# Useful Non-Elementary Techniques

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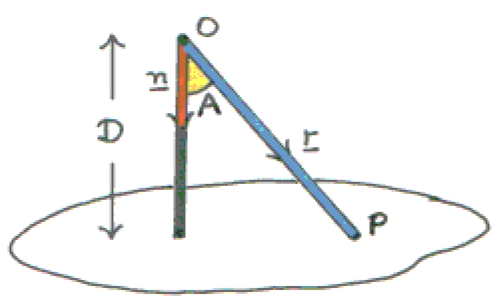
## 1 Introduction

High school mathematics competitions are generally limited in scope to "elementary" mathematics. With the sole exception of HMMT, the use of any methods based on calculus or more advanced topics is usually not needed at any high school level competition. However, such techniques can often be quite useful in simplifying or even trivializing problems intended to be solved with an elementary approach.

## 2 Distances and the Dot Product

In any situation in which we wish to determine the distance between a point and a line, between a point and a plane, between a line and a line, between a line and a plane, or between a plane and a plane, we can employ the dot product as a projective operator to quickly and easily determine the desired value.

As an illustrative example, consider the distance between a point and a plane. We can find the vector *normal* to the plane by taking the cross product of two linearly independent vectors lying in the plane. We can then take any vector from a point on the plane to the point of interest and compute its dot product with a unit vector in the direction of the normal.



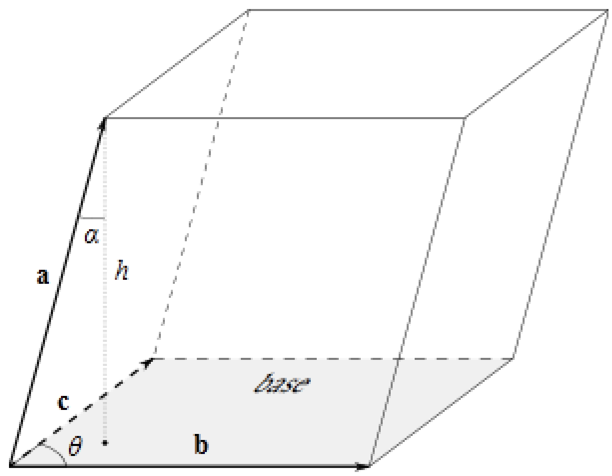

By projecting the arbitrary displacement vector from the plane to the point onto the normal vector, we eliminate the "sideways" portion of the displacement and reduce it to its perpendicular part. The magnitude of the resulting value is the distance we wished to determine.

## 3 Area and the Cross Product

Recall the well known formula for the area of a triangle, $A = \frac{ab\sin(C)}{2}$. Notice that this is exactly one half of the expression for the cross product of two vectors in terms of their magnitudes and the angle between them. In the case that the angle involved (or its sine) is not easily determined, such as in a three-dimensional situation, we can directly apply the cross product to vectors representing two sides of the triangle to determine its area. Keep in mind that the cross product will produce a vector; its magnitude is the value of interest in determining the area we desire. Of course, dropping the factor of $\frac{1}{2}$ yields the area of the parallelogram produced by reflecting this triangle across its third side.

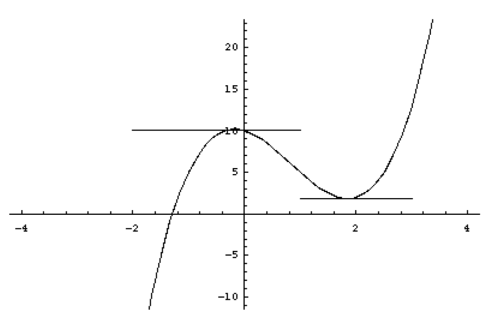
# 4 Vector Products and Volumes

We can extend the use of vector products further to determine the volume of an abritrary tetrahedron, and by extension the associated parallelepiped. Notice that if we use the technique described above to determine the area of one of the faces of a parallelepiped, simply multiplying by the distance to the opposite face will produce the full volume. Suppose our parallelepiped has sides represented by the vectors $a$, $b$, and $c$. The cross product of $a$ and $b$ will yield a vector normal to the face defined by $a$ and $b$ whose magnitude represents its area. Finally, taking the dot product of this resulting vector with $c$ results in a scalar whose magnitude gives the area of the parallelepiped defined by $a$, $b$, and $c$. Dividing out a factor of 2 produces the area of the tetrahedron defined by $a$, $b$, and $c$.




# 5 Extremization Over One Variable

The most obvious direct application of calculus is in the minimization or maximization of a continuous function of one variable. Most, if not all of you, are probably familiar with this technique. Essentially, we observe that at a minimum or maximum, the function in question must switch from increasing to decreasing, (or vice-versa). The value of the derivative of the function, which gives its instantaneous rate of change, will therefore be 0 at this point. This can be visualized as a horizontal tangent line on the graph of the function:

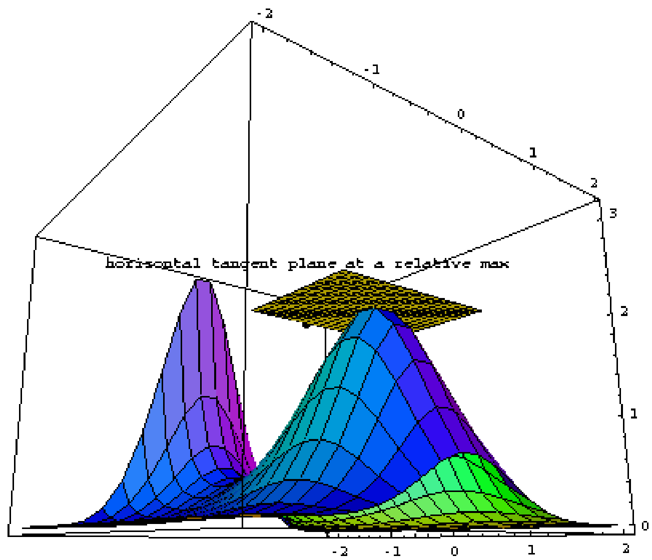


When we are given a function to extremize or have managed to reduce a problem to such a task and we wish to use this technique, we start by differentiating the function and solving for zeroes of the derivative. These zeroes, along with the endpoints of the range we wish to consider, are the candidates for maxima and minima of our function. We can simply plug these values in and compare the resulting function values to determine our answer.

# 6 Extension to Multiple Variables

The extension of extremization to functions of multiple variables is not much different from the single-variable case. For each variable of interest, we determine the partial derivative of the function with respect to the variable. A partial derivative describes the rate of change of a function with respect to the variable in question. For the purposes of determining the partial derivative, all other variables are treated as constants. In this sense, partial derivatives are not at all much different from "normal" derivatives.

As before, we wish to determine the roots of our derivative(s). For maxima and minima, every partial derivative will have a value of 0. In a function over two variables, for example, we can visualize extrema as locations at which the surface of interest has a horizontal tangent plane:




We now essentially have a system of equations in terms of our original variables to solve. But do not fear! Problems intended to be solved via an elementary approach usually result in an exceedingly simple system of equations when tackled using this technique.

# 7 Constraints and Lagrange Multipliers

Finally, we consider extremization of a function under some set of constraints. Will this might seem like a task that could become very complicated very quickly, a rather elegant approach due to the French mathematician Joseph-Louis Lagrange simplifies any problem of this type directly into a normal extremization task.

Suppose we wish to extremize the function $f(x, y, z)$ under the condition that $g(x, y, z) = 0$. It is not difficult to see that any constraint we wish to impose can be expressed in such a manner. We will essentially introduce a "dummy variable" $\lambda$ to produce a new function $h(x, y, z; \lambda)$ that will share its maxima with $f(x, y, z)$ and *must* satisfy our constraint.

To achieve this, we introduce the "dummy variable" $\lambda$, known as a Lagrange multiplier. The function $h(x, y, z; \lambda)$ is defined as $f(x, y, z) + \lambda(g(x, y, z))$. Notice that the maxima of $h$ must occur at points for which its partial derivative with respect to $\lambda$ has a root; that is, points at which our constraint $g(x, y, z) = 0$ is satisfied. This also means that at any of its maxima, $h$ will take on the same value as $f$, since the product of 0 and $\lambda$ is simply 0.

This technique can readily be extended to problems involving multiple constraints by adding on more multipliers, one per constraint. For extremization of $f(x, y, z)$ under the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, for example, we would consider the function $j(x, y, z; \lambda, \mu) = f(x, y, z) + \lambda(g(x, y, z)) + \mu(h(x, y, z))$.

# 8 Infinite Continuums of Possibilities

Suppose we wish to compute probabilities or expected values associated with an event with an infinite continuum of possibilities. Clearly, we cannot simply consider the number of ways for which some specific possibility occurs. In such cases, the elementary solution - if one exists - will usually involve some form of symmetry argument which may not be easy to see at all. We can tackle problems like this in an alternative manner by employing integration to deal with the infinity of possible outcomes.

As a introductory example, consider an ant placed at a random position on a meter stick. Once placed, the ant will select one end of the stick at random and travel towards it at a steady rate of 1 m/s. We wish to determine the expected value of the time taken for the ant to reach its chosen end of the stick. While the symmetry here is not difficult to see, it serves as an illustrative example of the application of integration to such problems. Without loss of generality, assume the ant is travelling to the right end of the stick. We then consider the following integral:

$$\int_0^1 x\mathrm{d}x$$

to arrive at the answer of $\frac{1}{2}$.

# 9 Problems

1. Let $PQ$ be the line passing through the points $P = (-1, 0, 3)$ and $Q = (0, -2, -1)$. Determine the shortest distance from $PQ$ to the origin.

2. Determine the shortest distance from the plane defined by the points $A = (-3, 4, -2)$, $B = (0, 1, 3)$, and $C = (4, -2, 3)$ to the point $D = (2, 2, 2)$.

3. Consider a rectangle $ABCD$ such that side $AB$ has length $n$ and side $BC$ has length $m$. A circle is drawn with center $E$ at the midpoint of side $BC$ such that it is tangent to the diagonal $AC$. Determine the radius of this circle in terms of $n$ and $m$.

4. Determine the area of the triangle with vertices at $A = (2, 5, 2)$, $B = (1, -4, 6)$, and $C = (0, 0, 5)$.

5. 1000 ants are distributed at random on a one-meter stick. They individually pick one end of the stick at random to travel towards, and begin crawling at a rate of 1 m/s. When two ants collide, they simply turn around and continue walking. Determine the expected value of the time it takes for all of the ants to fall off the stick.

6. Two people agree to meet at a given place between noon and 1 PM. By agreement, the first to arrive will wait 15 minutes for the second, after which he will leave. What is the probability that the meeting actually takes place if each of them selects his moment of arrival at random during the interval from 12 noon to 1 PM? (A.M. Yaglom and I.M. Yaglom)

7. Albert is experimenting with a curious new style of minting machine. When activated, the machine generates a random real number $p$ in the range $[0, 1]$ and produces a coin that is weighted to land heads with probability $p$. If Albert uses the machine to mint a coin and then flips that coin 2000 times, what is the probability that it lands heads exactly 1000 times? (Adapted from Bay Area Math Meet 2000)

8. Determine the expected value of the area of a triangle with circumradius 1.

9. Let $N$ be the positive integer with 1998 decimal digits, all of them 1; that is,

   $$N = 1111 \cdots 11$$

   Find the thousandth digit after the decimal point of $\sqrt{N}$. (Putnam 1998)

10. A piece is broken off at random from each of three identical rods. What is the probability that an acute-angled triangle can be formed from the three pieces? (A.M. Yaglom and I.M. Yaglom)