Algebraic Combinatorics

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Algebraic combinatorics is the field of mathematics in which algebra is used in combinatorics. Although in higher mathematics, *abstract algebra* instead of elementary algebra is used in algebraic combinatorics, but for this lecture, we will try to stay with elementary algebra by using generating functions and formal series.

1 Formal Series

As generating functions are not polynomials, we may not work in the ring of polynomials with real coefficients $\mathbb{R}[X]$. Hence, we must define a new ring to work in. Given a commutative ring A (generally we will work with formal series whose coefficients are in $\mathbb{R}$), we may define another ring called the ring of *formal series* with coefficients in A and denote this by $A[[X]]$. Every element of $A[[X]]$ can be written as $\sum_{n\geq 0} a_n X^n$, where a_i are in A. In this ring, we may define addition and multiplication analogously to those frequently encountered in polynomial rings:

$$\left(\sum_{n\geq 0} a_n X^n\right) + \left(\sum_{n\geq 0} b_n X^n\right) = \sum_{n\geq 0} (a_n + b_n) X^n$$

$$\left(\sum_{n\geq 0} a_n X^n\right)\left(\sum_{n\geq 0} b_n X^n\right) = \sum_{n\geq 0}\left(\sum_{p+q=n} a_p b_q\right) X^n.$$

In this ring, all polynomial identities belonging to the ring of polynomials are also true. In addition, for a formal series $f(X) = \sum_{n\geq 0} a_n X^n$, we may define its formal *derivative* $f'(X)$ as

$$f'(X) = \sum_{n\geq 0} n a_n X^{n-1}.$$

All properties of the derivative encountered in calculus are true in the ring of formal series; for instance, $(f(X) + g(X))' = f'(X) + g'(X)$ and $(f(X)g(X))' = f'(X)g(X) + f(X)g'(X)$. In addition, geometric expansions are valid; that is, $\frac{1}{1-X} = 1 + X + X^2 + \cdots$.

2 Recurrence Relations

Let $a_0, a_1, a_2, \cdots$ be a sequence of numbers. The *ordinary generating function*, or more commonly just *generating function* of this sequence is the formal series $\sum_{k\geq 0} a_k X^k$. For instance, if our sequence was $\{1, 2, 3, 5, 8, \dots\}$, then our generating function $f(X) = 1 + 2X + 3X^2 + 5X^3 + 8X^4 + \cdots$. The function encodes the information of the sequence inside a single function, and as such in many situations the generating function of a sequence is more desirable than an actual explicit formula.

Many recurrence relations can be formulated explicitly using generating functions. The general tactic is to multiply both sides of a relation by X^n, where n is the index of a variable, and summing up all terms of the relation. As a simple example, let us consider the Fibonacci recurrence:

Example 2.1. If $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$ with $a_0 = 0$ and $a_1 = 1$, find the explicit formula for a_n.

Solution. Multiplying both sides by X^n gives us $a_{n+2}X^n = a_{n+1}X^n + a_nX^n$. Now define $f(X) = \sum_{n\geq 0} a_nX^n$. If we sum the previous express for $n \geq 0$, we obtain

$$\sum_{n\geq 0} a_{n+2}X^n = \sum_{n\geq 0} a_{n+1}X^n + \sum_{n\geq 0} a_nX^n \implies \frac{f(X) - a_1X - a_0}{X^2} = \frac{f(X) - a_0}{X} + f(X).$$

Solving for $f(X)$ gives us $f(X) = \frac{X}{1-X-X^2}$. Partial fractions give us

$$\begin{aligned} f(X) &= \frac{1}{\phi - \phi^{-1}} \left(\frac{1}{1-\phi X} - \frac{1}{1-\phi^{-1}X} \right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_{n\geq 0} \phi^n X^n - \sum_{n\geq 0} \phi^{-n} X^n \right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_{n\geq 0} (\phi^n - \phi^{-n}) X^n \right). \end{aligned}$$

The coefficient of X^n in $f(X)$ gives us the nth Fibonacci number, so $a_n = \frac{1}{\sqrt{5}}(\phi^n - \phi^{-n})$. $\square$

As shorthand notation, it is customary to write $[X^n]f(X)$ as the coefficient of X^n in $f(X)$. Although obtaining Binet's Formula from generating functions is very elegant, obtaining the general formula for the so-called *Catalan numbers* is even more elegant.

Example 2.2 (Catalan)**.** In how many different ways can we parenthesize a non-associative product $c_1c_2\dots c_n$?

Solution. Note that the Catalan numbers are governed by the recurrence relation $a_{n+1} = \sum_{k=0}^n a_ka_{n-k}$ for $n \geq 0$, and $a_0 = 1$. Define $f(X) = \sum_{n\geq 0} a_nX^n$. If we multiply the recurrence by X^n and sum up all terms, we obtain

$$\sum_{n\geq 0} a_{n+1}X^n = \sum_{n\geq 0} \left(\sum_{k=0}^n a_ka_{n-k} \right) X^n \implies \frac{f(X) - 1}{X} = f(X)^2.$$

Thus $Xf(X)^2 - f(X) + 1 = 0$, so we may expand $f(X)$ as a power series using

$$f(X) = \frac{1 - \sqrt{1-4X}}{2X}.$$

Note that $\sqrt{1-4X} = \sum_{k\geq 0} \binom{\frac{1}{2}}{k}(-4X)^k = 1 - 2\sum_{k\geq 1} \binom{2k-2}{k-1} \left(-\frac{1}{4}\right)^k \frac{(-4X)^k}{k}$ by the Generalized Binomial Theorem. Thus

$$f(X) = \frac{1 - \left(1 - 2\sum_{k\geq 1} \binom{2k-2}{k-1} \left(-\frac{1}{4}\right)^k \frac{(-4X)^k}{k}\right)}{2X} = \sum_{k\geq 1} \binom{2k-2}{k-1} \frac{X^{k-1}}{k} = \sum_{k\geq 0} \binom{2k}{k} \frac{X^k}{k+1}.$$

Then $a_n = [X^n]f(X) = \frac{1}{n+1}\binom{2n}{n}$. $\square$

The method listed out in Herbert S. Wilf's *generatingfunctionology*[1]is as follows:

1. Make sure that the set of values of the free variable k for which the given recurrence relation is true, is clealy delineated.
2. Give a name to the generating function that you look for, and write out that function in terms of an unknown sequence.
3. Multiply both sides of the recurrence by X^k and sum over all values of k for which the recurrence holds.
4. Express both sides of the resulting equation explicitly in terms of your generating function $f(X)$.
5. Solve the resulting equation for the unknown generating function $f(X)$.
6. Expand $f(X)$ into a power series by any method you can think of to get an exact formula for the sequence. In particular, if $f(X)$ is a rational function (quotient of two polynomials), then success will result from expanding in partial fractions and then handling each of the resulting terms separately.

Trickier recurrence relations may involve derivatives and having to solve a polynomial differential equation.

3 Combinatorial Identities

Combinatorial identities are solved in a very similar manner; multiplying through by X^n, summing over all possible n, then proving that the two resulting generating functions are equivalent. The Generalized Binomial Theorem $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ helps simplify the terms.

Example 3.1 (Iran 2008)**.** Prove that for each positive integer n,

$$\sum_{k=1}^{n} \binom{n+k-1}{2k-1} = F_{2n},$$

where F_n is the Fibonacci sequence (with $F_1 = F_2 = 1$).

Solution. We only have to prove that

$$\sum_{n\geq 0} \sum_{k=1}^{n} \binom{n+k-1}{2k-1} X^n = \sum_{n\geq 0} F_{2n} X^n.$$

The left-hand side can be simplifying by swapping the order of summation:

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n+k-1}{2k-1} &= \sum_{k\geq 1} \sum_{n\geq 0} \binom{n+k-1}{2k-1} X^n \\
&= \sum_{k\geq 1} X^{1-k} \sum_{n\geq 0} \binom{n+k-1}{2k-1} X^{n+k-1} \\
&= \sum_{k\geq 1} X^{1-k} \sum_{r\geq 0} \binom{r}{2k-1} X^r \\
&= \sum_{k\geq 1} X^{1-k} \frac{X^{2k-1}}{(1-X)^{2k}} = \sum_{k\geq 1} \frac{X^k}{(1-X)^{2k}} = \frac{X}{1-3X+X^2}.
\end{aligned}$$

[1] `http://www.math.upenn.edu/~wilf/gfologyLinked2.pdf`

The right-hand side is easier; we know that $F_n = \frac{1}{\sqrt{5}}(\phi^n - \phi^{-n})$, so

$$\begin{aligned}\sum_{n\geq 0} F_{2n}X^n &= \sum_{n\geq 0}\frac{1}{\sqrt{5}}(\phi^{2n}-\phi^{-2n})X^n \\ &= \frac{1}{\sqrt{5}}\left(\sum_{n\geq 0}\phi^{2n}X^n - \sum_{n\geq 0}\phi^{-2n}X^n\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi^2X} - \frac{1}{1-\phi^{-2}X}\right) \\ &= \frac{1}{\sqrt{5}}\frac{\phi^2X-\phi^{-2}X}{(1-\phi^2X)(1-\phi^{-2}X)} = \frac{X}{1-3X+X^2}\end{aligned}$$

and since the generating functions of both sides are equal, their coefficients must also be equal, as desired. $\square$

Note that we never actually had to expand the generating functions. Since we worked in the ring of formal series, we know that they have a power series representation, and thus only need to confirm that the two functions are equal.

4 Other Applications of Formal Series

Generating functions are also used frequently to count objects that seem difficult to count using bijections or ordinary counting methods, or to prove facts about sequences that are easily done with formal series.

Example 4.1 (USAJMO 2013)**.** Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

Solution. Let $g(n,k)$ be the number of ways to write n as a sum of exactly k powers of 2. Then

$$g(n,k) = [X^n](X + X^2 + X^4 + X^8 + \cdots)^k.$$

Since $f(n) = \sum_{k\geq 1} g(n,k)$, we have

$$\begin{aligned}\sum_{n\geq 1} f(n)X^n &= (X + X^2 + X^4 + \cdots)^1 + (X + X^2 + X^4 + \cdots)^2 + \cdots \\ &= \frac{1}{1-X-X^2-X^4-\cdots} - 1\end{aligned}$$

We claim that $f(n)$ is odd if and only if $n = 2^m - 1$ for some positive integer m. It remains to prove that the coefficients of X^{2^m-1} in the generating function $\sum_{n\geq 1} f(n)X^n$ are all odd. Working in the ring $\mathbb{F}_2[[X]]$, this is equivalent to proving that

$$\begin{aligned}\frac{1}{1-X^1-X^2-X^4-\cdots} - 1 &= X^1 + X^3 + X^7 + X^{15} + \cdots \\ \iff 1 &= (1 + X^1 + X^3 + X^7 + \cdots)(1 + X^1 + X^2 + X^4 + \cdots).\end{aligned}$$

which is easy to see by a parity argument for each term. Thus our answer is $2^{11} - 1 = 2047$. $\square$

In the previous example, since we were trying to prove that the coefficient were either odd or even, it makes the most sense to work in the ring $\mathbb{F}_2[[X]]$, since $1 = -1$.

Example 4.2. Suppose that the set of nonnegative integers is partitioned into a finite number of infinite arithmetical progressions with common differences $r_1, r_2, \dots r_n$ and first terms $a_1, a_2, \dots, a_n$. Then prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} = 1.$$

Solution. Since the set of nonnegative integers is partitioned by the progressions starting with a_i with difference r_i,

$$(X^{a_1} + X^{a_1+r_1} + X^{a_1+2r_1} + \cdots) + \cdots + (X^{a_n} + X^{a_n+r_n} + X^{a_n+2r_n} + \cdots) = X^0 + X^1 + X^2 + \cdots.$$

Using the formula for a geometric series, we have

$$\frac{X^{a_1}}{1 - X^{r_1}} + \frac{X^{a_2}}{1 - X^{r_2}} + \cdots + \frac{X^{a_n}}{1 - X^{r_n}} = \frac{1}{1 - X}.$$

Multiplying both sides by $1 - X$ gives us

$$\frac{X^{a_1}}{1 + X + X^2 + \cdots + X^{r_1 - 1}} + \frac{X^{a_2}}{1 + X + X^2 + \cdots + X^{r_2 - 1}} + \cdots + \frac{X^{a_n}}{1 + X + X^2 + \cdots + X^{r_n - 1}} = 1.$$

Now taking the limit as X approaches 1 gives us $\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} = 1$, as desired. □

The most natural way to proceed in the previous problem was to use an infinite power series, as the fact that the integers is partitioned into arithmetic sequences is a giveaway that a geometric series may be used. Note that in this example, the information was encoded in the exponents, not in the coefficients.

5 Combinatorial Nullstellensatz

The *Combinatorial Nullstellensatz* (also called the *polynomial method*) is a technique in combinatorics developed by Noga Alon. The word *nullstellensatz* is German for "theorem of zeros." *Hilbert's Nullstellensatz* is also a very well-known theorem that is not related to the Combinatorial Nullstellensatz.

Although there are two parts to the original Combinatorial Nullstellensatz, we are interested in the latter:

Theorem 5.1 (Combinatorial Nullstellensatz). *Let F be an arbitrary field, and let $f(x_1, x_2, \dots, x_n)$ be a polynomial in $F[x_1, x_2, \dots, x_n]$. Suppose the degree $\deg f$ of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then, if $S_1, S_2, \dots, S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$ so that*

$$f(s_1, s_2, \dots, s_n) \neq 0.$$

A *field*, for our purposes, may simply be defined as a set in which every element has a multiplicative inverse. More formally, however, a field is a commutative ring in which each element contains a multiplicative inverse. Examples of fields include the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the integers modulo p $\mathbb{F}_p$ (where p is prime), and the algebraic numbers $\overline{\mathbb{Q}}$.

The proof of this theorem is shown in Noga Alon's renowned paper itself[2], and the reader is advised to take a look at the various applications of this theorem in the article. One of the most elegant applications of the Nullstellensatz is to prove the Cauchy-Davenport Theorem.

[2] `http://www.math.tau.ac.il/~nogaa/PDFS/null2.pdf`

Theorem 5.2 (Cauchy-Davenport). *If p is prime and A and B are nonempty subsets of $\mathbb{F}_p$, then*

$$|A+B| \geq \min\{p, |A|+|B|-1\}.$$

Proof. If $|A|+|B|>p$, for every $x \in \mathbb{F}_p$ the sets A and $x-B=\{x-b, b \in B\}$ must intersect by the Pigeonhole Principle. Thus there exists $a \in A$ and $b \in B$ such that $a=x-b$, so $a+b=x$. Since x goes through the elements of $\mathbb{F}_p$, we see that $|A+B|=p$.

If $|A|+|B| \leq p$, then suppose the result is false and so $|A+B| \leq |A|+|B|-2$. There must exist a subset C of $\mathbb{F}_p$ such that $A+B \subseteq C$ and $|C|=|A|+|B|-2$. We now implement the polynomial method. Let

$$f(x,y) = \prod_{c \in C}(x+y-c).$$

By the definition of C, we have that $f(a,b)=0$ for all $a \in A$ and $b \in B$. Set $t_1=|A|-1$, $t_2=|B|-1$, $n=2$, $S_1=A$, and $S_2=B$ in Theorem 5.1. Note that the coefficient of $x^{|A|-1}y^{|B|-1}$ is $\binom{|A|+|B|-2}{|A|-1}$, which is nonzero in $\mathbb{F}_p$ because $|A|+|B|-2 \leq p-2$. The Combinatorial Nullstellensatz tells us that there exists an $a \in A$ and $b \in B$ such that $f(a,b) \neq 0$, a contradiction. Therefore, $|A+B|>|A|+|B|-2$ and our result is proven. $\square$

The crux of the Combinatorial Nullstellensatz is to choose the "right" function f. Oftentimes, there may be only one function that actually proves a problem. The following problem from IMO 2007 was criticized for being a #6, but was routine if one knew how to apply the polynomial method; in fact, Noga Alon's paper itself had a very similar example involving n-hypercubes.

Example 5.1 (IMO 2007/6). Let n be a positive integer. Consider

$$S = \{(x,y,z) | x,y,z \in \{0,1,\ldots,n\}, x+y+z>0\}$$

as a set of $(n+1)^3-1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0,0,0)$.

Solution. Suppose that planes $P_1, P_2, \ldots, P_k$ satisfy the conditions, where P_i are planes. Let us consider planes of the form $a_ix+b_iy+c_iz=d_i$, where $d_i \neq 0$ (otherwise, $(0,0,0)$ would pass through this plane, which we do not want).

The most "obvious" polynomial is to use

$$f(x,y,z) = \prod_{i=1}^{k}(a_ix+b_iy+c_iz-d_i).$$

This vanishes everywhere where the planes cover, so this polynomial seems like a good choice. However, it never actually addresses the fact that the point $(0,0,0)$ is disallowed - $f(0,0,0)$ does not evaluate to 0. So we have to think of another polynomial that vanishes at every point in the cube (including $(0,0,0)$).

This motivates us to append another term to $f(x,y,z)$ so that it remains the same at every value in the cube other than $(0,0,0)$. Let us take

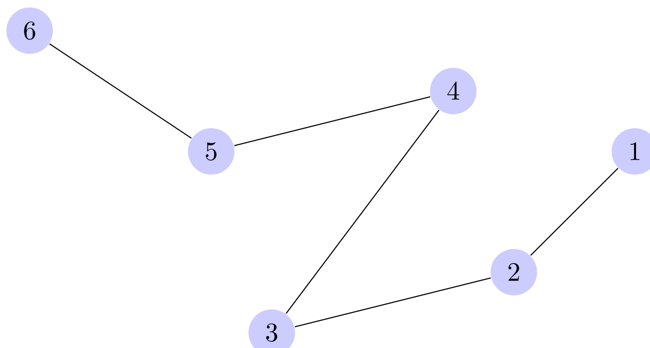
$$f(x,y,z) = \prod_{i=1}^{k}(a_ix+b_iy+c_iz-d_i) - \prod_{i=1}^{k}(-d_i)\prod_{i=1}^{n}\frac{(x-i)(y-i)(z-i)}{-i^3}.$$

At every point other than $(0,0,0)$ in the cube, the added term evalutes to 0, so the polynomial still vanishes. At $(0,0,0)$, the added term evaluates to $-(-d_k)^k$, so it cancels out the remaining term in our original polynomial. Now we are free to use the Combinatorial Nullstellensatz.

Suppose that $k < 3n$. Then the leading coefficient of $f(x, y, z)$ is $-\frac{\prod_{i=1}^{k}(-d_i)}{(-1)^n (n!)^3}$, which is nonzero, contributed by the term that we added to the original polynomial. We use the subsets $S_1 = S_2 = S_3 = \{0, 1, 2, \ldots, n\}$, since $|S_1| = |S_2| = |S_3| > n$. There must exist $s_1, s_2, s_3 \in \{0, 1, 2, \ldots, n\}$ such that $f(s_1, s_2, s_3) = 0$ by the Combinatorial Nullstellensatz. But $f(0, 0, 0) = 0$ and every other point in the cube evaluates to 0, so we must have a contradiction. Hence $k \geq 3n$ and we may construct $3n$ such planes using $x + y + z = i$ for $i = 1, 2, \ldots, 3k$. $\square$

6 Counting Trees

A *tree* is a graph in which any two vertices are connected by exactly one simple path. Thus, a tree is *acyclic* and *connected*. A tree that connects all vertices of a graph is called a *spanning tree*. A *forest* is a disjoint union of trees.



A *complete graph* K_n on n vertices is a graph in which any two vertices are connected by an edge. Due to the abundance of edges in a complete graph, it is a natural question to ask how many spanning trees are in a complete graph. It is easy to check that for K_3, there are 3 spanning trees; for K_4, there are 16; for K_5, there are 125. Borchardt showed in 1860 that there is a very simple formula for this, which was eventually named under Cayley:

Theorem 6.1 (Cayley)**.** *The number of spanning trees in a complete graph K_n is n^{n-2}.*

Proof. Let t_n be the number of *unordered* trees on K_n (we will divide t_n by n after we obtain an explicit formula), and let $t_0 = 0$. Define the *exponential generating function*

$$f(X) = \sum_{n \geq 0} t_n \frac{X^n}{n!}.$$

It is a known consequence of viewing the tree as a *product structure*, which I will not get into this lecture, that the generating function can be defined recursively as

$$f(X) = xe^{f(X)}.$$

In order to extract the power series of $f(X)$, we use a technique called *Lagrange inversion:*

Theorem 6.2 (Lagrange)**.** *Suppose f and g are formal series such that $g(f(X)) = X$. Then*

$$[X^n]f(X) = \frac{1}{n!} \left[\frac{d^{n-1}}{dX^{n-1}} \left(\frac{X}{g(X)} \right)^n \right]_{X=0}.$$

We are looking for the inverse of the function $\frac{X}{e^X}$. Substituting $g(X) = \frac{X}{e^X}$ gives us

$$[X^n]f(X) = \frac{1}{n!}\left[\frac{d^{n-1}}{dX^{n-1}}e^{nX}\right]_{X=0} = \left[n^{n-1}e^{nX}\right]_{X=0} = n^{n-1}.$$

Thus the number of *ordered* trees is $\frac{n^{n-1}}{n} = n^{n-2}$. $\square$

This gives a nice result for the number of spanning trees in a complete graph. A natural extension is to find the number of spanning trees in an arbitrary graph. A celebrated theorem of Kirchoff enumerates these spanning trees given a corresponding matrix called the *Laplacian matrix*. Before we define the Laplacian of a graph, we define the *degree matrix* and the *adjacency matrix* of a graph.

Given a graph G with n vertices, its *degree matrix* D is the $n \times n$ diagonal matrix defined by

$$d_{ij} = \begin{cases} \deg v_i, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Its *adjacency matrix* A is the $n \times n$ square matrix where its elements a_{ij} specifies the number of edges from vertex i to vertex j. Note that in an undirected graph, its adjacency matrix is always symmetric over the main diagonal. Its *Laplacian matrix* L is defined by

$$L = D - A.$$

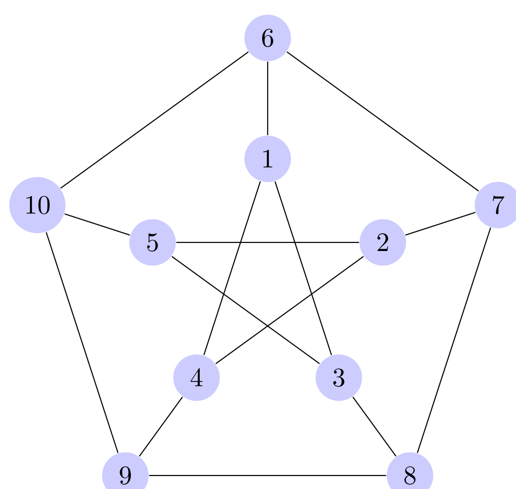
Kirchoff's Matrix Theorem then states:

Theorem 6.3 (Kirchoff)**.** *For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the nonzero eigenvalues of its Laplacian matrix. Then the number of spanning trees of G is*

$$\frac{\lambda_1\lambda_2\cdots\lambda_{n-1}}{n}.$$

Equivalently, the number of spanning trees is equal to any cofactor of the Laplacian matrix of G.

For example, consider the following graph:



This graph is called the *Petersen graph*[3]. How many spanning trees are in this graph? One example is the tree connecting vertices labeled 1, 3, 8, 9, 4, 2, 5, 10, 6, 7, in that order. To find the number of such trees in the graph, we first compute its degree matrix and its adjacentry matrix. It is not difficult to find them:

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus our Laplacian matrix is

$$L = \begin{pmatrix} 3 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

With sufficient computation, one can find the determinant of a cofactor matrix of L (it doesn't matter which cofactor you choose; they should all evaluate to the same answer). This graph has 2000 spanning trees, the most of any 10-vertex cubic graph - that is, all of its vertices having degree 3.

Note that we can easily obtain Cayley's Formula using Kirchoff's Matrix Theorem. I encourage the reader to actually expand the determinant out for small values of n.

7 The Probabilistic Method

The basic idea of the probabilistic method is to prove that an event happens with positive probability. Then, it can be concluded that such an event occurs. Before a few examples are introduced, the following lemma is useful in bounding the probability of an event $\mathbb{P}[X]$:

Lemma 7.1. *Let $A_1, A_2, \cdots, A_n$ be a collection of events. Then*

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \mathbb{P}[A_i].$$

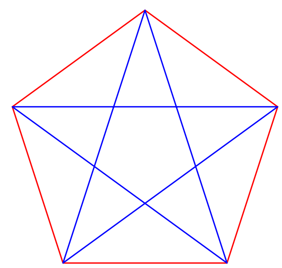
Proof. This follows from the Principle of Inclusion-Exclusion. □

The following example exemplifies the power of the method. *Ramsey theory*[4], the branch of mathematics

[3] `http://en.wikipedia.org/wiki/Petersen_graph`

[4] More information of Ramsey theory and its graph-theoretic results can be found in this article: `http://web.mat.bham.ac.uk/D.Kuehn/RamseyGreg.pdf`

that studies the conditions under which certain properties hold, often make use of the technique of the probabilistic method. We define the *Ramsey number* $R(k, \ell)$ as the smallest integer m such that in any two-coloring of K_n with red and blue, there is either a red K_k or a blue K_ℓ. For example, $R(3, 3) > 5$, since for K_5, there is a two-coloring such that there is no red or blue triangle:



Theorem 7.2. *If* $\binom{n}{k}2^{1-\binom{k}{2}} < 1$*, then* $R(k, k) > n$*.*

Proof. Consider a random two-coloring of K_n. Let R be a set of k vertices on K_n, and let A_R be the event that the induced subgraph of K_n on R is monochromatic. Note that

$$\mathbb{P}[A_R] = \frac{2}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}.$$

Thus the probability that at least one of the events A_R occuring is

$$\mathbb{P}\left[\bigcup_R A_R\right] \le \sum_R \mathbb{P}[A_R] = \binom{n}{k}2^{1-\binom{k}{2}} < 1.$$

Hence the probability that none of the events occur is greater than 0, so there exists a two-coloring of K_n without a monochromatic K_k. $\square$

Note that Theorem 7.2 implies that $R(k, k) > \lfloor 2^{k/2} \rfloor$ because

$$\binom{\lfloor 2^{k/2} \rfloor}{k}2^{1-\binom{k}{2}} < \frac{2^{1+k/2}}{k!} \cdot \frac{\lfloor 2^{k/2} \rfloor^k}{2^{k^2/2}} < 1.$$

Theorem 7.2 shows that there exists a two-coloring of K_n without a monochromatic $K_{2\log_2 n}$.

8 Incidence Matrices

In set theory problems, it is often useful to create a matrix (called an *incidence matrix*) to represent which elements belong to which set. Given a set $X = \{x_1, x_2, \ldots, x_n\}$ and $X_1, X_2, \ldots, X_k$ a family of subsets of X, we define the incidence matrix M by $(m_{ij})_{\substack{1\le i\le n \\ 1\le j\le k}}$, where

$$m_{ij} = \begin{cases} 1, & \text{if } x_i \in X_j; \\ 0, & \text{if } x_i \notin X_j. \end{cases}$$

Often, contradiction is used by considering the determinant or the rank of the matrix.

Example 8.1. Let n be even and let $A_1, A_2, \dots, A_n$ be distinct subsets of the set $\{1, 2, \dots, n\}$, each of them having an even number of elements. Prove that among these subsets there are two having an even number of common elements.

Solution. Suppose that $|A_i \cap A_j|$ is odd for all $i, j \in \{1, 2, \dots, n\}$. Let $M = (m_{ij})_{\substack{1\le i\le n \\ 1\le j\le n}}$ be the incidence matrix of the family $A_1, A_2, \dots, A_n$. Note that

$$M^T \cdot M = \begin{pmatrix} \displaystyle\sum_{k=1}^n m_{k,1}^2 & \displaystyle\sum_{k=1}^n m_{k,1}m_{k,2} & \cdots & \displaystyle\sum_{k=1}^n m_{k,1}m_{k,n} \\ \displaystyle\sum_{k=1}^n m_{k,2}m_{k,1} & \displaystyle\sum_{k=1}^n m_{k,2}^2 & \cdots & \displaystyle\sum_{k=1}^n m_{k,2}m_{k,n} \\ \vdots & \vdots & \ddots & \vdots \\ \displaystyle\sum_{k=1}^n m_{k,n}m_{k,1} & \displaystyle\sum_{k=1}^n m_{k,n}m_{k,2} & \cdots & \displaystyle\sum_{k=1}^n m_{k,n}^2 \end{pmatrix} = \begin{pmatrix} |A_1| & |A_1 \cap A_2| & \cdots & |A_1 \cap A_n| \\ |A_2 \cap A_1| & |A_2| & \cdots & |A_2 \cap A_n| \\ \vdots & \vdots & \ddots & \vdots \\ |A_n \cap A_1| & |A_n \cap A_2| & \cdots & |A_n| \end{pmatrix}.$$

We know that each term not on the main diagonal is odd, so let us work in the field $\mathbb{F}_2$. Then

$$\overline{M^T \cdot M} = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

The determinant of this matrix is 1 in $\mathbb{F}_2$. Thus $\det M^T \cdot M = (\det M)^2$ is odd, so $\det M$ is odd. But note that in M, if we add all of the rows to the first row, we obtain $|A_i|$ in the first row. Thus $\det M$ is even, a contradiction. $\square$

9 Problems

1. Determine an explicit formula for the Lucas numbers.
2. (Romania 1994) For $n \ge 3$ and $A \subset \{1, 2, \cdots, n\}$, say A is even if the sum of the elements of A is an even number. Otherwise, say that A is odd. By convention, the empty set is even.
   (a) Find the number of even, respectively odd subsets of $\{1, 2, \cdots, n\}$.
   (b) Find the sum of the elements of the even, respectively odd subsets of $\{1, 2, \cdots, n\}$.
3. (Putnam) Is there a way to number the sides of a 6-sided dice in such a way that the probability of rolling any particular sum is the same as if the dice were numbered in the ordinary fashion?
4. (MOP 2007) Prove that
$$\sum_{i=0}^n \binom{n}{2i+1}\binom{i}{m} = 2^{n-2m-1}\binom{n-m-1}{m}.$$
5. (China 1991) For any positive integer n, prove that the number of positive integers whose digits are only 1, 3, or 4 with digit sum $2n$ is a perfect square.

6. (St. Petersburg) Is it possible to partition the set of all 12-digit numbers into groups of four numbers such that the numbers in each group have the same digits in 11 places and four consecutive digits in the remaining place?

7. (TSTST 2013) A country has n cities, labelled $1, 2, 3, \ldots, n$. It wants to build exactly $n-1$ roads between certain pairs of cities so that every city is reachable from every other city via some sequence of roads. However, it is not permitted to put roads between pairs of cities that have labels differing by exactly 1, and it is also not permitted to put a road between cities 1 and n. Let T_n be the total number of possible ways to build these roads.

   (a) For all odd n, prove that T_n is divisible by n.

   (b) For all even n, prove that T_n is divisible by $n/2$.

8. Let $A_1, A_2, \ldots, A_{n+1}$ be distinct subsets of the set $\{1, 2, \ldots, n\}$, each having exactly three elements. Prove that there are two subsets among them that have exactly one common element.

9. (TST 2010) Let m, n be positive integers with $m \geq n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \ldots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_S 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1}(n-1)^m + \cdots + (-1)^{n-2}\binom{n}{2}2^m + (-1)^{n-1}\binom{n}{1}.$$

10. The set of nonnegative integers is partitioned into $n \geq 1$ infinite arithmetical progressions with common differences $r_1, r_2, \ldots, r_n$, and first terms $a_1, a_2, \ldots, a_n$. Then prove that

$$\frac{a_1}{r_1} + \frac{a_2}{r_2} + \cdots + \frac{a_n}{r_n} = \frac{n-1}{2}.$$

11. (USAMO 2011) Let A be a set with $|A| = 225$. Suppose further that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \cdots \cup A_{11}| \geq 165$, and give an example for which equality holds.

12. (China 2002) For which positive integers n can we find real numbers $a_1, a_2, \ldots, a_n$ such that

$$\{|a_i - a_j| | 1 \leq i < j \leq n\} = \left\{1, 2, \ldots, \binom{n}{2}\right\}?$$

13. (Kömal) Suppose that $a_0 = a_1 = 1$ and $(n+3)a_{n+1} = (2n+3)a_n + 3na_{n-1}$ for $n \geq 1$. Prove that all terms of this sequence are integers.

14. (OMO 2014) For a positive integer n, an *n-branch* B is an ordered tuple $(S_1, S_2, \ldots, S_n)$ of nonempty sets (where m is any positive integer) satisfying $S_1 \subset S_2 \subset \cdots \subset S_m \subseteq \{1, 2, \ldots, n\}$. An integer x is said to *appear* in B if it is an element of the last set S_m. Define an *n-plant* to be an (unordered) set of n-branches $\{B_1, B_2, \ldots, B_k\}$, and call it *perfect* if each of $1, 2, \ldots, n$ appears in exactly one of its branches.

    Let T_n be the number of distinct perfect n-plants (where $T_0 = 1$), and suppose that for some positive real number x we have the convergence

$$\ln\left(\sum_{n\geq 0} T_n \cdot \frac{(\ln x)^n}{n!}\right) = \frac{6}{29}.$$

    If $x = \frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.