ARML Lecture XII - Inequalities

VMT Math Team

April 29, 2004

1 Simple Identities

To begin, we present some inequalities that are well known and simple:

- $x^2 \geq 0$, for any real x, with equality iff $x = 0$. This is known as the *trivial inequality*, and is generally the basis for all other inequalities.
- $\frac{1}{x}$, where $x > 0$, is maximized where x is minimized, and vice-versa.
- Between any two points, the shortest distance is a line. This is a result of the *triangle inequality*.
- $\sin\theta$ and $\cos\theta$ take minimum and maximum values at -1 and 1 respectively. The minimum and maximum of $a\sin\theta + b\cos\theta$ where a, $b > 0$ are $\pm\frac{a^2+b^2}{\sqrt{a^2+b^2}}$ respectively.
- Extreme cases usually occur when all of the variables are equal or, though less frequently, when one variable is set to an extreme value and all the others are set to the other extreme

Using these basic facts, we can derive many inequalities.

2 Inequalities

Now we will investigate more substantial results:

AM-GM (Arithmetic Mean-Geometric Mean Inequality) - First consider maximizing the product ab of two positive reals a and b with the requirement that $a+b = k$, for some constant k. To obtain a constant expression involving ab, we have $(a+b)^2 = k^2 = (a-b)^2 + 4ab$. By the trivial inequality, $(a-b)^2$ is minimized where $a = b$. But the sum must always equal k^2, another constant. Thus, $4ab$ is maximized when $a = b$, and it follows that ab is maximized at $\frac{k^2}{4}$. Specifically, $\frac{a+b}{2} \geq \sqrt{ab}$ where a and b are positive. (Alternatively, we could argue that $a^2-2ab+b^2=(a-b)^2\geq 0$, which, with a little algebra and assuming $a,b\geq 0$ leads to the same base case.) This result can be generalized; the proof follows by induction on n:

$$\begin{aligned}
\text{Case } n\to 2n:\quad \frac{(a_1+\cdots+a_n)+(a_{n+1}+\cdots+a_{2n})}{2} &\geq \sqrt{(a_1+\cdots+a_n)(a_{n+1}+\cdots+a_{2n})}\\
&\geq \sqrt{(n\cdot\sqrt[n]{a_1\cdots a_n})(n\cdot\sqrt[n]{a_{n+1}\cdots a_{2n}})}\\
\Longrightarrow \frac{a_1+\cdots+a_{2n}}{2n} &\geq \sqrt[2n]{a_1\cdots a_{2n}}
\end{aligned}$$

$$\begin{aligned}
\text{Case } n\to n-1:\quad \frac{a_1+\cdots+a_{n-1}}{n-1} &= \frac{a_1+\cdots+a_{n-1}+\left(\frac{a_1+\cdots+a_{n-1}}{n-1}\right)}{n}\\
&\geq \sqrt[n]{a_1\cdots a_{n-1}}\left(\frac{a_1+\cdots+a_{n-1}}{n-1}\right)^{\frac{1}{n}}\\
\Longleftrightarrow \left(\frac{a_1+\cdots+a_{n-1}}{n-1}\right)^{\frac{n-1}{n}} &\geq \sqrt[n]{a_1\cdots a_{n-1}}\\
\Longleftrightarrow \frac{a_1+\cdots+a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1\cdots a_{n-1}}
\end{aligned}$$

With more effort, we can establish the AM-GM-HM (Harmonic Mean) Inequality, which states that for $a_1,\ldots,a_n\geq 0$:

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1a_2\ldots a_n}\geq\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$$

With equality iff $a_1=\cdots=a_n$. The rightmost expression is known as the harmonic mean.

Cauchy-Schwartz Inequality (pronounced Koshi) - To obtain this result, we re-introduce the discriminant principle. That is, if a quadratic $ax^2+bx+c=0$ has one root, then its discriminant (b^2-4ac) is 0. If it has two roots, its discriminant is positive. If it has no roots, then the discriminant is negative.

Let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be real numbers. What is the discriminant of the quadratic $(a_1x-b_1)^2+(a_2x-b_2)^2+\cdots+(a_nx-b_n)^2=0$? Expanding and plugging into the formula for the discriminant, we obtain $(2a_1b_1+\cdots+2a_nb_n)^2-4(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)$.

Let us consider other analysis of the quadratic first. It consists entirely of squares on the left. By the trivial inequality, each square is at least 0, with equality iff $x=\frac{b_i}{a_i}$. If any of the squares is not 0, then x is not a solution. Clearly, there is at most one x that solves this equation. This one possible x solves the equation iff $x=\frac{b_i}{a_i}$ for all $1\leq i\leq n$.

Returning to the discriminant, there are two possible cases. If there is one solution, $(2a_1b_1+\cdots+2a_nb_n)^2-4(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)=0$ or $(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)=(a_1b_1+\cdots+a_nb_n)^2$, this is applies iff all of the $\frac{b_i}{a_i}$ are equal. If there is no solution, then $(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)>(a_1b_1+\cdots+a_nb_n)^2$, that is, when there exist $1\leq i<j\leq n|\frac{b_i}{a_i}\neq\frac{b_j}{a_j}$. Summarizing, we have the Cauchy-Schwartz inequality, which states that for any two sets of reals $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$,

$$(a_1^2+\ldots+a_n^2)(b_1^2+\ldots+b_n^2)\geq(a_1b_1+\ldots+a_nb_n)^2$$

With equality iff $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$.

Rearrangement Inequality - Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be sequences sorted into increasing order. Then:

$$a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n \le S(a, b) \le a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Where $S(a, b)$ represents any pairwise-product summation. The expressions on the left and right are the minimum and maximum, respectively, of $S(a, b)$. This inequality is largely intuitable, and is easily proved by contradiction.

3 Practice

Now try to use these principles and inequalities.

1. The volume of a rectangular prism is 24. What is the least surface area that it could have?

2. Determine the minimum of $x^2 + y^2 + 4z^2 - 2x + 4y - 12z - 5$.

3. We have a billiard table shaped like an equilateral triangle of side s. We wish to hit the billiard ball from one corner pocket into a different corner pocket but must first hit a wall. How far must we hit the ball to accomplish this?

4. The sum of 3 positive numbers is 6. What is the minimum possible sum of their reciprocals?

5. The distance from the point (A, B, C) to the origin is 7. What is the maximum possible value of $(A + 2B + 3C)$?

6. Given a, b, $c > 0$ are real numbers, what is the minimum of $\frac{1}{a} + \frac{a}{2b} + \frac{b}{4c} + \frac{c}{8}$, and at what order triple(s) (a, b, c) does this minimum occur at?

7. In $\triangle ABC$, $AB = 4$, $BC = 5$, and $AC = 7$. Point X is in the interior of the triangle such that $AX^2 + BX^2 + CX^2$ is a minimum. What is X, and what is the value of this expression?

4 Answers

Hopefully you were able to solve all of the forementioned problems. Here are possible approaches:

1. Let the dimensions of the prism be x, y, and z. Then we wish to minimize $2(xy + xz + yz)$, given $xyz = 24$. We apply AM-GM and get $\frac{xy+xz+yz}{3} \ge \sqrt[3]{x^2y^2z^2} = \sqrt[3]{24^2} = 4\sqrt[3]{9}$, from which it follows that $2(xy + xz + yz) \ge 24\sqrt[3]{9}$, and it follows that the minimum surface area is $\mathbf{24}\sqrt[3]{9}$.

2. We complete the square: $x^2+y^2+4z^2-2x+4y-12z-5=(x-1)^2-1+(y+2)^2-4+4(z-\frac{3}{2})^2-9-5$. By the trivial inequality, the squares cannot be negative, so $x^2+y^2+4z^2-2x+4y-12z-5 \geq -1-4-9-5=-19$. So the minimum is **-19**.

3. If we must hit one wall, then it is obvious that we must hit at least two walls. We form a lattice by reflecting the equilateral billiard table over its sides and then the new tables over their sides. We see that we must hit three walls (why?), and using Pythagoras, we see that the ball must travel at least $\mathbf{s\sqrt{7}}$.

4. Let a, b, and c be the numbers with $a+b+c=6$. We apply AM-GM-HM to obtain $\frac{a+b+c}{3}=2 \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$. Multiplying by $\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{2}$, we get $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3}{2}$. Not surprisingly, the minimum of $\mathbf{\frac{3}{2}}$ occurs when the numbers are all equal.

5. The given distance implies $A^2+B^2+C^2=49$. We apply Cauchy: $(A^2+B^2+C^2)(1^2+2^2+3^2) \geq (A+2B+3C)^2$, from which $49*13 \geq (A+2B+3C)^2$. From here, we can find the maximum $\mathbf{7\sqrt{13}}$ easily.

6. We AM-GM and obtain $\frac{\frac{1}{a}+\frac{a}{2b}+\frac{b}{4c}+\frac{c}{8}}{4} \geq \sqrt[4]{\frac{1}{a}\frac{a}{2b}\frac{b}{4c}\frac{c}{8}}=\sqrt[4]{\frac{1}{64}}=\frac{1}{2\sqrt{2}}$, from which $\frac{1}{a}+\frac{a}{2b}+\frac{b}{4c}+\frac{c}{8} \geq \sqrt{2}$, and we have a minimum of $\mathbf{\sqrt{2}}$. Moreover, equality holds iff $\frac{1}{a}=\frac{a}{2b}=\frac{b}{4c}=\frac{c}{8}=\frac{1}{2\sqrt{2}}$, so we have only one ordered triple at which it occurs: $(\mathbf{2\sqrt{2}}, \mathbf{4}, \mathbf{2\sqrt{2}})$.