Polynomials and Their Applications

Greyson Lewis

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Abstract

The idea behind this lecture is that many otherwise annoying or multi-step problems involving polynomials can be solved with methods alternative to those such as the quadratic formula. This lecture will cover n-wise root sums, the binomial theorem, root power sums, uncommonly known but useful factorizations, and advice on approaching more difficult problems.

1 Introduction

1.1 Definition

So what is a polynomial?
For our purposes, we'll define it as
α: $P(x) = \sum_{i=0}^{n} a_n a_i x^i = x^n + ... + a_1 x + a_0$
β: $= a_n(x - r_1)(x - r_2)...(x - r_n)$

1.2 Viete's Formulae

Notably, this gives us a very helpful set of expressions for the various sums and products of roots. These have been named after a mathematician named Viete, though he was certainly not the first to recognize these properties.

Notice that, upon multiplying all of the terms of β, there is a term lacking any power of x, namely the product of all of the roots. However, what about multiplying by a_n? Well, the easiest thing to do is to simply divide the last term by a_n, giving us: $\prod_{i=1}^{n} r_i = a_0$

However, using the same logic, one can derive expressions for *f-wise* sums of roots, namely:
$r_1 + r_2 + \cdots + r_{n-1} + r_n = \frac{-a_{n-1}}{a_n}$
$(r_1 r_2 + r_1 r_3 + \cdots + r_1 r_n) + (r_2 r_3 + r_2 r_4 + \cdots + r_2 r_n) + \cdots + r_{n-1} r_n = \frac{a_{n-2}}{a_n}$
$\vdots$
$r_1 r_2 \ldots r_n = \frac{a_0}{a_n}$

The presence of a negative sign is determined by the parity of f, so if f is odd, there is a negative sign. This can be verified by expanding β and is left as an optional excercise to the reader.

1.3 The Binomial Theorem

Hopefully all of you have come across the binomial theorem, but just in case, a recap:
$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for all x and y.

2 And now for something completely different

There are multiple types of problems involving polynomials. The following are some methods of solving such problems.

2.1 Coefficient Sums

So what happens if you're looking for the sum of the coefficients of a polynomial? Well, since a polynomial involves an x term and some coefficient, the easiest way is to **set x equal to 1**. How about the sum of the coefficients of the even powers of x ? Well, setting x equal to -1 gives us $(-1)^n a_n + (-1)^{n-1} a_{n-1} + ... + a_0$, so if we add this to $P(1)$, we get $2\sum_{i=1,ieven}^{n} a_i$, so simply divide by 2 to get $\sum_{i=1,ieven}^{n} a_i = \frac{P(1)+P(-1)}{2}$
Notice that $\sum_{i=0,iodd}^{n} a_i = P(1) - \sum_{i=1,ieven}^{n} a_i = P(1) - \frac{P(1)+P(-1)}{2} = \frac{P(1)-P(-1)}{2}$

2.2 Root Power Sums

By Viete's formulae, we know all of the *n-wise* sums of the roots of any polynomial. However, what about sums of squares, cubes, etc. of roots? Unfortunately, this is a bit complicated; however, we will overcome our struggles. Pharoah will let my people go!

Anyways...

Let $s(r,k)$ be a function defined as $f(r,k) = \sum_{i=1}^{n} r_i^k = r_1^k + ... + r_n^k$, where n is the degree of the given polynomial, k is the power to which all of the roots of this polynomial $P(x)$ are raised, and all r_i are the roots of this polnomial $P(x)$

Also, let the $g(k)$ equal to the *k-wise* sum of r_i
Thus, $g(k) = (-1)^k \frac{a_{n-k}}{a_n}$ by Viete's formulae.

$s(r,k)$ can thus be expressed as follows:
$kg(k) = \sum_{i=1}^{k} (-1)^{i-1} g(k-i)s(r,k-1)$

However, this is pretty ugly, just like Ved's face, so let's write the first few terms of it and see what we can get.
$s(0) = 0$
$f(1) = s(1)$
$2f(2) = g(1)s(1) - s(2)$
$3f(3) = g(2)s(1) - g(1)s(2) + s(3)$
$4f(4) = g(3)s(1) - g(2)s(2) + g(1)s(3) - s(4)$
$\vdots$
etc.

Rearranging this mess, we get
$s(1) = g(1)$
$s(2) = s(1)g(1) - 2g(2)$
$s(3) = s(2)g(1) - s(1)g(2) + 3g(3)$
$s(4) = s(3)g(1) - s(2)g(2) + s(1)g(3) - 4g(4)$
$\vdots$
$s(n) = \sum_{i=1}^{n} (-1)^{i-1} s(n-i)g(i)$
The proof of this is time consuming and is thus left as an excercise to the reader.

3 Factorization

3.1 Big Table of Factorizations

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + ... + (-1)^{n-1}y^{n-1})$
$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + ... + y^{n-1}) = (x^{\frac{n}{2}} + y^{\frac{n}{2}})(x^{\frac{n}{2}} - y^{\frac{n}{2}})$
$(a + b)(c + d) = ac + ad + bc + bd$
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)!$
$a^2 + b^2 + c^2 + 2(ab + bc + ca) = (a + b + c)^2$
$(a + b)(b + c)(c + a) + abc = (a + b + c)(ab + bc + ca)$
$a^3(b - c) + b^3(c - a) + c^3(a - b) = -(a - b)(b - c)(c - a)(a + b + c)$
$(x^7 + 1) - (x + 1)^7 = -7x(x + 1)(x^4 + 2x^3 + 3x^2 + 2x + 1)$

3.2 Finding your Roots

Now, besides memorizing the above table, there are many ways of approaching factoring...

3.2.1 Rational Root Theorem

For all polynomials $P(x) = \sum_{i=0}^{n} a_i x^i$, all rational roots of $P(x)$ are of the form $\pm\frac{n_i}{n_j}$, where $n_i, n_j \in \mathbb{Z}$, $n_i|a_0, n_j|a_n$. Have fun, and don't forget to plug and chug!

3.2.2 Descartes' Rule of Signs

The number of changes in the sign of the coefficients of $P(x)$ is the max number of positive roots, in $P(-x)$ the max number of negative roots, and if the actual number of such roots is not equal to the max number, they differ by a multiple of 2.

3.2.3 Complex Root Theorem

If some of the roots of any polynomial $P(x)$, with all real coefficients, are complex, these complex roots come in pairs.

3.2.4 Completing the Square

Remember that $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + ... + y^{n-1})$
Hence, if you have some polynomial $P(x)$ of degree n, try explicitly adding and subtrating a term of degree n such that $P(x) + Q(x) - Q(x) = R(x)S(x)$

4 Transformations

4.1 Inverse Roots

If $P(x)$ has roots $r_1, r_2, ...r_n$ then clearly $P(r_i) = 0$. As a result, $P(\frac{1}{\frac{1}{r_i}}) = 0$

Consider the polynomial $Q(x) = P(1/x) = \sum_{i=0}^{n} a_i(\frac{1}{x})^i$ Notice that it is not a polynomial as n out of the $n + 1$ terms have degrees less than one. Thus, we must multiply through by the lowest degree of x possible to make Q(x) a polynomial.
$x^n Q(x) = \sum_{i=0}^{n} a_i x^{n-i} = a_n + a_{n-1}x + ... + a_0 x^n$
This is simply $P(x)$ with its coefficients in reverse order.

4.2 Multiplied Roots

So what happens if the roots of $Q(x)$ are m times greater than the roots of $P(x)$?
Well, $P(\frac{x}{m}) = \sum_{i=0}^{n} a_i(\frac{x}{m})^i \Rightarrow Q(x) = m^n P(\frac{x}{m}) = \sum_{i=0}^{n} a_i x^i m^{n-i}$

4.3 Shifted Roots

If the roots of $Q(x)$ are greater than the roots of $P(x)$ by some number z, $Q(x) = P(x - z)$. However, how does one determine the coefficients for $Q(x)$? The easiest way is to synthetically divide $P(x)$ by $x + z$, and the ith remainder of this division is the coefficient of x^{i-1} in $Q(x)$.

4.4 Turnip Roots

From the Wikipedia entry on Turnips:
The most common type of turnip is mostly white-skinned apart from the upper 16 centimeters, which protrude above the ground and are purple, red, or greenish wherever sunlight has fallen. This above-ground part develops from stem tissue, but is fused with the root. The interior flesh is entirely white. The entire root is roughly conical, but occasionally squircle in shape, about 520 centimeters in diameter, and lacks side roots. The taproot (the normal root below the swollen storage root) is thin and 10 centimeters or more in length; it is trimmed off before marketing. The leaves grow directly from the above-ground shoulder of the root, with little or no visible crown or neck (as found in rutabagas). Turnip leaves are sometimes eaten, and resemble mustard greens. Turnip greens are a common side dish in southeastern US cooking, primarily during late fall and winter. Smaller leaves are prefered; however, any bitter taste of larger leaves can be reduced by pouring off the water from initial boiling and replacing it with fresh water. Varieties specifically grown for the leaves resemble mustard greens more than those grown for the roots, with small or no storage roots. Varieties of B. rapa that have been developed only for use as leaves are called Chinese cabbage. Both leaves and root have a pungent flavor similar to raw cabbage or radishes that becomes mild after cooking.

5 Problems

1. Factor the following completely: $a^3 + b^3, a^3 - b^3, a^4 + 4b^4$

2. Find the sum of the fifth powers of the roots of $6x^4 - 17x^3 + 5x^2 + x - 19 = 0$

3. Let r, s, and t be the three roots of the equation $8x^3 + 1001x + 2008 = 0$. Find $(r + s)^3 + (s + t)^3 + (t + r)^3$ *Problem 7, AIME II 2008*

4. If α, β, and γ are the roots of $x^3 - x - 1 = 0$, find $\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma}$ *Canada 1996*

5. If the roots of $3x^3 - 14x^2 + x + 62 = 0$ are a, b, and c, determine $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$ *Mu Alpha Theta 1991*

6. Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes. *Problem 5, USAMO 2007*