Complex Numbers in Geometry

Robin Park

July 28, 2013

All geometry is algebra.

-Gabriel Dospinescu

Analytic geometry is the branch of geometry that uses algebra or analysis to solve problems. The use of complex numbers falls under the category of analytic geometry. Complex numbers are an extremely useful technique in geometry, and is the focus of this lecture. For more practice, the articles by Marko Radovanović[1] and Yi Sun[2] are recommended.

1 Preliminaries

The quote in the beginning of this lecture was meant to be facetious, but it can be proven that every geometry problem can be reduced to proving polynomial identities. The proof borrows a fundamental result of algebraic geometry:

Theorem 1 (Hilbert's Nullstellensatz). *Let $P_1, P_2, \cdots, P_k$, $Q \in \mathbb{C}[x_1, \cdots, x_n]$ be polynomials such that if $(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$ is a point where $P_1, P_2, \cdots, P_k$ all vanish, then Q also vanishes at that point. Then there exists a positive integer r and polynomials $C_1, C_2, \cdots, C_k \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ such that*

$$C_1P_1 + C_2P_2 + \cdots + C_kP_k = Q^r.$$

$\mathbb{C}[x_1, x_2, \cdots, x_n]$ refers to the ring of multivariate polynomials with coefficients in $\mathbb{C}$. Suppose that we translated a geometry problem using complex numbers and resulted in expressions $P_i(x_1, x_2, \cdots, x_n) = 0$ for $1 \le i \le k$, $P_i \in \mathbb{C}[x_1, x_2, \cdots, x_n]$. We wish to prove that $Q(x_1, x_2, \cdots, x_n) = 0$, but this is a direct consequence of Theorem 1.

2 Complex Numbers

Complex numbers are numbers that can be expressed as $z = a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. They can be plotted in the two-dimensional complex plane $\mathbb{C}$, which is similar to the Cartesian plane, except the x and y-axes are replaced by the real and imaginary parts of z, respectively.

The *complex conjugate* of a complex number z is denoted by $\overline{z}$. If $z = a + bi$, then $\overline{z} = a - bi$. Let $f : \mathbb{C} \to \mathbb{C}$ be the function that takes a complex number to its conjugate.

[1] `http://hoaxung.files.wordpress.com/2010/04/marko-radovanovic-complex-numbers-in-geometry.pdf`
[2] `http://web.mit.edu/yisun/www/notes/complex.pdf`

Lemma 2. *f is an automorphism.*

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers with $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Clearly f is surjective, and since $f(z_1) = f(z_2) \implies f(f(z_1)) = f(f(z_2)) \implies z_1 = z_2$, f is injective. It remains to prove that f is additive and multiplicative:

$$\begin{aligned} f(z_1 + z_2) &= f((x_1 + x_2) + i(y_1 + y_2)) \\ &= (x_1 + x_2) - i(y_1 + y_2) \\ &= (x_1 - iy_1) + (x_2 - iy_2) = f(z_1) + f(z_2); \end{aligned}$$

$$\begin{aligned} f(z_1 z_2) &= f((x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)) \\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \\ &= (x_1 - iy_1)(x_2 - iy_2) = f(z_1)f(z_2). \end{aligned}$$

□

The additive and multiplicative properties of complex conjugation are extremely important in conjugating large expressions. The following properties of conjugation are given without proof:

- $z = \overline{z} \iff z \in \mathbb{R}$.
- $z = -\overline{z} \iff z \in i\mathbb{R}$.
- $|z| = |\overline{z}|$.
- $|z|^2 = z\overline{z}$.
- $\overline{z} = \frac{1}{z} \iff |z| = 1$.

Complex numbers are an invaluable tool for solving geometry problems. The complex coordinate of a point is often denoted by the lowercase letter of that point (for instance, A, H, and Ω are denoted as a, h, and ω). There are many properties of complex numbers that easily reduce a geometry problem into proving an algebraic identity, which can be checked readily with computational fortitude and practice.

The following formulas are given without proof:

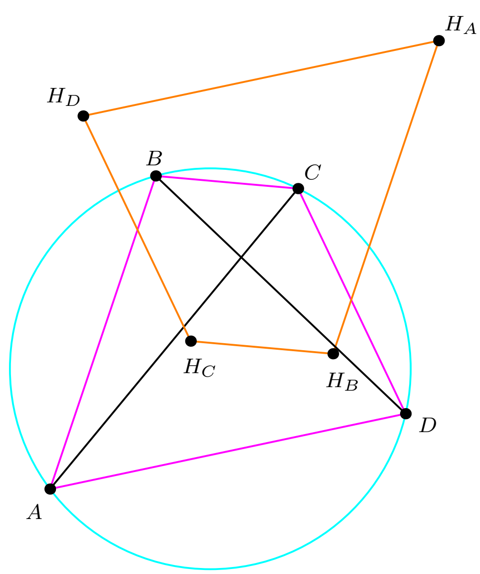
- $|AB| = |a - b|$
- $AB \parallel CD \iff \frac{a-b}{c-d} \in \mathbb{R} \iff \frac{a-b}{\overline{a}-\overline{b}} = \frac{c-d}{\overline{c}-\overline{d}}$.
- A, B, C are collinear $\iff \frac{a-b}{b-c} \in \mathbb{R} \iff \frac{a-b}{\overline{a}-\overline{b}} = \frac{b-c}{\overline{b}-\overline{c}}$.
- $AB \perp CD \iff \frac{a-b}{c-d} \in i\mathbb{R} \iff \frac{a-b}{\overline{a}-\overline{b}} = -\frac{c-d}{\overline{c}-\overline{d}}$.
- A, B, C, D are concyclic $\iff \frac{(a-b)(c-d)}{(a-d)(b-c)} \in \mathbb{R} \iff \sum_{cyc} a\overline{a}b\overline{c} = 0$.
- X is on the angle bisector of $\angle BAC \iff \frac{(x-a)^2}{(b-a)(c-a)} \in \mathbb{R}$.
- If G is the centroid of $\triangle ABC$, then $g = \frac{a+b+c}{3}$.

- The projection P of an arbitrary point C onto chord AB of the unit circle is $p = \frac{1}{2}(a + b + c - ab\overline{c})$.
- The tangents of the unit circle at A and B intersect at $\frac{2ab}{a+b}$.
- $\triangle ABC \sim \triangle PQR \iff \frac{a-b}{a-c} = \frac{p-q}{p-r}$.
- If $\triangle ABC$ lies on the unit circle:
  - If H is the orthocenter of $\triangle ABC$, then $h = a + b + c$.
  - If O is the circumcenter of $\triangle ABC$, then $o = 0$.
  - If I is the incenter of $\triangle ABC$, then $i = \pm\sqrt{ab} \pm \sqrt{bc} \pm \sqrt{ca}$.
  - If N is the nine-point center of $\triangle ABC$, then $n = \frac{a+b+c}{2}$.
- If $ABCD$ lies on the unit circle:
  - If X is the intersection of chords AB and CD, then $x = \frac{ab(c+d)-cd(a+b)}{ab-cd}$.
  - If $AD \parallel BC$, and Y is the intersection of chords AB and CD, then $y = \frac{ac-bd}{a+c-b-d}$.
- If $\triangle XYZ$ has a vertex Z at the origin,
  - $h = \frac{(\overline{x}y+x\overline{y})(x-y)}{x\overline{y}-\overline{x}y}$.
  - $o = \frac{xy(\overline{x}-\overline{y})}{\overline{x}y-x\overline{y}}$.

3 Applications

Example (BMO 1984)**.** Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C, H_D be the orthocenters of $\triangle BCD, \triangle CDA, \triangle DAB$, and $\triangle ABC$ respectively. Prove that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

Solution.



Let the circumcircle of $ABCD$ be the unit circle. Then we have:

$$\begin{aligned} h_a &= b + c + d \\ h_b &= c + d + a \\ h_c &= d + a + b \\ h_d &= a + b + c \end{aligned}$$

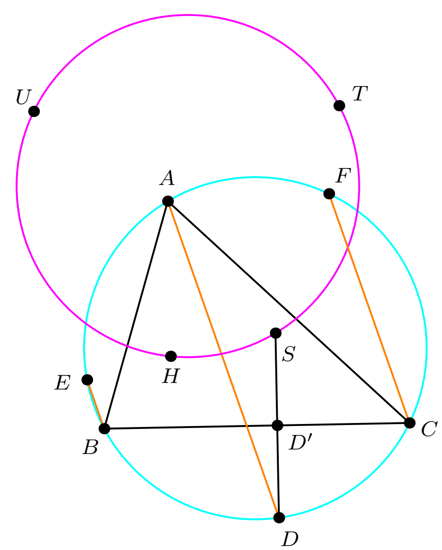
Note that $|h_x - h_y| = |x - y|$ for all $x, y \in \{a, b, c, d\}$. Hence $ABCD$ and $H_A H_B H_C H_D$ are congruent. $\square$

Remark. Although there is a purely synthetic solution involving centroids and homothety, that solution is essentially identical to the analytic solution as the formula for the orthocenter is obtained by the Euler Line. However, the analytic solution is cleaner and intuitive than the synthetic one.

The next example lends itself to an analytic solution.

Example (MOP 2006)**.** Point H is the orthocenter of triangle ABC. Points D, E, and F lie on the circumcircle of triangle ABC such that $AD \parallel BE \parallel CF$. Points S, T, and U are the respective reflections of D, E, and F across the lines BC, CA, and AB. Prove that S, T, U and H are concyclic.

Solution.



Rotate the figure until AD, BE, CF are perpendicular to the real axis. Then $d = \overline{a}, e = \overline{b}, f = \overline{c}$. (Why?) Let the circumcircle of $\triangle ABC$ be the unit circle. We then have $d = \frac{1}{a}, e = \frac{1}{b}, f = \frac{1}{c}$.

Note that BC is a chord of the unit circle. Let D' be the projection of D onto BC. Then

$$d' = \frac{1}{2}(b + c + d - bc\overline{d}) = \frac{1}{2}\left(b + c + \frac{1}{a} - abc\right).$$

Since D' is the midpoint of DS, we have $d + s = 2d'$. Hence,

$$s = 2d' - d = \left(b + c + \frac{1}{a} - abc\right) - \frac{1}{a} = b + c - abc.$$

Similarly,

$$t = c + a - abc;$$
$$u = a + b - abc.$$

We wish to prove that $STUH$ is cyclic. It suffices to show that $\frac{(s-t)(h-u)}{(s-u)(h-t)} \in \mathbb{R}$. Note that

$$\frac{(s-t)(h-u)}{(s-u)(h-t)} = \frac{(b-a)(c+abc)}{(c-a)(b+abc)}$$

A complex number is real iff it is equal to its conjugate. It remains to verify that

$$\frac{(b-a)(c+abc)}{(c-a)(b+abc)} = \overline{\frac{(b-a)(c+abc)}{(c-a)(b+abc)}} = \frac{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{abc}\right)}{\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{b}+\frac{1}{abc}\right)}$$

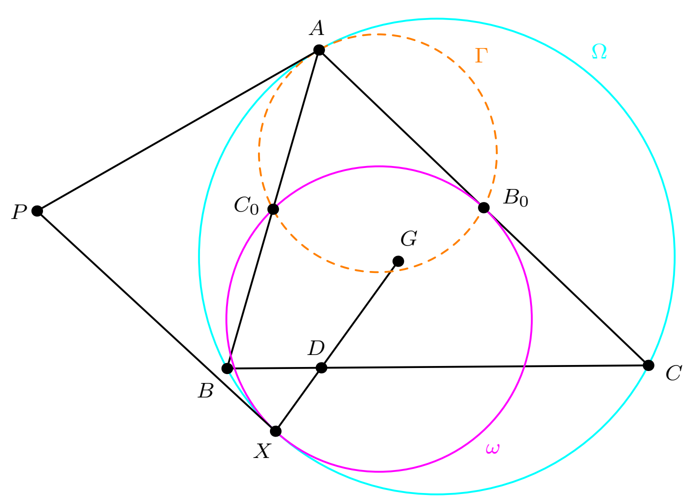
which can be checked by cross-multiplying and expanding. $\square$

Remark. Note that we rotated the figure so that the parallel lines became perpendicular to the real axis. This gives us a lot of information that allows us to compute the coordinates of s, t, u quickly. Also notice that the expressions are mostly symmetric, which is usually a good sign. The last identity involving a, b, c can be checked by multiplying everything through, but several terms can be canceled out (for instance, the $b - a$ and $\frac{1}{b} - \frac{1}{a}$). In an actual proof, it is a good idea to multiply everything out so that the grader won't hate you for making him do the work.

The next example combines synthetic and analytic geometry, which is generally more powerful than analytic geometry alone.

Example (ISL 2011/G4)**.** Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Solution.



Let Γ be the circle passing through A, B_0, C_0. This circle is tangent to Ω at A (Why?). Let P be the intersection of the tangents to Ω at A and X. By the Radical Axis Theorem, the pairwise radical axes of Ω, ω, Γ are concurrent. Note that the radical axis of Ω and ω is the tangent to Ω at X; the radical axis of Ω and Γ is the tangent to Ω at A; the radical axis of ω and Γ is line B_0C_0. Hence P must lie on the radical axis of ω and Γ, so P, C_0, B_0 are collinear.

Let the circumcircle of $\triangle ABC$ be the unit circle. Then we have $p = \frac{2ax}{a+x}, c_0 = \frac{a+b}{2}, b_0 = \frac{a+c}{2}, g = \frac{a+b+c}{3}, d = \frac{1}{2}\left(a+b+c-\frac{bc}{a}\right)$. By the collinearity of P, C_0, B_0, we have

$$\frac{b_0 - c_0}{\overline{b_0} - \overline{c_0}} = \frac{b_0 - p}{\overline{b_0} - \overline{p}} \iff -bc = \frac{a+b-2p}{\frac{1}{a}+\frac{1}{b}-2\overline{p}} \iff p + bc\overline{p} = \frac{(a+b)(a+c)}{2a}.$$

$PA \perp AO$, where O is the circumcenter of Ω, so we have

$$\frac{p-a}{\overline{p}-\overline{a}} = \frac{a-o}{\overline{a}-\overline{o}} \iff p + a^2\overline{p} = 2a.$$

Solving for p in the previous equations, we obtain

$$p = \frac{a^3 + a^2b + a^2c - 3abc}{2a^2 - 2bc}$$

Since $p = \frac{2ax}{a+x}$, we can solve for x:

$$p = \frac{2ax}{a+x} \iff x = \frac{ap}{2a-p} = \frac{a \cdot \frac{a^3+a^2b+a^2c-3abc}{2a^2-2bc}}{2a - \frac{a^3+a^2b+a^2c-3abc}{2a^2-2bc}} = \frac{a^3+a^2b+a^2c-3abc}{3a^2-bc-ab-ac}$$

It suffices to prove that $\frac{g-d}{\overline{g}-\overline{d}} = \frac{g-x}{\overline{g}-\overline{x}}$. This can be checked under 15 minutes, and is left as an exercise to the reader. $\square$

Remark. At first glance, a complex number solution seems unfeasible; there is no (clean) expression for tangent circles. However, if we borrow a result from synthetic geometry, we can reduce the problem of finding X into finding P, then using tangencies to find X. Such indirect methods of finding desired points is a great tactic in complex geometry. The expression found in the end is still quite ugly, and may seem extremely tedious. Nevertheless, actual olympiads are $4\frac{1}{2}$ hours, and the computations will only take up a small part of the total time given. In addition, this problem was from the IMO Shortlist in 2011, and so tedious computations are expected in complex number solutions.

4 Problems

It is advised that the reader solves the following problems using complex numbers, but combining synthetic observations and complex numbers when possible is also recommended.

1. (Varignon's Theorem) Prove that the midpoints of a quadrilateral form a parallelogram.
2. (Yugoslavia 1990) Let S be the circumcenter and H the orthocenter of $\triangle ABC$. Let Q be the point such that S bisects HQ and denote by T_1, T_2, and T_3, respectively, the centroids of $\triangle BCQ, \triangle CAQ$, and $\triangle ABQ$. Prove that
$$AT_1 = BT_2 = CT_3 = \frac{4}{3}R,$$
where R denotes the circumradius of $\triangle ABC$.

3. The diagonals AC and BD of a convex quadrilateral are perpendicular to each other. Draw a line that passes through point M, the midpoint of AB, and is perpendicular to CD; draw another line through point N, the midpoint of AD, and is perpendicular to CB. Prove that the point of intersection of these two lines lies on AC.

4. Let $ABCD$ be a cyclic quadrilateral and let N_A, N_B, N_C, N_D be the nine-point centers of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, $\triangle ABC$ respectively. Prove that the quadrilaterals $ABCD$ and $N_AN_BN_CN_D$ are similar, and find $\frac{[N_AN_BN_CN_D]}{[ABCD]}$.

5. (MOP 2013) In cyclic quadrilateral $ABCD$, diagonals AC and BD meet at E. Rays DA and CB meet at F. Point G is such that $ECGD$ is a parallelogram, and point H is the reflection of E across line AD. Prove that D, H, F, G lie on a circle.

6. (Romania 1984) Let z_1, z_2, z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = R$ and $z_2 \neq z_3$. Prove that
$$\min_{a \in \mathbb{R}} |az_2 + (1-a)z_3 - z_1| = \frac{1}{2R}|z_1 - z_2| \cdot |z_2 - z_3|.$$

7. (APMO 2013) Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

8. (Iran 2013) Let M be the midpoint of minor arc BC in the circumcircle of triangle ABC. Suppose that the altitude drawn from B intersects the circumcircle at N. Draw two lines through the circumcenter O of ABC parallel to MB and MC, which intersect AB and AC at K and L, respectively. Prove that $NK = NL$.

9. (Russia 2010) Let O be the circumcenter of acute non-isosceles triangle ABC. Let P and Q be points on the altitude AD such that OP and OQ are perpendicular to AB and AC respectively. Let M be the midpoint of BC and S be the circumcentre of triangle OPQ. Prove that $\angle BAS = \angle CAM$.

10. (TST 2011) In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $AD \perp BC, BE \perp CA, CF \perp AB$. Altitudes AD, BE, CF meet at orthocenter H. Points P and Q lie on segment EF such that $AP \perp EF$ and $HQ \perp EF$. Lines DP and QH intersect at point R. Compute HQ/HR.

11. (ISL 1998/G8) Let ABC be a triangle such that $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to the circumcircle ω of triangle ABC meets the line BC at D. Let E be the reflection of A in the line BC, let X be the foot of the perpendicular from A to BE, and let Y be the midpoint of the segment AX. Let the line BY intersect the circle ω again at Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.