Geometry 3 – Putting It All Together

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With geometry, we could teach you all the most obscure little theorems in the world, but they'd be completely meaningless without some practice and exposure to different types of problems as examples of how to use them. The focus of this lecture is actual problem-solving.

1 Before We Start

Here are some things you hopefully remember from the Geo 1 (The Basics) and Geo 2 (Collinearity and Concurrency) Lectures:

- Properties of cyclic quads, how they're useful, and Ptolemy's Theorem
- Similar triangles, areas, ratios, and how they're useful
- Menelaus and Ceva (also Trig Ceva)
- The Radical Axis Theorem
- The phantom point technique

Here are some tips to keep in mind when solving geometry problems and writing up proofs:

- If you're stuck, DON'T JUST STARE AT A DIAGRAM; label what you know, try making various constructions (parallel lines, extensions, reflections,...), make conjectures, do something.
- Work both ways.
- Work with a large, neat diagram (Bring compass and straightedge!); multiple diagrams are often helpful.
- When writing up your formal proof, start off with a diagram, and BE SURE TO DEFINE ALL NEW POINTS/VARIABLES YOU INTRODUCE, even if they're labelled in your diagram.
- Proofs should be written going forwards.

2 Problems

Disregarding the first few, nearly all the rest of these problems come from actual olympiads or olympiad training materials (USAMO and the like), so this is what you should expect to see. If you've been keeping up with the past lectures (cough.), very many of these problems should look familiar to you.

1. Circles C_1, C_2, and C_3 lie in a plane. The common external tangents of C_1 and C_2 intersect at X. Similarly, points Y and Z are the intersections of the common external tangents of C_2 and C_3, and C_3 and C_1, respectively. Prove that X, Y, and Z are collinear.

2. The isogonal conjugate Y of point X with respect to a triangle ABC is the point obtained by reflecting the lines AX, BX, and CX about the angle bisectors of angles A, B, and C, respectively. Prove that these three lines do indeed concur at a point.

3. The incircle of triangle ABC has center O and is tangent to sides AB, BC, and CA at points X, Y, and Z, respectively. BZ intersects the circle again at P. M is the midpoint of XY. Show that $PMOZ$ is cyclic.

4. Acute triangle ABC has altitudes AX, BY, and CZ. Ray YZ meets the circumcircle of $\triangle ABC$ at M, and XZ and BM intersect at N. Prove that AMN is an isosceles triangle.

5. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and then line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent.

6. In quadrilateral $ABCD$, AC bisects $\angle BAD$. A point E is chosen on CD, and BE intersects segment AC at F. DF intersects BC at G. Prove that $\angle GAC = \angle CAE$.

7. The diagonals of cyclic hexagon $A_1A_2A_3A_4A_5A_6$ concur at a point. Given that $A_1A_2 = A_3A_4 = A_5A_6$ and that A_3A_5 and A_1A_4 intersect at P, prove that $\frac{A_3P}{PA_5} = (\frac{A_1A_3}{A_3A_5})^2$.

8. Let O be a point inside equilateral traingle ABC. Let P, Q, and R be the intersections of AO with BC, BO with CA, and CO with AB, respectively. Prove that $\frac{PQ}{AR} \cdot \frac{QR}{BP} \cdot \frac{RP}{CQ} \geq 1$.

9. The diagonals of quadrilateral $ABCD$ intersect at O. Let M and N be the midpoints of sides AD and BC, respectively. Distinct points P and Q are the orthocenters of triangles ABO and CDO, respectively. Prove that PQ is perpendicular to MN.