ARML 2013 - Algebraic Recursion

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Problems often come up where, given a function $f(x)$, it's helpful to know $f(f(f(...f(f(x))...))) = f^{(n)}(x)$. The notation "$f^{(n)}(x)$" (where there are n f's) is the nth iteration of f. Another version of the same idea is a recursive sequence expressing a_{n+1} in terms of a_n, such as $a_{n+1} = 2a_n + 1$. To see why these are the same, let $x = a_0$. Then $f(x) = a_1$, $f^{(2)}(x) = a_2$, and so on.

This lecture covers techniques on how to find the nth iteration of a function or the general term of a recursive sequence.

Below are three basic functions that we know how to handle:

$$\begin{aligned} f(x) = x + k \quad &\Rightarrow \quad f^{(n)}(x) = x + nk \\ f(x) = kx \quad &\Rightarrow \quad f^{(n)}(x) = k^n x \\ f(x) = x^k \quad &\Rightarrow \quad f^{(n)}(x) = x^{k^n} \end{aligned}$$

And some more to keep in mind:

$$\begin{aligned} a_n = a_{n-1} + kn \quad &\Rightarrow \quad a_n = a_0 + k\Big(\frac{n(n+1)}{2}\Big) \\ a_n = a_{n-1} + kn^2 \quad &\Rightarrow \quad a_n = a_0 + k\Big(\frac{n(n+1)(2n+1)}{6}\Big) \\ a_n = a_{n-1} + kn^3 \quad &\Rightarrow \quad a_n = a_0 + k\Big(\frac{n(n+1)}{2}\Big)^2 \end{aligned}$$

0 Telescoping

Like yesterday's NYCIML!

1 Finding b_n (or $g(x)$)

The idea is that if we can find a series b_n, such that we can easily convert between a_n and b_n, and we know how to find the general term b_n, then we have a way to find a_n. This will make a lot more sense with examples.

1.1 Example

Given that $a_{n+1} = a_n^2 + 2a_n$ for all integers $n \geq 0$, if $a_0 = 4$, find a_{150}.

Solution: The right side reminds us of perfect squares. Adding one to both sides yields $a_{n+1} + 1 = a_n^2 + 2a_n + 1 = (a_n + 1)^2$. Making the substitution $b_n = a_n + 1$, we have $b_{n+1} = b_n^2$, so $b_n = b_0^{2^n}$. Converting back from b_n to a_n, we have $a_n + 1 = (a_0 + 1)^{2^n} \Rightarrow a_n = (a_0 + 1)^{2^n} - 1 \Rightarrow a_{150} = 5^{2^{150}} - 1$.

1.2 The Common $a_{n+1} = ca_n + d$

Linear relations are easy to transform by adding a constant λ to both sides of the equation: $a_{n+1} + \lambda = c(a_n + \frac{d+\lambda}{c})$. If we could find a λ so that $\lambda = \frac{d+\lambda}{c}$, then the equation would look like the second of the three basic functions. Solving for λ, we get $\lambda = \frac{d}{c-1}$. Replacing λ, we then have $(a_{n+1} + \frac{d}{c-1}) = c(a_n + \frac{d}{c-1})$, and letting $b_n = (a_n + \frac{d}{c-1})$, we have $b_{n+1} = cb_n$. From the second basic equation, we then have $b_n = c^n b_0$. Replacing b_n we have

$$a_n + \frac{d}{c-1} = c^n(a_0 + \frac{d}{c-1})$$

$$a_n = c^n(a_0 + \frac{d}{c-1}) - \frac{d}{c-1}.$$

Example: (HMMT2013) Values $a_1, a_2, ...a_{2013}$ are chosen independently and at random from the set $\{1, 2, ..., 2013\}$. What is the expected number of distinct values in the set $\{a_1, a_2, ..., a_{2013}\}$?

2 Linear Recurrences and Characteristic Equations

The recurrence relation $a_n = c_1a_{n-1}+c_2a_{n-2}+\cdots+c_da_{n-d}$ has the general term $a_n = k_1r_1^n+k_2r_2^n+\cdots+k_dr_d^n$, where $r_1, r_2, ..., r_d$ are the distinct roots of the *characteristic polynomial* $f(x) = x^d-c_1x^{d-1}-c_2x^{d-2}-\cdots-c_d$, and $k_1, ..., k_d$ are constants. (Note that if the roots of the characteristic polynomial are not distinct then the form changes slightly.)

Example: (AoPS) Let there be a sequence $\{a_n\}$ with initial values $a_0 = 0, a_1 = 38, a_2 = -90$ given by the recurrence relation $a_{n+1} = 19a_{n-1} - 30a_{n-2}$ for $n \geq 2$. Prove that a_{2010} is divisible by 2011.

3 Other Tips

1. Ranges are helpful. If, for instance, a problem states that the function only applies for values of x between -1 and 1, think sine or cosine!

   Example: Let $a_{n+1} = 4a_n(1 - a_n)$, $0 \leq a_n \leq 1$ for all a_n. Find the general term a_n if $a_0 = \frac{1}{4}$.

   Solution: The given range perfectly allows for the substitution $a_n = \sin^2(b_n)$. Substituting, we get $\sin^2(b_{n+1}) = 4\sin^2(b_n)(1-\sin^2(b_n)) = 4\sin^2(b_n)\cos^2(b_n) = \sin^2(2b_n)$, so $b_{n+1} = 2b_n$. That means that $b_n = 2^nb_0$, and it follows that $a_n = \sin^2(b_n) = \sin^2(2^nb_0) = \sin^2(2^n\arcsin(\sqrt{a_0}))$. The last step comes from the fact that $a_n = \sin^2(b_n)$ implies $b_n = \arcsin(\sqrt{a_n})$. Substituting in the fact that $a_0 = \frac{1}{4}$, we then have $a_n = \sin^2(\frac{2^n\pi}{6}) = \sin^2(\frac{2^{n-1}\pi}{3})$.

2. List out the first few terms of a sequence if you don't know what to do; easy problems will usually reveal a pattern.

4 Problems

1. Five monkeys discover a pile of peaches by a lake. They agree to divide the pile evenly the next day. The next morning, the first monkey comes and tries to split the pile into five equal parts, but fails because there's one left over. His solution is to toss one peach into the lake, and then takes one pile (1/5 of what's left) and leaves. The second monkey comes, but doesn't know the first one already took his share, so he tries splitting the pile into fifths, but ends up with one left over. He tosses a peach into the lake, takes his share, and leaves. The third, fourth, and fifth monkeys all have the same problem and do the same thing. What is the least number of peaches that could've been in the original pile?

2. Let $f(x) = 19x + 89$. Find the ones digit of $f^{(100)}(5)$.
3. Given that $f(x) = 2x^2 + 2x$ for $-\frac{1}{2} \leq x$, find $f^{(n)}(x)$.
4. If $a_0 = 1$ and $a_{n+1} = (n+1)a_n + (n+2)!$ for $n \geq 0$, find a_n.
5. Let $f(x) = \frac{x-3}{x+1}$ for $x \neq \pm 1$. Find $f^{(27)}(x)$.
6. Find the general term F_n of the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
7. Given that $a_{n+1} = 2a_n^2 - 1$ and -1 $\leq a_n \leq 1$ for all a_n, find the general term a_n.
8. Let $f(x) = \sqrt{2+x}$, -2 $\leq x \leq 2$. Find $f^{(n)}(x)$. (**Hint:** Half angle formula! Also, the range being from -2 to 2 instead of -1 to 1 means something...)
9. Find $f^{(n)}(x)$ for $f(x)$ as defined in Question 6, but for the domain $x > 2$.