Introduction to Number Theory: Arithmetic Functions

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1 Introduction

Number theory is the study of integers. Arithmetic functions describe interesting properties about integers.

2 Arithmetic functions

The following functions describe basic questions we might ask about an integer n.

2.1 Number of divisors of n

We call $\tau(n)$ the number of divisors of an integer n.

Example 2.1.1 Evaluate $\tau(7000000)$.

Solution. First, we prime factorize $7000000 = 2^6 * 5^6 * 7$. We know that all the divisors of 7000000 can only have prime factors of 2, 5, and 7, so each divisor d can be expressed as $d = 2^{f_1} * 5^{f_2} * 7^{f_3}$ for some integers f_1, f_2, and f_3. But in order for d to divide 7000000, we must have $0 \leq f_1 \leq 6$, $0 \leq f_2 \leq 6$, and $0 \leq f_3 \leq 1$. Therefore we have 7 choices for f_1, 7 choices for f_2, and 2 choices for f_3, so the total number of divisors of 7000000 is $\tau(7000000) = 7 * 7 * 2 = \boxed{98}$.

In general, if we prime factorize $n = p_1^{e_1} * p_2^{e_2} * \ldots * p_k^{e_k}$, then each divisor d can be expressed as $d = p_1^{f_1} * p_2^{f_2} * \ldots * p_k^{f_k}$, where $0 \leq f_1 \leq e_1$, $0 \leq f_2 \leq e_2$, ..., $0 \leq f_k \leq e_k$. Therefore we have $e_1 + 1$ choices for f_1, $e_2 + 1$ choices for f_2, ..., $e_k + 1$ choices for f_k. Thus,

$$\boxed{\tau(n) = (e_1 + 1)(e_2 + 1) \ldots (e_k + 1)}.$$

This formula helps us compute $\tau(n)$, but this method also helps us solve similar problems.

Example 2.1.2 How many square divisors does 7000000 have?

Solution. Again, we prime factorize $7000000 = 2^6 * 5^6 * 7$. Each divisor d can be expressed as $d = 2^{f_1} * 5^{f_2} * 7^{f_3}$, where $0 \leq f_1 \leq 6$, $0 \leq f_2 \leq 6$, and $0 \leq f_3 \leq 1$. If d is square, then f_1, f_2, and f_3 must be even. So we have 4 choices for f_1 (0, 2, 4, or 6), 4 choices for f_2 (0, 2, 4, or 6), and 1 choice for f_3 (0). Thus, the number of square divisors is $4 * 4 * 1 = \boxed{16}$.

2.2 Sum of divisors of n

We call $\sigma(n)$ the sum of the divisors of n.

Example 2.2.1 Compute $\sigma(1152)$.

Solution. First, we prime factorize $1152 = 2^7 * 3^2$. We see that

$$\sigma(1152) = 2^0 * 3^0 + 2^1 * 3^0 + 2^2 * 3^0 + \ldots + 2^7 * 3^0$$
$$+2^0 * 3^1 + 2^1 * 3^1 + 2^2 * 3^1 + \ldots + 2^7 * 3^1$$
$$+2^0 * 3^2 + 2^1 * 3^2 + 2^2 * 3^2 + \ldots + 2^7 * 3^2$$

$$= (2^0 + 2^1 + 2^2 + \ldots + 2^7) * 3^0$$
$$+(2^0 + 2^1 + 2^2 + \ldots + 2^7) * 3^1$$
$$+(2^0 + 2^1 + 2^2 + \ldots + 2^7) * 3^2$$
$$= (2^0 + 2^1 + 2^2 + \ldots + 2^7) * (3^0 + 3^1 + 3^2) = 255 * 13 = \boxed{3315}.$$

In general, if we prime factorize $n = p_1^{e_1} * p_2^{e_2} * \ldots * p_k^{e_k}$, then

$$\boxed{\sigma(n) = (p_1^0 + p_1^1 + \ldots + p_1^{e_1})(p_2^0 + p_2^1 + \ldots + p_2^{e_2}) \ldots (p_k^0 + p_k^1 + \ldots + p_k^{e_k})}$$

2.3 Number of coprime integers less than or equal to n

We call two integers a and b *coprime* if the greatest common divisor of a and b is 1. We call the *Euler's Totient function,* $\phi(n)$ the number of coprime integers less than or equal to n. For example, $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, and $\phi(4) = 2$.

Example 2.3.1 Compute $\phi(43)$.

Solution. Since 43 is prime, each integer from 1 to 42 is coprime to 43, so $\phi(43) = \boxed{42}$.

In general, if p is prime, then $\phi(p) = p - 1$.

Example 2.3.2 Compute $\phi(49)$.

Solution. First, we prime factorize $49 = 7^2$. Thus, Euler's Totient function counts all integers less than or equal to 49, which are not multiples of 7. There are seven multiples of 7 less than or equal to 49, so $\phi(49) = 49 - 7 = \boxed{42}$.

In general, if p is prime, then $\phi(p^2) = p^2 - p$.

Example 2.3.3 Compute $\phi(n)$ where $n = p^k$ for a prime p and some positive integer k.

Solution. We count all integers less than or equal to n, which are not multiples of p. There are $\frac{n}{p}$ multiples of p less than or equal to n, so $\phi(n) = n - \frac{n}{p} = \boxed{n(1 - \frac{1}{p})}$.

Example 2.3.4 Compute $\phi(n)$ where $n = p_1^{e_1} * p_2^{e_2}$ for primes p_1, p_2 and some positive integers e_1, e_2.

Solution. We count all integers less than or equal to n, which are not multiples of p_1 or p_2. There are $\frac{n}{p_1}$ multiples of p_1 less than or equal to n and $\frac{n}{p_2}$ multiples of p_2 less than or equal to n. However, some of these overlap and are multiples of both p_1 and p_2. There are $\frac{n}{p_1 p_2}$ multiples of $p_1 p_2$ less than or equal to n. So the total number of integers which are multiples of p_1 or p_2 is $\frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p_1 p_2}$ by the Principle of Inclusion-Exclusion. Thus, $\phi(n) = n - (\frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p_1 p_2}) = \boxed{n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})}$.

In general, if we prime factorize $n = p_1^{e_1} * p_2^{e_2} * \ldots * p_k^{e_k}$, then we can use the Principle of Inclusion-Exclusion in a similar manner to show that

$$\boxed{\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k})}.$$

3 Summary

If we prime factorize $n = p_1^{e_1} * p_2^{e_2} * \ldots * p_k^{e_k}$, then

- $\tau(n) = (e_1 + 1)(e_2 + 1) \ldots (e_k + 1)$
- $\sigma(n) = (p_1^0 + p_1^1 + \ldots + p_1^{e_1})(p_2^0 + p_2^1 + \ldots + p_2^{e_2}) \ldots (p_k^0 + p_k^1 + \ldots + p_k^{e_k})$
- $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k})$

So now what do we do these formulas? These three formulas all come up frequently in problems. The third formula for ϕ is most useful when applying another theorem:

Euler's Totient Theorem. If a and m are relatively prime, then $a^{\phi(m)} \equiv 1 \pmod{m}$.

If you don't know what this theorem means, don't worry about it! You'll learn about it some other time, but just remember that the formulas for τ, σ, and ϕ are all important. So now, let's do some problems!

4 Problems

1. How many positive even divisors does 7000000 have?
2. Find the number of rational numbers r, $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and the denominator have a sum of 1000. (AIME I, 2014)
3. Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. What is the probability that exactly one of the selected divisors is a perfect square? (AIME I, 2010)
4. Compute the sum of all perfect square divisors of 1152.
5. How many positive integers have exactly three proper divisors (positive integral divisors excluding itself), each of which is less than 50? (AIME I, 2005)
6. How many positive perfect squares less than 10^6 are multiples of 24? (AIME I, 2007)
7. Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$. (AIME II, 2005)
8. Compute the smallest positive integer n such that $214n$ and $2014n$ have the same number of divisors. (ARML 2014)

9. How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers? (AIME II, 2004)

10. Find the number of ordered triples (a, b, c) where a, b, and c are positive integers, a is a factor of b, a is a factor of c, and $a + b + c = 100$. (AIME II, 2007)