TJHSST Math Team HMMT
Calculus Subject Test
Dr. Osborne 2010

All answers must be given exactly, but you may leave your answers in factorial form if you like.

1. [**2**] Determine the value of $\lim_{x\to 0}\frac{e^{x^3}-1}{\sin x - x}$.

2. [**2**] Determine the value of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+2)}$.

3. [**3**] Given the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n=\sum_{k=1}^{n}\frac{4n^2+4kn+k^2}{n^2+k^2}$, find $\lim_{n\to\infty}\frac{a_n}{n}$.

4. [**3**] Given the function $f(x)=\frac{\sin\sqrt{x}}{\sqrt{x}}$, find $\lim_{x\to 0^+} f^{(2010)}(x)$.

5. [**4**] The function $f(x)$ is the solution to the differential equation
$$2x\,f''(x)+f'(x)=3x$$
for which $f(1)=-2$ and $f'(1)=1$. Find all values of $x>1$ for which $f(x)=0$.

6. [**5**] Determine the value of $\int_0^{2\pi}\sin^6 t\cos^6 t\,dt$.

Solutions to the Calculus Test
Dr. Osborne 2010

1. Determine the value of $\lim_{x\to 0} \frac{e^{x^3}-1}{\sin x - x}$.

   Solution: It will be best to avoid l'Hôpital in this case. Using the expansions of the exponential function and the sine function gives
$$\lim_{x\to 0} \frac{e^{x^3}-1}{\sin x - x} = \lim_{x\to 0} \frac{x^3 + x^6/2 + \cdots}{-x^3/6 + x^5/120 - \cdots} = \mathbf{-6}.$$

2. Determine the value of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)}$.

   Solution: Using partial fractions, we can easily write this series as
$$\sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{2n+1} - \frac{1}{2n+2}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$
   These series are both well-known, so our answer is $\mathbf{\frac{\pi - 2\ln 2}{4}}$

   Alternate solution: As with many cases involving infinite series, we consider the function
$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)}.$$
   Clearly, the derivative of $f(x)$ is given by
$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \text{Arctan } x.$$
   Since the function is clearly zero at $x = 0$, its value at $x = 1$ is given by
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} = \int_0^1 \text{Arctan } x \, dx = x \text{ Arctan } x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\, dx = \mathbf{\frac{\pi}{4} - \frac{1}{2}\ln 2}.$$

3. Given the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \sum_{k=1}^{n} \frac{4n^2 + 4kn + k^2}{n^2 + k^2}$, find $\lim_{n\to\infty} \frac{a_n}{n}$.

   Solution: We recognize this limit as a Riemann sum, as
$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{(k/n)^2 + 4k/n + 4}{(k/n)^2 + 1} = \int_0^1 \frac{x^2 + 4x + 4}{x^2 + 1}\, dx.$$
$$= \int_0^1 \left(1 + \frac{4x}{x^2+1} + \frac{3}{x^2+1}\right) dx = \mathbf{1 + 2\ln 2 + 3\pi/4}$$

4. Given the function $f(x)=\dfrac{\sin\sqrt{x}}{\sqrt{x}}$, find $\lim\limits_{x\to 0^+} f^{(2010)}(x)$.

Solution: The Maclaurin expansion of $f(x)$ is easily seen to be given by

$$f(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{(2k+1)!}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n ,$$

where the derivatives should be replaced by limits of the derivatives if, as in this case, 0 cannot just be 'plugged in'. The desired derivative is therefore given by 2010! times the coefficient of x^{2010} in this expansion, or $\mathbf{2010!/4021!}$.

5. The function $f(x)$ is the solution to the differential equation

$$2x\, f''(x)+f'(x)=3x$$

for which $f(1)=-2$ and $f'(1)=1$. Find all values of $x>1$ for which $f(x)=0$.

Solution: We first recognize that this differential equation can be written in the form

$$\left[2\sqrt{x}\, f'(x)\right]'=3\sqrt{x}$$

on dividing by the integrating factor $\sqrt{x}$. Integrating from 1 to t then gives

$$2\sqrt{t}\, f'(t)-2=2t^{3/2}-2 \quad , \quad \text{or} \quad f'(t)=t .$$

Integrating from 1 to x gives $f(x)=x^2/2-5/2$, so the zero occurs at $x=\sqrt{\mathbf{5}}$.

Alternate solution: It is obvious by inspection that a polynomial of second degree can be forced to satisfy this differential equation. The second derivative of such a polynomial is given by twice its leading coefficient a, and the first derivative is given by $2ax$. Therefore, we have $4a+2a=3$, or $a=1/2$. This immediately implies that the coefficient of x is 0 and the constant is $-5/2$, so the zero is attained at $x=\sqrt{\mathbf{5}}$.

6. Determine the value of $\int_0^{2\pi}\sin^6 t\cos^6 t\, dt$.

Solution: We first use a trigonometric identity to write this integral as

$$\int_0^{2\pi}\sin^6 t\cos^6 t\, dt=\frac{1}{2^6}\int_0^{2\pi}\sin^6 2t\, dt .$$

To compute this integral, we appeal to the exponential form of the sine function,

$$\sin t=\frac{e^{it}-e^{-it}}{2i} .$$

This gives

$$\int_0^{2\pi}\sin^6 t\cos^6 t\, dt=\frac{(-1)^3}{2^{12}}\int_0^{2\pi}\sum_{k=0}^{6}\binom{6}{k}(-1)^k e^{2ikt}e^{-2i(6-k)t}\, dt=\frac{(-1)^3}{2^{12}}\int_0^{2\pi}\sum_{k=0}^{6}\binom{6}{k}(-1)^k e^{4i(k-3)t}\, dt .$$

The integral of the exponential of it times any nonzero integer obviously gives zero, as can easily be shown just by doing one, so the value of this integral is given only by the term for $k=3$,

$$\frac{2\pi}{2^{12}}\binom{6}{3}=\frac{\pi\, 6\cdot 5\cdot 4}{2^{11}\cdot 3\cdot 2}=\frac{5\pi}{2^9}=\frac{\mathbf{5\pi}}{\mathbf{512}} .$$