b+1}} + \sqrt{\frac{c+a}{c+1}} \geq 3.$$

8. Let a, b, c be non-negative real numbers with sum 2. Prove that

$$a^2b^2 + b^2c^2 + c^2a^2 \leq 2.$$