Algebraic Tricks for Solving Equations

1 Warm-Ups

1. Given that $x + \frac{1}{x} = 3$, compute $x^3 + x^2 + \frac{1}{x^2} + \frac{1}{x^3}$.
2. If x is positive and $x^4 + \frac{1}{x^4} = 34$, compute $x - \frac{1}{x}$.
3. What is the value of $\sqrt{15 - 6\sqrt{6}} + \sqrt{15 + 6\sqrt{6}}$?
4. What is the greatest possible value of x for the equation $\left(\frac{4x-16}{3x-4}\right)^2 + \left(\frac{4x-16}{3x-4}\right) = 12$?

2 Problems

5. Find all real solutions to $x^4 - 8x^3 + 17x^2 - 8x + 1 = 0$.
6. Find all real solutions to $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$.
7. Find the sum of the real roots of the polynomial $f(x) = x^6 + x^4 - 115x^3 + x^2 + 1$.
8. Define the function $f(x) = x^6 - 10x^5 + 37x^4 - 70x^3 + 74x^2 - 40x + 8$. What is the sum of the real solutions to the equation $f(x) = 0$?
9. Find all real solutions to $\sqrt{3x^2 - 4x + 34} + \sqrt{3x^2 - 4x - 11} = 9$
10. Find all real solutions to $\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x$
11. Find all real numbers x such that $\sqrt[3]{20 + x\sqrt{2}} + \sqrt[3]{20 - x\sqrt{2}} = 4$.
12. Solve: $\sqrt[3]{3x - 5} + \sqrt[3]{2x - 4} = \sqrt[3]{5x - 9}$.
13. Solve the equation $\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}} = \frac{3}{2}\sqrt{\frac{x}{x+\sqrt{x}}}$.
14. Solve the equation $\sqrt[m]{(1 + x)^2} - \sqrt[m]{(1 - x)^2} = \sqrt[m]{1 - x^2}$.
15. Solve the equation $\sqrt[3]{x - 1} + \sqrt[3]{x + 1} = x\sqrt[3]{2}$.

16. For what real values of x is $\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=A$ for: *i)* $A=\sqrt{2}$; *ii)* $A=1$; *iii)* $A=2$?

17. Find all real solutions to $2\left(\sqrt{x}-\frac{1}{\sqrt{x}-\frac{1}{\sqrt{x}-\frac{1}{\sqrt{x}-\cdots}}}\right)^2=x+\frac{1}{x+\frac{1}{x+\frac{1}{x+\cdots}}}$.

18. Find all real solutions to $\frac{x+5}{x+4}-\frac{x+6}{x+5}=\frac{x+7}{x+6}-\frac{x+8}{x+7}$

19. Find all real solutions to $\frac{x+1}{x+2}+\frac{x+6}{x+7}=\frac{x+2}{x+3}+\frac{x+5}{x+6}$

20. Find a ratinal expression for x in terms of a and b, reduced as far as possible, satisfying the conditions $\frac{x}{x-a}+\frac{x-b}{x-a-b}=\frac{x-a}{x-2a}+\frac{x+a-b}{x-b}$ and $2a>x>b>a>0$.

21. Find all real solutions to $\frac{2x^2+1}{x+2}+\frac{2x+4}{2x^2+1}=3$.

22. Solve in terms of a and b: $\frac{a+x}{b+x}+\frac{b+x}{a+x}=\frac{5}{2}$.

23. Find all real solutions to $x^4+(2-x)^4=34$.

24. Compute the four complex solutions of $(x-1)^4+(x-5)^4+14=0$.

25. Let m be the largest real solution to the equation $\frac{3}{x-3}+\frac{5}{x-5}+\frac{17}{x-17}+\frac{19}{x-19}=x^2-11x-4$. There are positive integers a,b,c such that $m=a+\sqrt{b+\sqrt{c}}$. Find $a+b+c$.