ARML Lecture II - Geometry of Triangles

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February 5, 2004

1 Computing Area

In the work of previous officers, it has been shown that there are roughly $10^{10^{34}}$ ways to compute the area of a triangle. In the following equations, we let K be the area of triangle ABC, which has side lengths of a, b, and c for BC, AC, and AB respectively.

$\mathbf{K = \frac{1}{2}aH_a}$, where a is a side of the triangle, and H_a is the height to side a.

$\mathbf{K = rs}$, where s is the semiperimeter (half of the perimeter), and r is the radius of the inscribed circle – sometimes called the inradius.

$\mathbf{K = \sqrt{s(s-a)(s-b)(s-c)}}$, where s is again the semiperimeter. This relationship is known as *Heron's Formula*.

$\mathbf{K = \frac{abc}{4R}}$, where R is the circumradius of ABC. That is, R is the radius of the smallest circle that can be drawn containing triangle ABC.

$\mathbf{K = \frac{1}{2}ab\sin C}$, where C is the angle ACB of triangle ABC.

$\mathbf{K = \frac{|x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_1y_3-x_3y_2|}{2}}$, where ABC has vertices at (x_1, y_1), (x_2, y_2), (x_3, y_3).

$\mathbf{K = I + \frac{B}{2} - 1}$, where I is the number of lattice points contained within ABC, and B is the number of points on the boundary of ABC, and the vertices of ABC are lattice points.

Of course, these represent just a few ways to find the area of ABC.

2 Special Points

There are many different centers of triangles, and typically, they are not the same point. Here are the most common centers:

- **Orthocenter** - This is the center of the triangle at which the three altitudes intersect. The name is in part derived from the term *orthogonal*, which means perpendicular.
- **Incenter** - The is the center of the triangle at which the three angle bisectors intersect. This center is also the center of the inscribed circle of the triangle, a property from which its name is derived.

- **Circumcenter** - This is the center at which the three perpendicular bisectors of the three sides of the triangle intersect. It is the center of the circumscribed circle of the triangle, and hence its name.

- **Centroid** - This is the intersection of the three medians of triangle ABC. This point divides each median into two segments of length x and $2x$ for some x. Also, this point is the center of mass of the triangle. That is, one could cut out ABC and balance it on a needle from this point.

- **Excenter(s)** - This is the term given to the center of an *excircle*, each of which is formed by extending two sides of ABC and constructing a circle tangent to the extensions of the sides and the third side of ABC.

The *Euler line* is the line that passes through the circumcenter (typically denoted as O), centroid (G), and orthocenter (H), in that order. Specifically, 2OG = GH. This can be remembered by considering an isosceles right triangle.

3 Computing Sides

There are many formulas that relate lengths in triangles. The two most common are trigonometric in nature.

$\mathbf{c^2 = a^2 + b^2 - 2ab\cos C}$, the *Law of Cosines*. This is useful in a number of different situations, including proving identities that surface from time to time as some of the theorems below. REMARK - The Pythagorean Theorem turns out to be a special case where C = 90^o or $\frac{\pi}{2}$.

$\mathbf{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$, the [Extended] *Law of Sines*. This often comes into play when we have circles involved, and can be used in conjunction with the Law of Cosines.

If these two fail, then we draw upon other tools.

- **Stuart's Theorem** - Let ABC be a triangle labeled as normal. Pick D on BC and let $AD = d$, $BD = m$, and $CD = n$. Then we have $dad + man = bmb + cnc$. This is read as "Dad plus man equals bomb plus sink." Letting $m = n$ and solving for the length of the median produces $AD^2 = \frac{2AB^2+2AC^2-BC^2}{4}$, which is worth memorizing.

- **Angle Bisector Theorem** - Let ABC be a triangle with D on BC such that AD bisects angle BAC. Then: $\frac{DB}{AB} = \frac{DC}{AC}$. This comes up frequently and helps avoid use of trigonometric functions.

- **Menelaus' Theorem** - Given triangle ABC, extend AB beyond B to point D. Pick point E on BC and extend DE to F on AC. Then we have $AD{\cdot}BE{\cdot}CF = BD{\cdot}CE{\cdot}AF$. This is less common but still appears in official solutions, including #10 on the 2003

Geometry Subject HMMT Exam. This can be generalized to an iff statement by setting a ratio of -1 and using directed distances.

- **Ceva's Theorem** - Given triangle ABC and three points D, E, and F on BC, AC, and AB respectively such that AD, BE, and CF are concurrent, we have $\frac{DB}{DC}\frac{EC}{EA}\frac{FA}{FB} = 1$. This is derived from mass points, a topic we will discuss later, and shows up frequently. REMARK - The converse and inverse are true as well.

4 Applying Formulas

Most problems draw upon only the aforementioned theorems, but the hard part is knowing how to use them. Here are examples that should give a sense of how to solve triangle geometry:

1. ABC is a triangle in which AC = 13 and AB = 14. AB is extended beyond B to D such that BD = 6. E is chosen on BC, and DE is extended to F on side AC of triangle ABC such that AF = 4. Compute the numerical value of $\frac{CE}{BE}$.

   One solves this problem with an application of Menelaus' Theorem after finding FC = 9. We have $20 \cdot BE \cdot 9 = 6 \cdot CE \cdot 4$. Dividing both sides by 24$\cdot$ BE produces $\frac{15}{2} = \frac{CE}{BE}$, the desired answer.

2. A point D on side BC of triangle ABC is selected such that CD = 3 and BD = 2. A point E is selected on AC such that EC = 1. If AD is an angle bisector of BAC, and AB = 6, compute the value of DE.

   To tackle this problem, one first applies the Angle Bisector Theorem to obtain $\frac{3}{AC} = \frac{2}{6}$, from which AC = 9. One then reflects triangle ABD over AD to map B onto B', giving DB' = 2 and AB' = 6. Furthermore, EB' = 2 by subtraction. One then applies Stuart's Theorem to obtain $3DE^2 + 6 = 18 + 4$, from which one computes DE = $\frac{4\sqrt{3}}{3}$.

3. Triangle ABC has AB = 15, AC = 37, and BC = 44. Points E and F are chosen on AC and AB respectively such that AE = 7 and AF = 5. Let the intersection of BE and CF be P. AP is extended to D on side BC of triangle ABC. Let the angle bisectors of BDA and DBA intersect at P_2. Compute the distance from P_2 to AB.

   To solve this, we first find CE = 30 and BF = 10. Letting CD = x and BD = 44 - x, and applying Ceva's Theorem, we have $\frac{7}{30}\frac{10}{5}\frac{x}{44-x} = 1$, From which we find CD = 30 and BD = 14. Then using Stuart's Theorem, we find AD = 13. Recognizing that the intersection of two angle bisectors is at the incenter of a triangle, we reason that the desired distance is the inradius of triangle ABD. To find the inradius, we compute the area of triangle ABD using Heron's formula and the side lengths of 13, 14, and 15. Obtaining K = 84, we use the identity K = rs and substitute K = 84 and s = 21 to obtain the desired answer, 4.