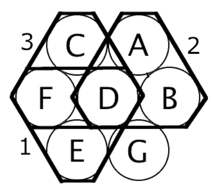
Answer Key

1. 288
2. 3
3. $300\sqrt{3}$
4. 2.93
5. $\sqrt{14}/7$*
6. 10004
7. 110592

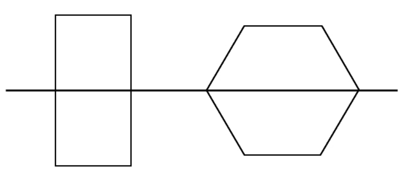
*See below for an alternate acceptable answer.

1. For a positive integer to be divisible by 9 the sum of its digits must also be a multiple of 9. But if all the digits are even then the sum of the digits will be also, making this sum at least 18. Since each digit is at most 8 we will need at least three digits; the first time the digits add to 18 is for **288**.

2. Note that any move changes the position of two of the outer six letters. Since five of those letters will need to be moved (all but the G) we require at least three moves. In fact, the puzzle can be solved in three moves as shown at right: rotate each of the highlighted triangular blocks in a counterclockwise direction in the order indicated. Hence the minimum number of moves is **3**.



3. Contrary to intuition, a square does *not* give the region of maximal area in this sort of situation. To understand why, imagine drawing a mirror image of the enclosed region on the other side of the barn wall.



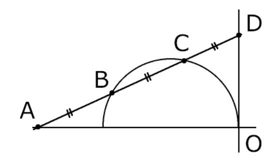
This will simply double the area of the chicken coop, so an equivalent problem would be to enclose as large an area as possible with six straight lengths of fencing. It makes sense to come as close to a circle as possible, which is achieved by a regular hexagon. Therefore the answer to the original problem is "half a hexagon." Its area is equal to that of three equilateral triangles of side length 20, or $3(20^2\sqrt{3}/4) = \mathbf{300\sqrt{3}} \approx 519.6$, considerably larger than the 400 we would obtain with a square.

4. We wish to approximate the solution to $3^x + 2x = 31$. In order to match the form of the equation whose solution we are given, we multiply both sides by 3, then add 6, yielding

$$3^x + 2x = 31 \quad\Longrightarrow\quad 3^{x+1} + 6x = 93 \quad\Longrightarrow\quad 3^{x+1} + 6(x+1) = 99.$$

In order to satisfy this latter equation we are told to take $x + 1 \approx 3.93$. Therefore we want $x = \mathbf{2.93}$.

5. Scale the figure so that the length of each congruent segment is 1. Note that this transformation does not change the slope of the line, so it is a valid simplification of the problem. For convenience, we label the various points in the diagram as shown. Now apply the Power of a Point Theorem to point D, which states that $(DC)(DB) = (DO)^2$. Since $DC = 1$ and $DB = 2$ we find that $DO = \sqrt{2}$.



Next the Pythagorean Theorem implies that $(AO)^2 + (DO)^2 = (AD)^2$, hence $(AO)^2 = 3^2 - (\sqrt{2})^2 = 7$, so $AO = \sqrt{7}$. We are now in a position to determine the slope, which is equal to $\sqrt{2}/\sqrt{7} = \mathbf{\sqrt{14}/7}$ (or $\sqrt{\mathbf{2/7}}$).

6. Let r and s be the roots of the quadratic $x^2 + bx + c$, so $b = -(r+s)$ and $c = rs$. One can readily verify that $|b^2 - 4c| = |r - s|^2$. Therefore it is equivalent to choose values of b and c that maximize the distance between the roots. Observe that the graph of $y = x^2 + bx + c$ will be congruent to the graph of $y = x^2$ regardless of b and c, which only serve to translate the parabola in the plane without changing its shape. To maximize $|r - s|$ we should arrange to translate $y = x^2$ as far down as possible, since such a translation spreads out the roots the most. This can be accomplished by choosing a parabola passing through $(-1, 100)$ and $(1, -100)$, so we need $1 - b + c = 100$ and $1 + b + c = -100$. Solving for the coefficients leads to $b = -100$, $c = -1$, giving an optimal value of $(-100)^2 - 4(-1) = \mathbf{10004}$.

7. Since each square is located within exactly two bands, we will need six colored squares. Four of these will need to be on the left/right or front/back faces to accommodate the four horizontal bands. Similar considerations apply for the two types of vertical bands; together these conditions imply that we need two colored squares on the top/bottom faces, two on the left/right faces, and two on the front/back faces. Furthermore, each square on the top face has a corresponding square directly beneath it that occupies precisely the same bands, and similarly for the left/right faces and front/back faces. Hence we can count the number of ways to place six colored squares on the top, front, and right faces and then multiply our answer by 2^6.

There are $\frac{1}{2}(16 \cdot 9) = 72$ ways to choose a pair of colored squares on the top face that are not in the same row or column. This pair rules out two columns of the front face, leaving $\frac{1}{2}(8 \cdot 3) = 12$ ways to choose a pair of colored squares from the remaining eight squares that do not occupy the same row or column. These choices in turn narrow down the options on the right face to just four square, which can be filled by the remaining two colored squares in 2 ways. In total we have found

$$2^6 \cdot 72 \cdot 12 \cdot 2 = \mathbf{110592}$$

ways to color six squares in the prescribed fashion.

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⋆ National Level ⋆

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