TJUSAMO 12-13: Geometry #1

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Geometry problems appear frequently on Olympiads. At least one problem or sometimes even two are given to us to solve. When people first confront a geometry problem, they tend to back away and dislike it. That is expected because unlike algebra, you cannot try random things and expect to get it right. However, as you continue to study and solve more geometry problems, you'll get addicted, and eventually, you will be able to write a beautiful solution. But getting there is the hard part... let's start with the basics.

1 Triangles

A triangle is defined by three non-collinear points. Triangles are the simplest polygons, and also the most important ones. Most of the geometry problems you'll see will be based on triangles.

These are some key triangle terms one must recognize.

- cevian
- median
- centroid
- incircle
- excircle
- circumcircle
- altitude
- orthocenter

1.1 Similar & Congruent Triangles

I'm going to assume you guys already know this, but in case you don't know, feel free to visit the VMT wiki and look at some past lectures.

1.2 Angle Chasing

Now that we are armed with tools, we can start angle chasing. This is the most important skill a geometer must learn. Here are some tips for how one should go about angle chasing.

- Angle chasing is not just making trivial observations such as: "hey these two angles are vertical, they must be the same!" Instead, you must use similar triangles, cyclic quads, angle bisectors, etc. to discover how angles are related. Angle chasing can be difficult!

- Figure out what you want to chase and what you want to end up with. Don't angle chase purposelessly.

- Try splitting an angle into two different angles that you have more knowledge of.

- Introduce a few parameters and identify which angles can be easily expressed in terms of those parameters. If the whole diagram can be expressed in terms of one or two parameters, you've pretty much solved the problem.

2 THEOREMS

There are multiple theorems you should know to solve geometry problems for triangles. Let's start with the simplest one.

- Pythagorean Theorem: If $\triangle ABC$ is a right triangle with $\angle C$ as a right angle, then

$$AC^2 + BC^2 = AB^2$$

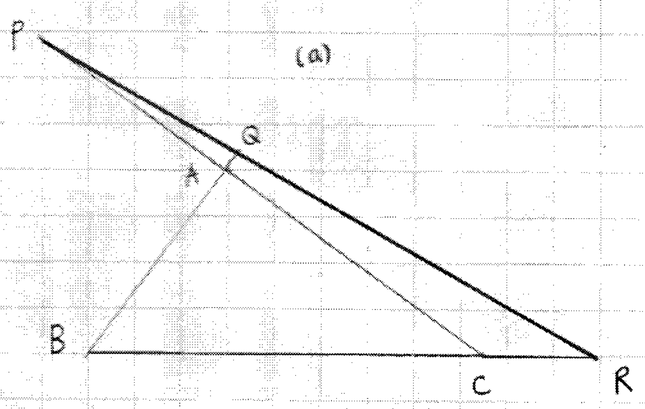
Although this theorem is very simple, it is very widely used in Olympiads AND AIME-type problems. So, now, the next two theorems I'm going to list are a little harder.

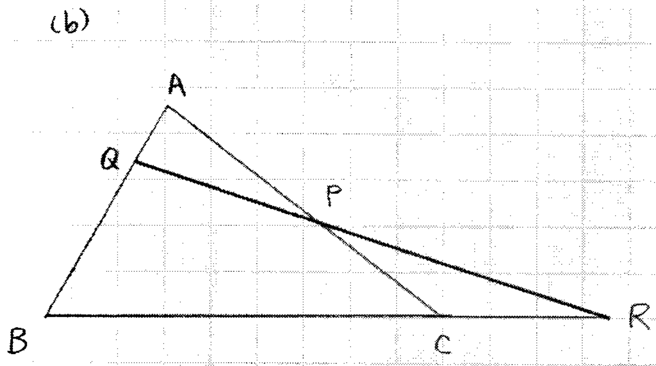
• Menelaus' Theorem :

Points P, Q, and R are taken on sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ (extended if necessary) of $\triangle ABC$. If these points are collinear,

$$\frac{AQ}{QB} \cdot \frac{BR}{RC} \cdot \frac{CP}{PA} = -1$$

The converse of this also included in the theorem.

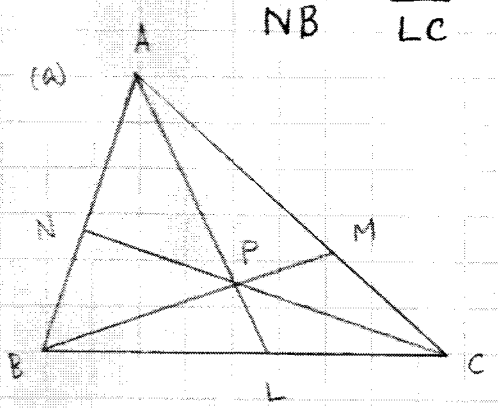


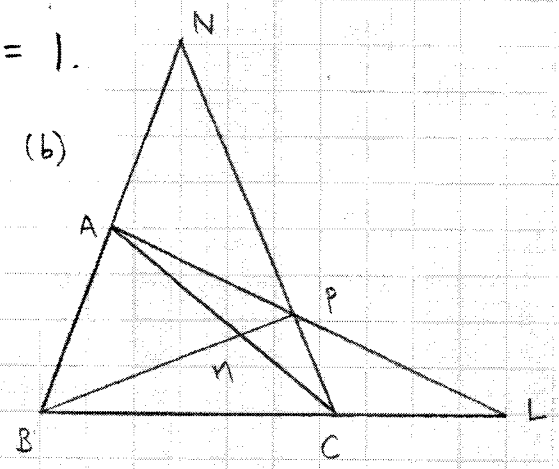


• Ceva's theorem :

Three lines drawn from the vertices A, B, and C of $\triangle ABC$ meeting the opposite sides in points L, M, and N, respectively, are concurrent if and only if

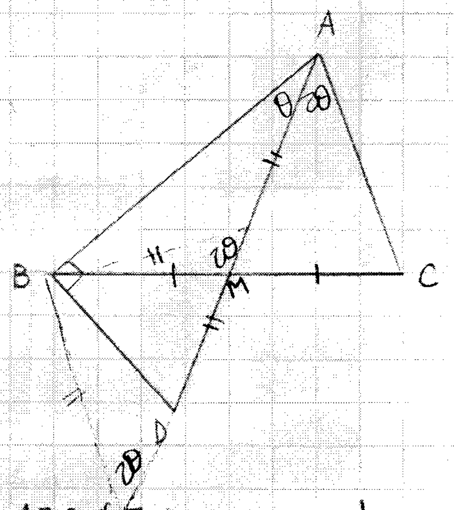
$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1.$$





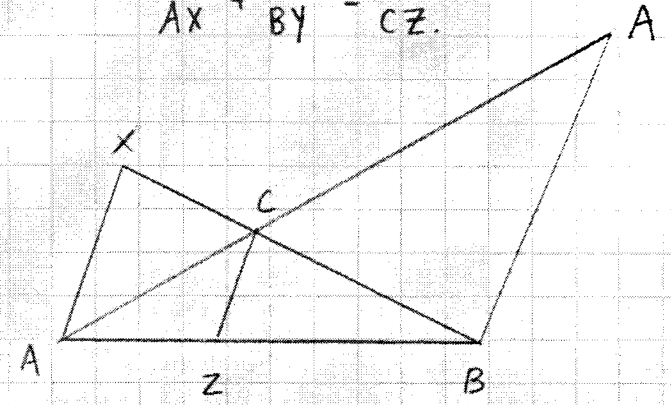
3 Problems

1. Prove the following: if, in $\triangle ABC$, median $\overline{AM}$ is such that $m\angle BAC$ is divided in the ratio 1:2, and $\overline{AM}$ is extended through M to D so that $\angle DBA$ is a right angle, then $AC = \frac{1}{2}AD$.



$AC = BX$ b/c $\triangle AMC \cong \triangle XMB$

2. In $\triangle ABC$, Z is any point on base $\overline{AB}$. $\overline{CZ}$ is drawn. A line is drawn through A parallel to $\overline{CZ}$ meeting $\overleftrightarrow{BC}$ at X. A line is drawn through B parallel to $\overline{CZ}$ meeting $\overleftrightarrow{AC}$ at Y. Prove that $\frac{1}{AX} + \frac{1}{BY} = \frac{1}{CZ}$.



3. For $\triangle ABC$ with medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, let $m = AD + BE + CF$, and let $s = AB + BC + CA$. Prove that $\frac{3}{2}s > m > \frac{3}{4}s$.

4. Let D be a point in the interior of $\triangle ABC$, and let E, F, G be points on AB, BC, CA, respectively. Given that AEBD and BFCD are cyclic, prove that CGAD is also cyclic.

5. Let M and N be the midpoints of sides AD and BC of rectangle ABCD, respectively. Let P be a point on ray CD but not on side CD. Let point Q be the intersection of AC and PM. Prove that $\angle MNQ = \angle MNP$.

6. Prove that the medians of any triangle are concurrent.

7. Prove that the interior angle bisectors of two angles of a non-isosceles triangle and the exterior angle bisector of the third angle meet the opposite sides in three collinear points.

8. Sides $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$ of quadrilateral ABCD are cut by a straight line at points K, L, M and N, respectively. Prove that $\frac{BL}{LC} \cdot \frac{AK}{KB} \cdot \frac{DN}{NA} \cdot \frac{CM}{MD} = 1$.

9. (USAMO 2012) Let P be a point in the plane of triangle ABC, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear

* Parts taken from Haitao Mao's Geometry Lecture in 2008.