Symmedians

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September 1, 2013

A *symmedian* is the reflection of a median over the corresponding angle bisector. There are many articles on symmedians online, such as the Mathematical Reflections article[1] by Sammy Luo and Cosmin Pohoata.

1 Definition

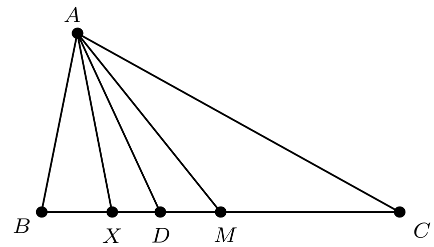

Figure 1: The A-symmedian AX.

In triangle ABC, its A-symmedian is the reflection of the median from A over the angle bisector of A.

2 Properties

We begin with a simple yet important lemma:

Lemma 1. *The A-, B-, and C-symmedians of $\triangle ABC$ concur at the symmedian point K.*

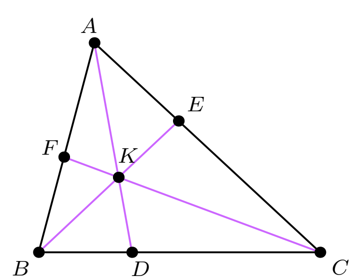

Figure 2: The symmedian point K.

[1] `https://www.awesomemath.org/assets/PDFs/MR4_Symmedians.pdf`

Proof. Let D, E, and F be the intersections of the symmedians with their corresponding sides, and let M be the midpoint of segment BC. Notice that $\angle BAD = \angle CAM$ and $\angle CAD = \angle BAM$ because D is the reflection of M over the angle bisector of $\angle A$. Hence

$$\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{\sin \angle CAM}{\sin \angle BAM} = \frac{AB}{CA}.$$

It follows that

$$\frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} = \frac{AB}{CA} \cdot \frac{BC}{AB} \cdot \frac{CA}{BC} = 1,$$

so AD, BE, and CF concur by Trig Ceva. □

Actually, there is a more general result, which states that:

Theorem 2 (Isogonal Conjugate)**.** *Let P be a point in the plane of $\triangle ABC$. If AP, BP, and CP are reflected over the angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively, then these three lines concur at the isogonal conjugate of P.*

Proof. Again, Trig Ceva to mimic the proof of Lemma 1. □

The following property of the symmedian is so well-known that it is basically called "the symmedian lemma."

Lemma 3. *Let ABC be a triangle, and let P be the intersection of the tangents to the circumcircle of $\triangle ABC$ at B and C. Then AP is the A-symmedian of $\triangle ABC$.*

Proof.

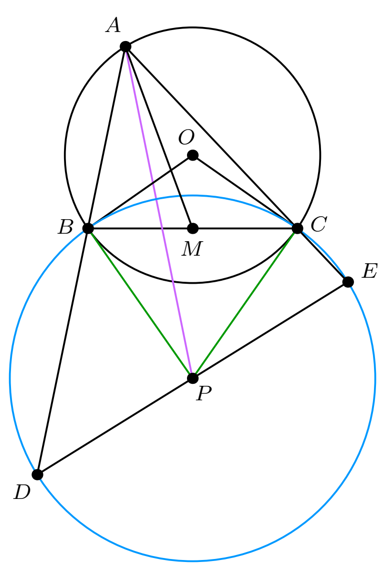

Figure 3: Lemma 3.

Let ω be the circle centered at P with radius PB. This circle passes through C because $PB = PC$. Now let D and E be the intersections of ω with AB and AC, respectively. Finally, let M be the midpoint of segment BC and O the circumcenter of $\triangle ABC$.

Note that

$$\begin{aligned}\angle DBE &= \angle BAE + \angle AEB \\ &= \angle BAC + \angle CEB \\ &= \frac{1}{2}(\angle BOC + \angle CPB) = 90^\circ.\end{aligned}$$

Hence DE is a diameter of ω, and so P is the midpoint of segment DE. Observe that $\triangle ABC \sim AED$, and so $\triangle AMC \sim APD$. Thus $\angle CAM = \angle DAP = \angle BAP$, implying that AP is the A-symmedian. $\square$

There is also a nice proof involving projective geometry, but since we haven't studied projective geometry yet, I will go over the second proof when I lecture on it.

The next lemma is very important, as it provides a nice ratio relationship between the distances from X to B and C:

Lemma 4. *Let X be a point on BC such that AX is the A-symmedian of $\triangle ABC$. Then*

$$\frac{BX}{CX} = \frac{AB^2}{AC^2}.$$

Proof. Note that

$$\frac{\sin\angle BAX}{BX} = \frac{\sin\angle AXB}{AB} \text{ and } \frac{\sin\angle CAX}{CX} = \frac{\sin\angle AXC}{AC}.$$

Dividing these two gives us

$$\frac{BX}{CX} = \frac{AB}{AC}\frac{\sin\angle BAX \sin\angle AXC}{\sin\angle AXB \sin\angle CAX} = \frac{AB}{AC}\frac{\sin\angle BAX}{\sin\angle CAX} = \frac{AB^2}{AC^2}$$

as desired. $\square$

Another interesting property of the symmedian is that it is the locus of the midpoints of *antiparallels*. We say that two lines/segments ℓ_1 and ℓ_2 are antiparallel with respect to an angle if the angle formed by ℓ_1 with one side of the angle is equal to the angle formed by ℓ_2 with the other side.

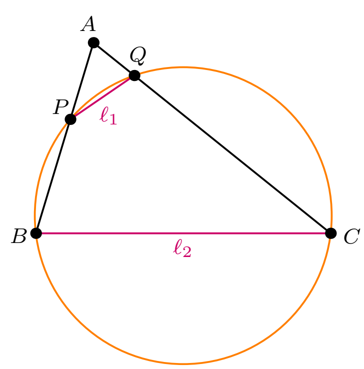

Figure 4: Segments PQ and AB are antiparallel with respect to $\angle BAC$.

In Figure 4, $\angle AQP = \angle ABC$ and $\angle APQ = \angle ACB$. Notice that this immediately implies that $BCQP$ is cyclic, since $\angle ABC + \angle PQC = \angle AQP + \angle PQC = 180^\circ$.

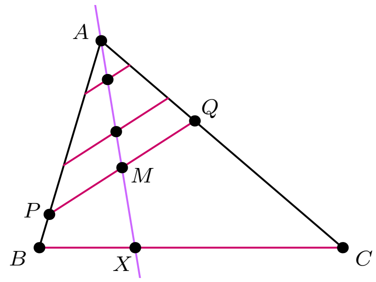

Figure 5: Lemma 5.

Lemma 5. *The A-symmedian of $\triangle ABC$ is the locus of the midpoints of the antiparallels to BC with respect to BAC.*

Proof. Let P and Q be points on AB and AC such that PQ is antiparallel to BC, and let M be the midpoint of segment PQ. Let X be the intersection of AM and BC. By the Generalized Angle Bisector Theorem,

$$1 = \frac{MP}{MQ} = \frac{AP}{AQ}\frac{\sin\angle MAP}{\sin\angle MAQ}.$$

Hence

$$\frac{BX}{CX} = \frac{AB}{AC}\frac{\sin\angle XAB}{\sin\angle XAC} = \frac{AB}{AC}\frac{\sin\angle MAP}{\sin\angle MAQ} = \frac{AB}{AC}\frac{AQ}{AP} = \frac{AB^2}{AC^2}$$

and so by Lemma 4, AX is the A-symmedian. □

3 Applications

There are two circles that correspond to the symmedian point of a triangle:

Theorem 6 (First Lemoine Circle)**.** *Let K be the symmedian point of triangle ABC. Prove that the six intersections formed by the three parallels with respect to the sides of $\triangle ABC$ passing through K and the sides themselves lie on a circle.*

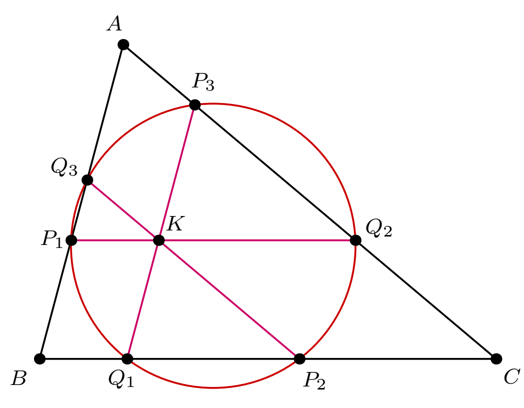

Figure 6: The First Lemoine Circle.

Proof. Let P_1, Q_3; P_2, Q_1; and P_3, Q_2 be points on AB, BC, and CA, respectively, such that $P_1Q_2 \parallel BC$, $P_2Q_3 \parallel CA$, and $P_3Q_1 \parallel AB$.

Notice that AP_3KQ_3 is a parallelogram, so the midpoint of P_3Q_3 lies on AK. However, AK is the A-symmedian of $\triangle ABC$, implying that P_3Q_3 is antiparallel to BC. Therefore, $\angle AP_3Q_3 = \angle ABC = \angle Q_3P_1Q_2$ and so $P_1Q_2P_3Q_3$ is cyclic. Similarly, $P_1Q_1P_2Q_3$ and $Q_1P_2Q_2P_3$ are cyclic.

Assume that the three circumcircles are distinct. Then by the Radical Axis Theorem, their pairwise radical axes concur. However, their radical axes are AB, BC, and CA, which do not concur. Hence the circumcircles are not distinct and so they coincide. $\square$

Theorem 7 (Second Lemoine Circle)**.** *Let K be the symmedian point of triangle ABC. Prove that the six intersections formed by the three antiparallels with respect to the sides of $\triangle ABC$ passing through K and the sides themselves lie on a circle.*

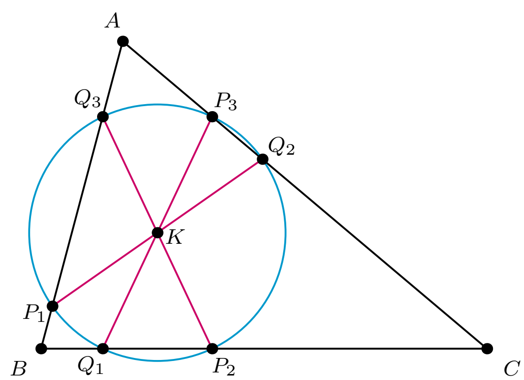

Figure 7: The Second Lemoine Circle.

Proof. Define points as in Theorem 6, except with antiparallels. By Lemma 5, K is the midpoint of P_1Q_2, P_2Q_3, and P_3Q_1. Now note that $\angle KQ_1P_2 = \angle BAC = \angle Q_3P_2Q_1$ because AC and AB are antiparallel to Q_3P_2 and Q_1P_3, respectively. Thus $KQ_1 = KP_2$ and so by symmetry the circle centered at K with radius KP_1 passes through all six points P_1, P_2, P_3, Q_1, Q_2, and Q_3. $\square$

The First Lemoine Circle is actually a special case of the more general Tucker Circle, which we will not go over in this lecture. However, feel free to peruse the WolframMathWorld article on them here[2].

4 Problems

1. Prove that the symmedian point of a triangle is the centroid of the pedal triangle of the symmedian point.

2. Let ABC be a triangle, and let ℓ be the A-median. Prove that the inverse of ℓ with respect to A is the A-symmedian of $\triangle AB'C'$, where B' and C' are the inverses of B and C, respectively.

3. Let PQ be a diameter of circle ω. Let A and B be points on ω on the same arc $\widehat{PQ}$, and let C be a point such that CA and CB are tangent to ω. Let ℓ be a line tangent to ω at Q. If $A' = PA \cap \ell$, $B' = PB \cap \ell$ and $C' = PC \cap \ell$, prove that C' is the midpoint of segment $A'B'$.

[2] `http://mathworld.wolfram.com/TuckerCircles.html`

4. Let K be the symmedian point of triangle ABC, and let X be the intersection of AK and BC. Prove that
$$\frac{AK}{XK} = \frac{AC^2 + AB^2}{BC^2}.$$

5. (PAMO 2013) Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively. Prove that if $\angle ABP = \angle CBD$, then $\angle BCQ = \angle ACD$.

6. (Iran 2013) Let P be a point outside of circle C. Let PA and PB be the tangents to the circle drawn from C. Choose a point K on AB . Suppose that the circumcircle of triangle PBK intersects C again at T. Let P' be the reflection of P with respect to A. Prove that $\angle PBT = \angle P'KA$.

7. (Russia 2010) Let O be the circumcenter of the acute non-isosceles triangle ABC. Let P and Q be points on the altitude AD such that OP and OQ are perpendicular to AB and AC respectively. Let M be the midpoint of BC and S be the circumcenter of triangle OPQ. Prove that $\angle BAS = \angle CAM$.

8. (Vietnam 2001) In the plane let two circles be given which intersect at two points A and B. Let PT be one of the two common tangent lines of these circles. Tangents at P and T to the circumcircle of triangle APT intersect at S. Let H be the reflection of B over PT. Show that A, S, and H are collinear.

9. Prove that the Gergonne point of a triangle is the symmedian point of intouch triangle.

10. (USAMO 2008, Modified) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle. Prove that AF is the A-symmedian of $\triangle ABC$.

11. Let ABC be a triangle, M the midpoint of segment BC and X the midpoint of the A-altitude. Prove that the symmedian point of $\triangle ABC$ lies on MX.