TJUSAMO - Geo 3

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1 Geometric Inequalities

(Po-Ru Loh) Common techniques include:

- Turning the inequality purely into an algebra problem. Common choices for the variables in the inequality are sidelengths, trig functions of angles, or areas. Keep the triangle substitution in mind if you'd like to remove the triangle inequality constraint.
- The Triangle Inequality. The giveaway is summed lengths; try to move the segments so that they share an endpoint.
- Ptolemy's Inequality: For **any** quadrilateral $ABCD$, $AB \cdot CD + BC \cdot DA \geq AC \cdot BD$. Consider this when you see the products or ratios of lengths, or when the equality case involves cyclic quads.
- Projection: The projection of a segment on a line is at most as long as the original segment.
- Sometimes geometry problems involving equalities are solved by realizing that the equality is the equality-case of a inequality. A hint to this is when it seems like there's too much freedom in a diagram to guarantee the result.
- Look at equality cases to guess at which techniques you can and cannot use.

1.1 Problems

1. In triangle ABC with $\angle A, \angle B, \angle C < 120°$, the point P that minimizes the value of $AP + BP + CP$ is the Fermat point F such that $\angle AFB = \angle BFC = \angle CFA = 120°$. Show that this point F exists and also prove this property.
2. (Erdös-Mordell inequality) For acute triangle ABC and point P in its interior, $AP + BP + CP \geq 2(PD+PE+PF)$ where D, E, F are the feet of the perpendiculars from P to BC, CA, AB, respectively.
3. Points D, E, and F are on sides BC, CA, and AB of triangle ABC, respectively. Show that if the perimeter of triangle DEF is to be minimal, then D, E, and F should be the feet of the altitudes (i.e., triangle DEF is the orthic triangle of ABC.).
4. (Euler's inequality) In any triangle, $R \geq 2r$.
5. Let ABC be a triangle. Prove that
$$\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} \geq \left(\frac{BC}{CA} + \frac{CA}{BC}\right) + \left(\frac{CA}{AB} + \frac{AB}{CA}\right) + \left(\frac{AB}{BC} + \frac{BC}{AB}\right).$$
6. Let O be a point inside equilateral traingle ABC. Let P, Q, and R be the intersections of AO with BC, BO with CA, and CO with AB, respectively. Prove that $\frac{PQ}{AR} \cdot \frac{QR}{BP} \cdot \frac{RP}{CQ} \geq 1$.

7. Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengtsh are all positive integers with no common divisor, and determine those integers.

2 A Conglomeration of... STUFF

1. Let P be a point in the interior of a triangle ABC, and let D, E, F be the point of intersection of the line AP and the side BC of the triangle, of the line BP and the side CA, and of the line CP and the side AB, respectively. Prove that the area of triangle ABC must be 6 if the area of each of the triangles PFA, PDB, and PEC is 1.

2. In any quadrilateral, the lines joining the midpoints of the diagonals and those of the opposite sides are concurrent.

3. If $ABCD$ is a cyclic quadrilateral, then prove that the incenters of the triangles ABC, BCD, CDA and DAB are the vertices of a rectangle.

4. Consider triangle ABC with circumcircle ω. Let M be the midpoint of BC and H the triangle's orthocenter. Let D, E, and F be where the altitudes from A, B, and C intersect the circumcircle respectively. Let I be the intersection of MH with ω such that H is bewteen I and M. Let T be the incenter of DIE and S the incenter of DIF. Prove that TS is parallel to BC.