2012 ARML - Advanced Geometry

14VXia

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1 Common Tips and Strategies

- If a problem is non-specific ("rectangle"), feel free to assume specifics to make your life easy ("square").
- Consider drawing two diagrams: one quick, one precise.
- Label everything in your diagram as you go along.
- Angle chase: parallel lines, isoscles triangles, similar triangles, cyclic quads, etc.
- Similar triangles: If all linear parameters are related by a constant k, then all two-dimensional parameters are related by k^2.
- Area of a triangle: $= \frac{1}{2}ah_a = \frac{1}{2}ab\sin C = rs = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$
- Cyclic quads: opposite angles sum to $180°$, equal angles cut off equal arcs.
- Don't be afraid of algebra: If lengths are in a given ratio, assign them variables. If you're not sure where to start, assign something to be x and start length-chasing (with the help of theorems).
- Circles: In a given circle, a chord of a fixed length will always cut off the same arc and angle. Power of a Point.
- Construction: If there's not enough to work with in the given diagram, try dropping altitudes, adding parallel lines, bisecting double angles, extending lines to intersect (especially if you have two $60°$ angles), adding radii to circles (to tangent points in particular), reflecting points,...
- Coordinates: Sadly, yes. It works.
- Trig: A very good idea if the diagram has plenty of right angles. See theorems below.
- Finally, READ THE PROBLEM CAREFULLY. For example, diameter is not the same thing as radius, and make sure you're answering the right question.

2 Useful Theorems

The key to these theorems is having intuition as to when to apply what, not just knowing them.

- Pythagoren Theorem: Very useful!
- Angle Bisector Theorem
- Extended Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.
- Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$.

- Stewart's Theorem: If D is on side BC of $\triangle ABC$ and a, b, and c are the sides of the triangle, and $BD = m$, $CD = n$, $AD = d$, then $dad + man = bmb + cnc$.
- Ptolemy's Inequality: $AB \cdot CD + BC \cdot DA \geq AC \cdot BD$.
- Ceva's Theorem: Given a triangle ABC and points D, E, and F on lines BC, AC, and AB, respectively, lines AD, BE, and CF are concurrent if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.
- Menelaus' Theorem: Given a triangle ABC and points D, E, and F on lines BC, AC, and AB (possibly extended), respectively, points D, E, and F are collinear if and only if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$.
- Fermat Point: It can be shown that the point F in the plane of triangle ABC such that the expression $AF + BF + CF$ is minimum is the unique point F such that $\angle AFB = \angle BFC = \angle CFA = 120°$. This point is known as the Fermat Point.

3 Problems

1. In convex pentagon $ABCDE$, $[ABC] = [BCD] = [CDE] = [DEA] = [EAB] = 1$, where $[ABC]$ denotes the area of triangle ABC. Compute the area of the pentagon.
2. Hexagon $ABCDEF$ is inscribed in a circle such that $AB = BC = CD = 2$ and $DE = EF = FA = 6$. Find the diameter of the circle.
3. Triangle XYZ is such that $XY = 5$, $YZ = 7$, and $ZX = 6$. Find the minimum value of $(XP + YP + ZP)^2$ for any point P in the interior of $\triangle XYZ$.
4. (Mandelbrot, AoPS) Find the length of the median of a trapezoid whose diagonals are perpendicular segments of lengths 7 and 9.
5. Given triangle ABC, point D is on BC such that $\frac{BD}{DC} = 5$ and E is on AC such that $\frac{AE}{EC} = 2$, and let G be the intersection of AD and BE. If $[GDC] = 5$, find $[AGE]$.