ARML Complex Numbers

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1 What is a Complex Number?

Say you wanted to solve the equation $x^2 = 4$. You should quickly see that $x = \pm 2$. How about $x^2 = 2$? There are no integer solutions x, but using the real numbers we can say that $x = \pm\sqrt{2}$. What about $x^2 = -2$? Taking the square root of both sides, we see that $x = \pm\sqrt{2} \cdot \sqrt{-1}$. But a question arises: what is the square root of -1? Right now, we don't know, so we'll just call it i. Then, a complex number is a number in the form $z = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

2 Some Basic Definitions

Much of what you can do with regular numbers you can do with complex numbers as well. For these definitions, let $u = a + bi$, $v = c + di$, and α be a real number.

- $u = v$ iff $a = c$ and $b = d$.
- Addition: $u + v = (a + bi) + (c + di) = (a + c) + (b + d)i$
- Scalar multiplication: $\alpha \cdot u = \alpha(a + bi) = (\alpha a) + (\alpha b)i$
- Multiplication: $u \cdot v = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- The real part of u, usually denoted as $\mathrm{Re}(u)$, is equal to a. Likewise, the imaginary part of u, $\mathrm{Im}(z)$, is equal to b.
- The complex conjugate of u, written as u^*, is equal to $a - bi$. Why is this useful? Well, $u \cdot u^* = (a + bi)(a - bi) = a^2 + abi - abi - i^2 b = a^2 + b^2$. So multiplying a complex number by its conjugate will always give a real number.
- Notice that I haven't mentioned division yet; it might not work the way you think it does. Say we want to find $\frac{u}{v}$. If we want to simplify it, we're not allowed complex numbers in the denominator, just like we can't leave square roots in the denominator. So what do we do? Well, $\frac{u}{v} = \frac{u \cdot v^*}{v \cdot v^*} = \frac{u \cdot v^*}{c^2 + d^2}$, which gets rid of the complex number in the denominator!

2.1 The Complex Plane

Complex numbers can also be interpreted geometrically. A complex number $u = a + bi$ can be thought of as the vector $\mathbf{v} = (a, b)$ in the Cartesian plane. This brings us to some new definitions:

- The *magnitude* of u is the distance from u to the origin, which is equal to $\sqrt{a^2 + b^2}$. Note that this also equals $\sqrt{uu^*}$. The magnitude of u is usually written as $|u|$.
- The *argument* of u, written as $\arg(u)$, is the angle u makes with the positive x-axis. This is equal to $\tan^{-1}\left(\frac{b}{a}\right)$, but be careful to make sure your angle is in the right quadrant!

If we call $r = |u|$ and $\theta = \arg(u)$, we can see that $u = r(\cos\theta + i\sin\theta)$.

2.2 Euler's Formula

I'm not going to go through the derivation (or maybe I will...), but as it turns out,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$e^{i\theta}$ is also sometimes written as cis θ. This is known as the polar representation of a complex number. Note that for a given complex number z, this means that

$$z = |z|e^{i\arg z}.$$

Exercise: find i^i.

2.3 de Moivre's Formula

de Moivre's formula states that

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx).$$

2.4 Roots of Unity

Say you're trying to solve $x^n - 1 = 0$. Back before you knew about complex numbers, you would say that the only solutions are $x = 1$ and $x = -1$ (when n is even). But we're smarter than that! By the Fundamental Theorem of Algebra, we know that this equation has exactly n solutions. But what are they?
Rewrite the equation as $x^n = 1$. Now, notice that $1 = e^0 = e^{2\pi i} = e^{4\pi i} = e^{2\pi k i}$, where k is an integer.
The equation is now solvable and we find that $x = e^{\frac{2\pi i k}{n}}$, where $0 \leq k < n$. These n values of x are called the n^{th} Roots of Unity. They all lie on the unit circle (defined by $|z| = 1$) and they're equally spaced around it.

3 Exercises

1. If $u = 2 + 3i$ and $v = 1 - 2i$, find:
   (a) $u + v$
   (b) $2u - 3v$
   (c) u^2
   (d) v^*
   (e) $|v|$
   (f) $\arg(13u)$
   (g) $\text{Re}(u) + \text{Im}(v)$
   (h) $\dfrac{u}{v}$
2. Find the sixth roots of unity.
3. Solve $z^2 + 2iz - 3 = 0$
4. Find $\left(\sqrt{3} - 1\right)^{10}$
5. What is i^i?

4 Problems

1. There is a complex number z with imaginary part 164 and a positive integer n such that
$$\frac{z}{z+n} = 4i.$$
Find n.

2. Let $P(z) = x^3 + ax^2 + bx + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

3. Let z_1, z_2, z_3, …, z_{12} be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\sum_{j=1}^{12} w_j$ can be written as $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

4. Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^o$, find the least integer that is greater than $z^{2000} + \dfrac{1}{z^{2000}}$.

5. For how many positive integers n less than or equal to 1000 is $(\sin t + i \cos t)^n = \sin nt + i \cos nt$ true for all real t?

6. There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = cos\theta_m + i \sin\theta_m$, where $0 \leq \theta_1 < \theta_2 < \ldots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \ldots + \theta_{2n}$.