Combinatorics I

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Combinatorics is the art of counting. Sounds simple, right? Today's lecture will introduce the basic pillars of Combinatorics, so that future lectures focusing on more advanced concepts (Principle of Inclusion-Exclusion, Recursion, etc.) will be easier to understand.

1 The Essentials: $\binom{i}{u}$, Pikachu!

- **Permutations:** The number of ways to select r objects from n when order matters is

$$_nP_r = (n)(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

- **Combinations:** The number of ways to select r objects from n when order doesn't matter is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

because it is necessary to divide $_nP_r$ by the $r!$ orderings of the chosen set.

1.1 Pascal's Triangle

Pascal's Triangle is a famous mathematical shape in which each number is the sum of the two directly above it. It begins at row 0.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Its application to combinatorics is extremely useful. Notice that Pascal's triangle above is equivalent to the following:

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

If each number is assigned a column, with the first number in each row assigned column number 0, then each number in row r and column c is $\binom{r}{c}$.

1.2 Binomial Theorem

The Binomial Theorem states that the coefficients of the expansion of

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

In other words,

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \cdots + \binom{n}{n} x^n y^0.$$

Notice that the coefficients of the polynomial are the same as those across Pascal's Triangle!

1.3 PIE: Principle of Inclusion-Exclusion

Let's take a blast to the past with Venn-Diagrams. For sets A and B, where $A \cup B$ denotes the set of elements that satisfy A OR B (union) and $A \cap B$ denotes the set of elements that satisfy A AND B (intersection),

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three sets, A, B, and C, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

2 Exercises

1. How many ways are there to seat 6 people around a round table? Why?
2. How many diagonals are there in a complex polygon with n sides?
3. How many intersection points of the diagonals are there?
4. Find the coefficient of in x^5 in $(4x-3)^{17}$.
5. Find the sum of the coefficients in the expansion of $(4x-3)^{17}$.
6. Prove that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.
7. Prove that $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$.
8. How many numbers less than 1000 are divisible by 7 or 13, but not 5?
9. State a generalized formula (in terms of n for n sets) for the Principle of Inclusion-Exclusion.

3 General Strategies

- READ THE QUESTION! Underline words like "distinct", "exactly", "at least", "and", "or", "not", etc. if you must.
- Always check to see if counting things that DON'T fit the criteria is actually easier. This technique is called **complementary counting**. If so, remember to subtract your answer from the total number of combinations. (Don't forget this!)
- Make sure you are not overcounting or undercounting! Make sure your casework is neat.

4 Problems

1. (2001 AMC 12 #12) How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

2. (2006 AMC 12 #9) How many even three-digit integers have the property that their digits, read left to right, are in strictly increasing order?

3. (2005 AIME II #4) Find the number of positive integers that are divisors of at least one of 10^{10}, 15^7, 18^{11}.

4. (2004 AIME II #4) How many positive integers less than $10,000$ have at most two different digits?

5. (2005 AIME I #5) Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of one face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

6. (2011 AIME II #12)Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sites nxt to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. Hint: Use PIE and complementary counting.

Sources: TJAIME Intro Combinatorics 2010, Elementary Combinatorics 2006, Beginner Combinatorics 2010