Triangle Geometry

Ivy Ren

March 7, 2013

1 Angle Bisector Theorem

On the triangle ABC, let D be the intersection of angle bisector from A on side BC. Then, the theorem states that $\frac{AB}{AC} = \frac{DB}{DC}$.

2 Multiple Centers of a Triangle

- *Centroid*: The medians of a triangle are concurrent at a point called the *centroid* of the triangle. The distance from the centroid to the opposite side is always half of the distance from the centroid to the vertex.
- *Orthocenter*: The three altitudes of a triangle are concurrent at a point called *orthocenter* of the triangle.
- *Circumcenter*: All three perpendicular bisectors of a triangle are concurren at a point called *circumcenter* of the triangle. This is the center of the *circumscribed circle*.
- *Incenter*: All three angle bisectors are concurrent at a point called *Incenter* of the triangle. This is the center of the *inscribed circle*.
- *Euler Line*: In any triangle, the centroid G is in the line segment connecting the orthocenter H to the circumcenter O, such that $2OG = GH$.

3 Trigonometry in Triangle Geometry

- *Law of Sine*: $\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C} = 2R$, where R is the radius of the circumcenter.
- *Law of Cosine*: $c^2 = a^2 + b^2 - 2ab \cos \angle C$. This is an equation you should commit to memory since it appears everywhere. You can use it to find difficult side lengths and angles.
- *Law of Tangents*: In $\triangle ABC$ with $a = BC$ and $b = AC$, $\frac{a-b}{a+b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$.
- *Ceva's Theorem*: Let D, E, and F be points on BC, AC, and AB, respectively. If and only if AD, BE, CF are concurrent, $\frac{DB}{DC}\frac{EC}{EA}\frac{FA}{FB} = 1$.

4 Finding Area of Triangles

- *Area of Triangle using Sine Formula*: $S_{\triangle ABC} = \frac{1}{2}ab \sin \angle C$.
- *Heron's Formula*: $S_{\triangle ABC} = \sqrt{s(s-a)(s-b)(s-c)}$, where $(s = \frac{a+b+c}{2})$.

5 Examples:

1. Prove the Angle Bisector theorem.

2. In triangle ABC, $\angle A$ and $\angle B$ measure 60° and 45°, respectively. The bisector of $\angle A$ intersects side BC at T, and $AT = 24$. Find the area of the triangle ABC.

3. Let $ABCD$ be a trapezoid with $AB||CD, AB = 92, BC = 50, CD = 19, DA = 70$. Point P lies on the side of AB such that a circle centered at P touches AD and BC. Find AP.

4. In triangle ABC, $AB = 14, BC = 16$, and $AC = 26$. Let M be the midpoint of side BC, and let D be a point on segment BC such that AD bisects $\angle BAC$. Compute PM, where P is the foot of perpendicular from B to line AD.

6 Problems

1. In triangle ABC, $a > b > c$. If
$$\frac{a^3 + b^3 + c^3}{\sin^3 A + \sin^3 B + \sin^3 C} = 7,$$
compute the maximum value of a.

2. Show that in $\triangle ABC$,
$$\sin \frac{\angle A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

3. Prove: if P and Q are points on sides AB and AC, respectively, of triangle ABC so that PQ is parallel to BC and if X is the point of intersection of BQ and CP then AX goes through the midpoint of BC.

4. Equilateral triangle ABC is inscribed in circle O. Point P lies on minor arc $\widehat{BC}$. Segment AP and BC meet at D. Given that $BP = 21$ and $CP = 28$, compute PD.

5. Let ABC be a triangle and let P be a point in its interior. Let AX, BY, CZ go through P and intersect sides BC, CA, AB at points X, Y, Z, respectively. Suppose that $PX = 5, PY = 6, PZ = 7, AP = 10$, and $BP = 9$. Compute CP.