ARML Lecture XI - Polynomials

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1 Polynomials of One Variable

Typically we write a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Such a polynomial is said to be of *degree* n and has a *leading coefficient* of a_n. Specifically, if $a_n = 1$, then we call f a *monic polynomial*.

By the fundamental theorem of Algebra, any polynomial of degree n can be expressed as $f(x) = k(x - r_1)(x - r_2)\cdots(x - r_n)$. In this form the leading coefficient is k, and the *roots*, the values of x for which the polynomial $f(x)$ is 0, are r_i. Furthermore, if $a_0, a_1, \ldots, a_n$ are real numbers, all of the roots are complex numbers such that the imaginary roots of f come in *complex conjugates*.[1]

1.1 Factoids

The *Division Algorithm* for polynomials states that if $f(x)$ and $p(x)$ are polynomials, then there exist unique polynomials $q(x)$ and $r(x)$ such that the degree of $r(x)$ is less than the degree of $p(x)$ and $f(x) = p(x)q(x) + r(x)$. This should somewhat resemble the division of numbers.

Now, how do we use polynomials to determine facts? Consider finding the sum of the coefficients of any polynomial. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$. What is the sum $\sum_{i=0}^{n} a_i$? We only need to remove the x^n's. Because $1^k = 1$ for any real k, setting $x = 1$ causes all of the x^n's to go away leaving $f(1) = \sum_{i=0}^{n} a_i$.

What about the constant term, how do we find it?

1.2 Polynomial Division

Now think about polynomial division. Let us divide $f(x)$ by a linear expression $(x - a)$. By the division algorithm, $f(x) = p(x)(x-a)+r(x)$, but because r has a lower degree than $(x-a)$,

[1] Pairs of complex numbers z and $\overline{z}$ such that $z = a + bi$ and $\overline{z} = a - bi$ where a and b are real, and $i = \sqrt{-1}$.

it must be constant. We set $x = a$ and get an interesting result: $f(a) = p(a)(a-a) + r = r$, and discover that the remainder when $f(x)$ is divded by $(x-a)$ is $f(a)$. This implies that f(0) is the constant term of the polynomial. (Why?)

What about division by non-linear expressions? Suppose we have $f(x) = x^4+3x^2-4x+8$, and wish to divide by $p(x) = x^2-3x+2$. We set up our equation: $x^4+3x^2-4x+8 = (x^2-3x+2)q(x)+r(x)$. Noting that $x = 1$ and $x = 2$ cause the RHS to be $r(1)$ and $r(2)$ respectively, we see that $r(1) = 8$ and $r(2) = 28$. We are dividing by a quadratic expression, so the remainder $r(x)$ must be linear or constant. We have two values of $r(x)$ and can compute $r(x) = 20x - 12$.

There is another beast called *polynomial long division* that is completely methodic although very tedious and slow. It works by computing the highest degree term in the quotient $q(x)$ by dividing the leading coefficients of $f(x)$ and $p(x)$, and subtracting with each step until we obtain a remainder of degree lower than $p(x)$. An example would be nice, so we show it applied to the previous problem:

$$\begin{gathered}
\tfrac{x^4+0x^3+3x^2-4x+8}{x^2-3x+2} \rightarrow \tfrac{x^4}{x^2} = x^2 \\
x^4+3x^2-4x+8-x^2(x^2-3x+2) = 3x^3+x^2-4x+8 \\
\tfrac{3x^3+x^2-4x+8}{x^2-3x+2} \rightarrow \tfrac{3x^3}{x^2} = 3x \\
3x^3+x^2-4x+8-3x(x^2-3x+2) = 10x^2-10x+8 \\
\tfrac{10x^2-10x+8}{x^2-3x+2} \rightarrow \tfrac{10x^2}{x^2} = 10 \\
10x^2-10x+8-10(x^2-3x+2) = 20x-12 \\
\text{Degree is less than divisor} \\
x^4+3x^2-4x+8 = (x^2+3x+10)(x^2-3x+2)+(20x-12)
\end{gathered}$$

If we wish only to find the remainder polynomial, $r(x)$, we should focus on the first method.

1.3 Root Trickery

Consider $f_1(x) = k(x-r_1)\cdots(x-r_n)$. What if we want a polynomial with roots $\frac{1}{r_i}$ instead? Let $f_2(x) = (x-\frac{1}{r_1})\cdots(x-\frac{1}{r_n}) = (\frac{r_1x-1}{r_1})\cdots(\frac{r_nx-1}{r_n}) = \frac{k}{r_1r_2\cdots r_n}(r_1x-1)\cdots(r_nx-1)$. Because the leading coefficient does not affect the roots, we simply discard the product of the roots under k in $f_2(x)$ when we write $f_3(x) = k(r_1x-1)\cdots(r_nx-1)$. Expanding this, we see that $f_3(x)$ has the same coefficients as $f_1(x)$ but in reversed order![2] What can we conclude about a palindromic polynomial?

What about a polynomial with the negative roots? Through similar substitution, it becomes apparent that the sign of every other coefficient (starting with a_{n-1}) changes.

Finally, possibly the most useful of everything polynomial, **Viéta's Formulas**. Let $f(x) = a_nx^n + \cdots + a_0 = a_n(x-r_1)\cdots(x-r_n)$. Then the j-wise sum S_j, is given by

[2]Be careful if 0 is a root of $f_1(x)$ - one of the algebraic steps we did is invalidated.

$$S_j = \sum_{C \in C_j} \left(\prod_{i=1}^{j} c_i \right) = (-1)^j \cdot \frac{a_{n-j}}{a_n}$$

where C_j is the set of all j-element subsets of $\{r_1, \dots r_n\}$. This can be applied in a number of ways, so memorize it.

1.4 Predicting Values

It is often necessary to predict values of a polynomial given specific facts about a polynomial. Here we present two techniques.

Method of Finite Differences - Suppose we have a polynomial f and the values of $f(x_0)$, $f(x_1)$, ..., $f(x_n)$ where the x_i (in order) form an arithmetic progression. We consider the sequence $d_{1,i} = f(x_i) - f(x_{i-1})$, and then sequences $d_{j,i} = d_{j-1,i} - d_{j-1,i-1}$. Then all of the terms in the sequence $d_{j,i}$ for a fixed $j \geq n$ are constant. Moreover, the smallest k such that $d_{k,i}$ is a constant sequence is the degree of the polynomial g of least degree such that $g(x_i) = f(x_i)$ for $i = 0, \dots, n$. Furthermore, the value in this constant sequence is equal to $a_k \cdot (k!) \cdot (\Delta x)^k$, where a_k is the leading coefficient of g and $\Delta x = x_i - x_{i-1}$.

This can be used to predict $f(x)$ where x is close to known terms in the extended arithmetic sequence x_i in either direction. Suppose we are given f, a cubic polynomial, and $f(1) = 2$, $f(2) = 10$, $f(3) = 30$, and $f(4) = 68$, and are asked to compute $f(5)$. Using the method of finite differences, we have $d_{1,1} = 8$, $d_{1,2} = 20$, $d_{1,3} = 38$; $d_{2,1} = 12$, $d_{2,2} = 18$; $d_{3,1} = 6$. Because f is cubic, $d_{3,1} = d_{3,2} = 6$. $d_{2,3} = d_{3,2} + d_{2,2} = 24$, $d_{1,4} = d_{2,3} + d_{1,3} = 62$, and finally $f(x_5) = f(x_4) + d_{1,4} = 130$, where $x_5 = 5$. So we have $f(5) = 130$.

Lagrange Interpolation - Given n distinct points $x_1, x_2, \dots, x_n$ and corresponding values $y_1, \dots, y_n$, the unique polynomial with degree less than n such that $f(x_i) = y_i$ for all i is given by:

$$f(x) = \sum_{i=1}^{n} y_i \frac{(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

This can be used to mindlessly compute the polynomial of least degree that satisfies several $f(a_i) = b_i$. It is not, however, very practical for larger i.

2 Higher Polynomials

Polynomials of two variables or more variables do not appear as often as polynomials of one variable, but they do prove useful in a few situations. In particular, we will consider the use of *symmetric polynomials*. These are polynomials in which given a set of parameters

x, y, etc. all permutations of the arguments, when plugged into f, yield the same result. I.e. $f(x,y) = f(y,x)$. Although we will not prove it here, any symmetric polynomial $f(x_1, x_2, \ldots, x_n)$ in n variables can be represented as $g(j_1, j_2, \ldots, j_n)$, a polynomial in j_i, the j-wise sums of x_i, where g is not necessarily symmetric.

2.1 Symmetric Basics

From here on out, we will work with $j_1 = x + y$, $j_2 = xy$, and $p_n = x^n + y^n$. We note the identities:

$$\begin{aligned} p_1 &= j_1 \\ p_2 &= j_1^2 - 2j_2 \\ p_3 &= j_1^3 - 3j_1j_2 \\ p_4 &= j_1^4 - 4j_1^2j_2 + 2j_2^2 \\ p_5 &= j_1^5 - 5j_1^3j_2 + 5j_1j_2^2 \end{aligned}$$

We have the identity $p_{n+1} = p_nj_1 - p_{n-1}j_2$, which follows from the algebra: $(x^n + y^n)(x + y) - (x^{n-1} + y^{n-1})(xy) = (x^{n+1} + x^ny + xy^n + y^{n+1}) - x^ny + xy^n = x^{n+1} + y^{n+1}$.

In particular, if we have one of j_1 or j_2 and the value of some p_i, we can obtain an easily manipulated equation in j_2 or j_1 respectively.

2.2 Applications

In addition to Viete's formulas, familiarity with symmetric polynomials can help to deduce other facts about the roots of polynomials. The sum of the squares of n numbers is the square of the sum of the numbers minus twice the 2-wise sum of the numbers. Given a polynomial this would be easy to compute.

We could also see something similar to: $\sqrt[4]{a} + \sqrt[4]{97-a} = 5$. How would we solve this?

We could begin by setting $x = \sqrt[4]{a}$ and $y = \sqrt[4]{97-a}$. Apparently, we have only one equation: $x + y = 5$. Is that all? No - We also have $x^4 + y^4 = 97$. We substitute $j_1 = 5 = x + y$, $j_2 = xy$, and $p_4 = 97$, and obtain:

$$\begin{aligned} p_4 &= j_1^4 - 4j_1^2j_2 + 2j_2^2 \\ 97 &= 625 - 100j_2 + 2j_2^2 \\ 2j_2^2 - 100j_2 + 528 &= 0 \\ j_2 &= \tfrac{100 \pm \sqrt{5776}}{4} = 25 \pm 19 = 6 \text{ or } 44. \end{aligned}$$

We proceed casewise. If we have $x + y = 5$ and $xy = 6$. $y = 5 - x$, so $x(5 - x) = 6$. We get $x^2 - 5x + 6 = 0$, from which, $x = 2$ or 3. Plugging back into our expressions with a, 2 or $3 = \sqrt[4]{a}$, $a = 16$ or 81. Solving the second case yields the two other, imaginary answers.

Of course we probably could have guessed the real values, but if the 97 were changed slightly we would still have been able to solve for x and a.

3 Practice

None of this will be helpful unless you are keenly familiar with it and can break down a wide variety of problems in a short time. The following problems can be solved via common polynomial analysis.

1. Compute the sum of the squares of the roots of the polynomial $f(x) = x^3 + 14x^2 - 30x + 15$. What is the sum of the cubes of the roots? What are the roots of f?

2. Determine the sum of the reciprocals of the roots of the polynomial $f(x) = x^3 + 6x + 15$.

3. If x, y, and z are complex numbers such that

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 7 \end{aligned}$$

   Determine the value of $xy + yz + zx$. What is $x^2(y + z) + y^2(z + x) + z^2(x + y)$? What is the value of xyz?

4. The roots of the polynomial $f(x) = x^3 + 4x^2 - 1$ are ω_1, ω_2, and ω_3. The polynomial with roots $2\omega_1$, $2\omega_2$ and $2\omega_3$ can be written as $g(x) = x^3 + ax^2 + bx + c$. Determine the ordered triple (a, b, c).

5. Let x, y, and z be as defined in problem 3. Let $S_n = x^n + y^n + z^n$. Determine the values of S_4 and S_5. Show that $S_{n+3} = S_{n+2} + S_{n+1} + S_n$.

6. (HMMT 2003) $P(x)$ is a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

   for all real x where the expressions are well defined. Compute $P(-1)$.

7. Solve for all real x:

$$\sqrt[3]{38 - x} + \sqrt[3]{38 + x} = 4$$

8. The graph of

$$2x^3 + (y+1)x^2 + (2y^2 - 8)x + (y^3 + y^2 - 4y - 4) = 0$$

consists of a line and a circle. Given that the circle is centered at the origin, determine the points of intersection between the line and the circle.

9. Determine all real x that satisfy $x^5 - 11x^4 + 36x^3 - 36x^2 + 11x - 1 = 0$.

10. (USAMO 1975) $P(x)$ is a polynomial of degree n such that $P(0) = 0$, $P(1) = \frac{1}{2}$, $P(2) = \frac{2}{3}$, $\ldots$, $P(n) = \frac{n}{n+1}$. Determine the value of $P(n+1)$ in terms of n.

4 Hints

Rather than a complete solution, only a guiding hint is provided for each of the above problems. By working out the details you will become better acquainted with polynomials.

1. Since f is monic, the sum of the roots is $r_1 + r_2 + r_3 = -14$ and the two-wise sum is $r_1r_2 + r_2r_3 + r_3r_1 = -30$. These can be used to find $r_1^2 + r_2^2 + r_3^2$. For the third powers, you will also need to use $r_1r_2r_3 = -15$. To find the roots of a cubic, you will usually need to find a "nice" root by inspection,[3] and then factor out $(x - r)$.

2. We can obtain a polynomial with roots equal to the reciprocals of the current roots, and work from there. Alternatively, we may employ $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{r_1r_2 + r_2r_3 + r_3r_1}{r_1r_2r_3}$.

3. Work in the suggested order. $xy + yz + zx$ is easy to obtain from the values of $x+y+z$ and $x^2 + y^2 + z^2$. $x^2(y+z) + y^2(z+x) + z^2(x+y)$ can be found by comparing the expansion of $(x+y+z)(x^2+y^2+z^2)$ with $x^3 + y^3 + z^3$. Using this, evaluate the expression $(x+y+z)^3$ to find xyz.

4. If the roots are doubled, what happens to the sum of the roots? To the two-wise sum? To the product?

5. First show that x, y, and z are the roots of $f(a) = a^3 - a^2 - a - 1$. Then do some rearranging in the equation to prove it is true using the fact that $f(x) = f(y) = f(z) = 0$.

6. If x is large, $\frac{P(2x)}{P(x+1)}$ approaches $2^{deg_P} = 8$, so $P(x)$ is cubic. Setting $x = 0$ shows that $P(0) = 0$. Setting $x = -.999\ldots$ causes $P(x+1)$ to approach 0, which implies $P(2x) = P(-1.999\ldots)$ approached 0 at the same time. (Why?) Setting $x = -2.999\ldots$ shows that -6 is also a root of $P(x)$. From here we can determine $P(x)$.

[3] Using the **Rational Root Theorem**: If $f(x) = a_nx^n + \cdots + a_0$ has all integer coefficients and a rational root $\frac{p}{q}$, where p and q are relatively prime, then p divides a_0 and q divides a_n.

7. Let $a = \sqrt[3]{38 - x}$ and $b = \sqrt[3]{38 + x}$. Then $a + b = 4$ and $a^3 + b^3 = 76$. (Why?) Use the fact that $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ to find ab, and then solve for a and b by substitution. It is then easy to determine x from the values of a and b. Why is it obvious that the two values of x are additive inverses?

8. The equation of the circle is $x^2 + y^2 = r^2$ and that of the line is $ax + by + c$. We conclude that the expression can be factored as $(x^2 + y^2 - r^2)(ax + by - c) = 0$. Expanding this, we can deduce a and b immediately, and then c and r. Solving for the intersection of the two figures is now algebra.

9. Determine and factor out a nice root and then factor the remaining palindromic fourth degree polynomial using the fact that it is of the form $(x^2 + ax + 1)(x^2 + bx + 1)$.

10. $P(x)$ is given in an unusual form roughly like that of $P(x) = \frac{x}{x+1}$. It is therefore logical to consider the polynomial Q of degree $n + 1$ that is given by $Q(x) = (x + 1)P(x) - x$. What can be deduced about Q? Then use $P(x) = \frac{Q(x)+x}{x+1}$ to find $P(n + 1)$.