2012 ARML - Advanced Geometry Solutions

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1. **Problem:** In convex pentagon $ABCDE$, $[ABC] = [BCD] = [CDE] = [DEA] = [EAB] = 1$, where $[ABC]$ denotes the area of triangle ABC. Compute the area of the pentagon.

   Solution: Let the intersection of BD and CE be X. Because $[EDC] = [DCB]$ and the two triangles share a base, their heights must be equal so $EB \parallel DC$. Similarly, $BD \parallel AE$ and $EC \parallel AB$. Thus, $ABXE$ is a parallelogram, so $[EXB] = [AEB] = 1$. Now if we could find $[DXC]$ then we'd be done, so let that be m. We have $[EDX] = [BCX] = 1 - m$. Thus,

   $$\frac{[DXC]}{[EDX]} = \frac{[XBC]}{[XEB]} \Rightarrow \frac{m}{1-m} = \frac{1-m}{1} \Rightarrow m = \frac{3-\sqrt{5}}{2}$$

   The final answer is therefore $\frac{5+\sqrt{5}}{2}$.

2. **Problem:** Hexagon $ABCDEF$ is inscribed in a circle such that $AB = BC = CD = 2$ and $DE = EF = FA = 6$. Find the diameter of the circle.

   Solution: Because equal chords cut off equal arcs, the diameter of the circle we seek is the same as the diameter of the circle circumscribing a hexagon with side lengths of 2, 6, 2, 6, 2, and 6, in that order. Connecting alternating vertices forms an equilateral triangle. The side length of this triangle can be found with the Law of Cosines: $\sqrt{2^2 + 6^2 - 24\cos 120^\circ} = 2\sqrt{13}$. The diameter of the circle is then $\frac{4\sqrt{39}}{3}$.

3. **Problem:** Triangle XYZ is such that $XY = 5$, $YZ = 7$, and $ZX = 6$. Find the minimum value of $(XP + YP + ZP)^2$ for any point P in the interior of $\triangle XYZ$.

   Solution: P is the Fermat Point of triangle XYZ, so $\angle XPY = \angle YPZ = \angle ZPX = 120^\circ$. Let $XP = a$, $YP = b$, and $ZP = c$. From the Law of Cosines, we have $25 = a^2 + b^2 + ab$, $49 = b^2 + c^2 + bc$, and $36 = a^2 + c^2 + ac$. Summing these three equations we have

   $$110 = 2a^2 + 2b^2 + 2c^2 + ab + bc + ca$$

   Notice that $[XYZ] = [XPY] + [YPZ] + [ZPX] = \frac{1}{2}(ab + bc + ca)\sin 120^\circ$. From Heron's Formula this total area equals $6\sqrt{6}$ so $ab + bc + ca = 24\sqrt{2}$. Adding three times this new equation to our previous one,

   $$110 + 72\sqrt{2} = 2a^2 + 2b^2 + 2c^2 + 4ab + 4bc + 4ca \Rightarrow (a + b + c)^2 = 55 + 36\sqrt{2}$$

4. **Problem:** (Mandelbrot, AoPS) Find the length of the median of a trapezoid whose diagonals are perpendicular segments of lengths 7 and 9.

   Solution: Let the trapezoid be $ABCD$ such that $AB \parallel CD$, $AC = 9$, and $BD = 7$. Construct a line through A parallel to BD so that this line intersects the extension of CD at E. Now $ABDE$ is a parallelogram, so $DE = AB$ and AE is perpendicular to AC so $CE = \sqrt{130}$. The length of the median we seek is given by $\frac{1}{2}(AB + CD) = \frac{1}{2}(DE + CD) = \frac{1}{2}CE = \frac{\sqrt{130}}{2}$.

5. **Problem:** Given triangle ABC, point D is on BC such that $\frac{BD}{DC} = 5$ and E is on AC such that $\frac{AE}{EC} = 2$, and let G be the intersection of AD and BE. If $[GDC] = 5$, find $[AGE]$.

   Solution: Extend CG so it intersects AB at F. From Ceva's Theorem, $\frac{BF}{FA} = \frac{5}{2}$. Because $\frac{[BGD]}{[DGC]} = \frac{BD}{DC}$, $[BGD] = 25$. Because $\frac{[BGC]}{[AGC]} = \frac{BF}{FA}$, $[AGC] = 12$. It can then be computed that $[AGE] = 8$.