Beginner Combinatorics

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March 3, 2011

Combinatorics describes the branch of mathematics that deals with counting. I am going to assume throughout this lecture that you know how to calculate combinations and permutations and are familiar with Pascal's Triangle.

1 Counting

Combinatorics problems usually involve either counting, probability, or both. There is also a bit of overlap between combinatorics problems and set theory. First, let's start with counting.

1. In how many ways can n objects be arranged?

   Since there are n objects to place in the first slot, $n-1$ objects to place in the second slot, $n-2$ objects to place in the third slot, and so on, so that there are $n-r+1$ objects to choose to place in the rth slot. Thus, the total number of ways in which n objects can be arranged is $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$

2. Of those n objects, how many different ways can we pick an arrangement of r objects, if the order in which we pick the objects matters?

   This is similar to the previous problem. However, we only want to choose r objects, so instead of $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$, we only go as far as $n-r+1$, so the formula becomes $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+2) \cdot (n-r+1)$. If we multiply by $\frac{(n-r)\cdot(n-r-1)\cdot\ldots\cdot3\cdot2\cdot1}{(n-r)\cdot(n-r-1)\cdot\ldots\cdot3\cdot2\cdot1}$, we end up with the familiar formula for permutations: ${}_nP_r = \frac{n!}{(n-r)!}$

3. In how many ways can we pick r objects from n objects if the order in which we pick them does not matter?

   Using the results from the previous two problems, we know that there are $\frac{n!}{(n-r)!}$ ways to choose r objects from n objects. However, we note that, since order does not matter, $r!$ of these ways essentially choose the same elements, just ordered differently! So, we divide by $r!$, and get the familiar formula for combinations: $\frac{n!}{r!(n-r)!}$

4. How many subsets can be formed from n elements?

   Note that every object can have one of two states: it is either part of the subset, or it is not. Since these events are independent of one another, there are 2^n subsets of n objects. Using a similar argument for objects with r states, there are r^n subsets of n objects, each capable of r states.

2 Probability

Above, we derived the described basic methods of counting. Now, we will dive into the field of probability.

Probability essentially means, "What is the likelihood of even x occurring after 1 attempt?" As such, probability is generally a value between 0 and 1, inclusive. A probability of 1 means that the event will definitely occur, a probability of 0 means that the event will never occur, and anything outside the range means you've either done the problem wrong, or the problem isn't a probability problem. That said, let's take a look at some problems:

1. What is the probability that you will choose the only red marble from a bag of only 10 marbles?

Since there are 10 marbles, and only 1 red marble, the probability of that red marble being chosen is $\frac{1}{10} = 0.1$.

2. Let's now say that I put 2 more red marbles. Now, what is the probability of picking 2 red marbles (out of 2 times) from the bag (without replacement)?

   Since there are 3 red marbles in the bag, and 10+2=12 marbles total, the probability of the first marble being red is $\frac{3}{12}$. The probability of the second marble being red is $\frac{2}{11}$. Thus, the overall probability of picking red marbles both times is $\frac{3\cdot 2}{12\cdot 11} = \frac{6}{132} = \frac{1}{22}$. Note that this is also equivalent to $\frac{{}_nC_r}{{}_nP_r}$, where $n = 12$, and $r = 2$.

3. Suppose that I want to confiscate your 3 red marbles, because you were making loud noises with them during my lecture. What is the probability that you will keep at least red marble, if I randomly pick 3 marbles without replacement?

   Since the probability of you keeping at least 1 red marble is the same as me not getting all the red marbles in 3 tries, the probability of you having at least 1 red marble is:

$$1 - (\textit{probability of me confiscating all the red marbles}) = 1 - \frac{3\cdot 2\cdot 1}{12\cdot 11\cdot 10} = 1 - \frac{1}{220} = \frac{219}{220}$$

One last thing you should know: the binomial theorem. You have probably heard of this before, but I will put it here as a reminder:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

3 Pitfalls and Traps in Counting and Probability

One of the most common – and surely the most annoying – pitfall for probability is failing to account for the right number of objects. Always make sure that you have the right number of total objects. If you are given 10 objects, and of those x is supposed to be chosen, then your initial total number of objects is 10. However, if you are in a room with 10 other people, then there are not 10 people in the room, but 11.

Another trap is forgetting to divide out rearrangements of the same objects when counting combinations. You simply have to be vigilant, and not make these mistakes, as there's no sure-fire way to check for this, except to be careful (unless your answer is obviously wrong...such as probabilities greater than 1).

4 Pathfinding

For the past two weeks now, we have seen problems involving paths. For this lecture, we will only concern ourselves with the 2-D case of up-and-right pathfinding. However, there are two types of "pathfinding" problems.

4.1 Lattice-Point Pathfinding

Albert and Krishnan are playing a game of desk tag in 240. The desks have been arranged so that the paths between them forms an 6×6 lattice of paths. Due to Albert's height, he is required to be right next to a desk, or else Krishnan won't be able to catch him. At this point, Kevin walks in, and ties Albert down at the far end of the classroom. If Krishnan is at (0,0), and the gagged Albert is at (6,6), how many paths can Krishnan take to get to Albert, and ungag him, if Krishnan can only move one unit at a time in the $+x$ and $+y$ directions?

Since Krishnan will make 12 moves anyways, let us denote a move in the $+x$ direction as A and a move in the $+y$ direction as B, so the string representation of Krishnan's movement would be a permutation of "$ABABABABABAB$". Then, the number of paths that Krishnan can take is simply the number of distinct permutations of that string, which is $\binom{12}{6}$. This is a very large number, and thus ARML would likely not ask you to calculate it. However, this sets us up for the next problem:

After Krishnan got to Albert, he realizes that he should've picked up a pair of scissors to cut Albert's gag. He spots a pair of scissors lying on the floor, at (3,3). How many paths can Krishnan take going to (3,3) and coming back to (6,6)? Assume that Krishnan will take the shortest length path possible.

Note that this is essentially the same problem, except that we're finding smaller-length paths twice, and then multiplying them together. So, WLOG, let's re-coordinate where Albert is to be (0,0), and the scissors to be at (3,3). Thus, the square is now a 3x3 square. Using the same argument above, we find that there are $\binom{6}{3} = 120$ ways to get to the scissors, and $\binom{6}{3} = 120$ ways to get back. Thus, there are a total of $120 \cdot 120 = 14400$ paths which Krishnan can travel to retrieve the scissors and get back to Albert.

The derivation of finding the number of paths from point A(ax,ay) to B(bx,by) to be

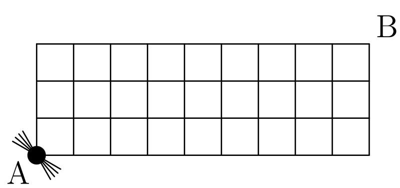
$$\binom{(by-ay)+(bx-ax)}{by-ay} = \binom{(by-ay)+(bx-ax)}{bx-ax}$$

requires the use of bijections, so for now, take this formula as a gift from God, and use it well.

4.2 Counting Paths

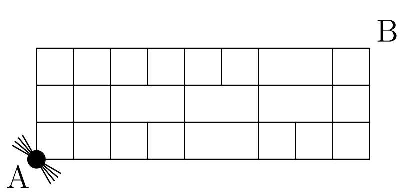
Another type of similar-looking, but completely different problem is counting the number of paths from A to B, given weights for a graph. For now, we're going to work with weights of 1, and you can easily extrapolate this method to work with other weights by scaling.

So, the problem is, given the grid below, the arachnid (Fluffy) at point A wants to move to where Sam is sleeping during English class, at point B:



Let A have a value of 1. Let every intersection of lines have a value equivalent to the sum of the values of the vertices directly to the left and below it. By moving upward and right through the grid, we eventually end up with a value of 220 at point B. Thus, there are 220 paths from A to B, with Fluffy moving strictly up or right. This will be much better explained in the lecture (hopefully).

The advantage of this method is that, if a connection is removed, then this method still works. For example, if we removed a few connections, as below, we could still solve the problem.



5 Set Theory

To add a little bit of flavoring in here, we will also go over set theory in combinatorics.
There are three very important topics here: the Principle of Inclusion-Exclusion, Pigeon Hole Principle, and Partitioning. Partitioning will not be covered here, since we simply don't have the time.

5.1 Inclusion-Exclusion

Suppose that every day during lunch, 30 people (mainly freshmen) play StarCraft in the SysLab, 20 people do work, and 15 people play other games. In addition, of those people, 10 people both play StarCraft and do work, 12 people play StarCraft **and** other games, and 8 people play other games and do work. Given that there are a total of 50 people in the SysLab, how many people do all three?

The Principle of Inclusion-Exclusion states that the total number n of objects in a set is equal to the number of objects in subsets of size 1, minus the number of objects in subsets of size 2, plus the number of objects in subsets of size 3, and so on, until you reach $n-1$. Thus, our problem quickly becomes trivialized into:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ \Rightarrow 50 &= 30 + 20 + 15 - 10 - 12 - 8 + |A \cap B \cap C| \\ \Rightarrow |A \cap B \cap C| &= 50 - (30 + 20 + 15) + (10 + 12 + 8) \\ &= 50 - (65) + (20) \\ &= 5 \end{aligned}$$

Thus, there are 5 people who play StarCraft, do work, **and** play other games in the SysLab during lunch.

5.2 The Pigeon Hole Principle

The Pigeon Hole Principle is really intuitive to understand, and I'm sure you've already figured this out on your own. Basically, it says that, if there were n holes, and k pigeons, then the minimum number of pigeons you would need to ensure that at least 1 hole has more than one pigeon in it is $k = n + 1$. In other words, two of the pigeons would be forced to share a hole.

So, on TJ IMO day, there are 25 teams, and 40 people who have volunteered to coach or run. What is the maximum number of runners we can have, if we want at least one team to be double-coached?

Obviously, the answer is $40 - 25 - 1 = 14$. But, that's how the pigeon hole principle usually works. Remember to layer on more pigeons if it's asking for at least 3, or 4, etc. pigeons. :)

6 Problems

1. On the planet Foo, in the city of Foosboro, there live 100 little green people. Of these, there are 95 little green boys, and 5 little green girls. Back here on Earth, you accidentally fire a high-frequency laser into space, and it just coincidentially hits Foosboro, eliminating one little green person. What is the probability that this little green person was male?

2. The little green people are very angry that you have killed one of their males (though they would be angrier if you had killed one of their females), and decide to return the favor. Lucily for you, you are in a room with 100 other mathletes, and since the little green men have terrible eyesight, they cannot distinguish between you all. If they shoot the laser once, and hit one person randomly, what is the probability that you will **not** be vaporized?

3. Sam has fallen asleep during Differential Equations class yet again. Yuqing decides to prank Sam, and drops a spider on Sam's head. Sam wakes up with a start, and in his fright, his body converts into light. He is then transmitted to Foosboro, where the little green people have overlooked your vaporization of one of their males, and decide to return Sam back to his normal form. However, the machine that will be used is old, and has a 100% chance of fully returning Sam's head to normal, 100% chance of returning his torso to normal, 80% chance of returning his arms and hands to normal, and 75% chance of returning Sam's legs and feet to normal. What is the probability that Sam will be fully returned to normal? How about missing exactly one appendage?

4. Luckily for us, Sam was fully returned to normal, and was sent back towards Earth at 0.999999 times the speed of light, so that he arrived just before everyone left for HMMT. Unfortunately, by the time he got back, he had forgotten some of his math. If there were 6 possible tests at HMMT (General 1,

General 2, Algebra, Geometry, Calculus, and Combinatorics), and Sam had forgotten all of his Algebra and Geometry, what is the probability that he will be randomly assigned two tests which he can do well in? (Assume that the General topics are easy, easy problems, so that Sam can do them without much Algebra and Geometry).

5. Sam got lucky, and ended up with Calculus and Combinatorics. However, Mitchell Lee also got Calc and Combo. If Mitchell thinks that he got between 8-10 points on Calculus, and 9-10 points on Combinatorics, and Sam knows that he got at least 1 wrong total, and at most 5 wrong total, what is the probability that Sam and Mitchell will end up with the same score, if Mitchell and Sam are each equally likely to get any score within their indicated ranges (inclusive)?

6. After the HMMT awards ceremony, Sam chased Yuqing through the halls of Harvard, before Mrs. Gabriel can get a hold of them. At a small square auditorium, 9 desks were arranged so that Sam must cross a 4x4 grid, from one vertex to the opposite vertex, to catch Yuqing, who was taking a water break. How many ways can Sam pass through the maze of desks, and tackle Yuqing?

7. After Sam caught up to Yuqing, they began to walk back. Yuqing subtly dropped a plushie spider on the floor. Sam, stepping on the plushie, was frightened and ran into a crowd of 50 Koreans (including Sam). Yuqing tried to find Sam, but lost him in the crowd. However, Yuqing was able to see that 40 Koreans were wearing glasses, 30 of whom were busy talking about the Calculus test. There were also 20 loud people, 5 of whom were also talking about the Calculus test. If Sam wears glasses, and Yuqing distinctively heard Sam loudly arguing about the Calculus test, how many people could possibly be Sam, from the information given above?

8. Sin randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ and arranges them in descending order to form a 3-digit number. Andre randomly picks 3 distinct numbers from the set $\{1,2,3,4,5,6,7,8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Sin's number is larger than Andre's number? (AMC 10A, 2010)