Introduction to Bijections
WSun14 and TZhong14
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What is a Bijection?

In simplest terms, a bijection is a one-to-one correspondence between two sets. This in turn determines that the cardinalities of the two sets (sizes of each set) are the same. It is frequently used to simplify otherwise daunting combo problems. Essentially every combo problem involves a bijection.

Basic Strategies (Warm-up):

- Rearranging Words
  How many distinct rearrangements of the word COMBINATORICS exist?
- Counting Paths
  How many ways can a bug travel from (0,0) to (4,5) if it can only move one unit up or one unit to the right at any given step?
- Sticks and Stones
  Define a "good" set as one with non-negative, integer elements that sum to 6. How many "good" sets are there with 4 elements?

Classic Problems:

- Sequences
  A fair 6-sided dice is rolled 4 times. What is the probability that each roll is at least as large as the previous one?
- Positioning
  A gardener plants 3 maple trees, 4 oak trees, and 5 birch trees in a row. He plants them in equal order, with each arrangement being equally likely. What is the probability that no 2 birch trees are next to one another?

Examples:

i. Andy has six 1 x 2 x π blocks. He stacks them, one on top of another, to form a tower six bricks high. Each brick can be in any orientation as long as it rests on top of the brick below it (or on the floor). How many distinct heights of towers can he make?

ii. In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two. A marksman is to break all eight targets according to the following rules: (1) He first chooses a column from which a target is to be broken. (2) He then must break the lowest remaining unbroken target in the chosen column. If these rules are followed, in how many different orders can the eight targets be broken?

iii. A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers choose their spaces at random from among the available spaces. Auntie Em arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

Practice:

1. A frog is at the point (0,0). Every second, he can jump one unit either up or right. He can only move to points (x,y) where x and y are not both odd. How many ways can he get to the point (6,12)?

2. Let $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12})$ be a permutation of $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$ for which $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and $a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$. An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$ Find the number of such permutations.

3. Find the number of 5-element subsets of $\{1,2, \ldots ,30\}$ such that no two pairs of elements differ by less than 3.