Vectors and their Applications

Thomas Mildorf

2003

1 Introduction to Vectors

From the pestilence of mosquitos to particles traveling at superluminal speeds, vectors are everywhere. They are the basis of motion, and can be used in conjunction with trigonometry and geometry to tackle just about any problem. In particular, they are important in non-planar mathematics, where the third dimension complicates computations.

2 Vector Operators

A vector $\vec{v}$ is defined by a direction and a magnitude (length). We can defined them in terms of an angle θ and a length r in the plane, but considering a third dimension it is usually simplest to use component form, typically denoted $\vec{v} = < v_1, v_2, v_3 >$. In most places (including this lecture), the first component is x displacement, the second y, and the third is z displacement. Vectors connecting two points, here a vector from A to B, are often denoted $\overrightarrow{AB}$. The magnitude of vector v is denoted $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. Some people substitute v for $\vec{v}$.

Vectors can be added and subtracted easily: just add and subtract componentwise. Let $\vec{u} = < u_1, u_2, u_3 >$ and $\vec{v} = < v_1, v_2, v_3 >$. Then we have $\vec{u} + \vec{v} = < u_1 + v_1, u_2 + v_2, u_3 + v_3 >$, and $\vec{u} - \vec{v} = < u_1 - v_1, u_2 - v_2, u_3 - v_3 >$. Vectors cannot be multiplied together, but can be multiplied by a scalar, which is a pure number. E.G. $2\vec{u} = < 2u_1, 2u_2, 2u_3 >$. Of course, this follows intuitively from the addition. *Unit vectors* are vectors of magnitude 1; the unit vector of any non-zero vector $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$. By using a unit vector as a stepping stone, we can create a vector of magnitude m in the direction of $\vec{v}$ simply by evaluating $\frac{m}{|\vec{v}|}\vec{v}$.

Now we examine two other, more interesting vector operations, the dot product and the cross product. These are the operations that almost become a necessity in 3-space.

DotProduct - Denoted by $\vec{u} \cdot \vec{v}$, where $\vec{u} = < u_1, u_2, u_3 >$ and $\vec{v} = < v_1, v_2, v_3 >$, the dot product is defined to be: $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$. Additionally, if θ is the angle formed by $\vec{u}$ and $\vec{v}$, then $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$. From this equation, it follows that $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u}$ and $\vec{v}$ are perpendicular. Note that the dot product of two vectors is a pure number.

CrossProduct - Denoted by $\vec{u}$ x $\vec{v}$, where the vectors are defined as above, the cross product is defined to be $\vec{u}$ x $\vec{v} =< u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >$. If θ is defined as above, then $|\vec{u}$ x $\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$. Geometrically, the cross product creates a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$. The magnitude of the cross product vector is the area of the parallelogram with sides of the two argument vectors.

3 Direct Applications

Here we present applications of vectors with a focus on spatial applications. In particular, we consider the use of a *normal*, a vector perpendicular to a plane, frequently.

Planes - In three-space, a plane is defined by an equation of the form $Ax + By + Cz = D$. A plane can be determined by three (non-collinear) points, although this form is somewhat useless as other points in the plane cannot be easily computed. Two parallel lines or two askew lines passing through a common point also determine exactly one plane, although similar problems cause computing other points in the plane to be difficult. How do we create the plane equation given any of these setups?

We begin by considering a normal which we'll call $\vec{n}$ defined by $< n_1, n_2, n_3 >$ to the plane p defined by Ax + By + Cz = D, and a point $P(x_1, y_1, z_1)$ on p. $\vec{n}$ is perpendicular to all of the vectors in the plane p. Consider all points (x, y, z) in the plane. The vector $\vec{v}$ from P to any of these points is $< x - x_1, y - y1, z - z1 >$. Since this vector must be perpendicular to $\vec{n}$, we have $\vec{n} \cdot \vec{v} = n_1(x - x_1) + n_2(y - y_1) + n_3(z - z_1) = 0$. This generic expression is equivalent to the equation of the plane, and we discover the identity $\vec{n} =< A, B, C >$, the coefficients of x, y, and z in the equation of the plane. A direct and very useful consequence is *a normal and a point on a plane determine that plane and vice versa.*

All of the scenarios allow us to choose three points in the plane we need to compute. We construct two vectors in the plane, cross them to obtain a normal vector, and then plug into the formula using any of the points in the plane to determine the plane.

Lines - In three space, a line is defined by two equivalent sets of equations, the more simple of which is $x = x_0 + At$, $y = y_0 + Bt$, and $z = z_0 + Ct$, where t ranges over the real numbers defining all of the points on the line in order (parametric form). The other form (symmetric form) is obtained by solving for t is each equation and then setting all three to be equal. Vectors work more naturally with the former, so we will work with it. Any two distinct points determine a line, as do two non-parallel planes (by intersecting them). A line is also uniquely determined by projecting a point onto a plane, although this is rarely used. How do we find the line given any of these scenarios?

We consider the line as determined by a vector $\vec{v} =< v_1, v_2, v_3 >$ and a point $P(x_0, y_0, z_0)$. The line can be defined in parametric form by $x = x_0 + v_1 t$, $y = y_0 + v_2 t$, and $z = z_0 + v_3 t$.

We need that point P in any case. When given two planes, we must solve for a common point. This is easy to do, just substitute a value for x, y, or z and solve the two equation, two variable system of equations that results. To obtain the vector, given two points we subtract componentwise to obtain a vector. Given a plane and a point, a the necessary vector is the normal obtained by taking the coefficients from the equation. Given two planes, we merely need to find a vector parallel to the line of intersection. Such a vector can be found nicely by taking the normals to the two planes and then computing their cross product. After obtaining a point and a vector, the equation for the line follows easily. After computing parametric form, conformity might be a good reason to convert the equations into symmetric form (since $t = 0$ and the vector's scalar can vary widely.)

Projections - When dealing with a projection of one segment onto some surface, we form right triangles that are instrumental in computing the length of the projected segment. We often project a line segment onto another line segment and sometimes project a line segment onto a plane. When dealing with these projections we are usually concerned with length, and are sometimes concerned with the orientation of the projection.

If we can obtain the vector $\vec{v}$ to be projected and the vector $\vec{u}$ onto which it will be projected (such as a line contained in a plane) then all we have to do to obtain the projection is use the dot product and some algebra. Let θ be the angle between $\vec{u}$ and $\vec{v}$. The directed length of the projection will be $|\vec{v}| \cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$. After finding the directed length, we need to orient the projection. It lies on $\vec{u}$, and has the same direction as $\vec{u}$, so by multiplying by a unit vector of $\vec{u}$ we obtain $\text{Prog}_{\vec{v}\,to\,\vec{u}} = \frac{(\vec{u}\cdot\vec{v})\vec{u}}{|\vec{u}|^2}$.

If we are given a second segment, finding $\vec{u}$ is easy. Given a plane, we must work indirectly. We project $\vec{v}$ onto a normal to the plane, and then subtract the normal projection from $\vec{v}$ to obtain the projection onto the plane.

4 Mass Points

Vectors are an excellent tool for finding length ratios. Through vectors, you can prove that the following math is valid.

In a triangle ABC be a triangle, and D, E, and F points on sides BC, AC, and AB respectively. Let G be the point of intersection of AD, BE, and CF. Attach *weights* w_1, w_2, and w_3 to the points A, B, and C respectively. If any two of the following are true, then all of them are true:

$$w_1 AF = w_2 FB$$
$$w_1 AE = w_3 EC$$

$$
\begin{aligned}
w_2 BD &= w_3 DC \\
(w_1 + w_2)FG &= w_3 GC \\
(w_1 + w_3)EG &= w_2 GB \\
(w_2 + w_3)DG &= w_1 GA
\end{aligned}
$$

Where the sides referred to are their lengths. These equations represent balanced levers with fulcrums D, E, F, and G. A simple example is if D and E are medians, we have AE = EC, so $w_1 = w_3$. Similarly, $w_2 = w_3$. The other equations show that CF is a median, and that each of the medians is divided into segments of length ratio 2:1 by G.

5 Example

We end with a 3-D geometry problem that makes use of many of the concepts presented in this lecture:

"A parallelpiped has a vertex at (1,2,3), and adjacent vertices (that form edges with this vertex) at (3,5,7), (1,6,-2), and (6,3,6). Find the volume of this parallelpiped."

We first obtain three vectors $\vec{u} = < 2, 3, 4 >$, $\vec{v} = < 0, 4, -1 >$, and $\vec{w} = < 5, 1, 3 >$ that represent the edges of the parallel piped. We let the face determined by $\vec{u}$ and $\vec{v}$ be the base. Since volume = base * height, we compute area of the base by crossing the edges that form it, $\vec{u}$ x $\vec{v} = < -19, 2, 8 >$, and take note that the magnitude of this vector is that area. Because the cross product is a normal to the base, the height is the projection of $\vec{w}$ onto $< -19, 2, 8 >$. Let θ be the angle between $\vec{w}$ and $< -19, 2, 8 >$. The length of the projection is $|\vec{w}| \cos\theta = \frac{<-19,2,8> \cdot \vec{w}}{|<-19,2,8>|}$. Recalling that $V = bh$, we plug in these expressions and obtain $V = \frac{<-19,2,8> \cdot \vec{w}}{|<-19,2,8>|} | < -19, 2, 8 > | = < -19, 2, 8 > \cdot \vec{w}$. Taking the dot product yields -69. Since the volume is positive, we have a volume of 69.