Inversion

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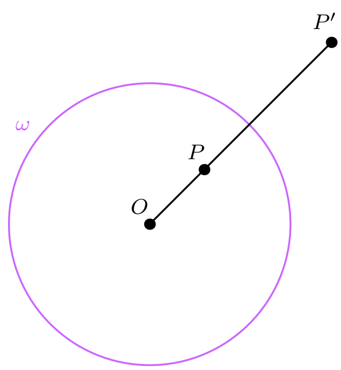
Inversion is a type of geometric transformation on the generalized Euclidean plane, also called the *inversive plane*. Many difficult geometry problems can be reduced into a simpler geometry problem when inversion is applied. Inversion can be generalized to higher dimensions, but we will only consider the two-dimensional case because Olympiad geometry problems are typically restricted to two dimensions.

1 Definition

An inversion with respect to a given circle ω with center O and radius r is the mapping that takes a point P to the point P', called the *inverse* of P, on ray $\overrightarrow{OP}$ such that

$$OP \cdot OP' = r^2.$$



Notice that when inversion is applied twice to the same circle, we are left with our original diagram. Hence inversion is an *involution*. However, we do not know where the center of inversion O is mapped to; there is no such point O' on the Euclidean plane such that $0 \cdot OO' = r^2$. Therefore, we introduce the *point at infinity*. By definition, the point at infinity is the inverse of O, and so O is the inverse of the point at infinity. The Euclidean plane equipped with the point at infinity compose the inversive plane, which we will work in throughout this lecture.

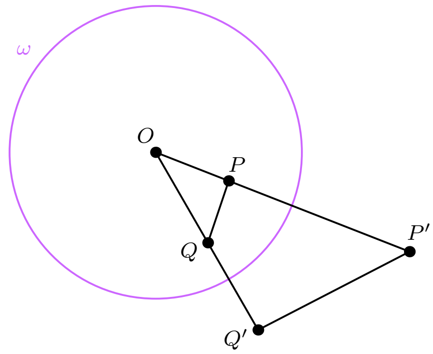
In addition, all lines pass through the point at infinity.

2 Properties

There are many properties of inversion that are applied frequently in solving problems.

Lemma 1. *Let ω be a circle with center O and radius r, and let P' and Q' be the inverses of P and Q, respectively, with respect to ω. Then we have*

$$\angle OQ'P' = \angle OPQ \text{ and } P'Q' = \frac{r^2}{OP \cdot OQ}PQ.$$



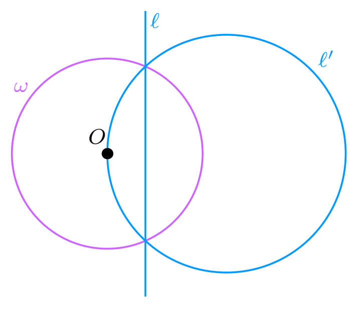
Proof. Since $\angle P'OQ' = \angle QOP$ and $\frac{OP'}{OQ'} = \frac{\frac{r^2}{OP}}{\frac{r^2}{OQ}} = \frac{OQ}{OP}$, triangles $P'OQ'$ and QOP are similar; hence $\angle OQ'P' = \angle OPQ$. Furthermore, note that the ratio of similitude is $\frac{r^2}{OP \cdot OQ}$, and so we have $P'Q' = \frac{r^2}{OP \cdot OQ}PQ$. $\square$

By Lemma 1, angles are preserved under inversion. Such transformations are called *conformal*, which means that directed angles are preserved under the transformation. We now proceed to the images of lines and circles under inversion.

Lemma 2. *A line through O is mapped to itself.*

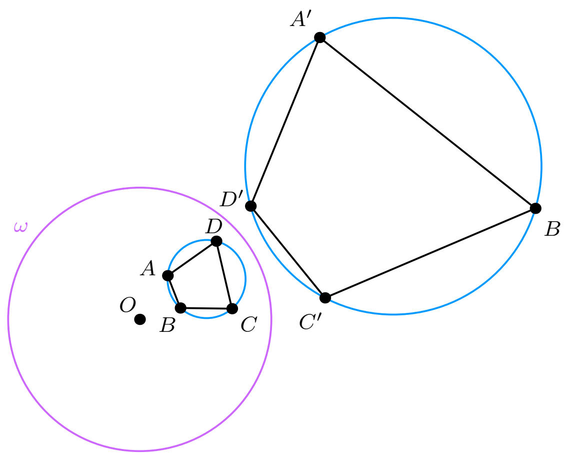
Proof. By definition, all points on the line are mapped to a point on the ray connecting O and itself, so it is mapped to itself. O itself is mapped to the point at infinity, which lies on the line, and the point at infinity is mapped to O, which also lies on the line. $\square$

Lemma 3. *A line not passing through O is mapped to a circle passing O, and vice versa.*



Proof. Let P be the projection of O onto line ℓ, and let $Q \in \ell$ be arbitrary. By Lemma 1, $\angle OPQ = \angle OQ'P' = 90^\circ$ and so Q' lies on a circle with diameter OP'. $\square$

Lemma 4. *A circle not passing through O is mapped to a circle not passing through O.*



Proof. We prove that A', B', C', and D', the inverses of four concyclic points A, B, C, and D with respect to ω, are concyclic. Applying Lemma 1 and angle chasing, we have (using directed angles)

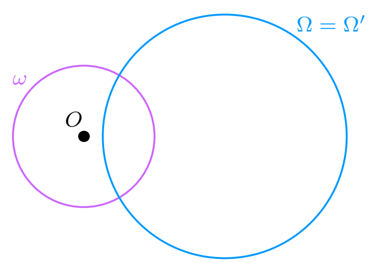
$$\begin{aligned}\angle A'D'B' - \angle A'C'B' &= (\angle OD'B' - \angle OD'A') - (\angle OC'B' - \angle OC'A') \\ &= (\angle OBD - \angle OAD) - (\angle OBC - \angle OAC) \\ &= \angle CBD - \angle CAD = 0.\end{aligned}$$

Hence $A'B'C'D'$ is cyclic. $\square$

By Lemmas 2, 3, and 4, we see that lines and circles are mapped to lines and circles. If we consider lines to be circles of infinite radius, we can say that inversion maps (generalized) circles to (generalized) circles.

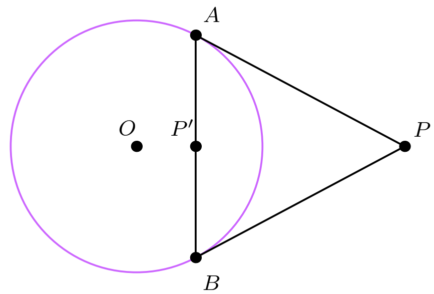
Two circles are said to be *orthogonal* if they intersect at right angles. Although orthogonal circles rarely appear in contest math, they hold a nice property under inversion.

Lemma 5. *If circle Ω is orthogonal to ω, Ω is mapped to itself.*



If we define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that takes takes a point P on the inversive plane to its inverse P' with respect to a circle, it is not difficult to see that f is continuous and bijective. Hence intersections are preserved, and so it follows that tangencies and concyclicities are also preserved.

The construction of an inverse is also helpful to notice in problems. If P is outside the circle of inversion, construct tangents PA and PB to the circle. Then P' is the midpoint of segment AB. If P is inside the circle of inversion, draw a chord AB perpendicular to OP that passes through P. The intersection P' of the tangents to the circle at A and B is the inverse of P.



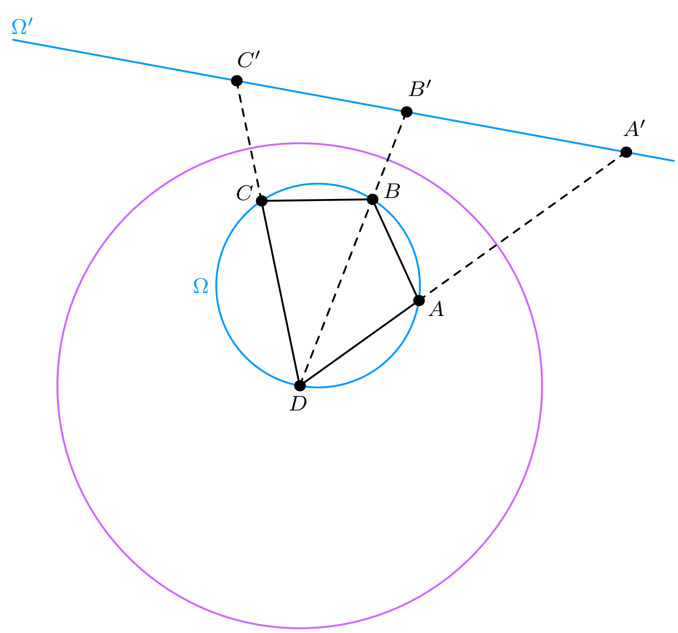
3 Applications

We begin by proving a well-known theorem:

Theorem 6 (Ptolemy)**.** *Let $ABCD$ be a cyclic quadrilateral. Then*

$$AD \cdot BC + AB \cdot CD = AC \cdot BD.$$

Proof. Let Ω be the circumcircle of $ABCD$. We now invert with respect to a circle centered at D with arbitrary radius:



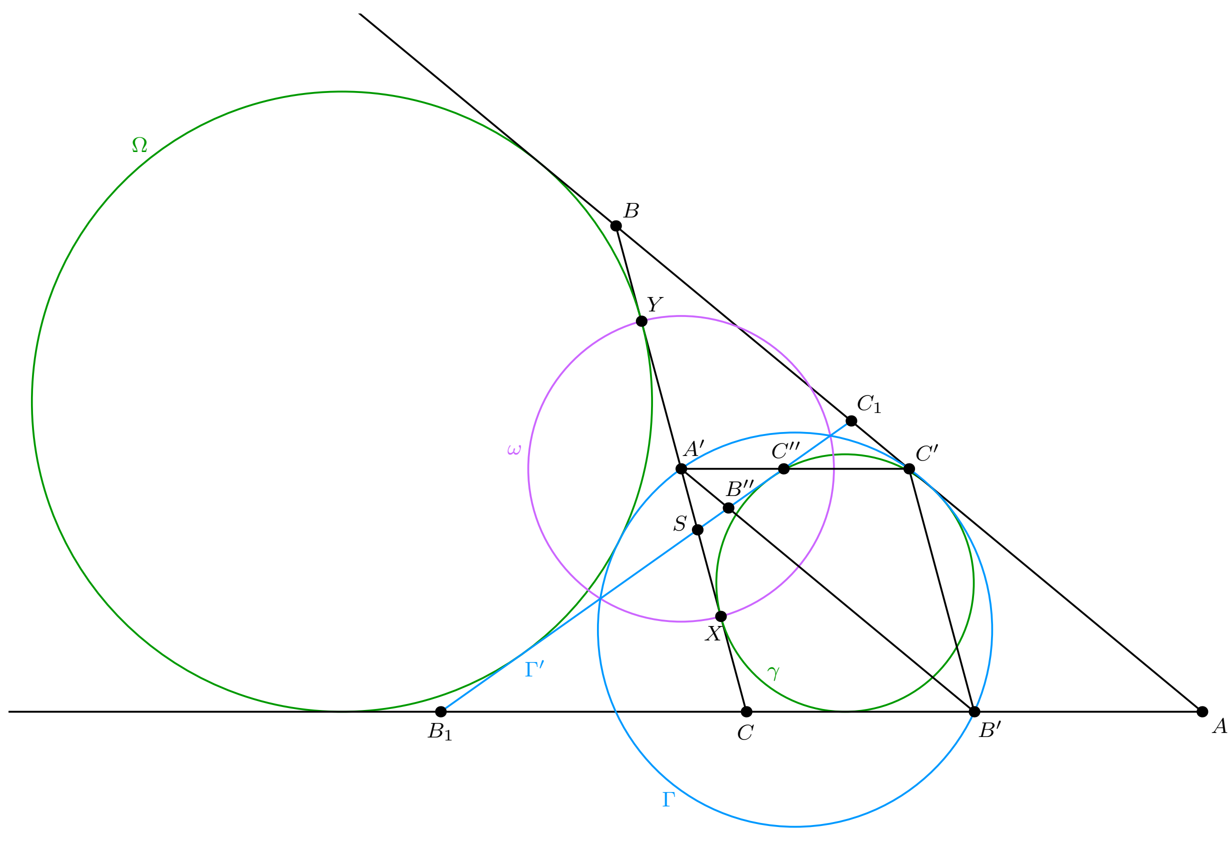
Note that Ω passes through the center of the circle of inversion. Hence, by Lemma 3, Ω is mapped to a line not passing through D. As a result, A', B', and C' are collinear, so $A'B' + B'C' = A'C'$. By Lemma 1,

$$\begin{aligned} A'B' + B'C' = A'C' &\iff \frac{r^2}{DA \cdot DB} AB + \frac{r^2}{DB \cdot DC} BC = \frac{r^2}{DA \cdot DC} AC \\ &\iff AB \cdot CD + BC \cdot AD = AC \cdot BD. \end{aligned}$$

The last equality follows from multiplying both sides by $\frac{DA \cdot DB \cdot DC}{r^2}$. □

The following theorem due to Feuerbach was discovered in 1822. Without inversion, proving this theorem is quite difficult. However, inversion with respect to the right circle simplifies tangency conditions.

Theorem 7 (Feuerbach)**.** *The nine-point circle of a triangle is tangent to its incircle and excircles.*



Proof. Let ABC be a triangle, and let $A'B'C'$ be its medial triangle. Since we can swap the order of the letters, we only have to prove that one of the excircles and the incircle are tangent to the nine-point circle.

Let X be the point at which the incircle γ touches BC, and let Y be the point at which the A-excircle Ω touches BC. Let B_1 and C_1 be on CA and AB respectively such that B_1C_1 is the internal tangent to the incircle and the A-excircle, and let $S = B_1C_1 \cap BC$, $B'' = B_1C_1 \cap A'B'$, and $C'' = B_1C_1 \cap A'C'$. Finally, let ω be the circle centered at A' with radius $r = A'X = A'Y$.

Notice that ω is orthogonal to the incircle and the A-excircle (Why?). If we let $\mathbf{I} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that maps to its inverse, by Lemma 5, we have

$$\mathbf{I}(\gamma) = \gamma \text{ and } \mathbf{I}(\Omega) = \Omega.$$

By definition, the nine-point circle Γ passes through A', the center of ω. By Lemma 3, $\mathbf{I}(\Gamma)$ is a line. We claim that this line is B_1C_1. In order to do this, we prove that B' and C' are mapped to points lying on B_1C_1, namely B'' and C''.

Observe that $\triangle SA'B'' \sim \triangle SBC_1$ and $\triangle SA'C'' \sim \triangle SCB_1$. Hence

$$\frac{A'B''}{BC_1} = \frac{SA'}{SB} \implies A'B'' = \frac{(b-c)^2}{2c}$$

and

$$\frac{A'C''}{CB_1} = \frac{SA'}{SC} \implies A'C'' = \frac{(b-c)^2}{2b}.$$

Therefore, $A'B' \cdot A'B'' = \frac{c}{2}\frac{(b-c)^2}{2c} = \left(\frac{b-c}{2}\right)^2$ and $A'C' \cdot A'C'' = \frac{b}{2}\frac{(b-c)^2}{2b} = \left(\frac{b-c}{2}\right)^2$. Notice that the radius of the circle of inversion is $\frac{b-c}{2}$. Hence $A'B' \cdot A'B'' = A'C' \cdot A'C'' = r^2$, and so by the definition of inversion, it follows that

$$\mathbf{I}(B') = B'' \text{ and } \mathbf{I}(C') = C''.$$

We have proven that Γ is mapped to line B_1C_1. Since B_1C_1 is tangent to γ and Ω, it follows that $\mathbf{I}(B_1C_1) = \Gamma$ is tangent to $\mathbf{I}(\gamma) = \gamma$ and $\mathbf{I}(\Omega) = \Omega$, as desired. $\square$

Remark. The *Feuerbach point* is the point at which the incircle and the nine-point circle are tangent. It is the Kimberling center X_{11}.

4 Problems

> *When should inversion be used? As always, the answer comes with experience and cannot be put on a paper. Roughly speaking, inversion is useful in destroying "inconvenient" circles and angles on a picture. Thus, some pictures "cry" to be inverted.*
>
> -Dušan Dukić, IMOmath

1. (Ptolemy) Prove that for any four points A, B, C, and D, the following inequality holds:

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD.$$

2. Let ω be a circle internally tangent to another circle Ω. Let AB be a chord of Ω that is tangent to ω at Q. If P is the point of tangency of ω and Ω, prove that PQ is the angle bisector of $\angle APB$.

3. Let ABC be a triangle and ω its circumcircle. Prove that the inverse of the A-median of $\triangle ABC$ with respect to ω is the A-symmedian of $\triangle A'B'C'$.

4. Let PQ be a diameter of circle ω. Let A and B be points on ω on the same side of line PQ, and let C be the intersection of the tangents to ω at A and B. Let ℓ be a tangent to ω at Q. If $A' = PA \cap \ell$, $B' = PB \cap \ell$ and $C' = PC \cap \ell$, prove that C' is the midpoint of segment $A'B'$.

5. Prove that any two non-intersecting circles that lie outside each other can be inverted into concentric circles.

6. (Isodynamic Point) Let ABC be triangle and let S be a point such that

$$AS \cdot BC = BS \cdot CA = CS \cdot AB.$$

What type of triangle is ABC mapped to under inversion with respect to a circle centered at S with arbitrary radius?

7. (Singapore TST 2004) Let D be a point in the interior of an acute-angled triangle ABC such that $AB = a \cdot b$, $AC = a \cdot c$, $AD = a \cdot d$, $BC = b \cdot c$, $BD = b \cdot d$, and $CD = c \cdot d$ for $a, b, c, d \in \mathbb{R}$. Prove that $\angle ABD + \angle ACD = \frac{\pi}{3}$.

8. (ISL 1999) The point M inside the convex quadrilateral $ABCD$ is such that
$$MA = MC,\ \angle AMB = \angle MAD + \angle MKD,\ \angle CMD = \angle MCB + \angle MAB.$$
Prove that $AB \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$.

9. Let ABC be a triangle. Prove that the inverse of the incenter of $\triangle ABC$ with respect to a circle centered at A with arbitrary radius is the A-excenter of $\triangle AB'C'$.

10. (MOP 2013) Let ABC be a triangle and let P be a point in its interior. Prove that
$$\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PC} \geq \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c},$$
where R_a, R_b, and R_c are the circumradii of triangles PBC, PCA, and PAB, respectively.

11. (ISL 1999) Let ABC be a triangle, Ω its incircle and $\Omega_a, \Omega_b, \Omega_c$ three circles orthogonal to Ω passing through $(B, C), (A, C)$ and (A, B) respectively. The circles Ω_a and Ω_b meet again in C'; in the same way we obtain the points B' and A'. Prove that the radius of the circumcircle of $A'B'C'$ is half the radius of Ω.

12. (IMO 1985) A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.

13. (TSTST 2013) Circle ω, centered at X, is internally tangent to circle Ω, centered at Y, at T. Let P and S be variable points on Ω and ω, respectively, such that line PS is tangent to ω (at S). Determine the locus of O, the circumcenter of triangle PST.

14. (Miquel) In triangle ABC, D, E, and F are points on sides BC, CA, and AB, respectively. Prove that the circumcircles of AEF, BFD, and CDE concur.

15. (ISL 2003) Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that
$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

16. Let C and C' be a pair of orthogonal circles. The points interior to C represent points of the hyperbolic plane and the arc of C' interior to C represents a line of the hyperbolic plane. Prove that inversion with respect to C' represents reflection in the hyperbolic plane.

5 Hints to Problems

1. Invert with respect to a circle centered at A with arbitrary radius.
2. Invert with respect to a circle centered at P with arbitrary radius.
3. Invert with respect to a circle centered at P with radius PQ.
4. Invert with respect to a circle centered at B with arbitrary radius, and use Problem 3.
5. Invert with respect to a circle centered at a point on the radical axis of the two circles.
6. Lemma 1.
7. Invert with respect to a circle centered at D with arbitrary radius.
8. Lemma 1.
9. Lemma 1.
10. Invert with respect to a circle centered at P with arbitrary radius. Then apply a well-known theorem.
11. Invert with respect to Ω.
12. Invert with respect to a circle centered at B with arbitrary radius.
13. Invert with respect to a circle centered at T with arbitrary radius.
14. Invert with respect to a circle centered at the second intersection point of the circumcircles of AEF and BFD.
15. Invert with respect to a circle centered at P with arbitrary radius.