Expected Value

Joseph Park

March 14, 2013

1 Definition

Let X be some random variable. Then,

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

where n is the total number of possible outcomes, x_i denotes the value (weight) of certain outcome, and p_i denotes the probability of that certain outcome. Note that expected value has nothing to do with most probable value. Rather, it is the weighted average of the outcomes.

2 Properties

For contest purposes, these may be useful in expected value calculations.

- $E(X + c) = E(X) + c$
- $E(X + Y) = E(X) + E(Y)$
- $E(cX) = cE(X)$
- $E(XY) = E(X)E(Y)$ if and only if X and Y are uncorrelated

3 Examples

1. A six-sided die with numbers 1-6 has probabilities in the ratio $1 : 2 : 3 : 4 : 5 : 6$ for landing with sides 1;2;3;4;5;6 facing up, respectively. Find the expected value of the roll.

2. What is the expected number of times the string TJ comes out in a permutation of $TJTJTJTJTJ$?

3. In a magic circuit, there are six lights in a series, and if one of the lights short circuit, then all lights after it will short circuit as well, without affecting the lights before it. Once a turn, a random light that isnt already short circuited is short circuited. What is the expected number of turns it takes to short circuit all the lights?

4 Problems

1. Olivia knits one sock every four seconds. Every fifth sock has a Mandelbrot pattern and all other socks have distinct colors. After each sock is made, Olivia flips a fair coin and gives the sock to Diana if the coin lands heads. What is the expected amount of time Diana has to wait to get a matching pair of Mandelbrot socks?

2. 20 math teamers are trying to form 10 pairs. If there are 5 students of each grade (5 freshmen, 5 sophomores, etc.), then what is the expected number of pairs with students of different grades?

3. We have 5 balls. Suppose that balls are tossed into 4 bins. Each toss is independent, and each ball is equally likely to end up in any bin. What is the expected number of ball tosses before at least one of the bins contains two balls?

4. Brian, Kee Young, Dan, and Greyson are playing tuolaji, which is played with two decks of 54 cards each (the standard 52 plus two jokers). Akshar, who doesn't understand the game at all but is watching anyway, counts the number of cards that show up before each joker (including cards before a previous joker and previous jokers, but not the current joker). What is the expected sum of the four numbers that he will obtain?

5. Suppose you and I play a game. You start with 500 dollars and I start with 1000 dollars. We ip a fair coin repeatedly. Each time it comes up heads, I give you a dollar. Each time it comes up tails, you give me a dollar. We continue playing until one of us has all the money. What is the probability you will win this game?

6. I give you a deck of n cards numbered 1 through n. On each turn, you take the top card of the deck and place it anywhere you choose in the deck. You must arrange the cards in numerical order, with card 1 on top and card n on the bottom. If I place the deck in a random order before giving it to you, and you know the initial order of the cards, what is the expected value of the minimum number of turns you need to arrange the deck in order?