Geometry 1 - Brocard Points

Matthew Park

October 15, 2012

1. Introduction - Approaching geometry problems

- Organized and neat diagrams allow you to see both the big picture and the small details.
- Working both ways is a merit.
- Staring is not the same thing as solving.
- Try making constructions! (e.g. parallel lines, circles, ... find "the" line)

2. Brocard Points

We can show that inside any triangle ABC, there exists a unique point P such that

$$\angle PAB = \angle PBC = \angle PCA$$

This point is called the Brocard point of triangle ABC.

2.1 Proof of the Brocard Point

Let S be the center of the circumcircle of $\triangle$ACP. Indeed if $\angle$PAB = $\angle$PCA, then the circumcircle of triangle ACP is tangent to the line AB at A. Then, S lies on the perpendicular bisector of segments AC, and the line SA is perpendicular to the line AB.

Therefore, point P lies on the circle centered at S with radius SA (note that this circle is not tangent to line BC unless BA=BC). We can use this equation $\angle$PBC = $\angle$PCA to construct the circle passing through B and tangent to line AC at C. The Brocard point P must lie on both circles and be different from C. Such a point is unique. Therefore, the third equation $\angle$PAB = $\angle$PBC clearly holds.

2.2 Example Problem #1

[AIME 1999] Point P is located inside triangle ABC so that angles PAB, PBC, and PCA are all congruent. The sides of the triangle have lengths, AB = 13, BC = 14, and CA = 15, and the tangent of $\angle$PAB is m/n, where m and n are relatively prime positive integers. Find $m+n$.

2.3 Example Problem #2

Let x, y, and z be positive real numbers satisfying the system of the equations

$$3x^2 + 3xy + y^2 = 75$$

$$y^2 + 3z^2 = 27$$

$$z^2 + xz + x^2 = 16$$

Evaluate $xy + 2yz + 3xz$

3. Exercises

1. Point O lies inside the irregular pentagon ABCDE. Let $\angle BAO = \angle BCO$, $\angle CBO = \angle CDO$, $\angle DCO = \angle DEO$, $\angle EDO = \angle EAO$. If $\angle AEO = 24°$, what are the possible values of the measure of $\angle ABO$ in degrees?

2. Diagonals AC and BD of a cyclic quadrilateral ABCD intersect at point E. Prove that if $\angle BAD = 60$ and $AE = 3CE$, then the sum of some two sides of the quadrilateral equals the sum of the other two.

3. Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

4. Point P lies inside triangle ABC such that $\angle PAB = \angle PBC = \angle PCA = \alpha$. Prove that

$$csc^2\alpha = csc^2(A) + csc^2(B) + csc^2(C)$$

5. Let S be an interior point of triangle ABC. Show that at least one of $\angle SAB$, $\angle SBC$, and $\angle SCA$ is less than or equal to $30°$.

6. Let P be a point inside triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D, E be the incenters of triangles APB, APC, respectively. Show that lines AP, BD, CE meet at a point.

7. Consider the five points A, B, C, D, and E such that ABCD is a parallelogram and BCED is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of segment CD at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.