Triangles!

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1 Special Lines and Points

There are a couple types of lines and their respective points that you should keep in mind for solving triangle problems.

1. **Medians:** These intersect the sides of a triangle at their midpoints; their common point of intersection is called the ***centroid***. Fun Fact: the ratio of the length of a centroid to a vertex is always twice that of its distance to the opposite side. Can you prove this?

2. **Angle Bisectors:** You guessed it, these bisect the angles of a triangle and intersect at a triangle's ***incenter***, the center of a triangle's incircle.

3. **Perpendicular Bisectors:** These perpendicularly bisect the sides of triangles. Their common point of intersection is called the ***circumcenter***, the center of a triangle's circumcircle.

4. **Altitudes:** These intersect at a triangle's ***orthocenter***.

2 Some Trig

Just a recap of the Law of Sines and the Law of Cosines

2.1 Law of Sines (Extended)

$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$, where R is the circumradius of $\triangle ABC$.

2.2 Law of Cosines

$c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is angle ACB.

3 Area

Finding the areas of triangles is often very important, so the following formulas can be quite handy...

1. $K = \frac{1}{2}bh$, This is the most basic one; area is half the base times the height.
2. $K = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter or $\frac{a+b+c}{2}$, also known as *Heron's formula.*
3. $K = \frac{abc}{4R}$, where R is again the circumradius of $\triangle ABC$.
4. $K = rs$, where r is the inradius of $\triangle ABC$.
5. $K = \frac{1}{2}absinC$, where C is angle ACB.

4 Useful Theorems

The following theorems will help you find the sidelengths of a triangle.

1. **The Angle Bisector Theorem:** If $\triangle ABC$ has an angle bisector going from A to point D on BC, $\frac{AB}{BD} = \frac{AC}{CD}$.
2. **Stewart's Theorem:** This time $\triangle ABC$ will have a random point D on BC. Let $AD = d$, $BD = m$, and $CD = n$. The sides a, b, and c still correspond to angles A, B, and C. We now have $dad + man = bmb + cnc$.
3. **Ceva's Theorem:** Let $\triangle ABC$ have points D, E, and F on sides BC, AC, and AB respectively. When AD, BE, and CF are concurrent, the following holds true: $\frac{AF}{AE}\frac{BD}{BF}\frac{CE}{CD} = 1$.

5 Keep in Mind...

While all of these formulas and theorems can help you solve problems, they will not solve them all. Finding similar triangles or angle chasing is often even more important, so do not rely on the formulas alone.

6 Practice Problems

Time to put your skills to good use!

1. (AHSME 1953) The base of an isosceles triangle is 6 inches and one of the equal sides is 12 inches. What is the radius of the circle through the vertices of the triangle?

2. (Russia Sharygin Geometry Olympiad 2008) Given right triangle ABC with hypothenuse AC and $\angle A = 50°$. Points K and L on side BC such that $\angle KAC = \angle LAB = 10°$. Determine the ratio CK/LB.

3. (AIME1 2009) Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{BC}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

4. (AHSME 1965) Let line AC be perpendicular to line CE. Connect A to D, the midpoint of CE, and connect B to E, the midpoint of AC. If AD and EB intersect at point F, and $\overline{BC} = \overline{CD} = 15$, then find the area of triangle DFE.

5. (AIMEI 1987) Triangle ABC has right angle at B, and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.