2013 JV Mass Points

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1 Introduction

Mass points is the method that implements and utilizes the center of mass of a triangle to quickly solve problems that otherwise have to be solved through similar trianges. The only **requirement** for this method is for there to be cevians (line segments that extend from one vertex to the opposite side) to work with.

1.1 Definitions and Properties

Definition 1 A mass point is a pair of weight, n, and point, P. This can be written as (n,P) or nP, signifying placing a weight n on point P.

Definition 2 Two mass points, mP and nQ, coincide if $m = n$ and $P = Q$.

Property 1 Given segment AB with point F on it, if the weights for A, B, and F are m, n, and (m+n) respectively, then F is the balancing point of segment AB. In addition, $\frac{AF}{FB} = \frac{n}{m}$. This is commonly written as mP + nQ = (m+n)F.

Property 2 You are allowed to multiply the above equation with a scalar. However, you **must** apply the scalar to all the mass points in the system.

2 Using Mass Points

2.1 Conventient Cevians

A problem involving only Cevians is the easiest to use mass points on. The begin, simply place an arbitrary mass on one of the vertex (pick wisely) and work from their, taing advantage of Property 1, and known ratios between special cevians.

Example: Point E is selected on side AB of $\triangle ABC$ in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Find $\frac{EF}{FC} + \frac{AF}{FD}$.

Solution: Let's start by giving point B a mass of 1. Following the ratios given, we can use Property 1 to quickly find that points A, C, D, and E have masses of 3, 2, 3, and 4, respectively. With this done, we can use Property 1 again to find the given ratios. $\frac{EF}{FC} + \frac{AF}{FD} = \frac{2}{4} + \frac{3}{3} = \boxed{\frac{3}{2}}$.

2.2 Tricky Transversals

Mass points becomes trickier once we introduce a transversal to the triangle. A transversal is a line intersecting a triangle without passing through one of the vertices. In this case, we have to use **split masses** for the vertex that lies on the two sides intersected by the transversal. In total there are three masses: one for each side, and one that is the sum of the other two *split*-masses for use with cevians stemming from that vertex.

Example: In triangle ABC, D, E, and F are on BC, CA, and AB, respectively, so that $AE = AF = CD = 2, BD = CE = 3$, and $BF = 5$. If DE and CF intersect at O, compute $\frac{OD}{OE} \times \frac{OC}{OF}$.

Solution: After drawing a diagram, we can see that vertex C will have split masses. So to start, we should pick a different vertex. Picking A, we assign it a mass of 15, as it will give us integer masses to work with. In terms of side BC, we can see that mass of B is 6 and the mass of C on that side is 9. Therefore D has mass 15. Looking at AC, we see that C has mass 10 on that side, which leads to E having a mass of 25. Now that we have the two split masses, we can add them up to get a "overall" mass of 19 for C. Adding up the masses of A and B, we get that the mass of F is 21. Now that all the required masses are found, we can easily find the ratio of the sides we need. $\frac{OD}{OE} \times \frac{OC}{OF} = \frac{25}{15} \times \frac{21}{19} = \boxed{\frac{35}{19}}$

3 More Tools

Often, the tecnhique of mass points on its own is not enough to solve problems. Here are some additional tools that may be useful.

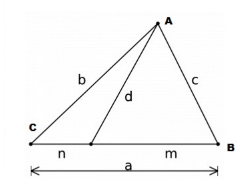
3.1 Angle Bisector Theorem

Given $\triangle ABC$ and angle bisector AD, where D is on side BC, then $\frac{c}{m} = \frac{b}{n}$. The converse of this theorem holds as well.

3.2 Stewart's Theorem

This comes in handy when dealing with cevians. We start with $\triangle ABC$ with sides of length a, b, c opposite vertices A, B, C respectively. If we draw cevian AD such that BD and DC have lengths m and n, we get the following relationship:

$$b^2m + c^2n = amn + d^2a$$



It can be easily remembered as "man + dad = bmb + cnc." (A man and his dad put a bomb in the sink.)

3.3 Apollonius's Theorem

Special case of Stewart's Theorem. If AD is a median, the following statement holds:

$$b^2 + c^2 = 2(m^2 + d^2)$$

3.4 Length of Angle Bisector

Special case of Stewart's Theorem. If AD is an angle bisector:

$$d^2 = bc - mn$$

$$d^2 = bc\left(1 - \frac{a^2}{(b+c)^2}\right)$$

3.5 Ceva's Theorem

Given $\triangle ABC$ and cevians AD, BE, and CF that concur, the following statement is true:

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

4 Problems

1. In $\triangle ABC$, angle bisectors AD and BE intersect at P. If $a = 3, b = 5, c = 7, BP = x$, and $PE = y$, compute the ratio $x : y$, where x and y are relatively prime integers. (AHSME)

2. In a triangle, segments are drawn from one vertex to the trisection points of the opposite side . A median drawn from a second vertex is divided , by these segments, in the continued ratio $x : y : z$. If $x \leq y \leq z$ then find $x : y : z$. (NYSML)

3. In $\triangle ABC$, A', B', and C' are on BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

4. In $\triangle ABC$, points D and E are on sides BC and CA, respectively, and points F and G are on side AB with G between F and B. BE intersects CF at point O_1 and BE intersects DG at point O_2. If $FG = 1, AE = AF = DB = DC = 2$, and $BG = CE = 3$, compute $\frac{O_1O_2}{BE}$. (Wikipedia :P)