Burnside's Lemma

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1 Introduction

Burnside's Lemma (also known as not-Burnside's lemma[1]) is an interesting result in group theory. Although it can only occasionally be applied in competition math problems, one never knows when it could be useful! Plus, Burnside's Lemma is just cool to learn about. :D

2 Burnside's Lemma

Burnside's Lemma states:

Let X be a finite set and G be a group of permutations of X, and let X^g, where $g \in G$, be the set of elements of X unaffected, or fixed, by g. Additionally, the *orbit* of an element $x \in X$ is defined as the set $Gx \subseteq X$ of elements of X to which x can be moved by the elements of G. If we denote the set of distinct orbits of G acting on X by $\frac{X}{G}$, then

$$\left|\frac{X}{G}\right| = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

In other words, Burnside's Lemma states that the number of distinct orbits of G acting on X is the average number of elements of X fixed by an element of G. (2014sare)

3 Applications

What can we use this seemingly contrived lemma for? In fact, since X, G, and g are arbitrary, we can use Burnside's lemma for quite a few applications. If we let g be rotations of X, we can find the number of rotationally distinct configurations in X. On the other hand, if we let g be reflections of X, we can find the number of 'reflectionally' distinct configurations in X. In fact, this can be extrapolated to any type of permutation of X.

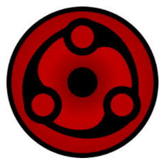
[1] Apparently, mathematicians Cauchy and Frobenius knew the lemma long before Burnside proved it.

4 2D Applications

4.1 Rotations

As mentioned before, one of the main types of Burnside's Lemma problems is finding rotationally distinct configurations. Examples are a great place to start for understanding Burnside's Lemma, so let's take a look at one.

Ex. 1 Some of you may recognize the picture below:



If you can fill in the open circles with three distinct colors, find the number of rotationally distinct ways to color the picture above.

Solution: We see that we have three different types of rotations: the identity (aka no rotation), rotation by 1 open circle, and rotation by 2 open circles. The identity gives us 3^3 unique configurations. Rotation by 1 open circle means that all three colors have to be the same, which gives us 3 unique configurations. Rotation by 2 open circles means that again, all three colors have to be the same, which gives us 3 unique configurations. The average number of unique configurations per rotation type is $\frac{3^3+3+3}{3} = 11$, which means that by Burnside's Lemma, the number of rotationally distinct configurations is 11.

4.2 Reflections

Let's take a look at another example:

Ex. 2 A 2x2 square is made up of four unit squares. If you have five distinct colors, how many different ways are there to color the unit squares if configurations obtained by reflection are not distinct?

Solution: We have four different reflections to consider: up-down, left-right, and the two diagonals. For the up-down reflections, the top two squares are free to have any color, so we have $5*5 = 25$ distinct configurations. The left-right reflections also similarly have 25 distinct configurations. For diagonal reflections, we have free choices for three out of four squares, while the fourth square is forced, so we have $5*5*5 = 125$ distinct configurations for each diagonal reflection. The average is $\frac{25+25+125+125}{4} = 75$. Therefore, by Burnside's Lemma, there are 75 distinct configurations.

5 3D Applications

This is one of the more difficult applications of Burnside's Lemma, as anything 3D is automatically (highly) difficult. Visualizing things like rotations of 3D shapes is quite difficult to begin with, and in 3D, you can play around with coloring vertices, faces, and edges, and these correspond to three different types of problems. To visualize rotations, I personally recommend building your own tetrahedrons and cubes :)

Since it's impossible to cover all of the types of 3-D Burnside's Lemma problems in a 45-minute lecture, I'll only provide one example here: tetrahedron rotation with edges.

Ex. 3 Find the number of rotationally distinct ways of coloring a tetrahedron's edges with three distinct colors.

Solution: There are 12 different types of rotations of a tetrahedron: the identity, eight 120-degree rotations, and three 180-degree rotations. The identity gives you 3^6 configurations, the 120-degree rotations give you 3^2 configurations (2 cycles of 3), and the 180-degree rotations give you 3^4 configurations (2 edges stay the same, and 2 pairs of edges are swapped.) The number of distinct configurations then becomes $\frac{1*3^6+3*3^4+8*3^2}{12} = 87$ configurations.

If you're interested in more types of Burnside's Lemma problems, or just want to find out more about it, google Burnside's Lemma – many of the documents that pop up are pretty helpful.

6 Problems

6.1 Standard Practice

1. How many rotationally distinct ways are there to color the vertices of a pentagon with 3 distinct colors?

2. How many distinct ways can you color the vertices of a square with 4 colors, if configurations obtained by reflection are not distinct?

3. How many distinct ways can you color the edges of a square with 5 colors, if configurations obtained by rotation OR reflection are not distinct?

6.2 Watch your step!

1. Find the number of rotationally distinct necklaces with 6 white beads and 3 black beads.

2. Find the number of rotationally distinct ways to color the vertices of a 15-gon, if you have four distinct colors.

3. How many ways can you fill a tic-tac-toe board with 5 Xs and 4 Os if rotations of the same board are not considered distinct?

4. Repeat the question above, but with rotations OR reflections not distinct. (This one is quite the challenge.)

6.3 Have fun (3D)

1. Robin Park, an avid *Team Fortress 2* player, has collected many hats. His favorite hat, the strange unique vintage genuine collector's self-made tetrahedron, has four colorable vertices: at the start of each game, he can pick any of 5 colors for each of the four vertices. Perry, who plays with Robin very often, wants to know how many games he must play before Robin is forced to repeat a hat. Help Perry by computing the number of possible configurations of Robin's strange unique vintage genuine collector's self-made tetrahedron. Rotations of the same hat are not considered distinct.

2. Find the number of rotationally distinct colorings of the faces of a cube using two colors.

6.4 Remember this?

1. Samuel is making fancy origami for a special someone ;) He starts out with a 2x2 square, and splits it up into 4 unit squares to plan out his intricate folding. He also draws one of the diagonals in each unit square (note that there are two different diagonals in a unit square). Finally, he arbitrarily colors blue any number of his newly-made 8 triangles. An example of one of Samuel's possible configurations is shown below. Find the number of distinct configurations Samuel can make, given that configurations that can be reached by rotation or reflection are not distinct. (ARML Practice 2014)