Circles

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1 Those Elusive Things

There really aren't many theorems involving circles that are required for ARML. You guys can probably all rattle them off. The hard part is actually seeing the circle and knowing when to use it. Here is a list of my favorite ways of seeing "those elusive things".

1. A circle is the set of points equidistant from the center of the circle in two dimensions.

2. All triangles have a circle circumscribed around it (3 points determine a circle) and inscribed in them. There are so many special circles in triangles, get familiar and comfortable with them!

3. Equal angles, especially equal right angles, often come with circles in the form of cyclic quadrilaterals.

4. There is only one circle, the unit circle. It always shows up when you coordinatize, use complex numbers, trig functions, or with polynomials relating to $x^n = 1$.

5. All regular polygons have a nice circle circumscribed around it.

After you spot the circle, it is time to apply what you know about circles.

2 Practice - Applying What You Know

The following problems involve circles: find the circles and then use the circles to do them. You'll notice that these problems generally wouldn't show up in ARML, but it is getting comfortable with circles that is more important. I did not include solutions here but they are all available online through kalva, google, etc. The astericked ones are used to prove Ptolemy's Theorem.

1. ***Law of Sines(extended)** -

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

2. **Law of Cosines** - $a^2 + b^2 - 2ab\cos C = c^2$

3. Given a triangle ABC, the orthic triangle of ABC has vertices at the bases of the three altitudes. Prove that the orthocenter of an acute triangle is the incenter of its orthic triangle. Label all the equal angles you can find.

4. Let d be the distance from a point P to the center of a circle of radius R. Prove power of a point and show that it is always equal to $|d^2 - R^2|$ (this formula actually is surprisingly useful). Once you know the proof, you'll realize that power of a point doesn't even capture the full power of the circle.

5. *Take a triangle ABC. Now take a "pedal point" P (Points where you drop perpendiculars from P to BC, AC, and AB to points A_1, B_1, C_1 respectively.) Prove that $B_1C_1 = a\frac{AP}{2R}$ and similar expressions for the other three, where R is the circumradius of ABC. The triangle formed by A_1, B_1, C_1 is known as a pedal triangle. It is a more general case of the medial triangle, orthic triangle, etc.

6. Take a pedal point and keep it fixed. Now take succesive pedal triangles. That is, take the original triangle and make a pedal triangle. Now make another smaller pedal triangle using the new triangle and the same point. Continue this process infinitely. Prove that every third triangle is similar.

7. *Prove that the pedal point P is on the cirumcircle of triangle ABC iff the pedal triangle is not a triangle, but is in fact degenerate. The line $A_1B_1C_1$ is known as the Simson Line.

8. *Prove Ptolemy's theorem. Given quadrilateral ABCD, $AB * CD + BC * DA \geq AC * BD$ with equality iff ABCD is cyclic.

9. (AIME 1983 Number 14) Two circles have radii of 6 and 8 and have centers a distance of 12 apart. They intersect at point P. A line through point P intersects the two circles at Q and R such that $QP = PR$. Compute QP^2.

10. (AIME 1990 Number 12) A regular 12-gon has circumradius 12. Find the sum of the lengths of all its sides and diagonals.

11. (AIME 1991 Number 14) A hexagon is inscribed in a circle. Five sides have length 81 and the other side has length 31. Find the sum of the three diagonals from a vertex on the short side.