Polynomials I

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Polynomials are a fundamental part of almost every algebra problem. A solid knowledge of polynomials can really go a long way.

1 Finding the Roots of a Polynomial

One of the most basic things you will have to be able to do with a polynomial is find its roots. I'm assuming all of you already know the quadratic formula, so finding the roots of a second-degree polynomial is pretty simple. But what about higher order polynomials? While explicit formulae do exist for finding the roots of third- and fourth- degree polynomials, the equations are unbelievably hideous and not practical at all for contest math (or anything really). Luckily for us, we have some tricks to help us find the roots of polynomials.

1.1 Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that a polynomial of degree n has at least 1 root and at most n roots in the complex plane. This means that a n-degree polynomial can always be written as $a(x-r_1)(x-r_2)(x-r_3)\dots(x-r_{n-1})(x-r_n)$, where r_1 through r_n are the (not necessarily real) roots of the polynomial.

1.2 Factoring

Factoring a polynomial and finding the roots of a polynomial often go hand in hand. Here are some special polynomial factorizations that you should memorize (or know how to derive).

$$\begin{aligned}
(x+y)^n &= \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i \quad \text{[also known as the Binomial Theorem]}\\
(x-y)(x^{n-1}+x^{n-2}y+\ldots+xy^{n-2}+y^{n-1}) &= x^n-y^n\\
(x+y)(x^{n-1}-x^{n-2}y+\ldots-xy^{n-2}+y^{n-1}) &= x^n+y^n \quad \text{[for odd } n\text{]}\\
(x+y+z)^2 &= x^2+y^2+z^2+2(xy+yz+zx)\\
(x+y+z)(x^2+y^2+z^3-xy-yz-zx) &= x^3+y^3+z^3-3xyz\\
(x+y)(y+z)(z+x)+xyz &= (x+y+z)(xy+yz+zx)\\
xy+x+y+1 &= (x+1)(y+1)\\
x^2+y^2+z^2-xy-yz-zx &= ((x-y)^2+(y-z)^2+(z-x)^2)/2
\end{aligned}$$

1.3 Root-Finding Theorems

Randomly guessing is an extremely inefficient way to determine the roots of a polynomial. Instead, we use some some nifty theorems to assist us.

- **Remainder Theorem:** Given a polynomial $P(x)$, $x-a$ is a factor of $P(x)$ if $P(a)=0$.
- **Intermediate Value Theorem:** Given a polynomial $P(x)$, if $P(a)<0$ and $P(b)>0$, there exists some c such that c is between a and b and $P(c)=0$. This may seem trivial but it can actually be useful!

- **Rational Root Theorem:** Given a polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$, all rational roots of $P(x)$ can be expressed in the form of $\frac{p}{q}$ where p divides a_0 and q divides a_n.

1.4 Viéta's Formulas

Viéta's Formulas are essential for many algebra problems. They relate the roots of a polynomial to the coefficients of that polynomial. Consider the polynomial $P(x) = a_n x^n + \ldots + a_0$. By the Fundamental Theorem of Algebra, $P(x) = a_n(x - r_1)\ldots(x - r_n)$. By equating with the first expression, we find that

$$\sum_{C \in C_j} \left(\prod_{i=1}^{j} c_i \right) = (-1)^j \cdot \frac{a_{n-j}}{a_n}$$

where C_j is the set of all j-element subsets of $r_1, \ldots, r_n$. This sum is sometimes called "j-wise sum" of $P(x)$. For example, if $n = 4$, $r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4 = \frac{a_2}{a_4}$.

2 Symmetric Polynomials

A polynomial is said to be symmetric if plugging in the variables in all possible permutations gives the same result. All symmetric polynomials can be represented as a polynomial of $j_1, j_2, \ldots, j_n$, where j_i is the j-wise sums of x_i.
For example, let $j_1 = x + y$, $j_2 = xy$, and $s_n = x^n + y^n$.

$$\begin{aligned} s_1 &= j_1 \\ s_2 &= j_1^2 - 2j_2 \\ s_3 &= j_1^3 - 3j_1 j_2 \\ s_4 &= j_1^4 - 4j_1^2 j_2 + 2j_2^2 \end{aligned}$$

While symmetric polynomials are not extremely common, they can often greatly simplify a problem.

3 Problems

1. The polynomial $x^3 - 2004x^2 + mx + n$ has integer coefficients and three distinct positive zeros. Exactly one of these is an integer, and it is the sum of the other two. How many values of n or possible?

2. Positive integers a, b, c, and d satisfy $a > b > c > d$, $a + b + c + d = 2010$, and $a^2 - b^2 + c^2 - d^2 = 2010$. Find the number of possible values of a.

3. Let P be the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$.

4. Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

5. Let $P(z) = x^3 + ax^2 + bx + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

6. Let C be the coefficient of x^2 in the expansion of the product $(1 - x)(1 + 2x)(1 - 3x)\ldots(1 + 14x)(1 - 15x)$. Find $|C|$.

7. For some integer m, the polynomial $x^3 - 2011x + m$ has the three integer roots a, b, and c. Find $|a| + |b| + |c|$.

8. Let $P(x) = x^2 - 3x - 9$. A real number x is chosen at random from the interval $5 \leq x \leq 15$. The probability that $\lfloor \sqrt{P(x)} \rfloor = \sqrt{P(\lfloor x \rfloor)}$ is equal to $\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}-d}{e}$, where a, b, c, d, and e are positive integers, and none of a, b, or c is divisible by the square of a prime. Find $a + b + c + d + e$.

Sources: AIME, AMC 12, Sin Kim's Polynomials Lecture