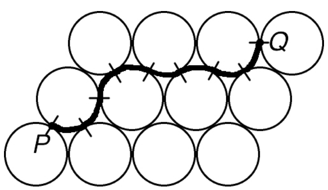
Answer Key		4.	$\tan\alpha = \cos\beta$
1.	18	5.	45 : 98
2.	$8\pi/3$	6.	310
3.	0.999964	7.	8191/32768

1. The two-digit numbers reading across could only be 14, 28, 42, 56, 70, 84, or 98. Since the numbers reading down are multiples of 3 this further restricts the possibilities to 14 and 28, 42 and 84, or 56 and 70. But if we use 14 and 28 then the numbers reading down are 12 and 48, which have a GCD of 12 rather than 3. The other order is no good either, and similar issues arises with 42 and 84. However, the pair 56 and 70 will work (in either order), giving a total digit sum of $5+6+7+0 = \mathbf{18}$.

2. There are several possible paths of minimal length through the circles; one is depicted at left. However, all such paths involve exactly eight arcs extending a sixth of the way around a circle. These arcs are indicated by the short divider segments in the diagram. Since each arc has length $\frac{1}{6}(2\pi) = \frac{1}{3}\pi$, the total length of the path is $\mathbf{8\pi/3}$.



3. Squaring both sides of the given equation and using the fact that $(a+b)^2 = a^2 + 2ab + b^2$ leads to

$$(\sqrt{1-x})^2 + 2\sqrt{1-x}\sqrt{1+x} + (\sqrt{1+x})^2 = 2.012.$$

The left-hand side then simplifies to

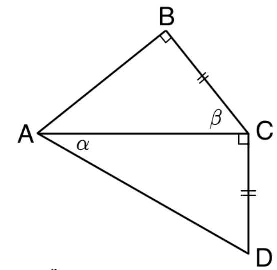
$$1 - x + 2\sqrt{(1-x)(1+x)} + 1 + x = 2.012.$$

Combining like terms and multiplying under the square root gives

$$2\sqrt{1-x^2} = 0.012 \quad \Longrightarrow \quad \sqrt{1-x^2} = 0.006.$$

Squaring again yields $1 - x^2 = 0.000036$, and rearranging finally brings us to the answer of $x^2 = \mathbf{0.999964}$.

4. An accurate diagram is reproduced at right. The two right angles are marked, as are angles α and β. Furthermore, since $m\angle CBD = m\angle CDB$ we deduce that $\triangle CBD$ is isosceles, meaning that $CB = CD$, so these congruent sides are also marked in the diagram. It now becomes clear that the ratio CD/AC in $\triangle ACD$ will be the same as the ratio BC/AC in $\triangle ABC$, since $CB = CD$. The trigonometric functions corresponding to these ratios are $\tan\alpha$ and $\cos\beta$, therefore we may conclude that $\mathbf{\tan\alpha = \cos\beta}$.



5. Suppose that we combine m cups of light red paint with n cups of pink paint. Since light red is $\frac{1}{5}$ white paint, while pink is $\frac{4}{7}$ white paint, we would have $\frac{1}{5}m + \frac{4}{7}n$ cups of white paint present in the mixture. Using the same reasoning, we would have $\frac{4}{5}m + \frac{3}{7}n$ cups of red paint. Therefore the ratio of white to red paint overall would be

$$\frac{\frac{1}{5}m + \frac{4}{7}n}{\frac{4}{5}m + \frac{3}{7}n} = \frac{7m + 20n}{28m + 15n}.$$

We want this expression to equal $\frac{5}{6}$. Cross-multiplying leads to the equality $42m + 120n = 140m + 75n$, or $45n = 98m$, so $\frac{m}{n} = \frac{45}{98}$. Therefore we should use a ratio $m\!:\!n$ of $45\!:\!98$ of white to red paint.

6. Let V, E and R be the number of vertices (points), edges (segments), and regions, including the region outside the polygon. Note that all regions are triangular, except for the outer region. Therefore if we were to cut out all the regions there would be $3(R-1) + 100 = 3R + 97$ edges in total. But every original edge contributes two edges to this total, since we cut along the edges, hence $3R + 97 = 2E$. Next recall that Euler's formula states that $V - E + R = 2$. Combining these two equalities to eliminate E gives $R + 101 = 2V$. Finally, observe that the average number of edges per vertex is $2E/V$, since each edge is counted

twice when adding up edges around each vertex. Thus $2E/V \leq 5$, or $2E \leq 5V$. Substituting in for E and V in terms of R leads to

$$3R + 97 \leq 5(\tfrac{1}{2}R + \tfrac{101}{2}) \quad \Longrightarrow \quad 6R + 194 \leq 5R + 505,$$

hence $R \leq 311$. We're not interested in the outer region, just the triangular ones, resulting in a maximum possible of **310** triangles.

7. There are 2^{28} outcomes possible when flipping 28 coins. Of these, precisely

$$\binom{28}{0} + \binom{28}{4} + \binom{28}{8} + \cdots + \binom{28}{28}$$

result in the number of heads being a multiple of 4. The challenge is to evaluate this sum. Fortunately, there is a handy shortcut that involves complex numbers. Observe that by the Binomial Theorem

$$\begin{aligned}
(1+1)^{28} &= \binom{28}{0} + \binom{28}{1} + \binom{28}{2} + \binom{28}{3} + \binom{28}{4} + \binom{28}{5} + \cdots, \\
(1-1)^{28} &= \binom{28}{0} - \binom{28}{1} + \binom{28}{2} - \binom{28}{3} + \binom{28}{4} - \binom{28}{5} + \cdots, \\
(1+i)^{28} &= \binom{28}{0} + \binom{28}{1}i - \binom{28}{2} - \binom{28}{3}i + \binom{28}{4} + \binom{28}{5}i + \cdots, \\
(1-i)^{28} &= \binom{28}{0} - \binom{28}{1}i - \binom{28}{2} + \binom{28}{3}i + \binom{28}{4} - \binom{28}{5}i + \cdots.
\end{aligned}$$

Adding together all four equalities causes every binomial coefficient except those involving multiples of four to cancel out, leaving

$$2^{28} + 0^{28} + (1+i)^{28} + (1-i)^{28} = 4\binom{28}{0} + 4\binom{28}{4} + \cdots + 4\binom{28}{28}.$$

But the complex number $(1+i)$ has length $\sqrt{2}$ and polar angle $\pi/4$, so by de Moivres Theorem $(1+i)^{28} = -2^{14}$. In the same way we find that $(1-i)^{28} = -2^{14}$ also. Dividing through by 4 now reveals that

$$\binom{28}{0} + \binom{28}{4} + \cdots + \binom{28}{28} = 2^{26} - 2^{12} - 2^{12},$$

for a probability of $(2^{26} - 2^{13})/2^{28} = (2^{13} - 1)/2^{15} = \mathbf{8191/32768}$.

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★ National Level ★

The Mandelbrot Competition

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