Projective Geometry

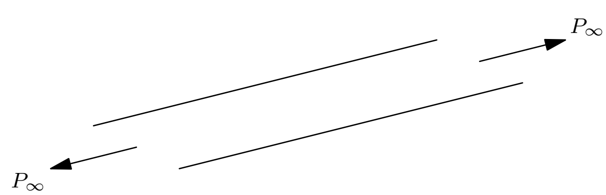
Robin Park

April 3, 2014

Projective geometry, in a general sense, is the branch of geometry that deals with ratios of segment lengths rather than the angles subtended by segments. In higher mathematics, projective geometry is seen as an algebraic tool rather than a purely geometric one by using homogenous coordinates, but in the realm of Olympiad/contest mathematics, projective geometry is used as a tool to solve various Euclidean geometry problems elegantly.

1 Definitions and Notation

Definition 1.1. We define the *projective plane* as the union of the Cartesian plane and the *line at infinity* $\mathcal{L}_\infty$, which consists of an infinite number of *points at infinity*. The line at infinity and the points of infinity lie infinitely far away from any point of the Cartesian plane. Using the projective plane, we can say that any two lines intersect, as parallel lines intersect at the unique point at infinity pointing in that direction.



Definition 1.2. Throughout the lecture, the convention that lines are denoted by lowercase letters and points are denoted by capital letters will be used. If P lies on both a and b, then P is its *intersection*, and we denote

$$P = a \cdot b.$$

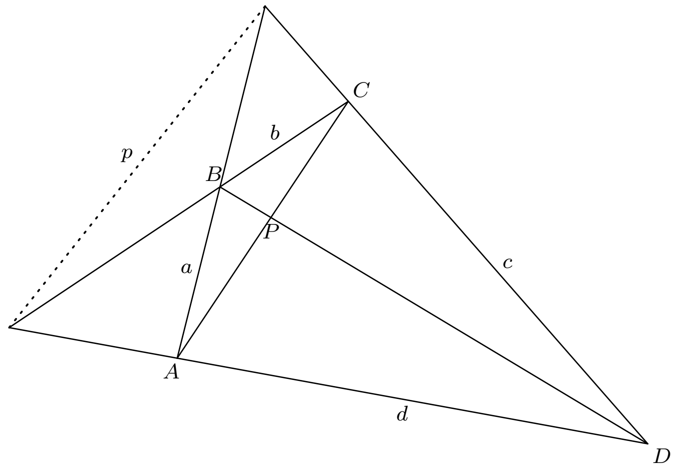
Similarly, the line a joining points P and Q is denoted by

$$a = PQ.$$

Definition 1.3. A *range* is the set of all points on a single line, and a *pencil* is the set of all coplanar lines that pass through a single point.

Definition 1.4. One of the most striking aspects of projective geometry is the concept of *duality*. The *dual* of an object is formed by interchanging the word *point* and *line*, *collinear* and *concurrent*, *vertex* and *side*, and so on. For example, the dual of the point $AB \cdot CD$ is the line $(a \cdot b)(c \cdot d)$.

The *principle of duality* asserts that the dual of any definition remains significant and the dual of every theorem remains true.



2 Projectivities and Perspectivities

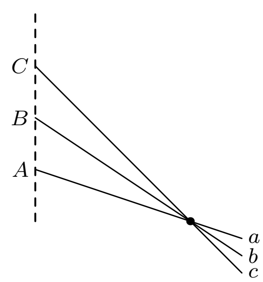
2.1 Definitions

When an element x of a pencil intersects a range at its corresponding element X, we say that x *projects* to X. This *elementary correspondence* is written as

$$x \barwedge X$$

and when x varies in particular positions a, b, and c, we can write

$$abc \barwedge ABC.$$



Similarly, the inverse transformation that sends X to x is written as $X \barwedge x$. A *projectivity* (also called a *homography*) is a product of these elementary correspondences. For instance,

$$X \barwedge x \barwedge X' \barwedge x' \barwedge X'' \barwedge x'' \barwedge \cdots \barwedge X^{(n)} \barwedge x^{(n)}$$

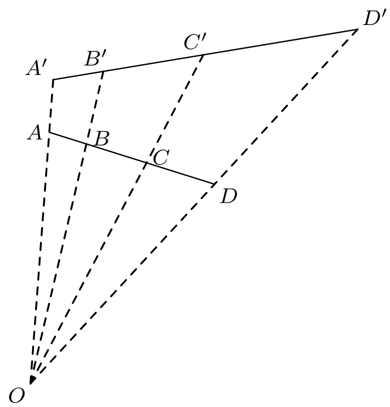
is a projectivity, which can also be written as $X \barwedge x^{(n)}$. Thus a projectivity is a bijection (or, an *isomorphism*) mapping lines to lines.

A *perspectivity* is the product of *two* elementary correspondences, and is indicated using the symbol $\doublebarwedge$ in place of $\barwedge$. Thus, we can say that two ranges are related by a perspectivity with *center* O if they are sections of one pencil (consisting of all lines through O) by two distinct lines o and o_1; that is, if the join XX' with corresponding points continually pass through the point O. We write this relation as

$$X \overset{=}{\barwedge} X' \quad \text{or} \quad X \overset{O}{\overset{=}{\barwedge}} X'.$$

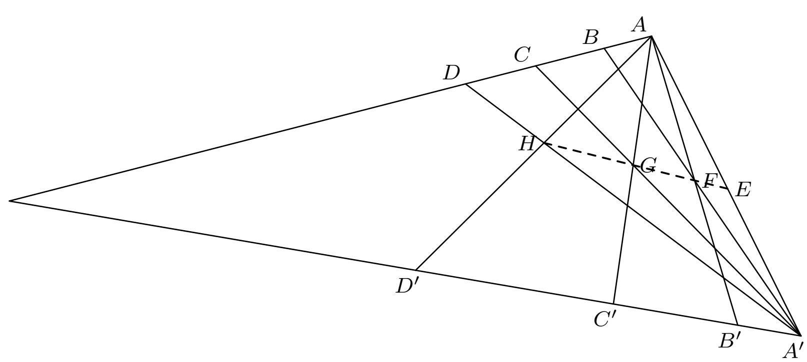
The latter is often used when the perspectivity is not obvious.

An easy way to think of perspectivities is to imagine the two ranges to be screens and the center to be a projector. We project the image on one screen to another using the projector, giving us a perspectivity.



2.2 Properties

Suppose that we have four points A, B, C, and D on one line and A', B', and C' on another. For how many positions of a fourth point D' on the second line is $ABCD \barwedge A'B'C'D'$?



As with almost every major subject in mathematics, there is a fundamental theorem associated with projective geometry:

Theorem 2.1 (Fundamental Theorem of Projective Geometry)**.** *A projectivity is determined when three collinear points and the corresponding three collinear points are given.*

Proof. Note that the following chain of perspectivities in the above diagram gives one such position of D':

$$ABCD \overset{A'}{\overset{=}{\barwedge}} EFGH \overset{A}{\overset{=}{\barwedge}} A'B'C'D'$$

We claim that this projectivity is unique. Suppose that there exist two distinct chain of perspectivities

$$ABCD \barwedge A'B'C'D' \quad \text{and} \quad ABCD \barwedge A'B'C'D''.$$

But this means that $A'B'C'D' \barwedge ABCD \barwedge A'B'C'D''$ is a projectivity, and so $D' = D''$, contradiction. □

A consequence of the Fundamental Theorem is the following lemma:

Lemma 2.2. *A projectivity relating ranges on two distinct lines is a perspectivity if and only if the common point of the two lines is invariant.*

Proof. A perspectivity obviously sends the common point to itself. Now suppose that there exists an invariant point X of the projectivity; this point must be the intersection of the two lines. If A and B are two points on the first range and A' and B' are their corresponding points, then there exists a perspectivity centered at $AA' \cap BB'$ such that

$$ABX \doublebarwedge A'B'X.$$

By the Fundamental Theorem, this perspectivity must identically be the given projectivity $ABX \barwedge A'B'X$, as desired. □

In addition, it turns out that two pairs of points can be interchanged by a suitable projectivity.

Lemma 2.3. *Any four collinear points can be interchanged in pairs by a projectivity.*

2.3 Involutions

A projectivity that interchanges pairs of points is called an *involution*. As a result, an involution maps a line to itself. A remarkable property of involutions is that if interchangeability is true for one position, it is true for all positions.

Lemma 2.4. *Any projectivity that interchanges two distinct points is an involution.*

We say that a projectivity $\pi : X \to Y$ from a conic or a line to another conic or a line *induces* another projectivity $\phi : AX \to BY$ from a pencil centered at a point A to a pencil centered at a point B. A must be on a conic if π maps from a conic and A must be on a line if π maps from a line; similar arguments hold for B. If ϕ is an involution, then the locus of the intersection Z of AX and BY is a line; otherwise, it is a conic.

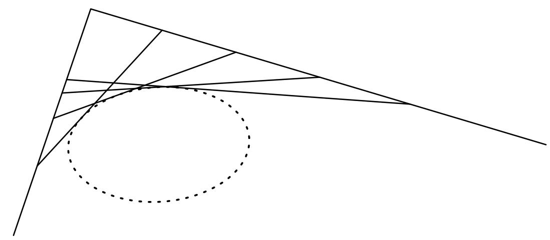
3 Conics

In Euclidean geometry, there are three types of conics: ellipses, parabolas, and hyperbolas. In projective geometry, however, due to the addition of the line at infinity, all conics turn out to be the same; there is no distinction between ellipses, parabolas, and hyperbolas.

How, then, is a conic defined in projective geometry? There are several definitions, but arguably the most useful one is Steiner's:

Theorem 3.1 (Steiner's Definition of a Conic)**.** *Let variable lines x and y pass through fixed points P and Q in such a way that $x \barwedge y$ but not $x \doublebarwedge y$. Then the locus xy is a conic through P and Q.*

The dual of Steiner's definition of a conic gives us the following fact:



Theorem 3.2. *Let X and Y vary on p and q such that $X \overline{\wedge} Y$ but not $X \overline{\overline{\wedge}} Y$. Then the envelope of XY is a conic touching p and q.*

This theorem allows us to prove the dual of a well-known fact:

Theorem 3.3. *Any five nonconcurrent lines determine a unique conic touching them.*

Proof. Let A, B, and C be three positions of X on line p, and let A', B', and C' be the three corresponding positions of X' on line q. By Theorem 2.1, there exists a unique projectivity $ABCX \overline{\wedge} A'B'C'X'$ from p to q. The envelope of XX' is a conic.

On the other hand, if five lines touch a conic, then any other tangent is part of the same projectivity. □

Dualizing this fact, we can derive the very well-known fact about the conic:

Theorem 3.4. *Five points determine a unique conic.*

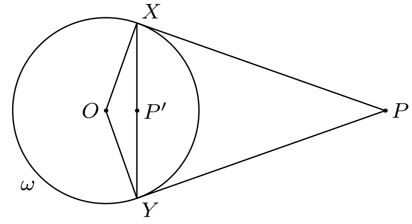
4 Poles and Polars

Now we stray from the more abstract regions of projective geometry and approach its more applicable regions. As mentioned before, projective geometry generally gives a very nice finish to an otherwise tricky problem, sometimes even trivializing the problem into a one-line solution (c.f. China 1996). Thus, poles and polars will be defined only for the specific case of the circle, although it should be noted that even conics have poles and polars, which are defined very similarly.

4.1 Inversion

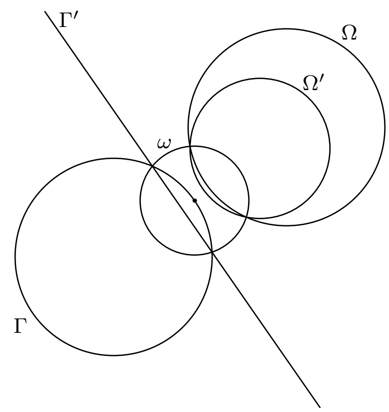
We will discuss inversion only briefly, as it detracts from the main content of the lecture.

The *inverse* of a point P with respect to a circle ω with center O and radius r is the point on line OP such that $OP \cdot OP' = r^2$. If P is outside of the circle, we can construct its inverse by taking the midpoint of the two points X and Y on ω such that PX and PY are tangent to ω.



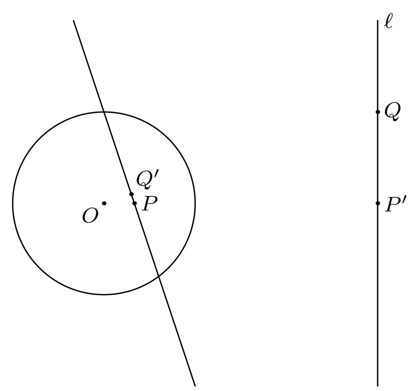
The center of ω O is sent to the point at infinity (in inversion, we only work with one point at infinity). If a circle passes through O, then it is mapped to a line not passing through O. If a circle does not pass through O, then it is mapped to a circle not passing through O. If a line passes through O, then it is mapped to itself. If a line does not pass through O, it is mapped to a circle passing through O.

If we call a line a "generalized circle" because it is simply a circle of infinite radius, we can concisely say that inversion maps generalized circles into generalized circles.



4.2 Definitions

Let ω be a circle and let P be a point and P' its inverse. The *polar* of P with respect to ω is the line perpendicular to OP passing through P'. The *pole* of ℓ with respect to ω is the inverse of the point on ℓ closest to the center of ω.



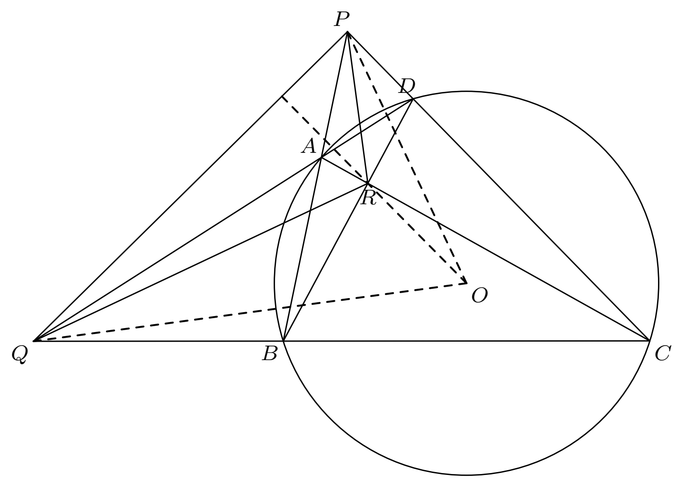
4.3 Properties

As you might have already guessed from the figure above, poles and polars have a very interesting relationship:

Theorem 4.1 (La Hire)**.** *If P lies on the polar of Q, then Q lies on the polar of P.*

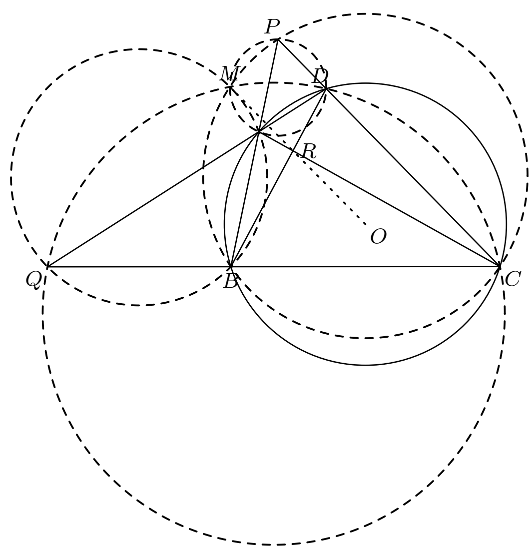
Although the statement of La Hire's Theorem is extremely simple, it is arguably the most often used in Olympiad geometry problems. Next to La Hire, Brokard's Theorem arises very frequently in a very common configuration:

Theorem 4.2 (Brokard)**.** *Let $ABCD$ be a cyclic quadrilateral, and let $P = AB \cap CD$, $Q = AD \cap BC$, and $R = AC \cap BD$. Then $\triangle PQR$ is self-polar; that is, QR is the polar of P, PR is the polar of Q, and PQ is the polar of R. Furthermore, O is the orthocenter of $\triangle PQR$.*



The configuration with a quadrilateral along with the intersections of its diagonals is iconic in Olympiad projective geoemtry. There are various other lemmas regarding this configuration, including a quadrilateral form of Miquel's Theorem:

Theorem 4.3 (Miquel)**.** *Let $ABCD$ be a convex quadrilateral, and let $P = AB \cap CD$, $Q = AD \cap BC$, and $R = AC \cap BD$. Then the circumcircles of $\triangle PAD$, $\triangle PBC$, $\triangle QAD$, and $\triangle QBC$ concur at a single Miquel point M. M lies on PQ if and only if $ABCD$ is cyclic, and $PQ \perp OM$, where O is the circumcenter of $ABCD$. Furthermore, M is the inverse of R with respect to the circumcircle of $ABCD$.*



A *pole-polar transformation* (also called *reciprocation*) with respect to some circle is a geometric transformation in which poles are sent to polars, and polars are sent to poles.

5 Cross-Ratios

In geometric transformations, there are almost always a quantity that is conserved. For instance, in inversion, angles are conserved. Similarly, in projective transformations, cross-ratios are conserved. Let A, B, C, and D be points on a line in this order. The *cross-ratio* of $ABCD$ (denoted by $(A, C; B, D)$) is the quantity

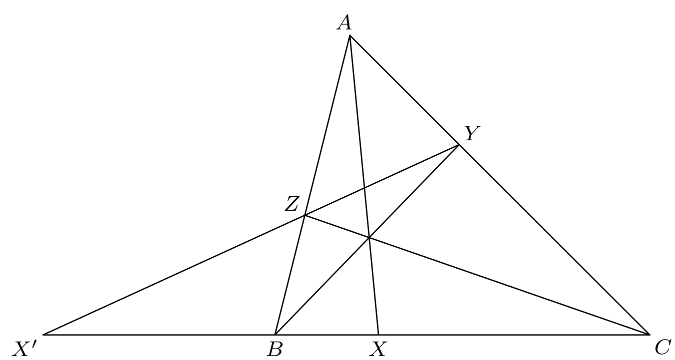
$$\frac{CA}{CB} \Big/ \frac{DA}{DB}$$

where lengths are directed. A quadruple of points A, B, C, and D is said to be a *harmonic division* (or simply *harmonic*) if $(A, C; B, D) = -1$. C is the *harmonic conjugate* of A, and vice versa.

If X is a point not lying on the same line as the harmonic quadruple, we say that the pencil $X(A, C; B, D)$ is harmonic.

The following three lemmas are very useful in solving problems using harmonic divisions.

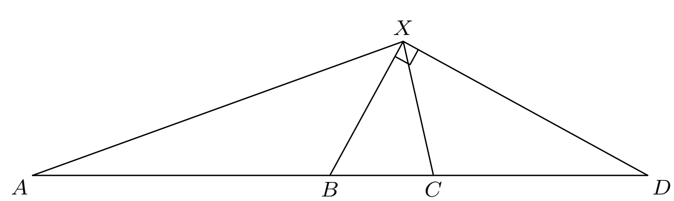
Lemma 5.1. *In $\triangle ABC$ let X, Y, and Z be points on segments BC, CA, and AB, respectively. If X' is the intersection of lines YZ and BC, then $(B, C; X, X') = -1$ if and only if AX, BY, and CZ are concurrent.*



This lemma is easily proven using Ceva and Menelaus' Theorems. The second lemma is less obvious, but comes up very frequently in Olympiad-caliber problems.

Lemma 5.2. *Let A, B, C, and D be four points lying on a line in this order, and let X be a point not lying on this line. Then, if two of the following are true, then the third is also true:*

- $(A, C; B, D) = -1$;
- *XB bisects $\angle AXC$;*
- $XB \perp XD$.



Finally, it turns out cross-ratio is invariant after a perspectivity:

Lemma 5.3. *Let A, B, C, and D be points on a line, and let A', B', C', and D' be the corresponding points mapped by a perspectivity to a second line. Then $(A, C; B, D) = (A', C'; B', D')$.*

We can prove this using the Law of Sines multiple times. In fact, this fact is not limited to only lines to lines. We may extend the definition of the cross-ratio to points on a circle, and not just points on a line. If A, B, C, and D are concyclic points such that $(A, C; B, D) = -1$, then $ABCD$ is called a *harmonic quadrilateral*.

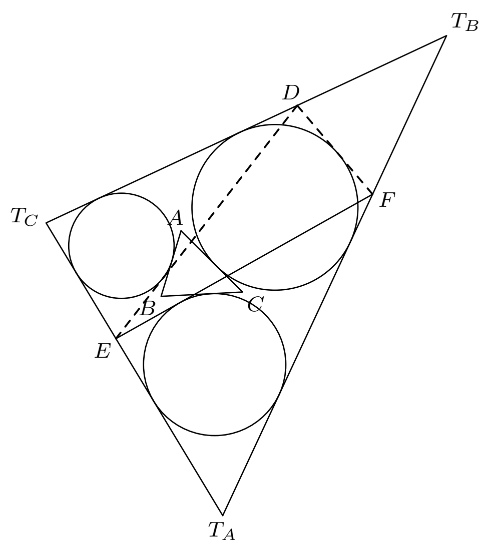
Harmonic quadrilaterals have various interesting properties. Suppose that $ABCD$ is a harmonic quadrilateral, let P be the intersection of its tangents to A and C, and let $R = AC \cap BD$. Then P, A, and C are collinear and R is the harmonic conjugate of P.

Finally, midpoints come in handy when dealing with cross-ratios:

Lemma 5.4. *Let M be the midpoint of AB, and let ∞ be the point at infinity pointing in the direction of AB. Then $(A, B; M, \infty) = -1$.*

6 Examples

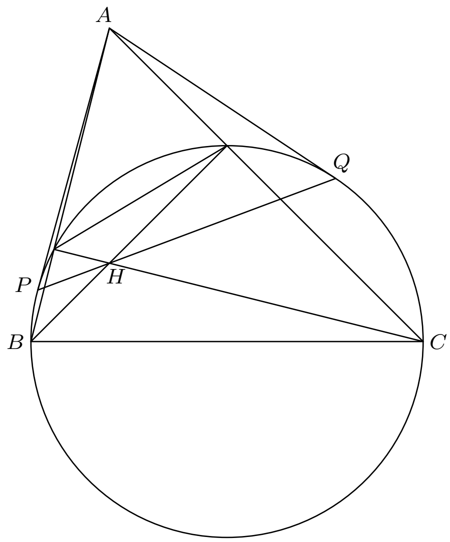
Example 6.1. Let $T_AT_BT_C$ be the triangle externally tangent to the excircles of $\triangle ABC$. Let D be an arbitrary point on T_BT_C. The tangent different from T_BT_C from D to the C-excircle intersects T_CT_A at E. The tangent different from T_BT_C from D to the B-excircle intersects T_AT_B at F. Prove that EF is tangent to the A-excircle.



Solution. Let E be the intersection of T_AT_C and the tangent line from D to the C-excircle, and let F be the intersection of T_AT_B and the tangent line from D to the B-excircle. As D varies on T_BT_C, E runs on T_AT_C and F runs on T_AT_B. Thus $D \to E$ is a projectivity from T_BT_C to T_AT_C and $D \to F$ is a projectivity from T_AT_B to T_BT_C. Thus $E \to F$ is also a projectivity, and so by Theorem 3.2, as D varies, EF envelops a conic tangent to T_AT_B and T_AT_C.

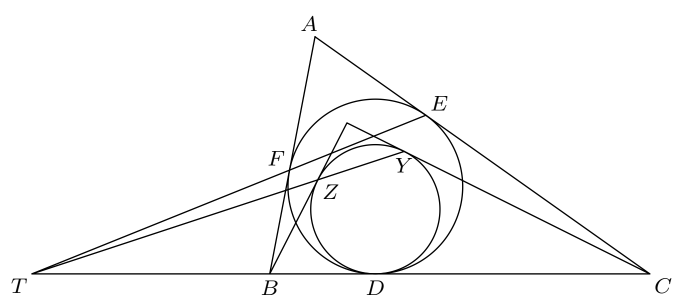
When $D = AB \cap T_BT_C$, EF coincides with AB. When $D = AC \cap T_BT_C$, EF coincides with AC. When $D = BC \cap T_BT_C$, EF coincides with BC. Therefore, EF is tangent to the unique conic inscribed in the pentagon formed by AB, BC, CA, T_AT_B, and T_AT_C, the A-excircle of $\triangle ABC$. $\square$

Example 6.2 (China 1996)**.** Let ABC be a triangle with orthocenter H. The tangent lines from A to the circle with diameter BC touch this circle at P and Q. Prove that H, P, and Q are collinear.



Solution. By Brokard's Theorem, A lies on the polar of H; by La Hire's Theorem, H lies on the polar of A, which is PQ. $\square$

Example 6.3 (ISL 1995)**.** Let ABC be a triangle, and let D, E, and F be the points of tangency of the incircle of $\triangle ABC$ with sides BC, CA, and AB, respectively. Let X be in the interior of $\triangle ABC$ such that the incircle of $\triangle XBC$ touches XB, XC, and BC in Z, Y, and D respectively. Prove that $EFZY$ is cyclic.



Solution. Let $T = BC \cap EF$. It is well-known that AD, BE, and CF concur at the Gergonne point of $\triangle ABC$, and so $(T, D; B, C) = -1$. Now let $T' = BC \cap YZ$. Again, XD, BY, and CZ concur at the Gergonne point of $\triangle XBC$, and so $(T', D; B, C) = -1$, implying that T and T' are coincident. Thus $TF \cdot TE = TD^2 = TZ \cdot YZ$, and so $EFZY$ is cyclic. $\square$

7 Miscellaneous Theorems

Theorem 7.1 (Pappus)**.** *Points A_1, A_2, and A_3 lie on a line and B_1, B_2, and B_3 lie on another line. Then $A_1B_2 \cap A_2B_1$, $A_1B_3 \cap A_3B_1$, and $A_2B_3 \cap A_3B_2$ are collinear.*

Theorem 7.2 (Pascal)**.** *Let $ABCDEF$ be a hexagon inscribed in a conic. Then $AB \cap DE$, $BC \cap EF$, and $CF \cap FA$ are collinear.*

Theorem 7.3 (Brianchon)**.** *Let $ABCDEF$ be a hexagon circumscribed about a conic. Then AD, BE, and CF concur.*

Theorem 7.4 (Desargues)**.** *Let ABC and $A'B'C'$ be two triangles. Then AA', BB', and CC' concur if and only if $AB \cap A'B'$, $BC \cap B'C'$, and $CA \cap C'A'$ are collinear.*

Theorem 7.5 (Monge)**.** *Let ω_1, ω_2, and ω_3 be circles. The exsimilicenters of ω_1 and ω_2, of ω_2 and ω_3, and of ω_3 and ω_1 are collinear.*

Theorem 7.6 (Monge d'Alembert)**.** *Let ω_1, ω_2, and ω_3 be circles. The exsimilicenter of ω_1 and ω_2 and the insimilicenters of ω_2 and ω_3, and of ω_3 and ω_1 are collinear.*

8 Problems

1. (Symmedian Lemma) Let ABC be a triangle and Γ its circumcircle. Let the tangent to Γ at B and C meet at D. Then AD coincides with the A-symmedian of $\triangle ABC$.

2. (USAJMO 2011) Points A, B, C, D, and E lie on a circle ω and point P lies outside of the circle. The given points are such that lines PB and PD are tangent to ω; P, A, and C are collinear; and $DE \parallel AC$. Prove that BE bisects AC.

3. (China 1997) Let $ABCD$ be a cyclic quadrilateral. Let P be the intersection of AB and CD and let Q be the intersection of AD and BC. Construct two tangents QE and QF to the circle, where E and F are the points of tangency. Prove that P, E, and F are collinear.

4. (Sharygin 2013) The incircle of $\triangle ABC$ touches side AB at C'; the incircle of $\triangle ACC'$ touches sides AB and AC at C_1 and B_1; the incircle of $\triangle BCC'$ touches sides AB and BC at C_2 and A_2. Prove that lines B_1C_1, A_2C_2, and CC' concur.

5. (Sharygin 2013) Let AD be a bisector of $\triangle ABC$. Points M and N are projections of B and C respectively onto AD. The circle with diameter MN intersects BC at X and Y. Prove that $\angle BAX = \angle CAY$.

6. (Austria-Poland 1998) Distinct points A, B, C, D, E, and F lie on a circle in this order. The tangents to the circle at points A and D and lines BF and CE are concurrent. Prove that AD, BC, and EF are either parallel or concurrent.

7. (HMMT 2008) Let ABC be a triangle with $BC = 2007$, $CA = 2008$, and $AB = 2009$. Let ω be an excircle of ABC that touches line segment BC at D, and touches the extensions of linse AC and AB at E and F, respectively (so that C lies on segment AE and B lies on segment AF). Let O be the center of ω. Let ℓ be the line through O perpendicular to AD. Let ℓ meet line EF at G. Compute the length DG.

8. (HMMT 2011) Collienar points A, B, and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x-axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $AB/BC = 2$. If $D = (a, a)$, the circumcircle of triangle ADC intersects $y = x$ again at E, and ray AE intersects $y = 2x$ at F, evaluate AE/EF.

9. (2013 Winter OMO) In $\triangle ABC$, $CA = 1960\sqrt{2}$, $CB = 6720$, and $\angle C = 45°$. Let K, L, and M lie on BC, CA, and AB, respectively, such that $AK \perp BC$, $BL \perp CA$, and $AM = BM$. Let N, O, and P lie on KL, BA, and BL, respectively, such that $AN = KN$, $BO = CO$, and A lies on line NP. If H is the orthocenter of $\triangle MOP$, compute HK^2.

10. Let P be a variable point on BC of $\triangle ABC$. The circle with diameter BP intersects the circumcircle of $\triangle ACP$ again at Q. Let H be the orthocenter of $\triangle ABP$, and M the intersection of PQ and AC. Prove that MH passes through a fixed point as P varies.

11. A circle is inscribed in quadrilateral $ABCD$ so that it touches sides AB, BC, CD, and DA at E, F, G, and H, respectively.

    (a) Show that lines AC, EF, and GH are concurrent.

    (b) Show that lines AC, BD, EG, and FH are concurrent.

12. Let ω be a circle with center O and let C be a point outside of ω. Lines CA and CB are tangent to ω. Let S be a point on ω and let ℓ be the line perpendicular to OS passing through O. Given that SA, SB, and SC intersect ℓ at A_1, B_1, and C_1, respectively, prove that C_1 is the midpoint of A_1B_1.

13. (TSTST 2012) In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, and C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, and F be the respective midpoints of sides BC, CA, and AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.

14. (China 2006) AB is a diameter of a circle with center O. Let C be a point on the extension of ray AB. A line through C cuts the circle at points D and E such that D is closer to C than E. OF is a diameter of the circumcircle of $\triangle BOD$ with center O_1. Line CF intersects the circumcircle of $\triangle BOD$ again at G. Prove that O, A, E, and G are concyclic.

15. (JBMO 2007) Let ABC be a right triangle with $\angle A = 90^\circ$ and let D be a point on side AC. Denote by E the reflection of A over line BD and F the intersection of CE and the perpendicular to BC at D. Prove that AF, DE, and BC are concurrent.

16. (Romania TST 2004) The incircle of a non-isosceles $\triangle ABC$ is tangent to sides BC, CA, and AB at A', B', and C'. Lines AA' and BB' intersect at P, AC and $A'C'$ at M, and $B'C'$ and BC at N. Prove that $IP \perp MN$.

17. (China TST 2002) Let $ABCD$ be a convex quadrilateral. Let $E = AB \cap CD$, $F = AD \cap BC$, $P = AC \cap BD$, and let O be the foot of the perpendicular from P to EF. Prove that $\angle BOC = \angle AOD$.

18. Let $ABCDE$ be a convex pentagon that satisfies $\angle BAC = \angle DAE$. Let H_1 and H_2 be the orthocenters of $\triangle BAC$ and $\triangle DAE$, respectively. If the midpoints of segments BE and CD are M and N, prove that $H_1H_2 \perp MN$.

19. Let P and Q be two arbitrary points on the circumcircle of $\triangle ABC$. Let P_1, P_2, and P_3 be the reflections of P across BC, CA, and AB, respectively. QP_1, QP_2, and QP_3 meet BC, CA, and AB at D, E, and F, respectively. Show that D, E, and F are collinear.

20. Let ABC be a triangle and let D and E be points on the exterior of $\triangle ABC$ such that $\angle ADB = \angle AEC = 90^\circ$ and that $\triangle ADB$ is inversely similar to $\triangle AEC$. Let CD and BE intersect at R. Prove that $AR \perp DE$.

21. (Iran TST 2007) The incircle ω of $\triangle ABC$ is tangent to AC and AB at E and F, respectively. Points P and Q are on AB and AC such that $PQ \parallel BC$ and PQ is tangent to ω. Prove that if M is the midpoint of PQ, and T is the intersection point of EF and BC, then TM is tangent to ω.

22. (ISL 2004/G8) In cyclic quadrilateral $ABCD$, let E be the intersection of AD and BC (so that C is between B and E) and F be the intersection of AC and BD. Let M be the midpoint of CD, and $N \neq M$ be a point on the circumcircle of $\triangle ABM$ such that $\frac{AM}{MB} = \frac{AN}{NB}$. Show that E, F, and N are collinear.