2012 ARML - Advanced Algebra

14VXia

March 1st, 2012

NOTE: This lecture incorporates material from a good number of previous ARML lectures, all up on the Wiki. That's a great source of materials and problems.

1 Good Things to Know

- Don't forget to check all the technical things in algebra; a common mistake is to have extranneous solutions or leave some out for reaons such as forgetting that logs and square roots can't take negative arguments. Similarly, before you divide through by an expression, check to see when it's zero, don't forget the plus or minus when taking a square root, etc.
- The whole point of algebra is using variables to represent things; look for clever substitutions, especially in ARML.
- A common trick is to make whatever you're working with symmetrical. For example, to deal with $(x-9)^4+(x-8)^4+(x-7)^4+(x-6)^4+(x-5)^4$, let $y=x-7$. To deal with $x^4+3x^3+8x^2+3x+1=0$, try dividing through by x^2 and letting $y=x+\frac{1}{x}$. (You should actually try solving for the roots of the quartic using this technique to make sure you understand.)
- When expanding or combining fractions is necessary, look for patterns that make the work nicer; group terms logically, often the smallest paired with the largest. For example, to expand $(x-5)(x-4)(x-3)(x-2)$, group the outer terms first to get $(x^2-7x+10)(x^2-7x+12)$. Depending on what the question would be asking for, consider letting $y=x^2-7x$ at this point.
- Don't forget about trig substituions; $2x\sqrt{1-x^2}$ is screaming for $x=\sin y$. But check to make sure your variables fall into the appropriate ranges!
- ARML usually isn't terribly bashy... I think. Instead of trying to force your way through pages and pages of expansion and squaring in the very limited time ARML allows, look for something more clever. Of course, this doesn't mean that if your solution involves expanding $(x^2+x-2)^2$ or something like that, then it's wrong.

1.1 Sequences and Series

You should be very familiar with arithmetic and geometric sequences: the general term, how to sum them, etc. Though these are extremely useful, here we'll cover techniques you may be less familiar with.

Set $S =$ your desired sum. For example, find $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots$. A trickier variant of this technique is to let $S_n=a_1+a_2+\cdots+a_n$. Here's an example from DMM 2002: Compute a_{16} given $a_1,a_2,...,a_{50}$ with the property that for positive integers $n\leq 50$,

$$n(a_1+a_2+\cdots+a_n)=1+(a_{n+1}+a_{n+2}+\cdots+a_{50})$$

Solution: Defining S_n as suggested above, the given equation becomes $nS_n = 1 + S_{50} - S_n$, so $S_n = \frac{1+S_{50}}{n+1}$. Plugging in $n = 50$ gives $S_{50} = \frac{1}{50}$, and then a_{16} can be calculated by subtracting S_{15} from S_{16}.

Telescoping: Telescoping is a general technique used for evaluating series in which each term of the series is broken up into two or more pieces, such that these pieces cancel upon being added, making for easy computation at the end. For example:

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots+\frac{1}{9900}=(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{5})+\cdots+(\frac{1}{99}-\frac{1}{100})=1-\frac{1}{100}=\frac{99}{100}$$

You may be wondering how we possibly thought of this solution. Through experience, after you've tried your other series techniques and found that none of them really work, you think of telescoping and specifically try to break apart each term. It's particularly helpful that the denominator in each of the terms factored, as this tells you where to start. (This technique is called partial fractions.)

Here's a trickier example: $\sum_{k=1}^{99}(k^2+k+1)k! = \sum_{k=1}^{99}((k+1)\cdot(k+1)! - k\cdot k!) = 100\cdot 100! - 1$.

Also note that terms may not cancel with the terms directly to their left or right; they may cancel with terms two or even three terms away from them.

1.2 Algebraic Recursion

There's a lot to say here, but sadly not enough space. Therefore, today we will only discuss recursions similar to the Fibonacci sequence, i.e., linear recursive sequences of the form $a_n = ma_{n-1} + na_{n-2}$ for $n \geq 2$. We wish to find a general formula for a_n to make computation of large-numbered terms practical. To do this, we find the zeros of the characteristic polynomial for the particular sequence, which is $x^2 - mx - n$ in our case. Let the two zeros be x and y. The general term a_n is then of the form $a_n = A\cdot x^n + B\cdot y^n$ where A and B are constants that can be found by plugging in the given values of a_0 and a_1 and solving a system of two equations. Practice by using this technique to find the general term of the nth Fibonacci number using the recursion $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. You should get $F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$. It's worthwhile to note that this same technique extends to recurrences with more than just two terms. For example, the characteristic polynomial of the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ is $x^3 - x^2 - x - 1$.

2 Problems

1. Compute $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\cdots$.
2. Find all real x such that $x^{x^3-5x^2+6x} = 1$.
3. (OMO) The sequence $\{a_n\}$ satisfies $a_0 = 201$, $a_1 = 2011$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let
$$S = \sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}$$
What is $\frac{1}{S}$?
4. Given arithmetic sequences $\{a_n\}$ and $\{b_n\}$ such that for all positive integers n, $\frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n} = \frac{4n+27}{27n+4}$, find $\frac{b_{25}}{a_{25}}$.
5. (ASHME) Let $R_n = \frac{1}{2}(x^n + y^n)$, where $x = 3+2\sqrt{2}$ and $y = 3-2\sqrt{2}$. Compute the units digit of R_{12345}.
6. Given non-zero complex numbers x, y, and z, such that $\frac{x}{y} = \frac{y}{z} = \frac{z}{x}$, find all possible values of $\frac{x+y-z}{x-y+z}$.